Static and dynamical decoherence
Why CTP?

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Traditional use of QFT: transition amplitudes between pure states
scattering amplitude (Feynman,...)

Closed Time Path formalism: expectation values in mixed states
time dependent observables (Schwinger, Keldysh,...)

$\nwarrow$

needed for **open systems** as well: - classical limit
environment $\Longrightarrow$ decoherence $\leftarrow$
- renormalization S. Nagy
elimination $\Longrightarrow$ mixed states

1. Decoherence
2. CTP formalism
3. Dynamical, static and instantaneous decoherence
4. Effective Lagrangian of a test particle in an ideal gas
5. Static decoherence, time scales
6. Dynamical decoherence of harmonic systems
7. Summary

Decoherence

Necessary condition of the classical limit

System and its environment: $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_e$

Preferred system basis: $\{|\psi_n\rangle\}$, $|\Psi\rangle = \sum_n c_n |\psi_n\rangle \otimes |\chi_n\rangle$, $\langle\chi_n|\chi_{n'}\rangle \neq 0$

Entanglement $\Longrightarrow$mixed state:

$$\begin{aligned}
\rho &= |\Psi\rangle\langle\Psi| = \sum_{nn'} c_n c^*_{n'} |\psi_n\rangle \otimes |\chi_n\rangle\langle\chi_{n'}| \otimes \langle\psi_{n'}| \\
\rho_s &= \mathrm{Tr}_e \rho = \sum_{nn'} c_n c^*_{n'} \langle\chi_{n'}|\chi_n\rangle |\psi_n\rangle\langle\psi_{n'}|
\end{aligned}$$

Decoherence: - suppression of interference (Schrödinger's cat)
- orthogonality of the environment states
- diagonal ρ_s in the preferred basis
- additive expectation value in the preferred basis
  $A = \sum_n |\psi_n\rangle a_n \langle\psi_n| \to A \otimes 1$

$$\begin{aligned}
\langle\Psi|A|\Psi\rangle &= \sum_{nn'} c^*_{n'} c_n \langle\psi_{n'}|A|\psi_n\rangle\langle\chi_{n'}|\chi_n\rangle \\
\text{(full decoherence)} &= \sum_n |c_n|^2 a_n
\end{aligned}$$

Decoherence

Liouville space propagator

$$
\begin{aligned}
\rho(x_f^+, x_f^-; t_f) &= \mathcal{U}(t)\rho(x_i^+, x_i^-; t_i) \\
\text{(closed sys.)} &= \int_{x^\pm(t_f)=x_f^\pm} D[x^+]D[x^-]e^{\frac{i}{\hbar}S[x^+]-\frac{i}{\hbar}S[x^-]}\rho(x^+(t_i), x^-(t_i); t_i) \\
\text{(open sys.)} &= \int_{x^\pm(t_f)=x_f^\pm} D[x^+]D[x^-]e^{\frac{i}{\hbar}S_{eff}[x^+,x^-]}\rho(x^+(t_i), x^-(t_i); t_i)
\end{aligned}
$$

Coupling of x^+ and x^-: - dissipative forces
- system-environment entanglement

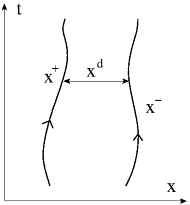
$x^\pm = x + \frac{x^d}{2}$ - x: physical coordinate
- x^d: quantum fluctuations

Decoherence:

Suppression by $e^{-\frac{1}{\hbar}\Im S_{ef}[x,x^d]}$

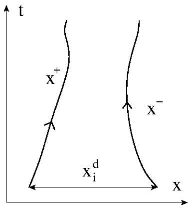
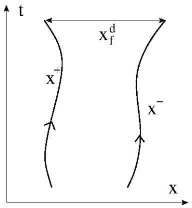
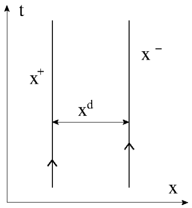
Opposite time arrow for x and x^d

Unstable quantum fluctuations



Decoherence

Dynamical, instantaneous, and static

Dynamical	**Instantaneous**	**Static**
		
Decay of Schrödinger's cat	Time dependent suppression of interference	System dynamics suppressed
ℓ_{dd}, τ_{dd}	ℓ_{id}, τ_{id}	ℓ_{sd}, τ_{sd}

Scales: $e^{-\frac{1}{\hbar}\Im S_{ef}[x,x^d]} \sim e^{-\frac{x^{d2}}{2\ell^2(t)}}$, $\ell = \ell(\frac{t}{\tau})$

$\tau_{sd} \ll \tau_{diss}$ (a collisional model *Joos, Zeh, 1985,...*)

Are dissipation and decoherence really so different processes?

CTP formalism for closed system

(Schwinger 1961, Keldysh, 1964,...)

Generating functional:

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{j}]} &= \mathrm{Tr} T[e^{-\frac{i}{\hbar}\int dt(H(t)-j^+(t)x(t))}]\rho_i T^*[e^{\frac{i}{\hbar}\int dt(H(t)+j^-(t)x(t))}] \\
&= \int D[\hat{x}] e^{\frac{i}{\hbar}S[x^+]-\frac{i}{\hbar}S[x^-]+\frac{i}{\hbar}\int dt\hat{j}(t)\hat{x}(t)} \\
&= \int D[\hat{x}] e^{\frac{i}{\hbar}S[\hat{x}]+\frac{i}{\hbar}\int dt\hat{j}(t)\hat{x}(t)}, \quad \hat{x} = (x^+, x^-)
\end{aligned}
$$

Reduplication of the degrees of freedom: $x^+ \to |\psi\rangle$, $x^- \to \langle\psi|$

Time inversion: $T : \begin{cases} |\psi\rangle \to \langle\psi| \\ x^\pm \to x^\mp \\ S[x^+, x^-] \to -S^*[x^-, x^+] \end{cases}$

CTP formalism for closed system

Free propagator

Harmonic system: $S_0[\hat\phi] = \frac{1}{2}\hat\phi\hat D^{-1}\hat\phi$

Feynman Wightman

$$
\begin{aligned}
i\hat D_{x,y} &= \begin{pmatrix} \langle T[\phi_x\phi_y]\rangle & \langle\phi_y\phi_x\rangle \\ \langle\phi_x\phi_y\rangle & \langle T[\phi_y\phi_x]\rangle^* \end{pmatrix} = i\begin{pmatrix} D^n + iD^i & -D^f + iD^i \\ D^f + iD^i & -D^n + iD^i \end{pmatrix} \\
\hat D_k &= \begin{pmatrix} \frac{1}{k^2-m^2+i\epsilon} & -2\pi i\delta(k^2-m^2)\Theta(-k^0) \\ -2\pi i\delta(k^2-m^2)\Theta(k^0) & -\frac{1}{k^2-m^2-i\epsilon} \end{pmatrix}
\end{aligned}
$$

Physical Green functions: $D_F = D^n + iD^i$, $D^{\overset{r}{a}} = D^n \pm D^f$

Off-shell: D^n, on-shell: D^f, D^i

CTP formalism for open system

(Wilsonian) Effective theory

Bare action: $S[x,y] = S_s[x] + S_e[x,y]$

$$e^{\frac{i}{\hbar}W[\hat{j}]} = \int D[\hat{x}]D[\hat{y}]e^{\frac{i}{\hbar}S[x,y]+\frac{i}{\hbar}\hat{x}\hat{j}} = \int D[\hat{x}]e^{\frac{i}{\hbar}S_{eff}[\hat{x}]+\frac{i}{\hbar}\hat{x}\hat{j}}$$

Effective action:

$$e^{\frac{i}{\hbar}S_{eff}[\hat{x}]} = e^{\frac{i}{\hbar}S_s[x^+]-\frac{i}{\hbar}S_s[x^-]}\int D[\hat{y}]e^{\frac{i}{\hbar}S_{se}[x^+,y^+]-\frac{i}{\hbar}S_{se}[x^-,y^-]}$$

Influence functional: *(Feynman, Vernon 1963)*

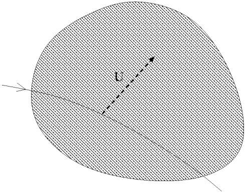
$$\begin{aligned} S_{eff}[\hat{x}] &= S_s[x^+] - S_s[x^-] + S_{infl}[\hat{x}] \\ &= S_1[x^+] - S_1[x^-] + S_2[x^+,x^-],\ S_2[0,x] = S_2[x,0] = 0 \end{aligned}$$

S_1: closed (conservative) interactions for quasi-particles
S_2: open (nonconservative) interactions: - dissipation
- entanglement
- decoherence

Ideal gas environment

Model

Particle interacting with the ideal gas by the potential $U(\boldsymbol{x})$:



$$S[\boldsymbol{x},\psi^\dagger,\psi] = S_p[\boldsymbol{x}] + S_g[\psi^\dagger,\psi] + S_i[\boldsymbol{x},\psi^\dagger,\psi]$$

$$\begin{aligned}
S_p[\boldsymbol{x}] &= \int dt \left[\frac{M}{2}\dot{\boldsymbol{x}}^2(t) - V(\boldsymbol{x}(t))\right] \\
S_g[\psi^\dagger,\psi] &= \int dt d^3y \psi^\dagger(t,\boldsymbol{y}) \left[i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta + \mu\right]\psi(t,\boldsymbol{y}) \\
S_i[\boldsymbol{x},\psi^\dagger,\psi] &= \int dt d^3y U(\boldsymbol{y}-\boldsymbol{x}(t))\psi^\dagger(t,\boldsymbol{y})\psi(t,\boldsymbol{y}) = \psi^\dagger\Gamma[\boldsymbol{x}]\psi
\end{aligned}$$

Ideal gas environment

Generating functional

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{\boldsymbol{j}}]} &= \int D[\hat{\boldsymbol{x}}]D[\hat{\psi}]D[\hat{\psi}^\dagger]e^{\frac{i}{\hbar}S_p[\hat{\boldsymbol{x}}]+\frac{i}{\hbar}\hat{\psi}^\dagger(\hat{F}^{-1}+\hat{\Gamma}[\hat{\boldsymbol{x}}])\hat{\psi}+\frac{i}{\hbar}\int dt\hat{\boldsymbol{j}}(t)\hat{\boldsymbol{x}}(t)} \\
&= \int D[\hat{\boldsymbol{x}}]e^{\frac{i}{\hbar}S_p[\hat{\boldsymbol{x}}]+\frac{i}{\hbar}S_{infl}[\hat{\boldsymbol{x}}]+\frac{i}{\hbar}\int dt\hat{\boldsymbol{j}}(t)\hat{\boldsymbol{x}}(t)}
\end{aligned}
$$

Influence functional:

$$
\begin{aligned}
e^{\frac{i}{\hbar}S_{infl}[\hat{\boldsymbol{x}}]} &= \int D[\hat{\psi}]D[\hat{\psi}^\dagger]e^{\frac{i}{\hbar}\hat{\psi}^\dagger(\hat{F}^{-1}+\hat{\Gamma}[\hat{\boldsymbol{x}}])\hat{\psi}} \\
S_{infl}[\hat{\boldsymbol{x}}] &= -i\hbar\mathrm{Tr}\ln[\hat{F}^{-1}+\hat{\Gamma}] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'=\pm}\sigma\sigma' j^\sigma G^{\sigma\sigma'} j^{\sigma'} + \mathcal{O}\left(\hat{j}^3\right)
\end{aligned}
$$

$\hat{G} =$  , $\qquad j^\sigma(t,\boldsymbol{y}) = U(\boldsymbol{y}-\boldsymbol{x}^\sigma(t))$

Ideal gas environment

Effective Lagrangian in $\mathcal{O}(\hbar)$, $\mathcal{O}\left(x^2\right)$, $\mathcal{O}\left(\partial_t^2\right)$ for an ideal fermi gas

$$S_{infl}[\hat{\boldsymbol{x}}] = -\frac{1}{2}\sum_{\sigma\sigma'=\pm}\sigma\sigma' j^{\sigma} G^{\sigma\sigma'} j^{\sigma'} + \mathcal{O}\left(\hat{j}^3\right)$$

$$\hat{G} =$$  $$, \quad j^{\sigma}(t,\boldsymbol{y}) = U(\boldsymbol{y} - \boldsymbol{x}^{\sigma}(t))$$

$$\begin{aligned}
L_{eff} &= \boldsymbol{x}^d\left[-m\ddot{\boldsymbol{x}} - \delta m\ddot{\boldsymbol{x}} - k\dot{\boldsymbol{x}} + id_0\boldsymbol{x}^d - id_2\ddot{\boldsymbol{x}}^d\right] \\
\delta m &= \frac{1}{48\pi^2 v_F^2}\int dk U_k^2 \partial_{ix}^2 G_{0k}^n(x,y) \qquad \leftarrow \quad \text{mass ren.} \\
k &= \frac{1}{24\pi^2 v_F}\int dk k U_k^2 \partial_{ix} G_{0k}^f(x,y) \qquad \leftarrow \quad \text{friction} \\
d_0 &= -\frac{1}{48\pi^2}\int dk k^2 U_k^2 G_{0k}^i(x,y) \qquad \leftarrow \quad \text{decoherence} \\
d_2 &= \frac{1}{96\pi^2 v_F^2}\int dk U_k^2 \partial_{ix}^2 G_{0k}^i(x,y) \qquad \leftarrow \quad \text{decoherence} \\
0 &= \langle \boldsymbol{x}^d \rangle \quad \leftarrow \quad \text{identity}, \qquad m_R\langle\ddot{\boldsymbol{x}}\rangle = -k\langle\dot{\boldsymbol{x}}\rangle \quad \leftarrow \quad \delta\boldsymbol{x}^d
\end{aligned}$$

Ideal gas environment

Effective Lagrangian in $\mathcal{O}(\hbar)$, $\mathcal{O}\left(\partial_t^2 x\right)$, $\mathcal{O}\left(\partial_t^0 x^d\right)$ for an ideal fermi gas

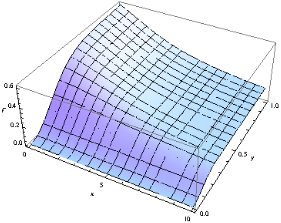
$$
\begin{aligned}
L_{eff} &= \boldsymbol{x}^d\left[-m\ddot{\boldsymbol{x}} - k\dot{\boldsymbol{x}} - \delta m\ddot{\boldsymbol{x}} + iU_d(\boldsymbol{x}) - id_2\ddot{\boldsymbol{x}}^d\right] \\
U_d(\boldsymbol{x}^d) &= \frac{1}{2\pi^2}\int_0^\infty dq q^2\left(\frac{\sin|\boldsymbol{x}^d|q}{q|\boldsymbol{x}^d|} - 1\right)\Gamma^i_{0q},
\end{aligned}
$$

qualitative similarity with the collisional model of Joos & Zeh

Decoherence: $e^{-\frac{1}{\hbar}\Im S_{ef}[x,x^d]}$

$$
\tau_{sd}(\boldsymbol{x}^d) = \frac{\hbar}{U_d(\boldsymbol{x}^d)}.
$$

$$
\begin{aligned}
\frac{\tau_{diss}}{\tau_{sd}(\boldsymbol{x}^d)} &= F\left(\frac{\epsilon_F}{k_B T}, \frac{|\boldsymbol{x}^d|}{\lambda_T}\right) < 1 \\
F(x,y) &= \frac{3}{4}\frac{\int_0^\infty du \frac{1-\frac{\sin 4\sqrt{\pi u}y}{4\sqrt{\pi u}y}}{1+e^{u-x}}}{\int_0^\infty du \frac{u}{1+e^{u-x}}}
\end{aligned}
$$



Limits of the static decoherence scenario

Effective theory: perturbation expansion

$$\frac{\text{decoherence}}{\text{kinetic energy}} = \frac{\frac{\hbar}{\tau_{sd}}}{k_B T} \quad \rightarrow \quad \frac{\hbar}{k_B T} \sim \frac{10^{-11}}{T[K]} < \tau_{sd}$$

Collisional model:

(i) Finite time step, Δt, in deriving the master equation:

$\Delta\rho < \rho$ during the build up of decoherence: $\tau_{coll} = \dfrac{r_0}{v_{env}} < \tau_{sd}$

- Ideal fermi gas: $\dfrac{m}{\hbar k_F^2} < \tau_{sd}$
- Air at room temperature: $\dfrac{3 \times 10^{-7} cm}{10^5 m/s} = 10^{-12} s < \tau_{sd}$

(ii) Multiple scatterings ignored: $\ell_{mfp} = \dfrac{1}{\sigma_{tot} n_g} < r_0 \Longrightarrow |\boldsymbol{x}^d| < r_0$

Dynamical decoherence

Harmonic model

Saddle point expansion (exact):

$$
\begin{aligned}
\rho(\hat{x}_f; t_f) &= \int_{\hat{x}(t_f)=\hat{x}_f} D[\hat{x}] e^{\frac{i}{\hbar} S_{eff}[\hat{x}]} \rho(\hat{x}(t_i); t_i) \\
&\approx \mathcal{N} e^{\frac{i}{\hbar} S_{eff}(\hat{x}_f, \hat{x}_i, T)}
\end{aligned}
$$

Effective Lagrangian:

$$
L_{eff} = x^d \left[-m\ddot{x} - k\dot{x} - m\omega_0^2 x + i d_0 x^d - i d_2 \ddot{x}^d \right]
$$

E.O.M.

$$
\begin{aligned}
\delta x^d : \quad m\ddot{x} &= -m_0\omega_0^2 x - k\dot{x} + i(d_0 x^d - d_2 \ddot{x}^d) \\
\delta x : \quad m\ddot{x}^d &= -m_0\omega_0^2 x^d + k\dot{x}^d
\end{aligned}
$$

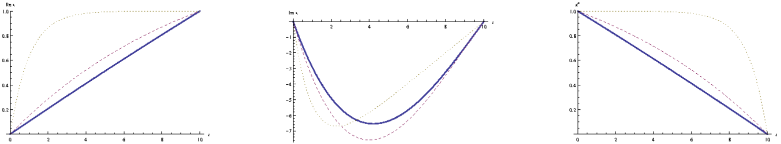
$\nearrow$

Exponentially increasing quantum fluctuations!

Dynamical decoherence

Brownian motion

$$\begin{aligned} m\ddot{x} &= -k\dot{x} + i(d_0 x^d - d_2 \ddot{x}^d) \\ m\ddot{x}^d &= k\dot{x}^d \end{aligned}$$



$Rex(t)$ $\Im x(t)$ $x^d(t)$

$\frac{k}{m} = 0.01$ (solid line), 0.1 (dashed line) and 1 (dotted line)

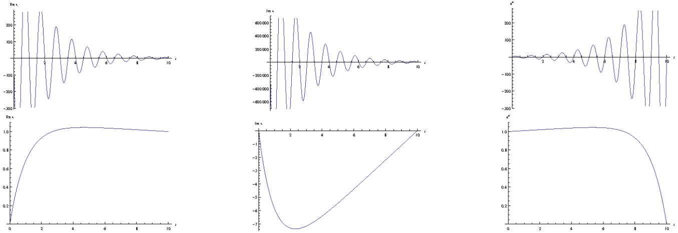
x^d: relaxation backward in time - static decoherence scenario applies

- $\tau_{id} = \tau_{diss}$

Dynamical decoherence

Harmonic oscillator

$$
\begin{aligned}
m\ddot{x} &= -m_0\omega_0^2 x - k\dot{x} + i(d_0 x^d - d_2\ddot{x}^d) \\
m\ddot{x}^d &= -m_0\omega_0^2 x^d + k\dot{x}^d
\end{aligned}
$$



$Re x(t)$ $\qquad$ $\Im x(t)$ $\qquad$ $x^d(t)$

$x^d(t)$: Overshooting backward in time

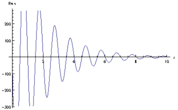
$$
e^{-\frac{1}{\hbar}\Im S_{eff}} \quad \sim \quad e^{-\frac{\boldsymbol{x}^{d2}}{\ell_{dd}^2(t)}} = e^{-c(t)e^{\frac{t}{\tau_{dd}}}} \leftarrow \text{double exponential}
$$

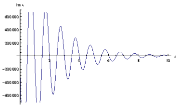
$$
\frac{1}{\ell_{dd}^2(t)} = \frac{1}{2\nu}(d_0 + d_2\omega_0^2)\left(\frac{\nu^2}{4\omega_0^2} - 1\right)\begin{cases} 4e^{t(\nu-\sqrt{\nu^2-4\omega_0^2})} & 4\omega_0^2 < \nu^2 \\ -\frac{e^{t\nu}}{\sin^2\omega_\nu t} & 4\omega_0^2 > \nu^2 \end{cases}
$$

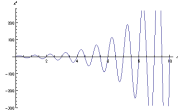
Dynamical decoherence

Anharmonic oscillator

H.O.:







Unstable quantum fluctuations $\Longrightarrow$harmonic regime left quickly

Relaxed state: strong, non-perturbative quantum fluctuations

Summary

1. Decoherence: two set of characteristic scales
   - Final state (instantaneous decoherence)
   - Initial state (dynamical decoherence)
2. Ignoring the system dynamics: static decoherence
3. Time scales: no separation (no small parameter, as in QCD)
   - $\tau_{id} = \tau_{diss}$
   - $\tau_{sd} \sim \tau_{diss}$
4. Qualitatively different cases:
   - Brownian motion: static decoherence scenario applies
   - Harmonic oscillator: Fast, double exponential decoherence
   - Anharmonic oscillator: Non-perturbative asymptotic state