Quantum renormalization group[a]

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[a] S. Nagy, J. Polonyi, I. Steib, *Quantum renormalization group*, Phys. Rev. D **93** (2016) 025008.

Motivation

- The functional renormalization group (RG) method, is suitable to eliminate the UV fluctuations systematically.
- In quantum theory the elimination of modes generates **mixed states**.
- The traditional **in-out** formalism has **pure** initial and final **states** implying that during the elimination we have only pure states
- We need an **in-in** formalism to take into account the effect of mixed states.
- The contribution of the mixed states can introduce
  - different phase structure,
  - new fixed points,
  - new relevant operators,
  - treatment of open systems

  even in text book example models.

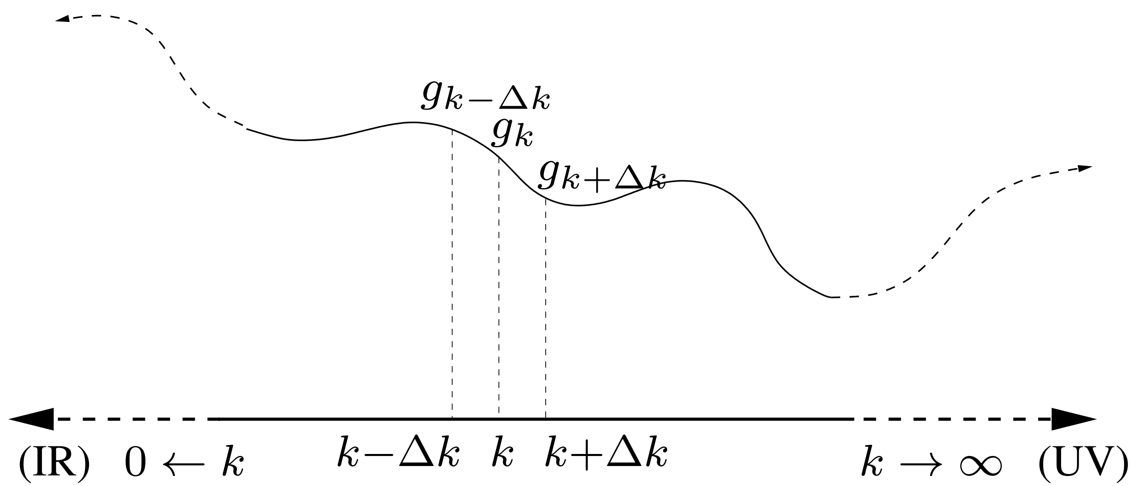
Renormalization group

- The functional renormalization group (RG) method is a non-perturbative method in quantum field theory.
- The RG method can elimininate the UV modes systematically and gives an IR description of the investigated model.

The evaluation of the path integral dresses up the values of the couplings with their corrections coming from the quantum fluctuations. The vacuum to vacuum transition amplitude is

$$Z[j] = \int \mathcal{D}\phi e^{\frac{i}{\hbar}S_k + \frac{i}{\hbar}j\phi} \equiv \int d\phi_0 \dots d\phi_{k-\Delta k} d\phi_k d\phi_{k+\Delta k} \dots d\phi_\infty e^{\frac{i}{\hbar}S_k + \frac{i}{\hbar}j\phi}$$

The path integral is performed by removing the modes one-by-one, which gives scale dependent couplings.



The CTP formalism

The path integral expressions for the generating functional for a scalar field theory can be written in the form

$$Z[j^+, j^-] = \mathrm{Tr}[U(t_f, t_i; j^+)\rho_i U^\dagger(t_f, t_i; -j^-)] = \int D[\hat\phi] e^{\frac{i}{\hbar}S[\hat\phi] + \frac{i}{\hbar}\int dx \hat{j}_x \hat\phi_x}.$$

The CTP doublets $\hat\phi = (\phi^+, \phi^-)$ have been introduced, and $S[\hat\phi] = S_0[\hat\phi] + S_i[\phi^+] - S_i[\phi^-]$. The bare CTP path integral is

$$e^{\frac{i}{\hbar}S_i[\hat\phi]} = \int D[\hat\chi] e^{\frac{i}{2\hbar}\int dx dy \hat\chi_x \hat K_{x,y}\hat\chi_y - \int dx [U_B(\phi^+_x + \chi_x) - U_B(\phi^-_x + \chi_x)]},$$

with integration over the UV field, χ, with spatial momentum $k < |\mathbf{p}|$ and the free inverse CTP propagator $\hat K$

$$\hat K = \begin{pmatrix} K^n + iK^i & K^f - iK^i \\ -K^f - iK^i & -K^n + iK^i \end{pmatrix}, \quad K^n_p = p^2 - m^2, \;\; K^f_p = i\epsilon \mathrm{sign}(p^0), \;\; K^i_p = i\epsilon.$$

The bare potential, $U_B(\phi)$, is an even polynomial, starting beyond the quadratic order

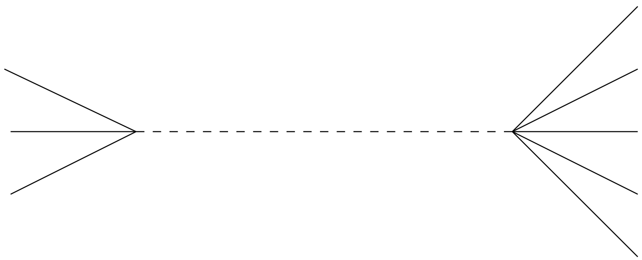
$$U_B(\phi) = \sum_{n=2}^{\infty} \frac{g_{B2n}}{(2n)!}\phi^{2n}.$$

CTP Feynman graphs

$$Z[j^+, j^-] = \text{Tr}[U(t_f, t_i; j^+)\rho_i U^\dagger(t_f, t_i; -j^-)].$$

The lines of a graph in a Feynman diagram can represent either the diagonal or the off diagonal CTP blocks of the free propagator. We have

1. homogeneous graphs: graphs with lines and vertices coming from the same copy
2. inhomogeneous graphs: graphs having external legs belonging exclusively to either ϕ^+ or ϕ^-, but have vertices from both copies
3. genuine CTP graphs has the CTP off diagonal lines and their external legs belong to both ϕ^+ and ϕ^-



- An $\mathcal{O}(\phi^{+3}\phi^{-5})$ contribution which couples the time axes by a fourth and and sixth order vertex connected by a dashed line, standing for D^{+-}.
- The interaction between the IR and the UV modes can take place easily if we consider e.g. models with compact field variables.

Blocking

The blocked action is obtained in the RG strategy by decreasing the gliding cutoff in infinitesimal steps, $k \to k - \Delta k$. An infinitesimal blocking step transfers the modes with momentum $k - dk < |\mathbf{p}| < k$ from the system to the environment. The effective action is given by

$$e^{\frac{i}{\hbar}S_{k-\Delta k}[\hat{\phi}]} = \int D[\hat{\chi}]e^{\frac{i}{\hbar}S_k[\hat{\phi}+\hat{\chi}]}.$$

- system modes: $\phi \to |\mathbf{p}| < k - \Delta k$
- environmental modes: $\chi \to k - \Delta k < |\mathbf{p}| < k$

We use the ansatz $S = S_1 + S_2$, where S_1 is kept in the local potential approximation,

$$S_1[\hat{\phi}] = \frac{1}{2}\int dxdy\hat{\phi}_x\hat{K}^d_{x,y}\hat{\phi}_y - \int dx[U(\phi^+_x) - U(\phi^-_x)],$$

The term S_2 represents the nonlocal part of the action, its bilocal cluster form is

$$S_2[\hat{\phi}] = -\int dxdyV_{x-y}(\hat{\phi}_x, \hat{\phi}_y),$$

with
$$V_{x-y}(\hat{\phi}, \hat{\phi}') = \sum_{\sigma,\sigma'}\sum_{mn\geq 3}\frac{1}{m!n!}\phi^{\sigma m}v^{\sigma,\sigma'}_{m,n,x-y}\phi'^{\sigma' n}.$$

Tree level evolution

To get saddle point we should solve the equation of motion for $\hat{\chi}_x$ for a given $\hat{\phi}_x$. It is enough to consider linearized equation of motion (higher order terms are of order $\sim \mathcal{O}(\Delta k^n)$).
The linearized equation of motion is

$$\hat{D}^{-1}\hat{\chi} = \hat{L},$$

with $(D^{-1})^{\sigma,\sigma'} = (D_0^{-1})^{\sigma,\sigma'} - \delta^{\sigma,\sigma'}\sigma U''(\phi^\sigma)$ and we introduce

$$L_x^\sigma = \sigma U'(\phi_x^\sigma) - 2\int dy \partial_{\phi_x^\sigma} V_{x-y}(\hat{\phi}_x, \hat{\phi}_y).$$

The substitution of the solution back into $S_k[\hat{\phi} + \hat{\chi}]$ yields $S_{k-\Delta k} = S_k + \Delta S_k$, with

$$S_k - S_{k-\Delta k} = \frac{\Delta k}{2}\int dx dy \hat{L}_x \hat{D}^{(k)}_{x-y}\hat{L}_y,$$

with the environment propagator

$$\hat{D}^{(k)}_{x-y} = \int \frac{d^4q}{(2\pi)^4}\delta(|\mathbf{q}| - k)\hat{D}_q e^{-i(x-y)q}.$$

Tree level evolution

Due to the ansatz we use the truncation

$$L_x^\sigma \to \bar{L}_x^\sigma = \sigma U'(\phi_x^\sigma) - 2\int dy W_{x-y}^\sigma(\hat{\phi}_y),$$

with

$$W_{x-y}^\sigma(\hat{\phi}) = \partial_{\phi'^\sigma_x} V_{x-y}(\hat{\phi}', \hat{\phi})_{|\phi'=0}.$$

It yields the following evolution equation

$$\begin{aligned}
\frac{dS}{dk} &= \frac{1}{2}\int dxdy \hat{\bar{L}}_x \hat{D}^{(k)}_{x-y} \hat{\bar{L}}_y \\
&= \frac{1}{2}\sum_{\sigma,\sigma'}\int dxdy \Big[\sigma U'(\phi_x^\sigma) D^{(k)\sigma,\sigma'}_{x-y} \sigma' U'(\phi_y^\sigma) \\
&\quad -4\int dz W^\sigma_{z-x}(\hat{\phi}_x) D^{(k)\sigma,\sigma'}_{z-y} \sigma' U'(\phi_y^\sigma) \\
&\quad +4\int dz dz' W^\sigma_{z-x}(\hat{\phi}_x) D^{(k)\sigma,\sigma'}_{z-z'} W^{\sigma'}_{z'-y}(\hat{\phi}_y)\Big].
\end{aligned}$$

Tree level evolution

The leading order in momentum space is

$$\frac{dS}{dk} = \frac{1}{2}\int\frac{d^{d+1}q}{(2\pi)^{d+1}}\sigma[U'(\phi^\sigma)]_{-q}D_q^{(k)\sigma,\sigma'}\sigma'[U'(\phi^\sigma)]_q$$

- The tree-level evolution accumulates the independent contributions from the gliding cutoff values.
- The solution shows that the local potential does not evolve on the tree level, ie. $U(\phi) = U_B(\phi)$.
- After performing the integration for the scale k the bilocal part S_2 becomes

$$V_{x-y}(\hat{\phi}_x, \hat{\phi}_y) = -\frac{1}{2}\sum_{\sigma,\sigma'}\sigma\sigma' U'(\phi_x^\sigma)D_{x-y}^{(k,\Lambda)\sigma\sigma'}U'(\phi_y^{\sigma'}),$$

where

$$\hat{D}_{x-y}^{(k,\Lambda)} = \int\frac{d^{d+1}q}{(2\pi)^{d+1}}\Theta(|\mathbf{q}|-k)\Theta(\Lambda-|\mathbf{q}|)\hat{D}_q e^{-i(x-y)q}$$

is the propagator, covering all eliminated modes.

Renormalized trajectory

The kernel of the bilocal action turns out to be

$$v^{\sigma,\sigma'}_{m,n,x-y} = g_{m+1} g_{n+1} \sigma D^{(k,\Lambda)\sigma,\sigma'}_{x-y} \sigma'.$$

- The evolution of the kernel is driven by the couplings belonging to the local part of the potential.
- We need $m, n \geq 3$ in order to get the mixed state contribution.
- the bilocal couplings become q dependent.
- The contributions related to the diagonal part of the CTP index $\sigma = \sigma'$ are covered by the traditional, single time axis formalism.
- The off diagonal elements ($\sigma = -\sigma'$) are related to the mixed state contributions.
- The off diagonal elements represent the IR-UV entanglement. They are genuine CTP contributions.

Outlook

- **Some questions related to the multilocal expansion**
  - can be non-trivial in the traditional Euclidean description
  - momentum dependent couplings
  - the classification of the couplings into relevant and irrelevant ones might be changed
  - many important results that are based on local potential description should be reconsidered

- **CTP RG equation beyond tree level**
  - proper functional ansatz for the potential containing local and bilocal parts
  - real and imaginary part of the local part is off- and on-shell, respectively, it is driven by the Wegner-Houghton RG equation in Minkowski spacetime, which contains the contribution of the zero momentum bilocal potential.
  - saddle point bilocal part, it evolves according to the tree level evolution with some generalizations. It mixes the tree level and the loop contributions in a non trivial manner.
  - the fluctuating bilocal part, its evolution is driven by two-cluster expansion of the Wegner-Houghton equation.

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