STOCHASTIC QUANTIZATION AND THE SIGN PROBLEM

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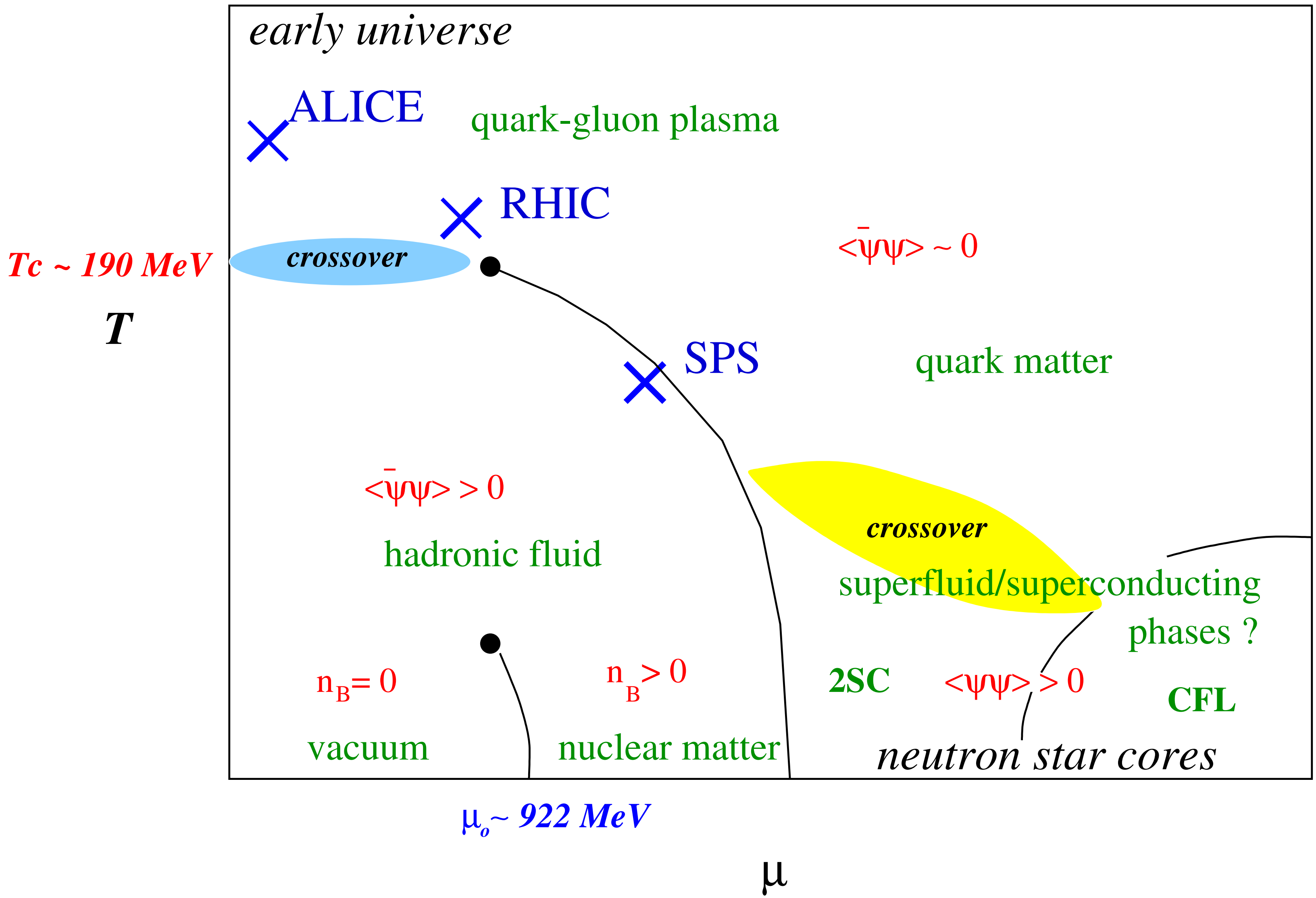
OUTLINE

- QCD phase diagram from the lattice ?
- sign problem at finite chemical potential

- a revived approach: stochastic quantization
- three QCD inspired models
- the Silver Blaze problem is not a problem

QCD PHASE DIAGRAM

A SKETCH



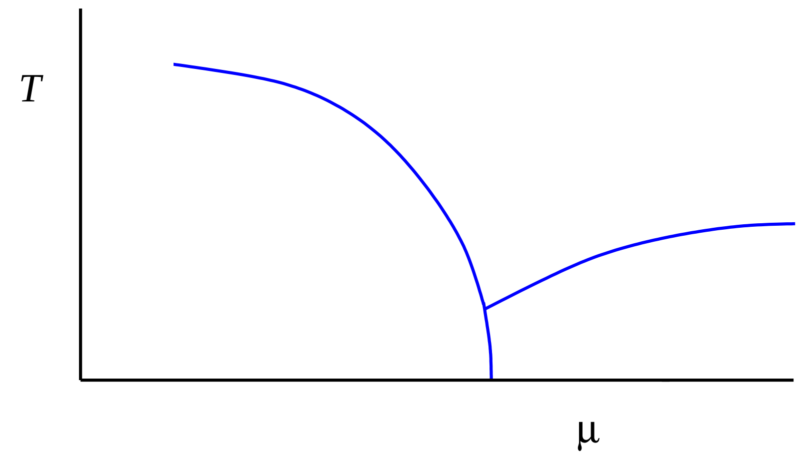
QCD PHASE DIAGRAM

NONPERTURBATIVE DETERMINATION

- QCD is confining at low temperature and chemical potential

⇒ nonperturbative study

$\boxed{\text{lattice QCD}}$



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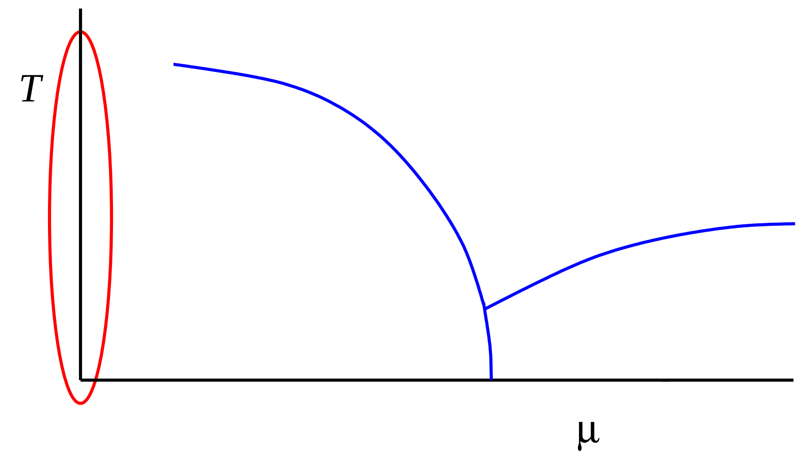

status:

- works well at $\mu = 0$

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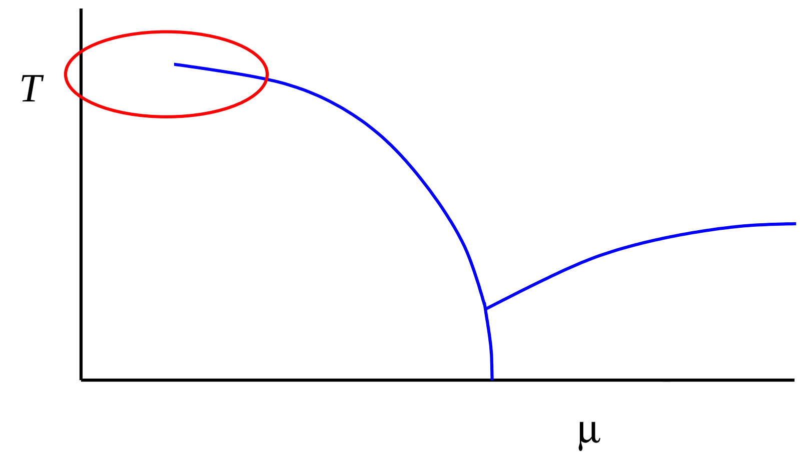

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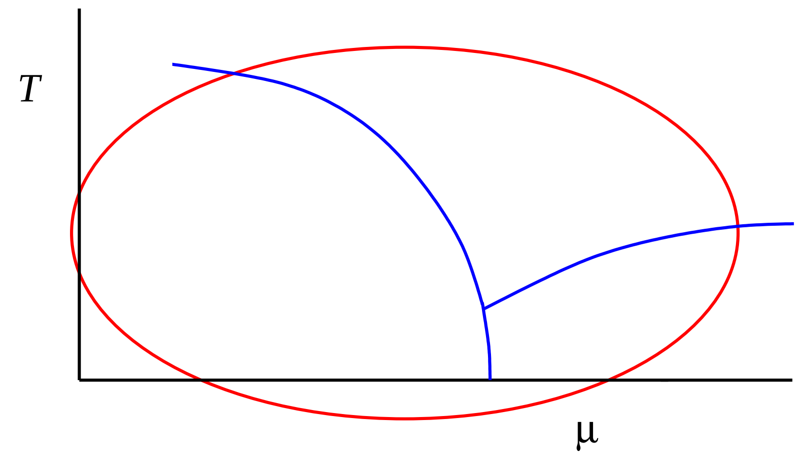

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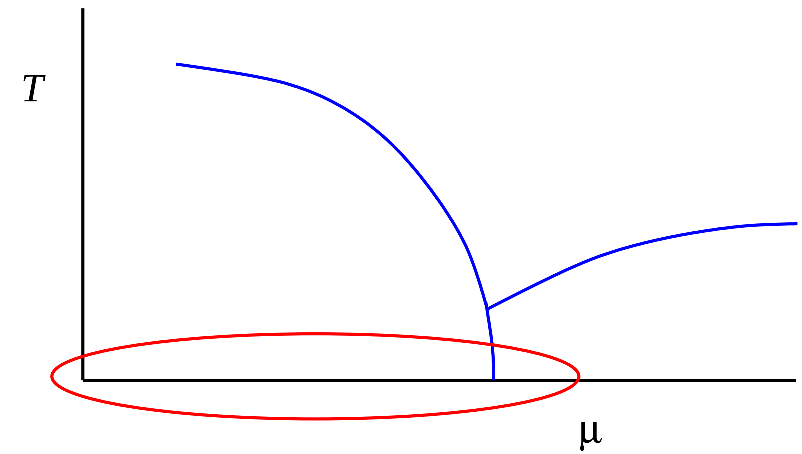

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in this talk: alternative lattice QCD approach
first results encouraging
potentially applicable in cold dense phase

Lattice QCD

Importance sampling

partition function: $Z = \int DU\, D\bar{\psi} D\psi\, e^{-S} = \int DU\, e^{-S_B} \det M$

- if $e^{-S_B} \det M > 0$, interpret as probability weight
- evaluate using importance sampling

LATTICE QCD

IMPORTANCE SAMPLING

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- evaluate using importance sampling

QCD at finite baryon chemical potential:

$$\det M(\mu) = [\det M(-\mu)]^*$$

fermion determinant is complex!

- importance sampling not possible

$$\boxed{\text{sign problem}}$$

- basic tool of all lattice QCD algorithms breaks down

WHY IS THE SIGN PROBLEM DIFFICULT?

PHASE QUENCHED THEORY

write $\det M = |\det M| e^{i\varphi}$

- phase quenched theory with weight $e^{-S_B} |\det M| > 0$

$$\langle O \rangle_{\text{full}} = \frac{\int DU \, e^{-S_B} \det M \, O}{\int DU \, e^{-S_B} \det M}$$

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Why is the sign problem difficult?

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$$= \frac{\langle e^{i\varphi} O \rangle_{\rm pq}}{\langle e^{i\varphi} \rangle_{\rm pq}}$$

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$$= \frac{\langle e^{i\varphi} O\rangle_{\rm pq}}{\langle e^{i\varphi}\rangle_{\rm pq}} \quad \to \quad \frac{0}{0} \quad \to \quad ??$$

- average phase factor

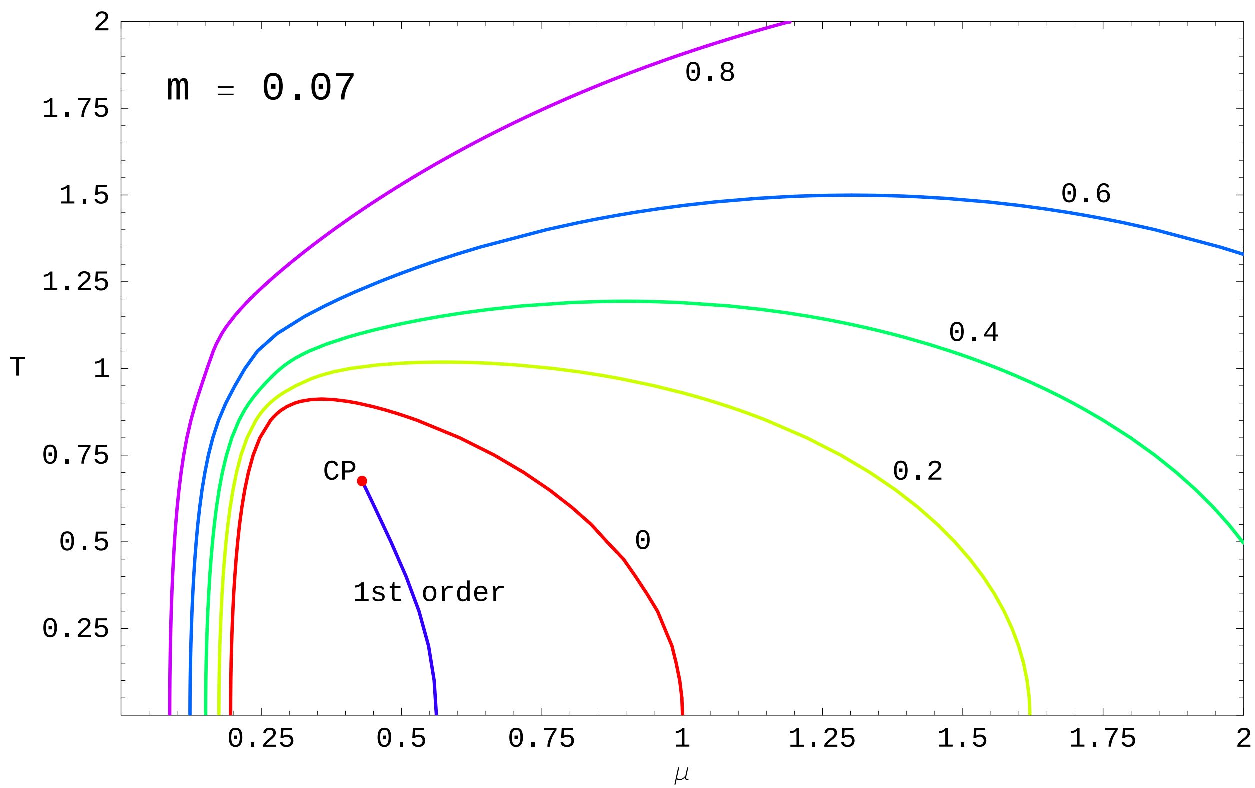
$$\langle e^{i\varphi}\rangle_{\rm pq} = \frac{\int DU\, e^{-S_B}|\det M|\, e^{i\varphi}}{\int DU\, e^{-S_B}|\det M|} = \frac{Z_{\rm full}}{Z_{\rm pq}} = e^{-\Omega\Delta f} \to 0$$

overlap problem, exponentially hard in thermodynamic limit

WHY IS THE SIGN PROBLEM DIFFICULT?

PHASE QUENCHED THEORY

average phase factor $\langle e^{i\varphi}\rangle_{\rm pq}$ in Random Matrix Theory



Han & Stephanov, A Random Matrix Study of the QCD Sign Problem, arXiv:0805.1939 [hep-lat]

Sign Problems and Complex Actions

workshop at ECT* Trento

Monday March 2 - Friday March 6 2009

organizers: Gert Aarts (Swansea University) & Shailesh Chandrasekharan (Duke University)

contact me if you are interested

QCD AT FINITE μ

SIGN PROBLEM

- configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$
- cancelation between configurations with ‘positive’ and ‘negative’ weight
- how to pick the dominant configurations in the path integral?

QCD AT FINITE μ

SIGN PROBLEM

- configurations differ in an essential way from those obtained at $\mu = 0$ or with $|\det M|$
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- how to pick the dominant configurations in the path integral?

radically different approach:

- complexifying all degrees of freedom: SU(3) $\rightarrow$ SL(3,$\mathbb{C}$)

$$\boxed{\text{stochastic quantization and complex Langevin dynamics}}$$

in collaboration with Nucu Stamatescu

initiated at the bottom of Death Valley (CA) based on

- with I.O. Stamatescu:
  stochastic quantization at finite chemical potential
  0807.1597 [hep-lat], JHEP

- can stochastic quantization evade the sign problem? –
  the relativistic Bose gas at finite chemical potential
  0810.2089 [hep-lat]

more reading

- with I.O.S.: Lattice proceedings, 0809.5527 [hep-lat]

- SEWM proceedings: 0811.1850 [hep-ph]

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

- alternative nonperturbative numerical approach
- weight $=$ equilibrium distribution of stochastic process

$$\boxed{\text{Brownian motion}}$$

particle in a fluid: friction (γ) and kicks (η)

- Langevin equation

$$\frac{d}{dt}\vec{v}(t) = -\gamma\vec{v}(t) + \vec{\eta}(t)$$

- Gaussian noise

$$\langle\eta_i(t)\rangle = 0 \qquad \langle\eta_i(t)\eta_j(t')\rangle = 2kT\gamma\delta_{ij}\delta(t-t')$$

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

- Langevin equation

$$\frac{d}{dt}v_i(t) = -\gamma v_i(t) + \eta_i(t) \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2kT\gamma\delta_{ij}\delta(t-t')$$

- analytical solution

$$v_i(t) = e^{-\gamma t}v_i(0) + \int_0^t dt'\, \eta_i(t')e^{-\gamma(t-t')}$$

- noise averaged kinetic theory in long time limit

$$\lim_{t\to\infty} \frac{1}{2}\langle v_i(t)v_j(t)\rangle = \frac{1}{2}\delta_{ij}kT$$

- Maxwell-Boltzmann distribution not used

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

application to field theory

Parisi & Wu '81

- path integral $Z = \int D\phi \, e^{-S}$
- Langevin dynamics in "fifth" time direction

$$\frac{\partial\phi(x,\theta)}{\partial\theta} = -\frac{\delta S[\phi]}{\delta\phi(x,\theta)} + \eta(x,\theta)$$

- Gaussian noise

$$\langle\eta(x,\theta)\rangle = 0 \qquad \langle\eta(x,\theta)\eta(x',\theta')\rangle = 2\delta(x-x')\delta(\theta-\theta')$$

- equilibrium distribution $P[\phi] \sim e^{-S}$

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

force $\partial S/\partial\phi$ complex: Parisi, Klauder '85

complexify Langevin dynamics

- example: real scalar field $\phi \to \phi^{\rm R} + i\phi^{\rm I}$
- Langevin eqs

$$\frac{\partial\phi^{\rm R}}{\partial\theta} = -{\rm Re}\,\frac{\delta S}{\delta\phi}\Big|_{\phi\to\phi^{\rm R}+i\phi^{\rm I}} + \eta$$

$$\frac{\partial\phi^{\rm I}}{\partial\theta} = -{\rm Im}\,\frac{\delta S}{\delta\phi}\Big|_{\phi\to\phi^{\rm R}+i\phi^{\rm I}}$$

- observables: analytic extension

$$\langle O(\phi)\rangle \to \langle O(\phi^{\rm R} + i\phi^{\rm I})\rangle$$

Stochastic quantization

Langevin dynamics

ultimate sign problem: dynamics in real time

- Minkowski path integral $Z = \int D\phi \, e^{iS}$
- Langevin equation

$$\frac{\partial \phi}{\partial \theta} = i \frac{\delta S}{\delta \phi} + \eta$$

- after complexification

$$\frac{\partial \phi^{\mathrm{R}}}{\partial \theta} = -\mathrm{Im} \left. \frac{\delta S}{\delta \phi} \right|_{\phi \to \phi^{\mathrm{R}} + i\phi^{\mathrm{I}}} + \eta$$

$$\frac{\partial \phi^{\mathrm{I}}}{\partial \theta} = +\mathrm{Re} \left. \frac{\delta S}{\delta \phi} \right|_{\phi \to \phi^{\mathrm{R}} + i\phi^{\mathrm{I}}}$$

Berges, Borsanyi, Sexty, Stamatescu '05-'08

INTERMEZZO

HISTORY OF STOCHASTIC QUANTIZATION

original suggestion

- Parisi & Wu '81, Parisi, Klauder '85

lots of activity in 80's

- Damgaard and Hüffel, Physics Reports '87

application to finite μ: three-dimensional spin models

- Karsch & Wyld '85, ...

stopped because of numerical problems (runaways, instabilities)

- renewed interest: Minkowski dynamics

  `Berges, Borsanyi, Sexty, Stamatescu '05-'08`

FINITE CHEMICAL POTENTIAL

TOWARDS QCD

consider three models with a partition function

$$Z = \int DU e^{-S_B} \det M \qquad\qquad \det M(\mu) = [\det M(-\mu)]^*$$

- QCD with static quarks
- SU(3) one link model
- U(1) one link model

observables:

- (conjugate) Polyakov loops
- density
- phase of determinant

THREE MODELS

I: QCD WITH STATIC QUARKS

$$Z = \int DU e^{-S_B} \det M$$

- bosonic action: standard SU(3) Wilson action

$$S_B = -\beta \sum_P \left(\frac{1}{6} \left[\mathrm{Tr}\, U_P + \mathrm{Tr}\, U_P^{-1} \right] - 1 \right)$$

- determinant $\det M$ for Wilson fermions

  fermion matrix:

$$M = 1 - \kappa \sum_{i=1}^{3} \text{space} - \kappa \left(e^{\mu} \Gamma_{+4} U_{x,4} T_4 + e^{-\mu} \Gamma_{-4} U_{x,4}^{-1} T_{-4} \right)$$

THREE MODELS

I: QCD WITH STATIC QUARKS

- hopping expansion:

$$\det M \approx \det \left[1 - \kappa \left(e^{\mu}\Gamma_{+4}U_{x,4}T_4 + e^{-\mu}\Gamma_{-4}U_{x,4}^{-1}T_{-4}\right)\right]$$
$$= \prod_{\mathbf{x}} \det \left(1 + h e^{\mu/T}\mathcal{P}_{\mathbf{x}}\right)^2 \det \left(1 + h e^{-\mu/T}\mathcal{P}_{\mathbf{x}}^{-1}\right)^2$$

  with $h = (2\kappa)^{N_\tau}$ and (conjugate) Polyakov loops $\mathcal{P}_{\mathbf{x}}^{(-1)}$

- static quarks propagate in temporal direction only: Polyakov loops

- full gauge dynamics included

THREE MODELS

II: SU(3) ONE LINK MODEL

$$Z = \int dU e^{-S_B} \det M \qquad \text{link } U \in \text{SU(3)}$$

$$S_B = -\frac{\beta}{6}\left(\text{Tr}\, U + \text{Tr}\, U^{-1}\right)$$

determinant:

$$\begin{aligned}\det M &= \det\left[1 + \kappa\left(e^{\mu}\sigma_+ U + e^{-\mu}\sigma_- U^{-1}\right)\right] \\ &= \det\left(1 + \kappa e^{\mu} U\right) \det\left(1 + \kappa e^{-\mu} U^{-1}\right)\end{aligned}$$

with $\sigma_\pm = (\mathbb{1} \pm \sigma_3)/2$

- det in colour space remaining
- exact evaluation by integrating over the Haar measure

THREE MODELS

III: U(1) ONE LINK MODEL

U(1) model: link $U = e^{ix}$ with $-\pi < x \leq \pi$

$$S_B = -\frac{\beta}{2}\left(U + U^{-1}\right) = -\beta\cos x$$

determinant:

$$\det M = 1 + \frac{1}{2}\kappa\left[e^{\mu}U + e^{-\mu}U^{-1}\right] = 1 + \kappa\cos(x - i\mu)$$

partition function:

$$Z = \int_{-\pi}^{\pi}\frac{dx}{2\pi}\, e^{\beta\cos x}\left[1 + \kappa\cos(x - i\mu)\right]$$

- all observables can be computed analytically

COMPLEX LANGEVIN DYNAMICS

Langevin update:

$$U(\theta+\epsilon) = R(\theta)\,U(\theta) \qquad\qquad R = \exp\left[i\lambda_a\left(\epsilon K_a + \sqrt{\epsilon}\eta_a\right)\right]$$

- drift term

$$K_a = -D_a S_{\rm eff} \qquad\qquad S_{\rm eff} = S_B + S_F \qquad\qquad S_F = -\ln\det M$$

- noise

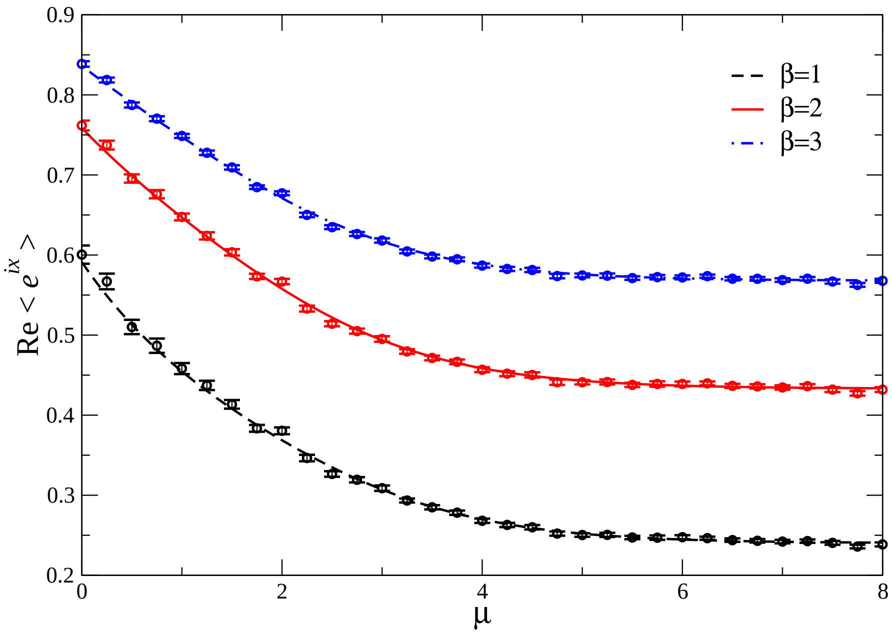
$$\langle\eta_a\rangle = 0 \qquad\qquad \langle\eta_a\eta_b\rangle = 2\delta_{ab}$$

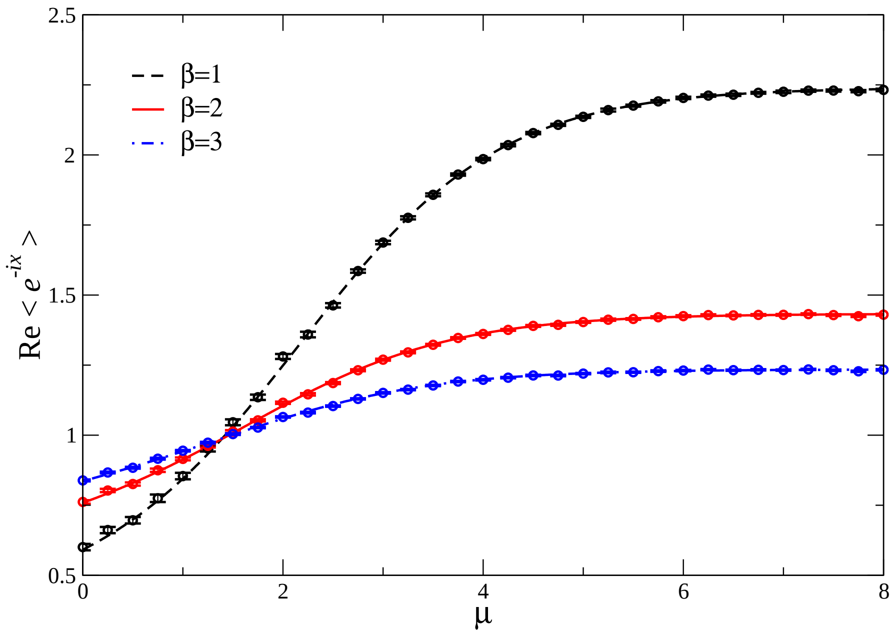
real action: $\Rightarrow K^\dagger = K \Leftrightarrow U \in$ SU(3)

complex action: $\Rightarrow K^\dagger \neq K \Leftrightarrow U \in$ SL$(3,\mathbb{C})$

(CONJUGATE) POLYAKOV LOOPS

U(1) ONE LINK MODEL



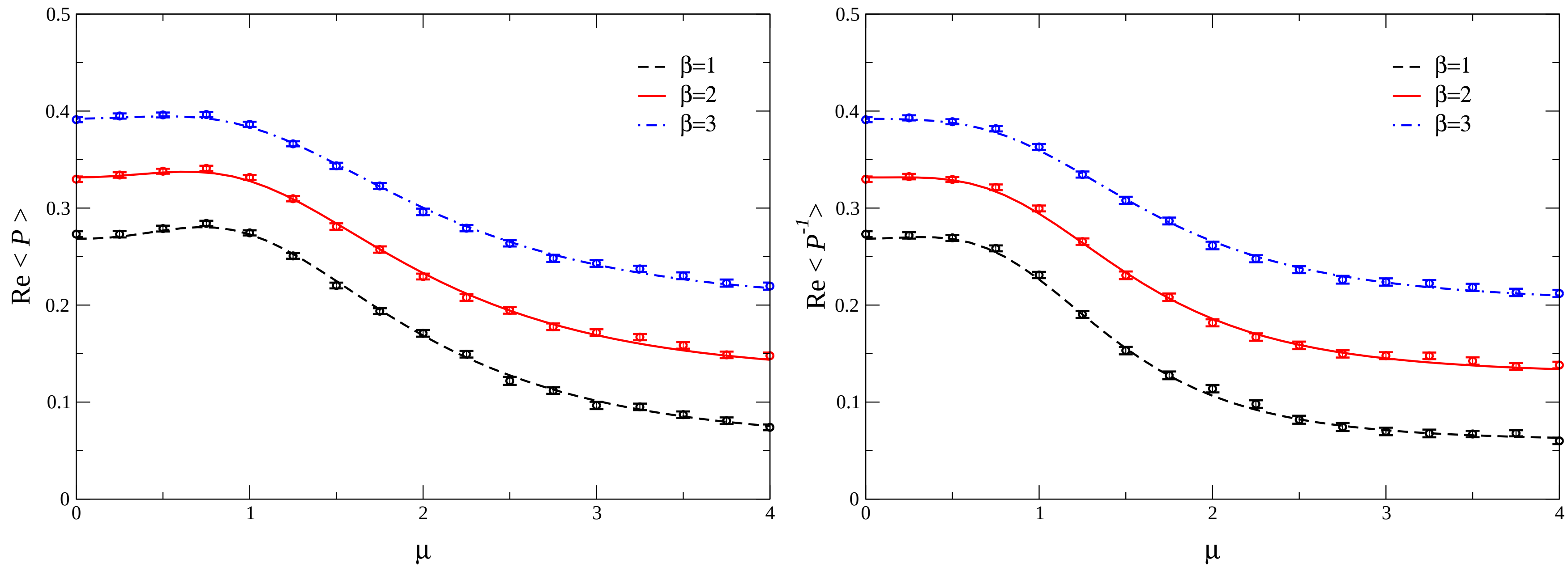


- data points: complex Langevin
  stepsize $\epsilon = 5 \times 10^{-5}$, 5×10^{7} time steps
- lines: exact results

excellent agreement for all μ

(CONJUGATE) POLYAKOV LOOPS

SU(3) ONE LINK MODEL



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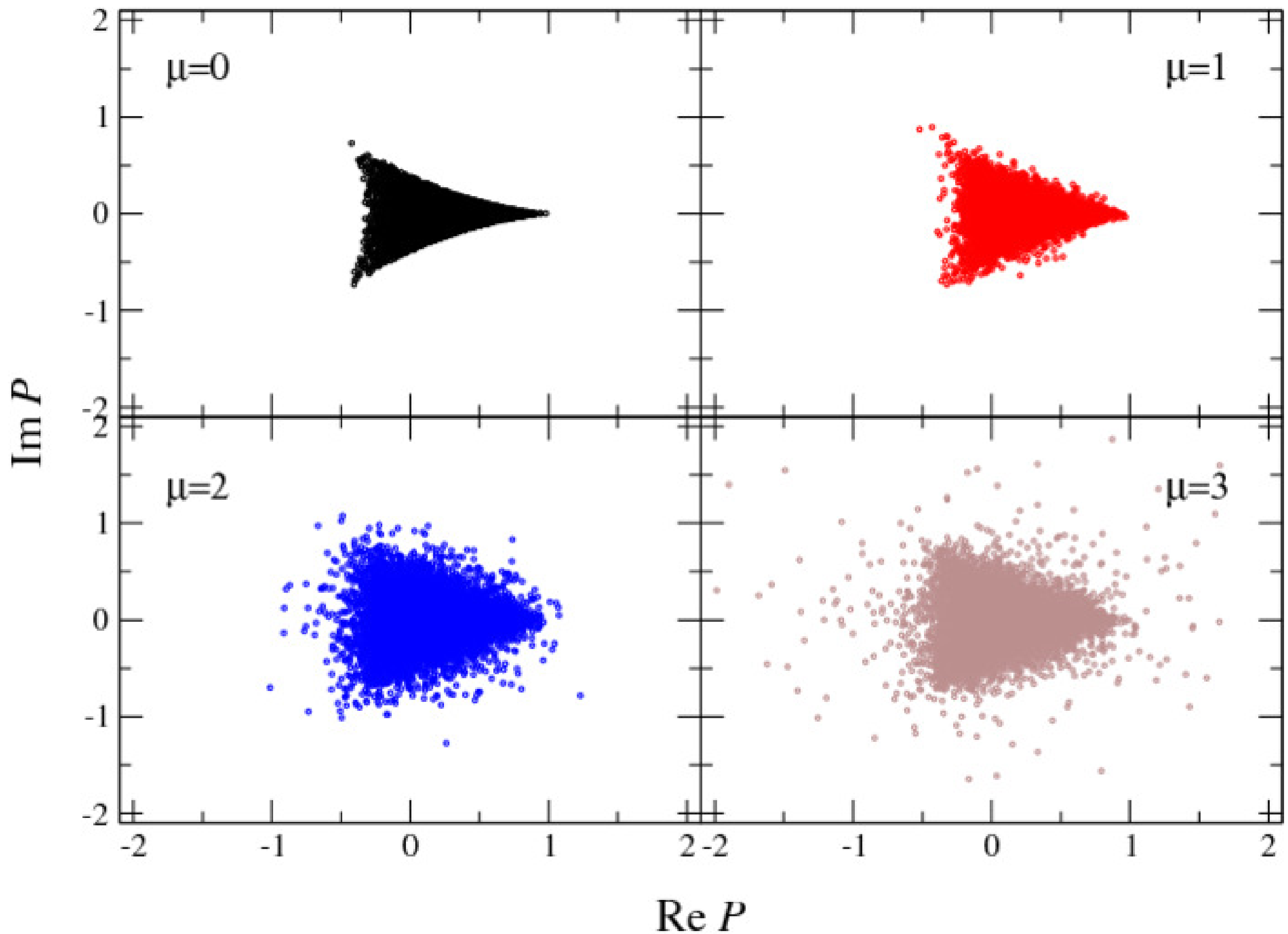
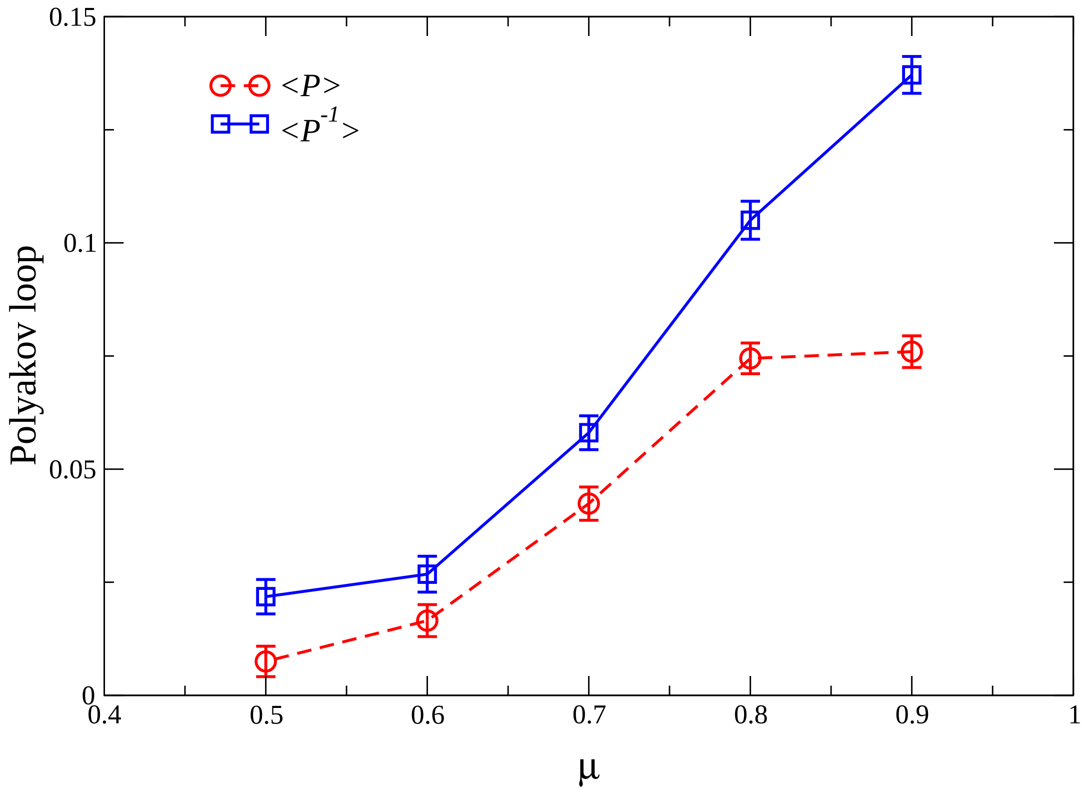


scatter plot of P during Langevin evolution

(CONJUGATE) POLYAKOV LOOPS

QCD WITH STATIC QUARKS

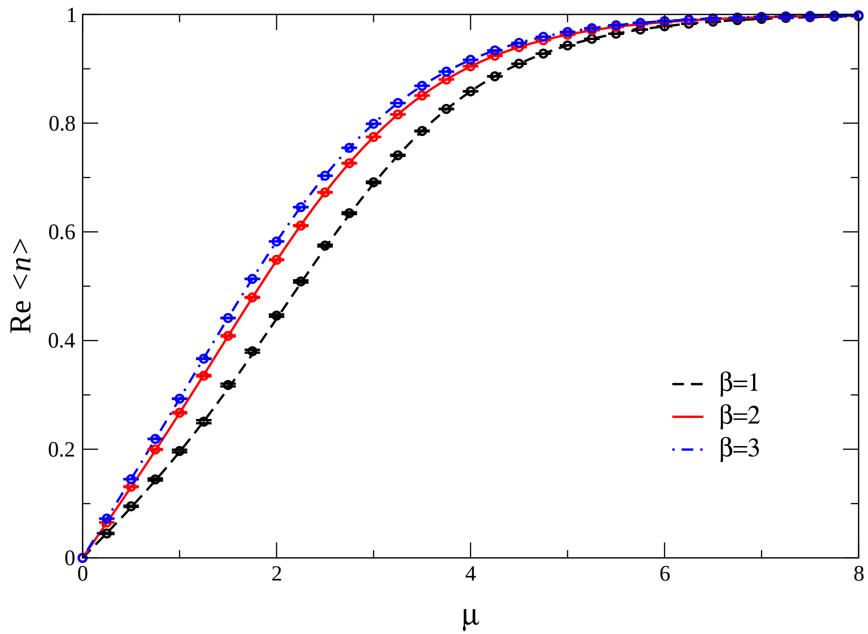
first results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$



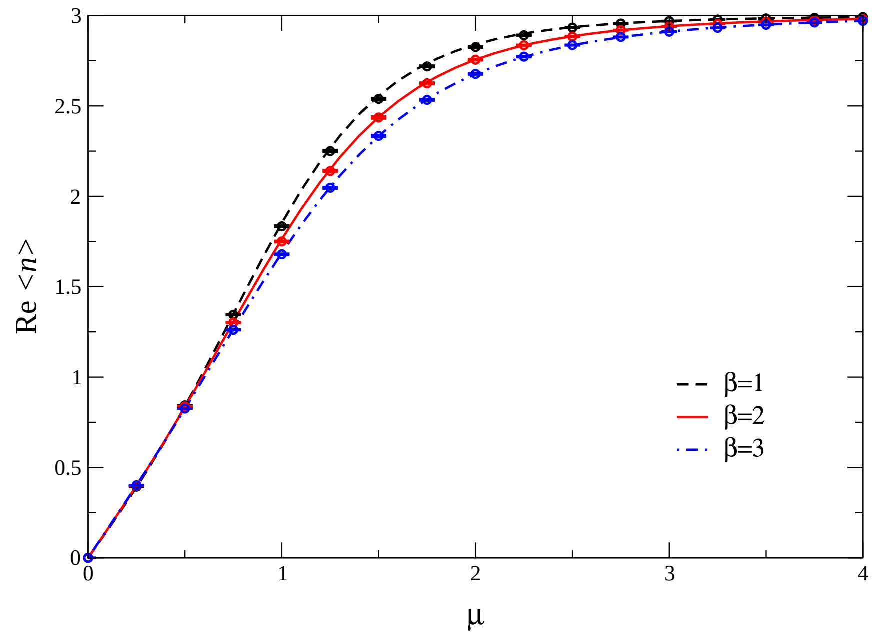
low-density “confining” phase $\Rightarrow$ high-density “deconfining” phase

DENSITY

U(1) ONE LINK MODEL



SU(3) ONE LINK MODEL

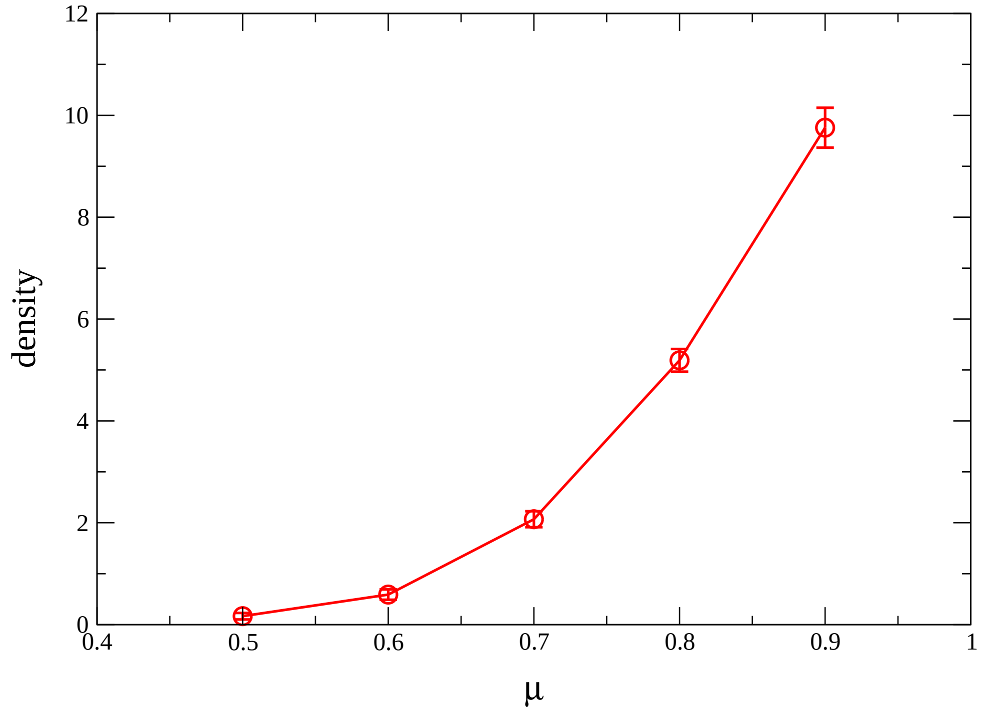


- linear increase at small μ
- saturation at large μ

excellent agreement for all μ

DENSITY

QCD WITH STATIC QUARKS



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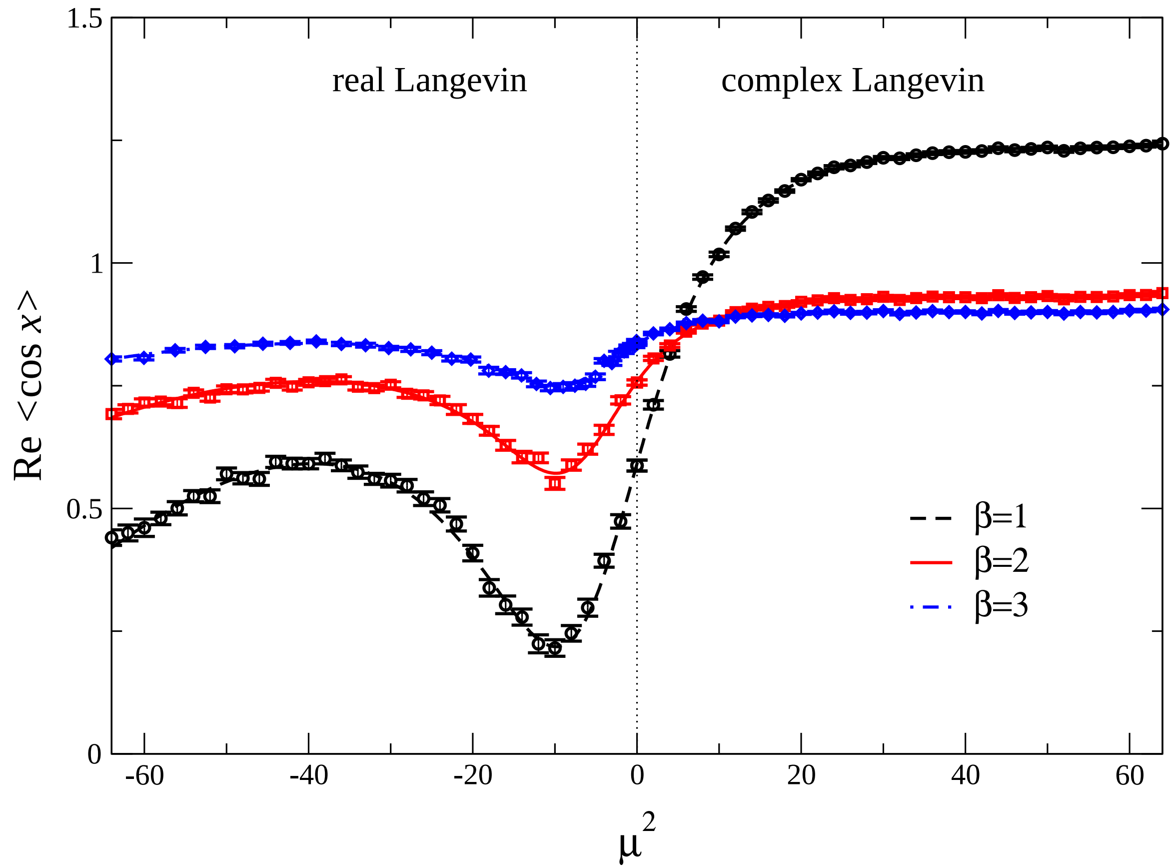
low-density phase $\Rightarrow$ high-density phase

REAL VS. COMPLEX LANGEVIN

U(1) ONE LINK MODEL



plaquette as a function of μ^2

$\mu^2 < 0$: imaginary chemical potential $\Leftrightarrow$ real action

NUMERICAL STABILITY/RUNAWAYS

PROBLEM IN THE 80'S

- one link models: no problem

- field theory: runaways (practically) eliminated
  careful with numerical precision and roundoff errors
  dynamical step size

SIGN PROBLEM

QCD WITH STATIC QUARKS

$$\det M(\mu) = [\det M(-\mu)]^* = |\det M(\mu)| e^{i\varphi}$$

average phase factor: $\langle e^{2i\varphi} \rangle = \left\langle \dfrac{\det M(\mu)}{\det M(-\mu)} \right\rangle$

SIGN PROBLEM

QCD WITH STATIC QUARKS

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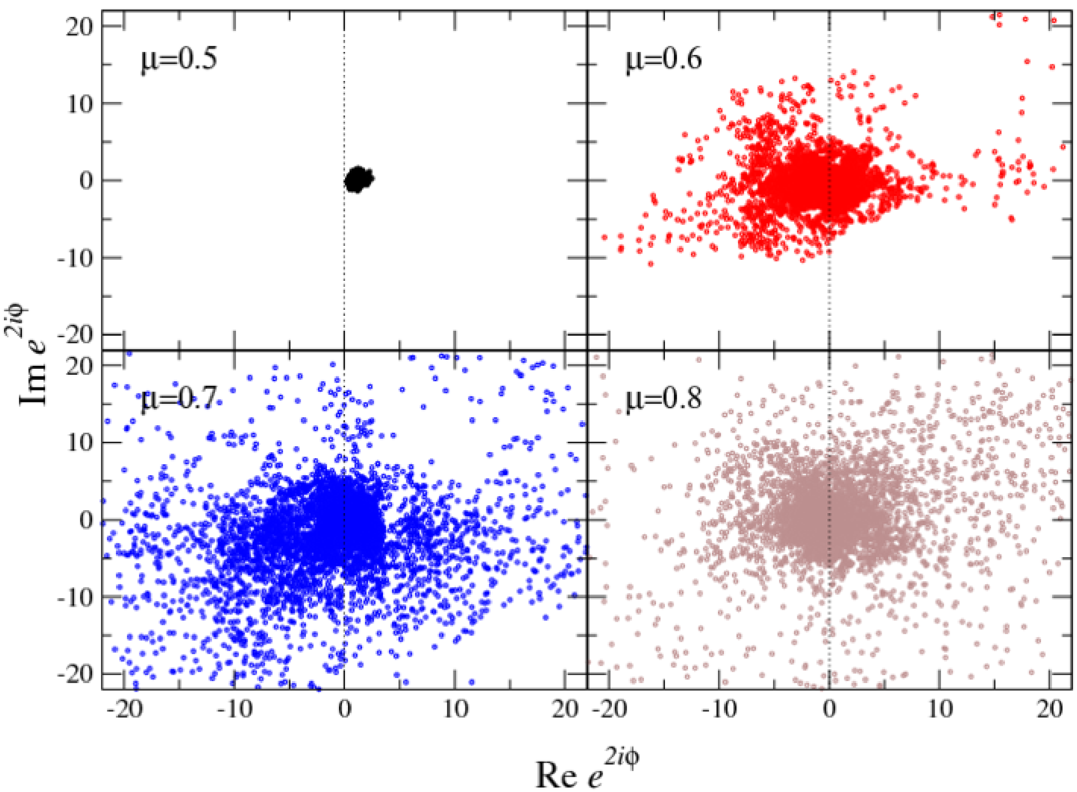


scatter plot of $e^{2i\varphi}$ during Langevin evolution

SIGN PROBLEM

QCD WITH STATIC QUARKS

$$\det M(\mu) = [\det M(-\mu)]^* = |\det M(\mu)| e^{i\varphi}$$

sign problem: expectations

$$\langle \varphi \rangle \sim e^{-\Omega f} \qquad \langle \varphi^2 \rangle - \langle \varphi \rangle^2 \sim \Omega \qquad \Omega = N_s^3 N_\tau$$

- exploding phase of fermion determinant
- yet observables under control (4^4 lattice)

$\mathbf{SU(3) \to SL(3,\mathbb{C})}$

QCD WITH STATIC QUARKS

- complex Langevin dynamics: no longer in SU(3)
- instead $U \in$ SL(3,$\mathbb{C}$)
- in terms of gauge potentials $U = e^{i\lambda_a A_a/2}$
  A_a is now complex
- how far from SU(3)?

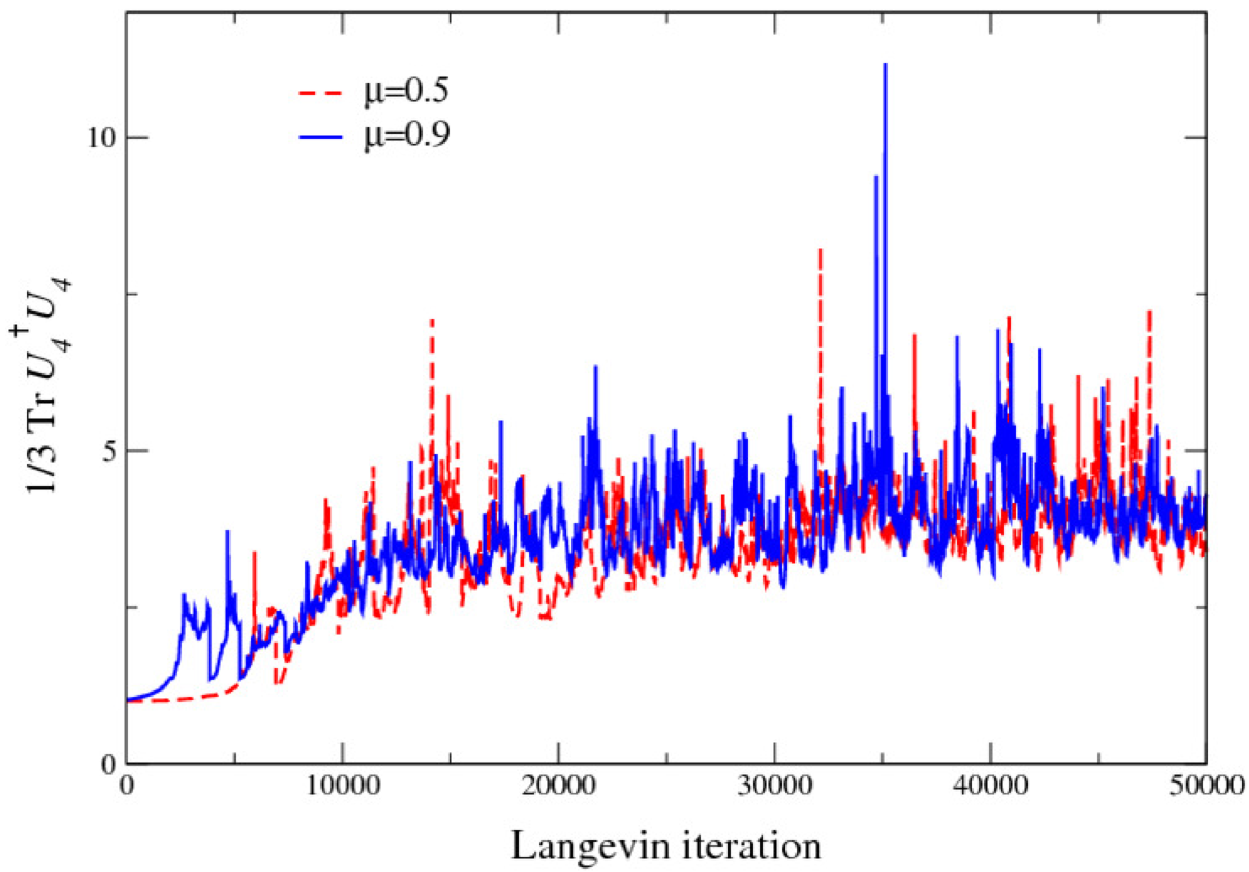
consider

$$\frac{1}{N}\mathrm{Tr}\, U^\dagger U \begin{cases} = 1 & \text{if } U \in \mathrm{SU}(N) \\ \\ \geq 1 & \text{if } U \in \mathrm{SL}(N,\mathbb{C}) \end{cases}$$

$\mathbf{SU(3)} \rightarrow \mathbf{SL(3,}\mathbb{C}\mathbf{)}$

QCD WITH STATIC QUARKS

$$\frac{1}{3}\mathrm{Tr}\, U^\dagger U \geq 1 \qquad\qquad = 1 \text{ if } U \in \mathrm{SU(3)}$$



Complexification of phase space

WHY DOES IT WORK?

- most approaches start from $\mu = 0$ or $|\det M(\mu)|$
- complex Langevin dynamics radically different

$\Rightarrow$ complexification of degrees of freedom

visualization in U(1) model

- understanding in terms of classical fixed points

CLASSICAL FLOW

U(1) ONE LINK MODEL

- link $U = e^{ix}$ complexification $x \to z = x + iy$
- Langevin dynamics:

$$\dot{x} = K_x + \eta \qquad \dot{y} = K_y$$

- classical forces:

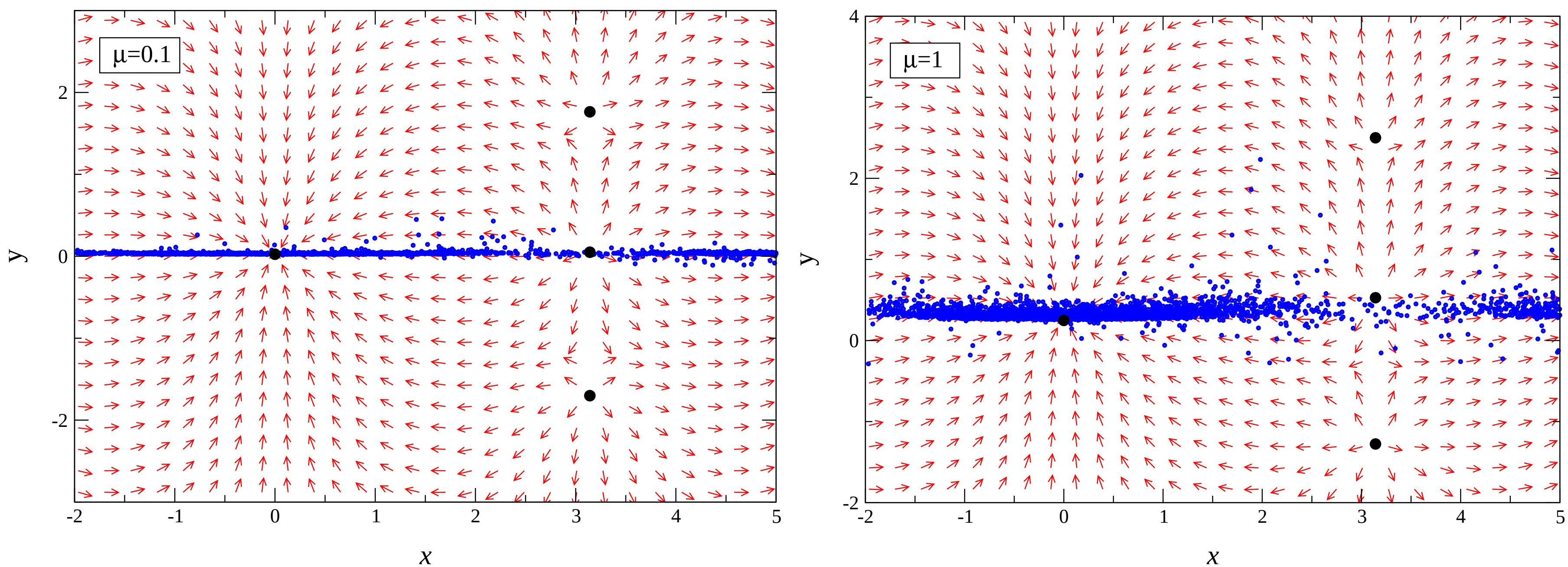
$$K_x = -\mathrm{Re}\frac{\partial S}{\partial x}\Big|_{x\to z} \qquad K_y = -\mathrm{Im}\frac{\partial S}{\partial x}\Big|_{x\to z}$$

- classical fixed points: $K_x = K_y = 0$

CLASSICAL FLOW

U(1) ONE LINK MODEL

flow diagrams and Langevin evolution

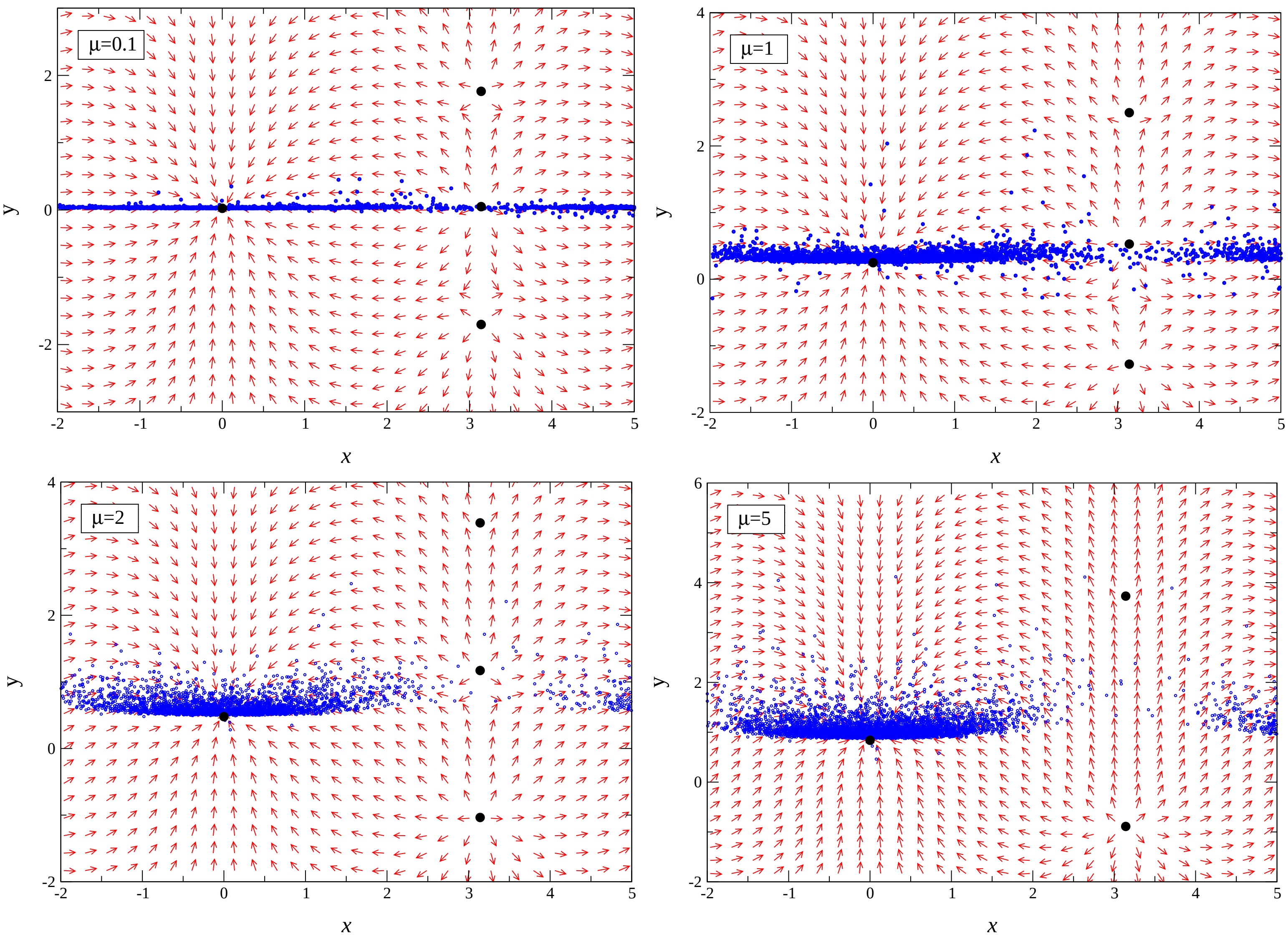


- black dots: classical fixed points
- $\mu = 0$: dynamics only in x direction
- $\mu > 0$: spread in y direction

CLASSICAL FLOW

U(1) ONE LINK MODEL

imag part of gauge potential →



real part of gauge potential →

CLASSICAL FLOW

U(1) ONE LINK MODEL

at finite chemical potential:

- one stable fixed point at $x = 0$, $y = y_s(\mu)$
- unstable fixed points at $x = \pi$, $y = y_u(\mu)$

$\Rightarrow$ fixed point structure is independent of μ!

for Minkowski dynamics:

- fixed point structure collapses at larger β
- Langevin dynamics no longer converges

Berges & Sexty '07

COMPLEX FOKKER-PLANCK EQUATION

U(1) ONE LINK MODEL

chemical potential vs real time

- one degree of freedom $U = e^{ix}$

complex Fokker-Planck equation

$$\frac{\partial P(x,\theta)}{\partial\theta} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{\partial S}{\partial x}\right) P(x,\theta)$$

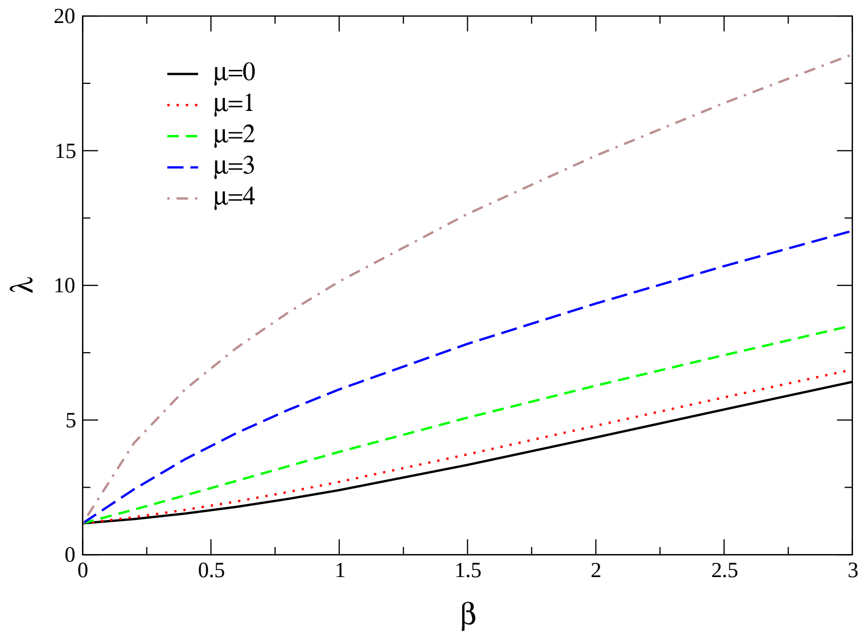
find eigenvalues of Fokker-Planck operator

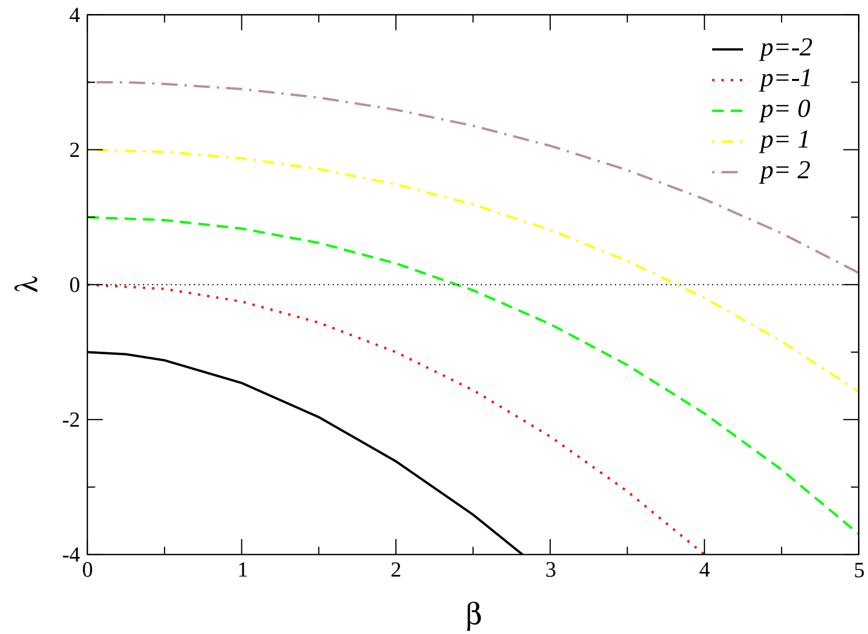
- real time: $iS = \beta\cos x + px$
- chemical potential: $S = -\beta\cos x - \ln\det M$
  with $\det M(\mu) = 1 + \kappa\cos(x - i\mu)$

Complex Fokker-Planck equation

U(1) one link model

smallest nonzero eigenvalue as a function of β





chemical potential	real time
all eigenvalues > 0	eigenvalues go negative

G.A. & Stamatescu ’08

PHASE TRANSITIONS AND THE SILVER BLAZE

intruiging questions:

- how severe is the sign problem?
- thermodynamic limit?
- phase transitions?
- Silver Blaze problem? Cohen '03
- ...

study in a model with a phase diagram with similar features as QCD at low temperature

$\Rightarrow$ relativistic Bose gas at nonzero μ or scalar O(2) model

RELATIVISTIC BOSE GAS AT NONZERO μ

PHASE TRANSITIONS AND THE SILVER BLAZE

- continuum action

$$S = \int d^4x \Big[\, |\partial_\nu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu \, (\phi^* \partial_4 \phi - \partial_4 \phi^* \phi) + \lambda |\phi|^4 \Big]$$

- complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu)$ as in QCD

RELATIVISTIC BOSE GAS AT NONZERO μ

PHASE TRANSITIONS AND THE SILVER BLAZE

- lattice action

$$S = \sum_x \Bigg[\left(2d + m^2\right) \phi_x^* \phi_x + \lambda \left(\phi_x^* \phi_x\right)^2$$
$$- \sum_{\nu=1}^{4} \left(\phi_x^* e^{-\mu\delta_{\nu,4}} \phi_{x+\hat{\nu}} + \phi_{x+\hat{\nu}}^* e^{\mu\delta_{\nu,4}} \phi_x \right) \Bigg]$$

- complex scalar field, $d = 4$, $m^2 > 0$

- $S^*(\mu) = S(-\mu)$ as in QCD

RELATIVISTIC BOSE GAS AT NONZERO μ

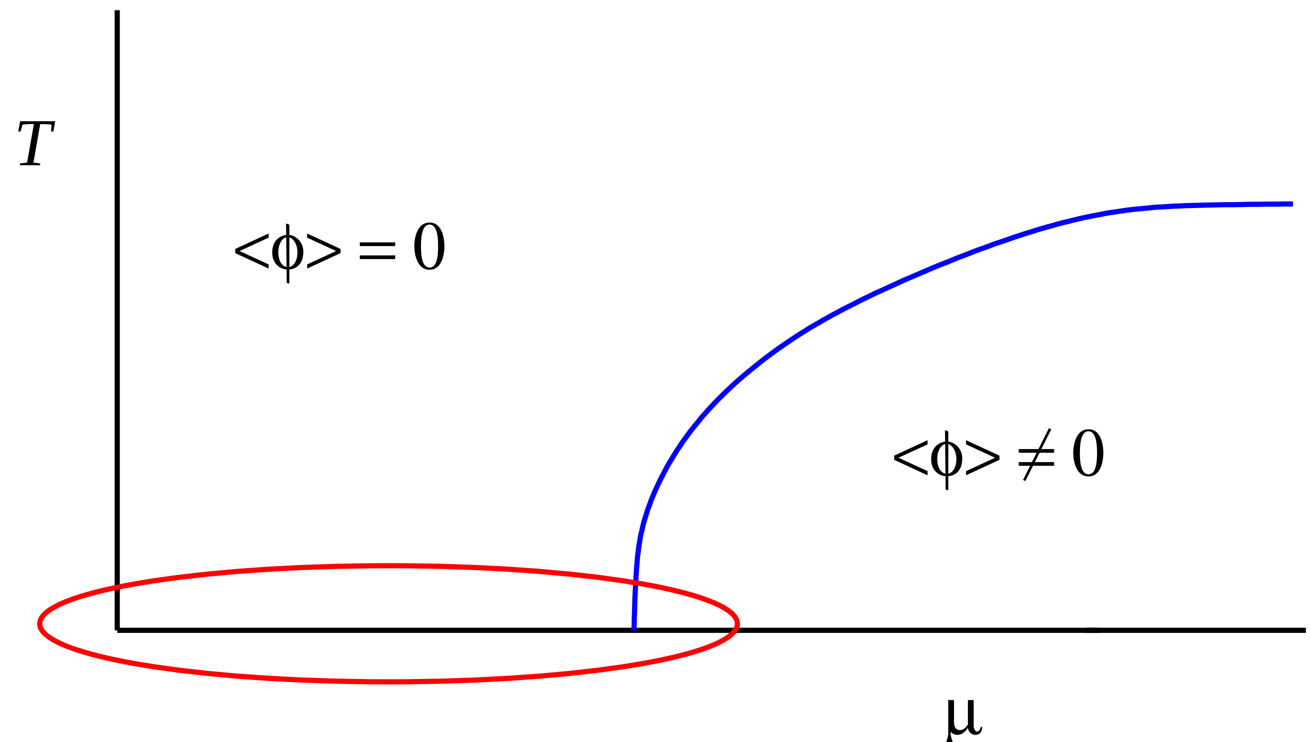
PHASE TRANSITIONS AND THE SILVER BLAZE

tree level potential in the continuum

$$V(\phi) = (m^2 - \mu^2)|\phi|^2 + \lambda|\phi|^4$$

condensation when $\mu^2 > m^2$, SSB

Silver
Blaze
problem



RELATIVISTIC BOSE GAS AT NONZERO μ

COMPLEX LANGEVIN

- write $\phi = (\phi_1 + i\phi_2)/\sqrt{2} \;\; \Rightarrow \;\; \phi_a \;\; (a = 1, 2)$
- complexification $\phi_a \;\; \to \;\; \phi_a^{\mathrm{R}} + i\phi_a^{\mathrm{I}}$
- complex Langevin equations

$$\frac{\partial \phi_a^{\mathrm{R}}}{\partial \theta} = -\mathrm{Re}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a \to \phi_a^{\mathrm{R}} + i\phi_a^{\mathrm{I}}} + \eta_a$$

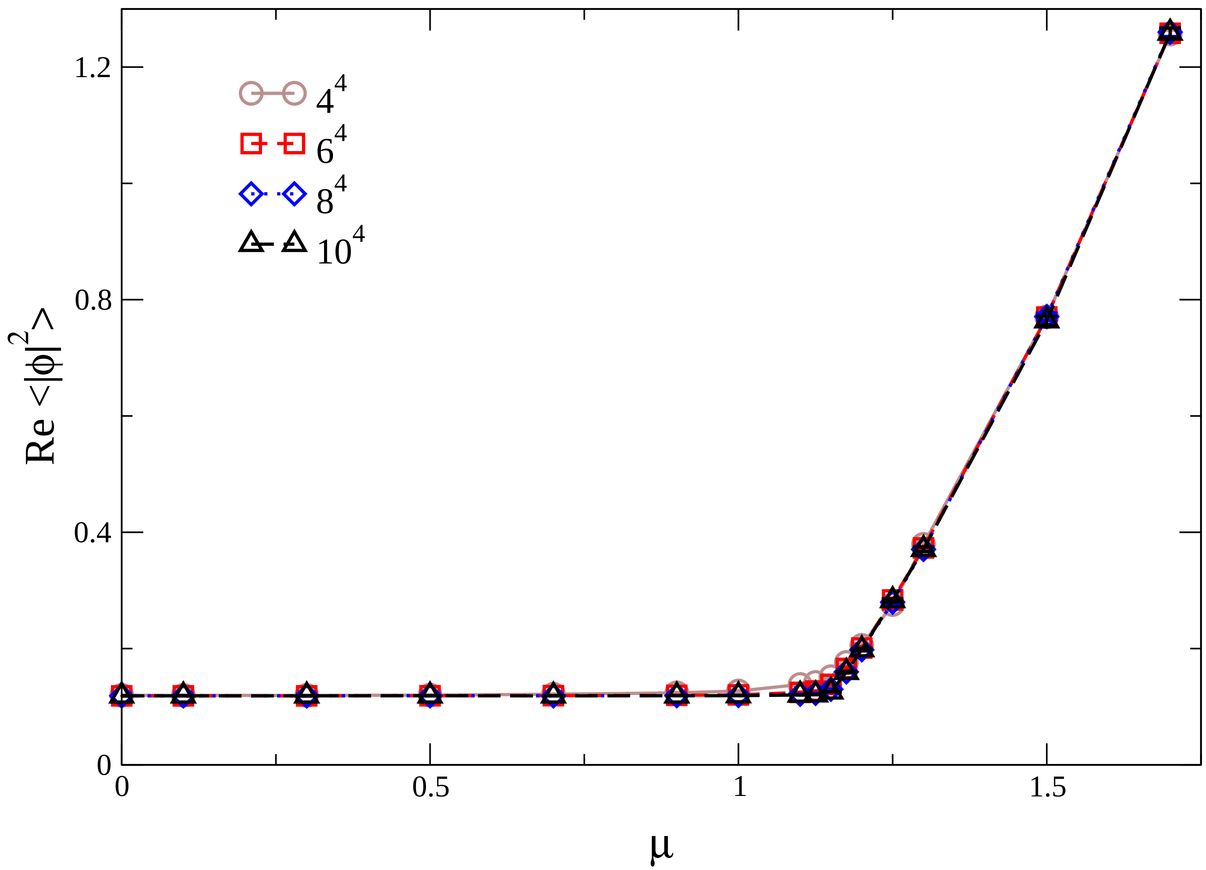
$$\frac{\partial \phi_a^{\mathrm{I}}}{\partial \theta} = -\mathrm{Im}\, \frac{\delta S}{\delta \phi_a}\Big|_{\phi_a \to \phi_a^{\mathrm{R}} + i\phi^{\mathrm{I}}}$$

- straightforward to solve numerically, $m = \lambda = 1$
- lattices of size N^4, with $N = 4, 6, 8, 10$
- no instabilities etc

Relativistic Bose gas

complex Langevin

field modulus squared $\quad |\phi|^2 \to {\phi_1^{\rm R}}^2 - {\phi_1^{\rm I}}^2 + {\phi_2^{\rm R}}^2 - {\phi_2^{\rm I}}^2$

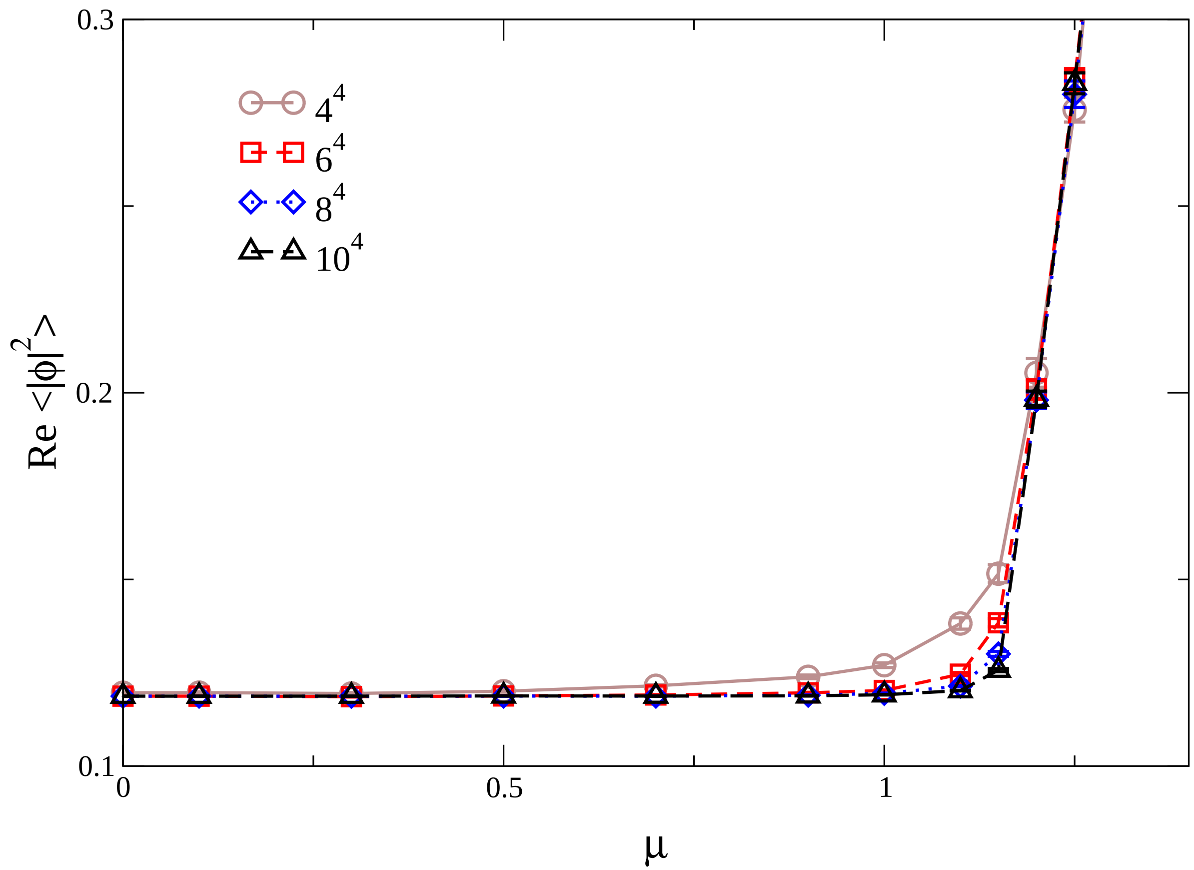


Silver Blaze!

RELATIVISTIC BOSE GAS

COMPLEX LANGEVIN

field modulus squared $\quad |\phi|^2 \to {\phi_1^{\mathrm{R}}}^2 - {\phi_1^{\mathrm{I}}}^2 + {\phi_2^{\mathrm{R}}}^2 - {\phi_2^{\mathrm{I}}}^2$

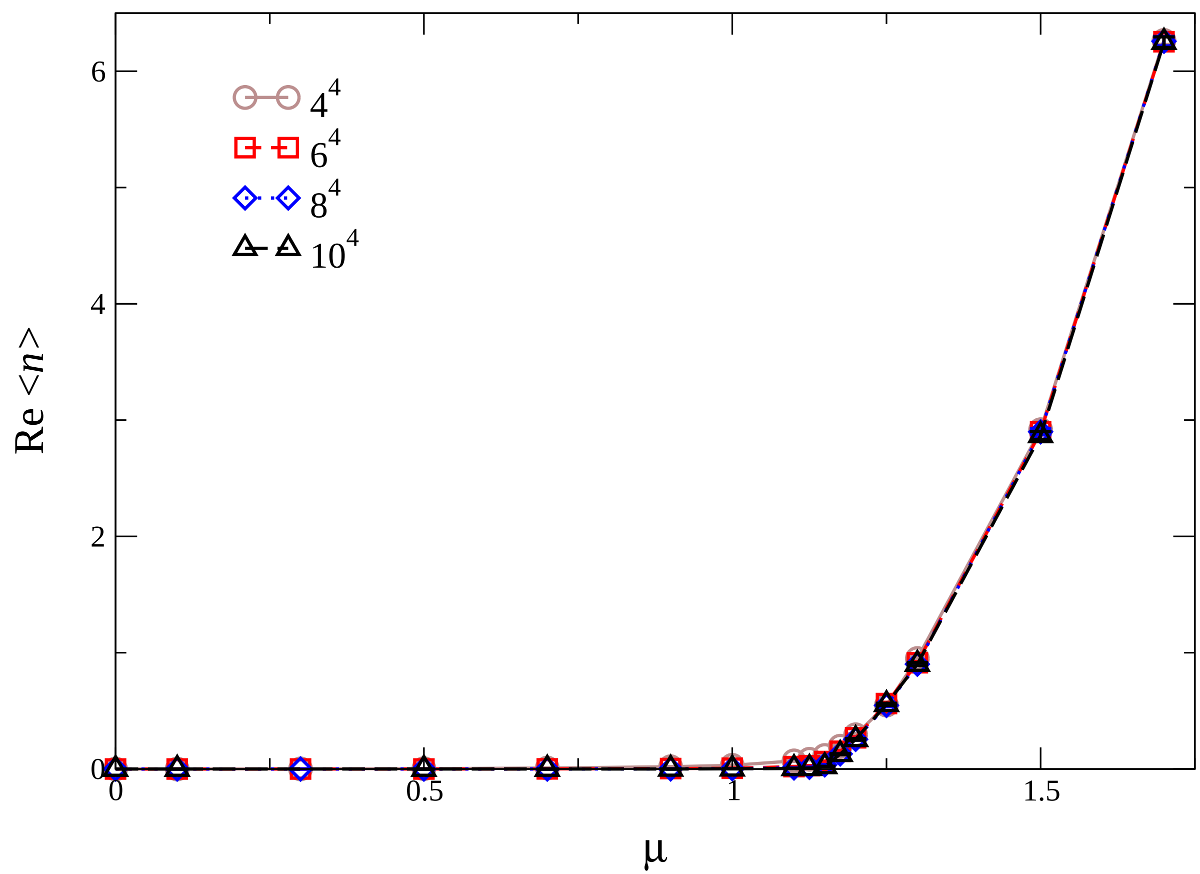


second order phase transition in thermodynamic limit

Relativistic Bose gas

complex Langevin

density $\langle n \rangle = (1/\Omega)\partial \ln Z/\partial\mu$

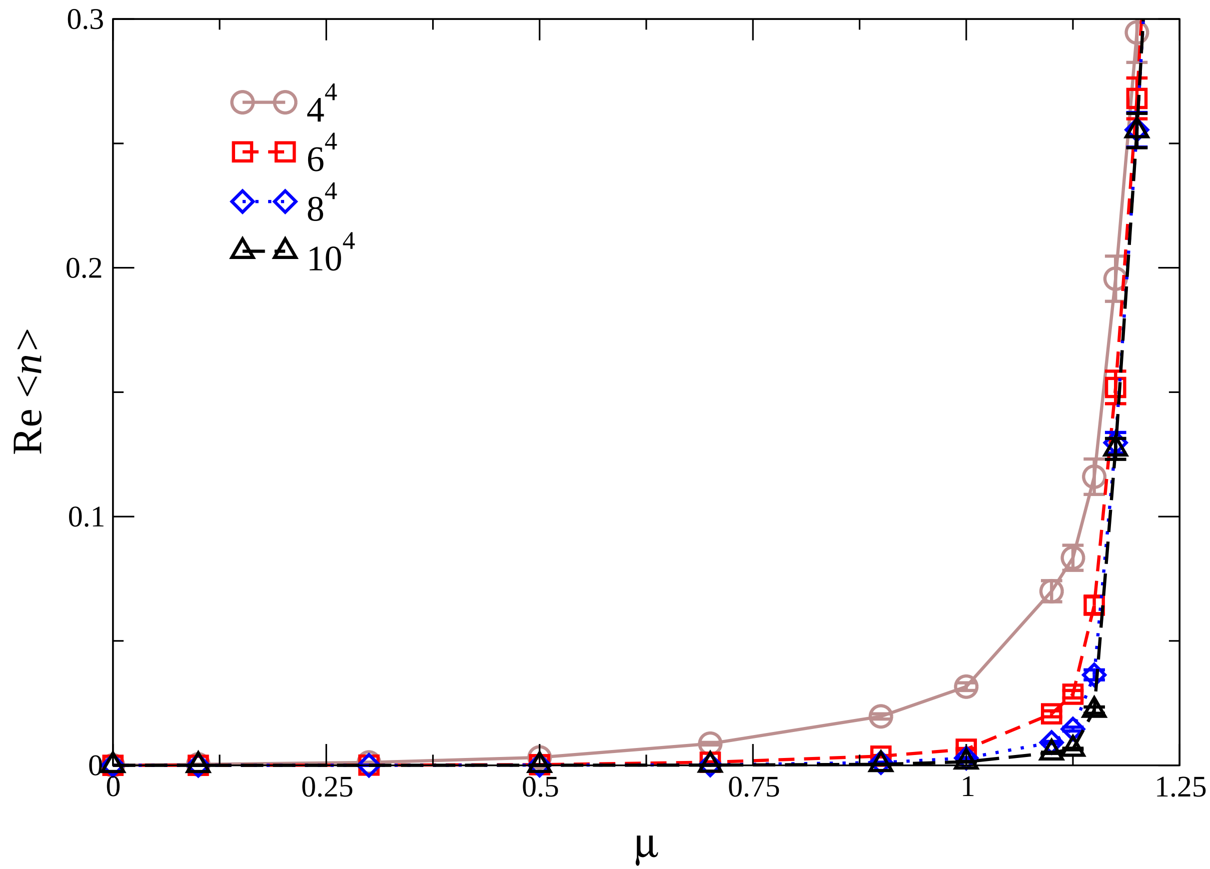


Silver Blaze

Relativistic Bose gas

complex Langevin

density $\langle n\rangle = (1/\Omega)\partial \ln Z/\partial\mu$



second order phase transition in thermodynamic limit
mean field approximation: $\mu_c \sim 1.15$

SILVER BLAZE AND THE SIGN PROBLEM

RELATIVISTIC BOSE GAS

Silver Blaze and sign problems are intimately related

- complex action

$$e^{-S} = |e^{-S}|e^{i\varphi}$$

- phase quenched theory

$$Z_{\rm pq} = \int D\phi |e^{-S}|$$

different physics

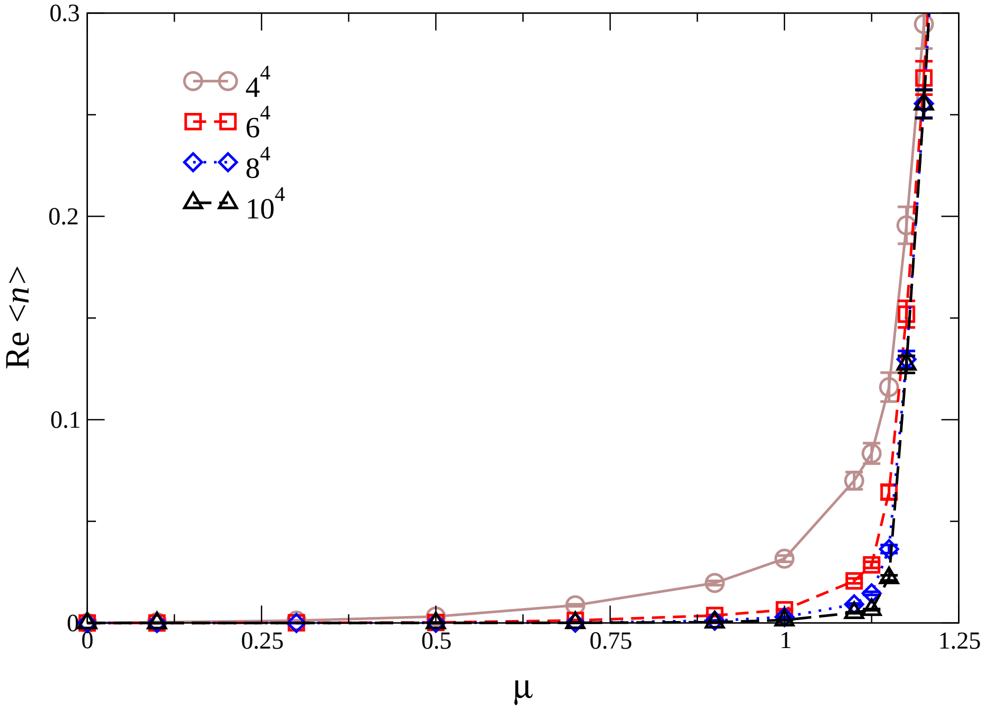
QCD: phase quenched = finite isospin chemical potential

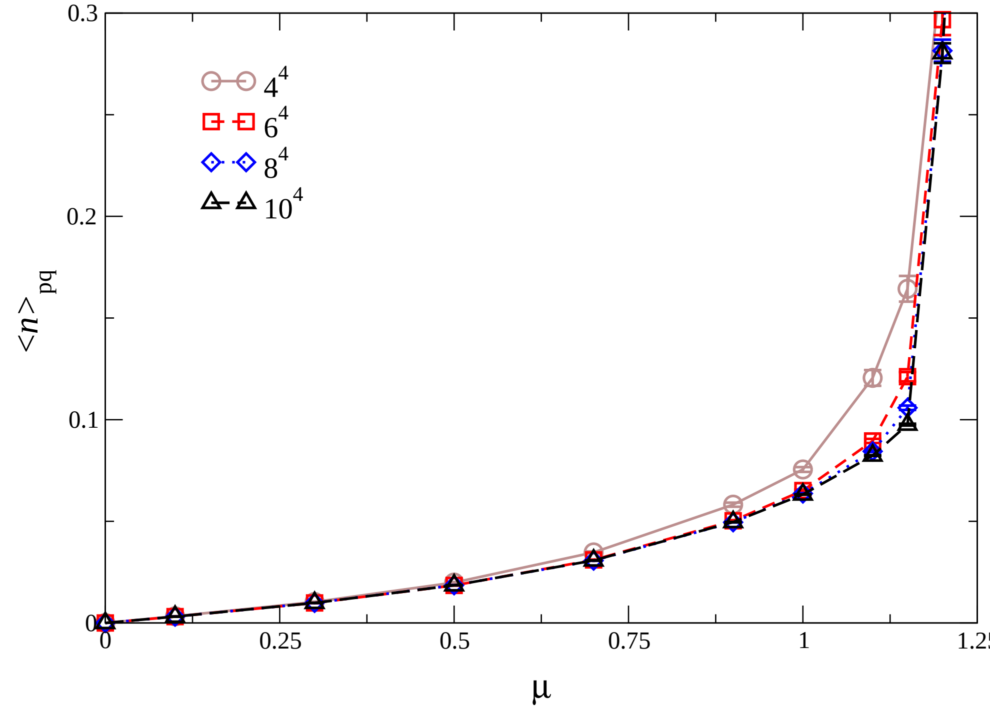
different onset: $m_N/3$ versus $m_\pi/2$

SILVER BLAZE AND THE SIGN PROBLEM

COMPLEX VS PHASE QUENCHED

density





complex   phase quenched

phase $e^{i\varphi} = e^{-S}/|e^{-S}|$ does precisely what is expected

How severe is the sign problem?

AVERAGE PHASE FACTOR

- complex action $e^{-S} = |e^{-S}| e^{i\varphi}$
- full and phase quenched partition functions

$$Z_{\text{full}} = \int D\phi \, e^{-S} \qquad\qquad Z_{\text{pq}} = \int D\phi |e^{-S}|$$

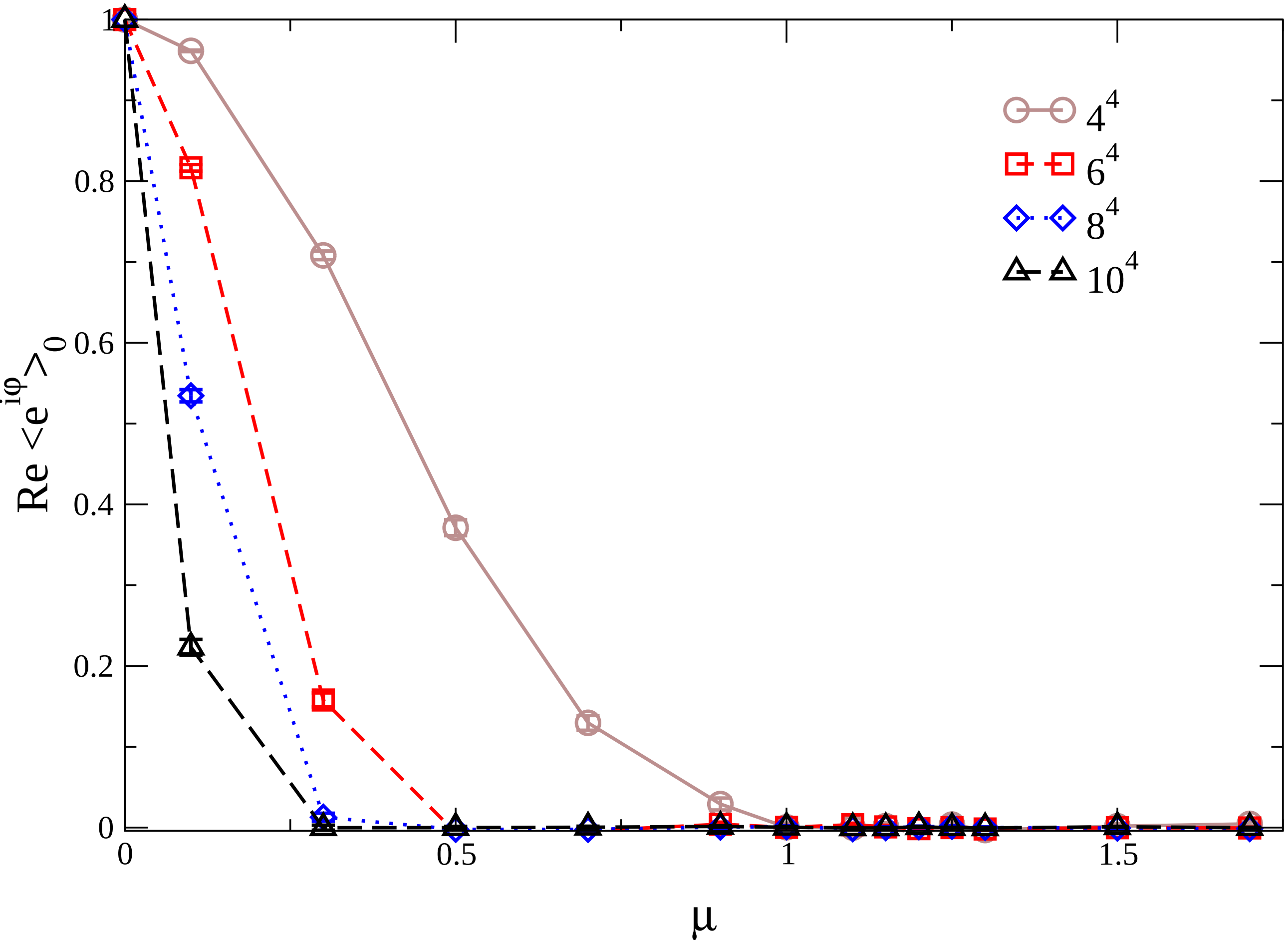
- average phase factor in phase quenched theory

$$\langle e^{i\varphi} \rangle_{\text{pq}} = \frac{Z_{\text{full}}}{Z_{\text{pq}}} = e^{-\Omega \Delta f} \to 0 \quad \text{as} \quad \Omega \to \infty$$

- exponentially hard in thermodynamic limit

How severe is the sign problem?

AVERAGE PHASE FACTOR



average phase factor $\langle e^{i\varphi}\rangle_{\mathrm{pq}}$

How severe is the sign problem?

AVERAGE PHASE FACTOR

- phase factor behaves exactly as expected
- for larger μ: phase factor $\to 0$ on all volumes
  - in the condensed phase: phase factor $= 0$
- at small μ, sign problem gets exponentially worse with increasing volume

yet, no problem in practice

SUMMARY & OUTLOOK

many stimulating results

- one link models: excellent agreement
- relativistic Bose gas: phase transition and Silver Blaze
- QCD with static quarks: encouraging

why does it work?
partly understood in simple models and relativistic Bose gas

in progress:

- analytical insight in the relativistic Bose gas
- QCD with static and dynamical quarks
- ...