Abelian color cycles: a new approach to strong coupling expansion and dualization of non-abelian lattice field theories

christof.gattringer@uni-graz.at

Work done together with Carlotta Marchis

Introductory comments

Solving the complex action problem with dual variables

- In recent years the complex action problem of several lattice field theories at finite density was completely solved by exactly mapping the partition sum to a dual representation.

- The dual variables are loops for matter fields and world sheets for gauge fields.

- All terms in the dual partition sum are real and positive and Monte Carlo simulations are done in terms of the dual variables.

- All abelian gauge Higgs theories and several spin systems were successfully dualized, but so far there exists no convincing dualization for non-abelian gauge fields.

- A key problem is the re-ordering of the non-abelian gauge links after the expansion of the gauge action Boltzmann factor.

- Here we propose a new approach: Decompose the gauge action into abelian color cycles (= paths through color space along plaquettes) that solve the re-ordering problem.

How does dualization of U(1) LGT work?

- Partition sum:

$$Z \;=\; \int D[U] \, e^{\,\beta \sum_{x,\mu<\nu} \text{Re}\, U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^\star_{x+\hat{\nu},\mu}\, U^\star_{x,\nu}} \quad , \quad \int D[U] \;=\; \prod_{x,\mu} \int_{U(1)} dU_{x,\mu}$$

- Expansion of the Boltzmann factor:

$$Z = \int D[U] \prod_{x,\mu<\nu} e^{\,\frac{\beta}{2}\, U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^\star_{x+\hat{\nu},\mu}\, U^\star_{x,\nu}} \; e^{\,\frac{\beta}{2}\, U^\star_{x,\mu}\, U^\star_{x+\hat{\mu},\nu}\, U_{x+\hat{\nu},\mu}\, U_{x,\nu}} \;=$$

$$\int D[U] \prod_{x,\mu<\nu} \sum_{p_{x,\mu\nu}} \sum_{\overline{p}_{x,\mu\nu}} \frac{\left(\frac{\beta}{2}\right)^{p_{x,\mu\nu}+\overline{p}_{x,\mu\nu}}}{p_{x,\mu\nu}!\; \overline{p}_{x,\mu\nu}!} \Big(U_{x,\mu} U_{x+\hat{\mu},\nu} U^\star_{x+\hat{\nu},\mu} U^\star_{x,\nu} \Big)^{p_{x,\mu\nu}} \Big(U^\star_{x,\mu} U^\star_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu} U_{x,\nu} \Big)^{\overline{p}_{x,\mu\nu}}$$

- Reordering the terms (use: $U^\star_{x,\mu} = U^{-1}_{x,\mu}$):

$$Z \;=\; \sum_{\{p,\overline{p}\}} \; \prod_{x,\mu<\nu} \frac{\left(\frac{\beta}{2}\right)^{p_{x,\mu\nu}+\overline{p}_{x,\mu\nu}}}{p_{x,\mu\nu}!\; \overline{p}_{x,\mu\nu}!} \; \prod_{x,\mu} \int_{U(1)} dU_{x,\mu} \, \Big(U_{x,\mu} \Big)^{\sum_{\nu:\mu<\nu} [d_{x,\mu\nu} - d_{x-\hat{\nu},\mu\nu}] - \sum_{\rho:\rho<\mu} [d_{x,\rho\mu} - d_{x-\hat{\rho},\rho\mu}]}$$

$$d_{x,\mu\nu} \equiv p_{x,\mu\nu} - \overline{p}_{x,\mu\nu}$$

How does dualization of U(1) LGT work?

- Integrating out the gauge fields ($\int dU\, U^n = \delta(n)$):

$$Z = \sum_{\{p,\overline{p}\}} \prod_{x,\mu<\nu} \frac{\left(\frac{\beta}{2}\right)^{p_{x,\mu\nu}+\overline{p}_{x,\mu\nu}}}{p_{x,\mu\nu}!\,\overline{p}_{x,\mu\nu}!} \prod_{x,\mu} \delta(\, J_{x,\mu}\,)$$

$$J_{x,\mu} = \sum_{\nu:\mu<\nu} [d_{x,\mu\nu} - d_{x-\hat{\nu},\mu\nu}] - \sum_{\rho:\rho<\mu} [d_{x,\rho\mu} - d_{x-\hat{\rho},\rho\mu}]\,, \quad d_{x,\mu\nu} \equiv p_{x,\mu\nu} - \overline{p}_{x,\mu\nu}$$

The partition function is exactly rewritten into a sum over configurations of the plaquette occupation numbers $p_{x,\mu\nu}, \overline{p}_{x,\mu\nu} \in \mathbb{N}_0$, which obey constraints giving rise to an interpretation as a sum over worldsheets.

- Key to success was the reordering of the abelian gauge links:

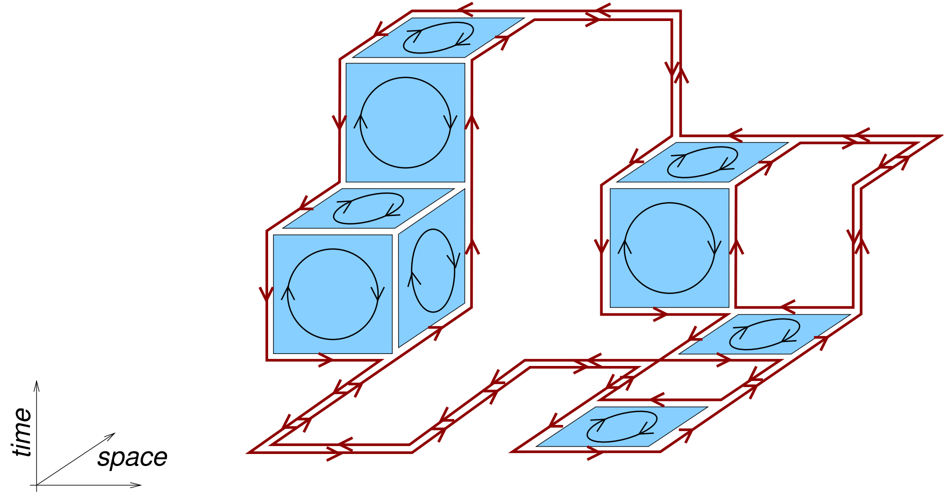
$$\prod_{x,\mu<\nu} \Big(U_{x,\mu}U_{x+\hat{\mu},\nu}U^\star_{x+\hat{\nu},\mu}U^\star_{x,\nu}\Big)^{p_{x,\mu\nu}} \Big(U^\star_{x,\mu}U^\star_{x+\hat{\mu},\nu}U_{x+\hat{\nu},\mu}U_{x,\nu}\Big)^{\overline{p}_{x,\mu\nu}}$$

$$= \prod_{x,\mu} \Big(U_{x,\mu}\Big)^{\sum_{\nu:\mu<\nu}[d_{x,\mu\nu}-d_{x-\hat{\nu},\mu\nu}]-\sum_{\rho:\rho<\mu}[d_{x,\rho\mu}-d_{x-\hat{\rho},\rho\mu}]}$$

Not possible for non-abelian theories!

Dual variables = worldsheets coupled to matter loops

Matter fields appear as loops that serve as boundaries for the gauge worldsheets.
Chemical potential couples to the temporal winding number of the loops.



Delgado Mercado, Gattringer, Schmidt. Phys.Rev.Lett. 111 (2013) 141601

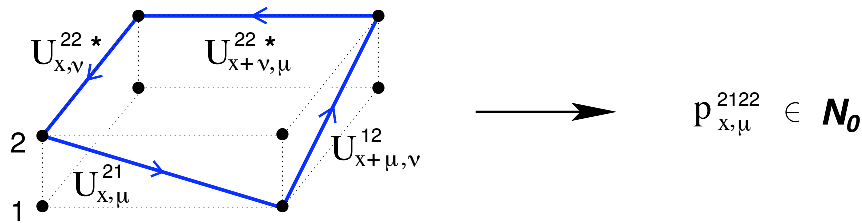
Abelian color cycles (ACC)

Decomposition of the non-abelian action into abelian color cycles:

- Action for SU(2) lattice gauge theory ($U_{x,\mu} \in \mathrm{SU}(2)$) :

$$S \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu} \mathrm{Tr}\; U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{\dagger}_{x+\hat{\nu},\mu}\, U^{\dagger}_{x,\nu} \;=\; -\frac{\beta}{2}\sum_{x,\mu<\nu} \;\sum_{a,b,c,d=1}^{2} U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$$

- The products $U^{ab}_{x,\mu}\, U^{bc}_{x+\hat{\mu},\nu}\, U^{dc\,\star}_{x+\hat{\nu},\mu}\, U^{ad\,\star}_{x,\nu}$ are the abelian color cycles (ACC) (= paths through color space along plaquettes) we use for expanding the Boltzmann factor. Example:



- Suitable parameterization:

$$U_{x,\mu} \;=\; \begin{bmatrix} \cos\theta_{x,\mu}\; e^{i\alpha_{x,\mu}} & \sin\theta_{x,\mu}\; e^{i\beta_{x,\mu}} \\ -\sin\theta_{x,\mu}\; e^{-i\beta_{x,\mu}} & \cos\theta_{x,\mu}\; e^{-i\alpha_{x,\mu}} \end{bmatrix} \qquad \theta_{x,\mu} \in [0, \pi/2]\;, \quad \alpha_{x,\mu}, \beta_{x,\mu} \in [-\pi, \pi]$$

Expansion in ACCs

- Partition sum:

$$Z = \int D[U] \; \exp\left(\frac{\beta}{2} \sum_{x,\mu<\nu} \sum_{a,b,c,d} U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu} \right) \; , \quad \int D[U] \; = \; \prod_{x,\mu} \int_{SU(2)} dU_{x,\mu}$$

- Expansion of the Boltzmann factor:

$$\begin{aligned} Z \;&=\; \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} e^{\frac{\beta}{2} U^{ab}_{x,\mu} U^{bc}_{x+\hat{\mu},\nu} U^{dc\,\star}_{x+\hat{\nu},\mu} U^{ad\,\star}_{x,\nu}} \\ &=\; \int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} \; \sum_{p^{abcd}_{x,\mu\nu}=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}\,!} \left(U^{ab}_{x,\mu} \, U^{bc}_{x+\hat{\mu},\nu} \, U^{dc\,\star}_{x+\hat{\nu},\mu} \, U^{ad\,\star}_{x,\nu} \right)^{p^{abcd}_{x,\mu\nu}} \end{aligned}$$

- Reordering the terms:

$$Z \; = \sum_{\{p\}} \prod_{x,\mu<\nu} \prod_{a,b,c,d} \frac{\left(\frac{\beta}{2}\right)^{p^{abcd}_{x,\mu\nu}}}{p^{abcd}_{x,\mu\nu}\,!} \prod_{x,\mu} \int d_H[\theta_{x,\mu}, \alpha_{x,\mu}, \beta_{x,\mu}] \prod_{ab} \left(U^{ab}_{x,\mu} \right)^{N^{ab}_{x,\mu}[p]} \left(U^{ab\,\star}_{x,\mu} \right)^{\overline{N}^{ab}_{x,\mu}[p]}$$

Remaining link integrals can be solved and give constraints and weights for the configurations $\{p\}$ of the cycle occupation numbers $p^{abcd}_{x,\mu\nu} \in \mathbb{N}_0$.

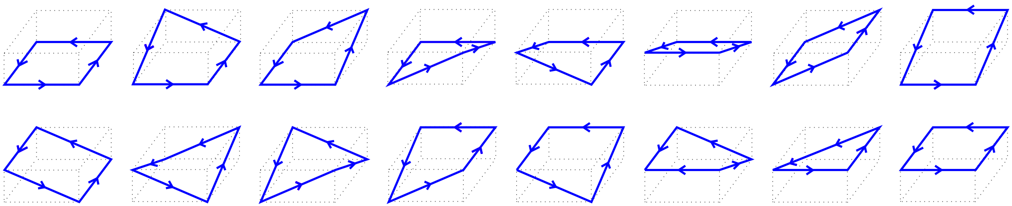
Partition function as sum over occupation numbers of ACCs

- Dual partition sum:

$$Z = \sum_{\{p\}} W_\beta[p] \, (-1)^{\sum_{x,\mu} J^{21}_{x,\mu}} \prod_{x,\mu<\nu} \delta\left(J^{11}_{x,\mu} - J^{22}_{x,\mu}\right) \, \delta\left(J^{12}_{x,\mu} - J^{21}_{x,\mu}\right)$$

$$J^{ab}_{x,\mu} = \text{ total flux from } a \text{ to } b \text{ along the link } x,\mu$$

- 16 possible ACCs that can be occupied (i.e., $p^{abcd}_{x,\mu\nu} > 0$):



- Constraints at each link:

Adding matter

Staggered fermions in an SU(2) background

- Fermionic partition sum:

$$Z_F[U] = \int D[\,\overline{\psi}, \psi]\; e^{-S_F[\,\overline{\psi},\psi,U]}$$

$$\overline{\psi}_x = (\,\overline{\psi}_x^{\,1}, \overline{\psi}_x^{\,2}\,)\;, \quad \psi_x = \begin{pmatrix} \psi_x^1 \\ \\ \psi_x^2 \end{pmatrix}$$

- Action and its decomposition into color bilinears:

$$S_F[\,\overline{\psi}, \psi, U] = \sum_x \Big[m\,\overline{\psi}_x \psi_x + \sum_\mu \frac{\gamma_{x,\mu}}{2} \Big(\overline{\psi}_x U_{x,\mu} \psi_{x+\hat{\mu}} - \overline{\psi}_{x+\hat{\mu}} U^\dagger_{x,\mu} \psi_x \Big) \Big]$$

$$= \sum_x \Big[\, m \sum_a \overline{\psi}_x^{\,a} \psi_x^a + \sum_\mu \frac{\gamma_{x,\mu}}{2} \sum_{a,b} \Big(\overline{\psi}_x^{\,a} U_{x,\mu}^{ab} \psi_{x+\hat{\mu}}^b - \overline{\psi}_{x+\hat{\mu}}^{\,b} U_{x,\mu}^{ab\,\star} \psi_x^a \Big) \Big]$$

Loop expansion

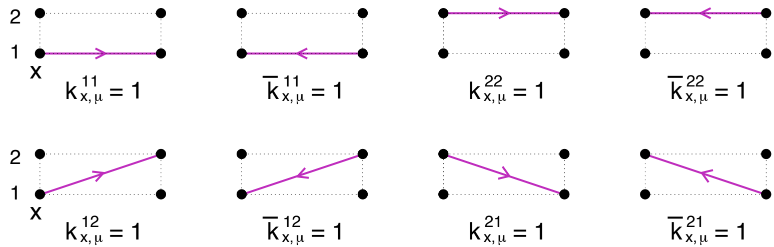
- Expanding the Boltzmann factors in the fermionic partition sum:

$$
\begin{aligned}
Z_F[U] &= \int D[\overline{\psi},\psi] \prod_x \prod_a e^{-m\overline{\psi}_x^a \psi_x^a} \prod_{x,\mu} \prod_{a,b} e^{-\frac{\gamma_{x,\mu}}{2}\overline{\psi}_x^a U_{x,\mu}^{ab} \psi_{x+\hat{\mu}}^b} \, e^{\frac{\gamma_{x,\mu}}{2}\overline{\psi}_{x+\hat{\mu}}^b U_{x,\mu}^{ab\star} \psi_x^a} \\
&= \int D[\overline{\psi},\psi] \prod_x \prod_a \sum_{s_x^a=0}^{1} (-m\overline{\psi}_x^a \psi_x^a)^{s_x^a} \\
&\quad \times \prod_{x,\mu} \prod_{a,b} \sum_{k_{x,\mu}^{ab}=0}^{1} (-\frac{\gamma_{x,\mu}}{2}\overline{\psi}_x^a U_{x,\mu}^{ab} \psi_{x+\hat{\mu}}^b)^{k_{x,\mu}^{ab}} \sum_{\overline{k}_{x,\mu}^{ab}=0}^{1} (\frac{\gamma_{x,\mu}}{2}\overline{\psi}_{x+\hat{\mu}}^b U_{x,\mu}^{ab\star} \psi_x^a)^{\overline{k}_{x,\mu}^{ab}} \\
&= \frac{1}{2^{2V}} \sum_{\{s,k,\overline{k}\}} (2m)^{\sum_{x,a} s_x^a} \prod_{x,\mu} \prod_{a,b} (U_{x,\mu}^{ab})^{k_{x,\mu}^{ab}} \, (U_{x,\mu}^{ab\star})^{\overline{k}_{x,\mu}^{ab}} \\
&\quad \times \int D[\overline{\psi},\psi] \prod_x \prod_a (\overline{\psi}_x^a \psi_x^a)^{s_x^a} \prod_{x,\mu} \prod_{a,b} (-\gamma_{x,\mu}\overline{\psi}_x^a \psi_{x+\hat{\mu}}^b)^{k_{x,\mu}^{ab}} (\gamma_{x,\mu}\overline{\psi}_{x+\hat{\mu}}^b \psi_x^a)^{\overline{k}_{x,\mu}^{ab}}
\end{aligned}
$$

The Grassmann integral is saturated by monomers ($s_x^a = 1$), dimers ($k_{x,\mu}^{ab} = \overline{k}_{x,\mu}^{ab} = 1$) and loops of $k_{x,\mu}^{ab} = 1$ and $\overline{k}_{x,\mu}^{ab} = 1$. Only loops introduce signs!

Interaction with the gauge fields

- $k_{x,\mu}^{ab}$ and $\overline{k}_{x,\mu}^{ab}$ introduce color flux on the links:



- Full partition sum:

$$Z \;=\; \sum_{\{p,k,\overline{k},s\}} C_{MDL}[s,k,\overline{k}\,]\; W_\beta[p]\, W_m[s] \prod_{x,\mu} (-1)^{\,J_{x,\mu}^{21}+k_{x,\mu}^{21}+\overline{k}_{x,\mu}^{21}} \prod_{L} \text{sign}\,(L)$$

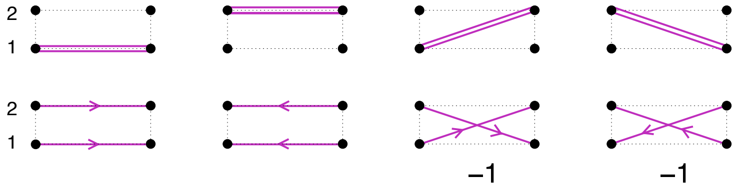
$$\times \prod_{x,\mu<\nu} \delta\Big(J_{x,\mu}^{11}+k_{x,\mu}^{11}-\overline{k}_{x,\mu}^{11} - [J_{x,\mu}^{22}+k_{x,\mu}^{22}-\overline{k}_{x,\mu}^{22}]\Big)\; \delta\Big(J_{x,\mu}^{12}+k_{x,\mu}^{12}-\overline{k}_{x,\mu}^{12} - [J_{x,\mu}^{21}+k_{x,\mu}^{21}-\overline{k}_{x,\mu}^{21}]\Big)$$

$$\text{sign}\,(L) \;=\; -\,(-1)^{\#\,plaquettes}\,(-1)^{length/2}\,(-1)^{temp.winding}$$

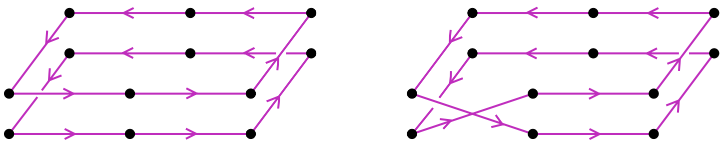
- Gauge constraints:

Strong coupling loops ($\beta = 0$)

- Strong coupling: $\beta = 0 \Rightarrow$ only fermion lines. The gauge constraints limit the number of admissible link elements:



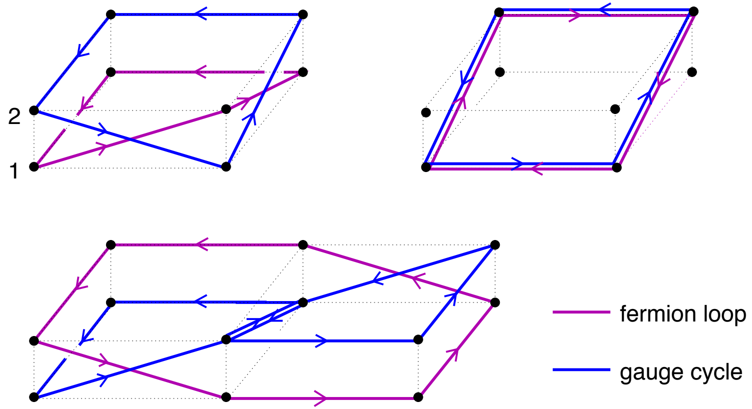
- Signs from color flips compensate fermion loop signs:



- In the strong coupling limit all contributions are positive.

Loops at $\beta > 0$

- At $\beta > 0$ one can activate gauge cycles to satisfy the gauge constraints for more general fermion loops. Examples of $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^2)$:



- One can show that all contributions up to $\mathcal{O}(\beta^3)$ are positive. From $\mathcal{O}(\beta^4)$ some configurations with negative signs appear. Chemical potential couples to total temporal winding number.

Summary

- The action of non-abelian gauge fields is decomposed into abelian color cycles (ACC) (= paths through color space along plaquettes).
- Expanding the Boltzmann factor introduces an occupation number for each ACC.
- The link contributions to the ACCs are $\mathbb{C}$-valued and the re-ordering problem is solved.
- The original link-degrees of freedom can be integrated out in closed form. This generates weights, constraints and signs for configurations of ACC occupation numbers.
- The ACC construction can be generalized by including matter fields.
- Weights for all terms of the strong coupling expansion are known in closed form.
- Up to $\mathcal{O}(\beta^3)$ only positive terms contribute.
- Generalization of the abelian color cycle decomposition to other gauge groups is possible.