Gluon distribution functions from lattice QCD

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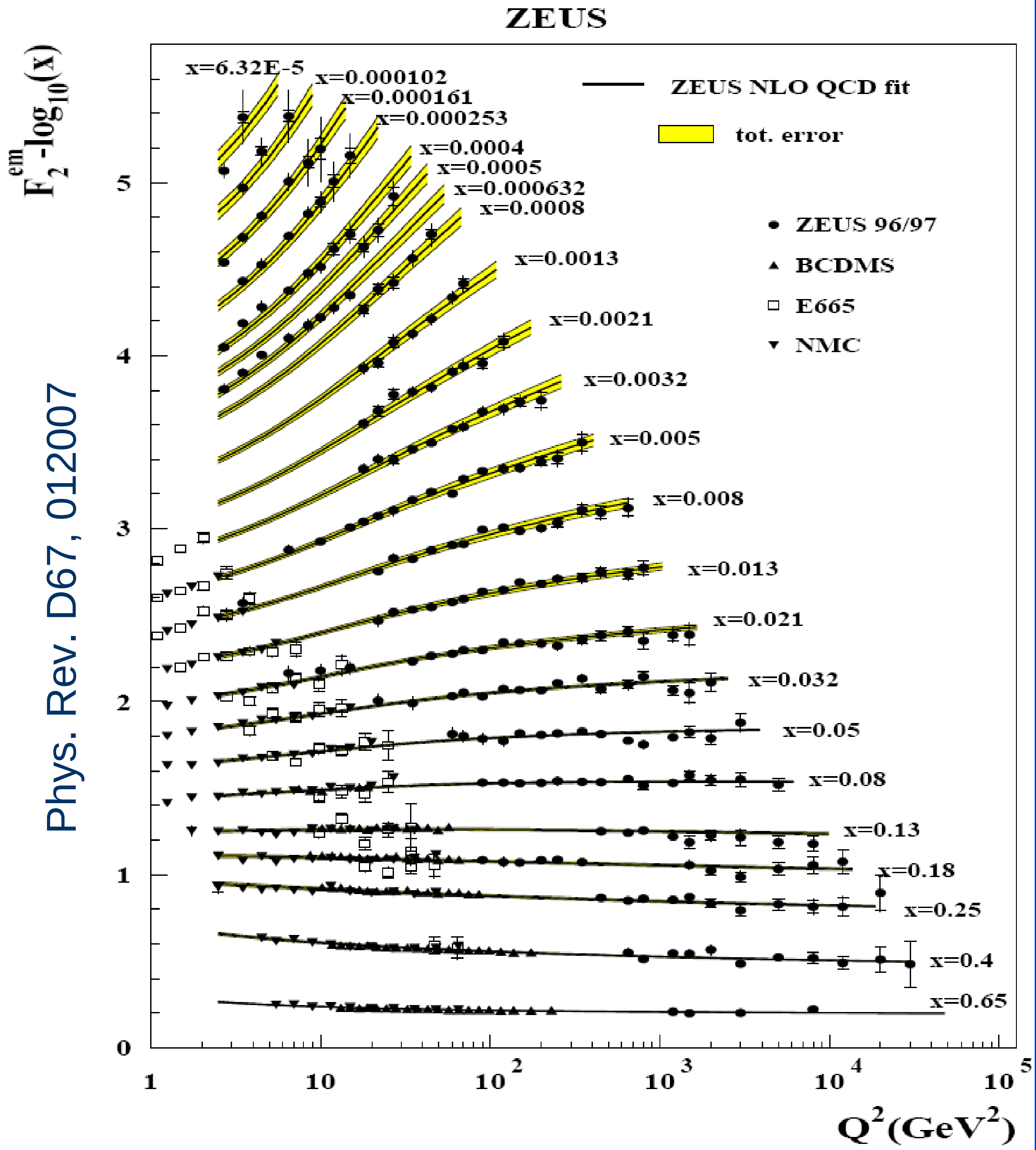
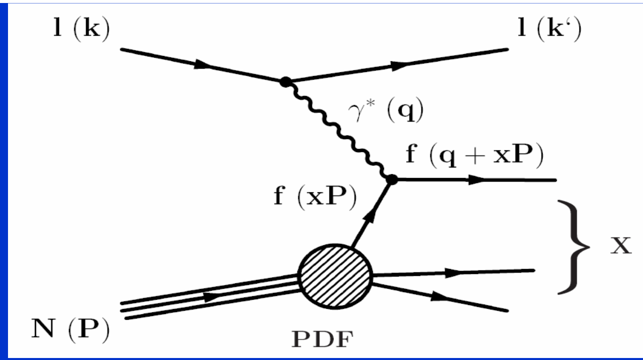
Outline

- Review and Motivation
- Introduction to Near-Light-Cone coordinates
- Hamiltonian lattice formulation near the light cone:
  - Hamiltonian
  - Trial wave functional optimization
- Determination of gluon distribution functions of a color dipole on the lattice
- Conclusions

Phys. Rev. D67, 012007





- QCD factorization theorem tells us separation of hard from soft physics
- Evolve u,d valence,g and total sea quark distributions by DGLAP

$$xf(x) = p_1 x^{p_2}(1-x)^{p_3}(1+p_5 x)$$

- Fit to data
- Excellent agreement
  Success story for perturbative QCD
- However: This is an ansatz. Is there a possibility to com-pute the structure function at some input scale directly?

Gluon distribution function

Collins and Soper, Nucl. Phys. B194, 445

$$g(x_B) = \frac{1}{x_B}\frac{1}{2\pi}\int_{-\infty}^{\infty} dz^- \, e^{-\mathrm{i}x_B p_- z^-} \frac{1}{p_-} \langle h(\vec{p})| G(z^-) |h(\vec{p})\rangle_c \Big|_{z^+=0}$$

$$\langle h(\vec{p})| \, G(z^-) \, |h(\vec{p})\rangle_c = \langle h(\vec{p})| \, G(z^-) \, |h(\vec{p})\rangle - \langle \Omega| \, G(z^-) \, |\Omega\rangle \, \langle h(\vec{p}) \, |h(\vec{p})\rangle$$

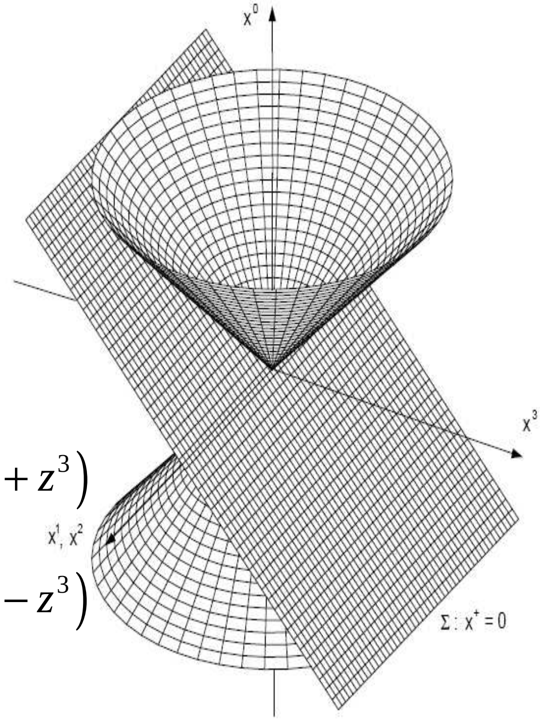
$$\langle h(\vec{p})| \, h(\vec{p})\rangle = 2 \, p_- V \, .$$

$$G(z^-) = \sum_{k=1}^{2} F^a_{-k}(z^-, \vec{0}_\perp) \, S^A_{ab}(z^-, 0\, ; \vec{0}_\perp) \, F^b_{-k}(0, \vec{0}_\perp)$$

$$S^A_{ab}(z^-, 0,) = \left[\mathcal{P}\exp\left\{\mathrm{i}\, g \int_0^{z^-} dv^- \, A^c_-(v^-, \vec{0}_\perp)\, \lambda^c_{adj}\right\}\right]_{ab}$$

$$z^+ = \frac{1}{\sqrt{2}}\left(z^0 + z^3\right)$$

$$z^- = \frac{1}{\sqrt{2}}\left(z^0 - z^3\right)$$



- Hadron is probed at equal light cone time
  $\Longrightarrow$ Static problem in light cone quantization

Motivation

- Structure functions at input scale not computable perturbatively (manifestly non-perturbative) $\Longrightarrow$ lattice methods
- Euclidean equal time lattice methods capable of computing moments by OPE (Martinelli and Sachrajda Nucl. Phys. B 306,865)
- Light cone quantisation seems to be natural to describe high energy scattering
- Is there a way to combine light cone quantization with lattice methods ?
  - Yes: transverse lattice method (Bardeen et al. Phys. Rev. D21,1037)
  - Yes: Lattice QCD near the light cone (Wilsonian approach)

    (D.G, E.-M. I., H.-J. P. and E.P.: Phys.Rev.D77:014512,2008)

Near light-cone coordinates

Prokhvatilov et. al, Sov. J. of Nucl. Phys.49 (688); Lenz et. al, Annals of Physics 208 (1-89)

- Transition to NLC coordinates is a two step process
  - Lorentz boost to a fast moving frame with relative velocity

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^3) \\ x'^3 &= \gamma(x^3 - \beta x^0) \end{aligned}$$

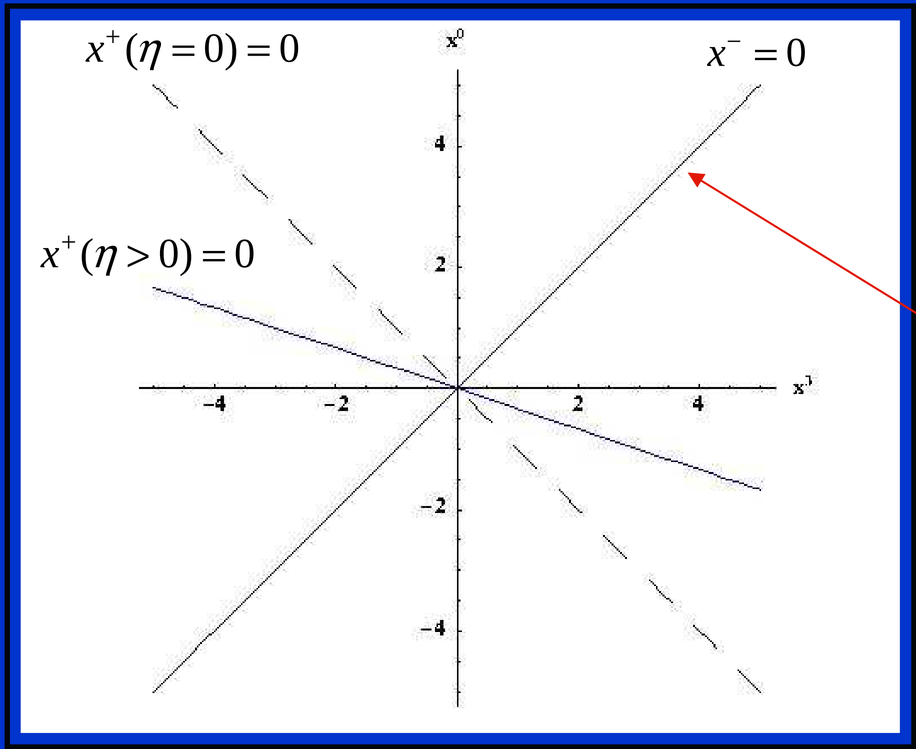
$$\beta = \frac{1-\eta^2/2}{1+\eta^2/2}$$

  - Rotation in the x'^0-x'^3-plane

$$\begin{aligned} x^+ &= \frac{1}{\sqrt{2}}\left[\left(1+\frac{\eta^2}{2}\right)x'^0 + \left(1-\frac{\eta^2}{2}\right)x'^3\right] \\ x^- &= \frac{1}{\sqrt{2}}\left[x'^0 - x'^3\right] \end{aligned}$$

- Allows interpolation between equal-time $\eta^2 = 2$ and light-cone quantization $\eta^2 = 0$
- Introduced to investigate light-cone quantization as a limiting procedure of equal time theories

Near light-cone coordinates



Light Cone time axis along which the system evolves

- scalar product

$$
\begin{aligned}
x_\mu y^\mu &= x^- y^+ + x^+ y^- - \eta^2 x^- y^- - \vec{x}_\perp \vec{y}_\perp \\
&= x_- y_+ + x_+ y_- + \eta^2 x_+ y_+ - \vec{x}_\perp \vec{y}_\perp
\end{aligned}
$$

- spatial distance ($\Delta x^+ = 0$)

$$\Rightarrow R^2 = -\eta^2 (\Delta x^-)^2 - (\Delta \vec{x}_\perp)^2$$

NLC correlation function

- Momentum sum rule of the gluon distribution function:

$$\langle x_B \rangle = \int_0^1 dx_B\, x_B\, g(x_B) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz^- \int_0^{\infty} \frac{dp_-^g}{p_-^2}\, e^{-\mathrm{i}p_-^g z^-} \langle h(\vec{p})|\, G(z^-)\, |h(\vec{p})\rangle_c$$

$$= \int_{-\infty}^{\infty} dz^- \frac{1}{2\,p_-^2}\delta(z^-) \langle h(\vec{p})|\, G(z^-)\, |h(\vec{p})\rangle_c$$

$$G(z^-)\big|_{z^-=0} = \mathcal{P}_-^{lc} \qquad \langle x_B \rangle = \frac{1}{2\,p_-^2} \langle h(p)|\, \mathcal{P}_-^{lc}\, |h(p)\rangle$$

- Same feature in nlc coordinates desirable

$\Longrightarrow$ introduce a generalization of the nlc longitudinal momentum operator

$$P_-^{NLC} = \frac{1}{2}\sum \int d^3x \left(\Pi_k^a F_{-k}^a + F_{-k}^a \Pi_k^a \right)(\vec{x})$$

$$g_{NLC}(x_B) = \frac{1}{2\pi x_B p\_L_-} \int dz^- dz^{-'}\, e^{-i x_B p_-(z^- - z^{-'})} \langle h(\vec{p}) | G_{NLC}(z^-, z^{-'}) | h(\vec{p}) \rangle_c$$

$$G_{NLC}(z^-, z^{-'}) = \frac{1}{4}\sum_k \Big(\Pi_k^a(z^-) S_-^{ab}(z^-, z^{-'}) F_{-k}^a(z^{-'})$$
$$+ \Pi_k^a(z^{-'}) S_-^{ab}(z^{-'}, z^-) F_{-k}^a(z^-) + h.c. \Big)$$

The lattice Hamiltonian

- $$U_i(x) \equiv \mathcal{P}\exp\left(\mathrm{i}\; g \int_x^{x+\widehat{e}_i} dy_\mu\; A_\mu^a(y)\; \frac{\sigma_a}{2}\right)$$ $$\xi \equiv \frac{a_-}{a_\perp}$$ $$\lambda \equiv \frac{4}{g^4} = \left(\frac{1}{2}\beta\right)^2$$
- Derivation of the lattice Hamiltonian from the path integral formulation with the transfer matrix-method (Creutz Phys. Rev. D 15, 1128):

$$\mathcal{H}_{\mathrm{lat}} = \frac{1}{N_- N_\perp^2} \frac{1}{a_\perp^4} \frac{2}{\sqrt{\lambda}} \sum_{\vec{x}} \left\{ \sum_a \frac{1}{2} \Pi_-^a(\vec{x})^2 + \frac{1}{2}\lambda \,\mathrm{Tr}\left[\mathbb{1} - \mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right] \right.$$
$$\left. + \sum_{k,a} \frac{1}{2} \frac{1}{\xi^2 \eta^2} \left[\Pi_k^a(\vec{x}) - \sqrt{\lambda}\, \mathrm{Tr}\left[\frac{\sigma_a}{2}\, \mathrm{Im}\Big(U_{-k}(\vec{x})\Big)\right]\right]^2 \right\} .$$

$$\left[\Pi_j^a(\vec{x}), U_{j'}(\vec{x}')\right] = \frac{\sigma_a}{2} U_j(\vec{x})\, \delta_{j,j'}\, \delta_{\vec{x},\vec{x}'} ,$$
$$\left[\Pi_j^a(\vec{x}), U_{j'}^\dagger(\vec{x}')\right] = -U_j^\dagger(\vec{x})\, \frac{\sigma_a}{2}\, \delta_{j,j'}\, \delta_{\vec{x},\vec{x}'}$$

$$U_{ij}(\vec{x}) = U_i(\vec{x}) U_j(\vec{x}+\vec{e}_i) U_i^\dagger(\vec{x}+\vec{e}_j) U_j^\dagger(\vec{x})$$
$$\mathrm{Im}\, U_{ij}(\vec{x}) = (U_{ij}(\vec{x}) - U_{ij}^\dagger(\vec{x}))/2i$$

- Hamiltonian does only depend on the product $\tilde{\eta} \equiv \xi \cdot \eta$
  - Only one effective parameter is needed
  - Light cone limit might be interpreted in two ways:
    - Light cone limit with equal lattice constants $\xi = 1$
    - Effective equal time theory with vanishing anisotropy $\eta^2 = 2$

Variational optimization

- Trial wavefunctional

$$\Psi_0(\rho,\delta) \;=\; \prod_{\vec{x}} \exp\left\{ \sum_{k=1}^{2} \rho\, \mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x}) \Big) \right] + \delta\, \mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x}) \Big) \right] \right\}$$

  - Restrict to product of single plaquette wavefunctionals

- Optimize the energy density with respect to ρ and δ
- The expectation values are computed via the prob. measure

$$dP(U) \;=\; |\Psi_0(a,b)|^2 \prod_{\vec{x},j} \mathcal{D}U_j(\vec{x})$$

- Fit of optimized wave functional parameters with extrapolation to $\eta \longrightarrow 0$ yields

$$\rho_0(\lambda,0) \;=\; \left(0.65 - \frac{0.87}{\lambda} + \frac{1.65}{\lambda^2}\right)\sqrt{\lambda}$$

$$\delta_0(\lambda,0) \;=\; \left(0.05 + \frac{0.04}{\lambda} - \frac{1.39}{\lambda^2}\right)\sqrt{\lambda}$$

$$\left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}\Big)\right]\right\rangle_{\Psi_0(\rho_0,\delta_0)} = \frac{I_2(4\,\rho_0)}{I_1(4\,\rho_0)} + \mathcal{O}\left(\rho_0^3, \rho_0\,\delta_0^2\right)$$

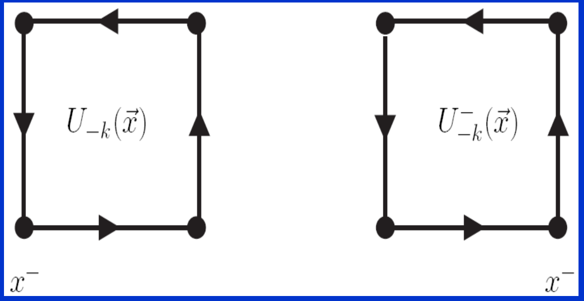
$$\Big\langle W_{ij}(n,m)\Big\rangle_{\Psi_0(\rho_0,\delta_0)} = \left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\Big(U_{ij}\Big)\right]\right\rangle^{n\cdot m}_{\Psi_0(\rho_0,\delta_0)}$$

- However, the trial wave functional does not allow for a continuum extrapolation

- Correlation function on the lattice:

$$
\begin{aligned}
&a_-^2\, a_\perp^2\, G_{\rm lat}(z^-, z'^-) \\
&= \frac{1}{4}\frac{1}{N_-}\sum_k \Big(2\,\Pi_k^a(z^-, \vec{0}_\perp)\, S_{ab}^A(z^-, z'^-; \vec{0}_\perp)\, {\rm Tr}\left[\frac{\sigma^a}{2}\,{\rm Im}\Big(\overline{U}_{-k}(z'^-, \vec{0}_\perp)\Big)\right] \\
&\qquad + 2\,\Pi_k^a(z'^-, \vec{0}_\perp)\, S_{ab}^A(z'^-, z^-; \vec{0}_\perp)\, {\rm Tr}\left[\frac{\sigma^a}{2}\,{\rm Im}\Big(\overline{U}_{-k}(z^-, \vec{0}_\perp)\Big)\right] + h.c. \Big)
\end{aligned}
$$

$$
\begin{aligned}
\overline{U}_{-k}(\vec{x}) &\equiv \frac{1}{2}\left(U_{-k}^{+}(\vec{x}) + U_{-k}^{-}(\vec{x})\right), \\
U_{-k}^{+}(\vec{x}) &\equiv U_-(\vec{x})U_k(\vec{x}+\hat{e}_-)U_-^\dagger(\vec{x}+\hat{e}_k)U_k^\dagger(\vec{x}), \\
U_{-k}^{-}(\vec{x}) &\equiv U_k(\vec{x})U_-^\dagger(\vec{x}+\hat{e}_k-\hat{e}_-)U_k^\dagger(\vec{x}-\hat{e}_-)U_-(\vec{x}-\hat{e}_-)
\end{aligned}
$$



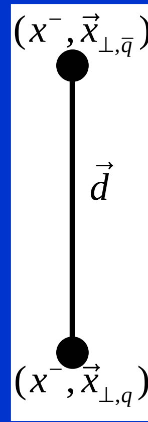
- Infinite momentum frame -> dipole state has maximal momentum on the lattice

$$p_- = \frac{2\pi}{N_-\, a_-}(N_-/2 - 1)$$

- $x_B\, p_-$ has to be a valid lattice momentum

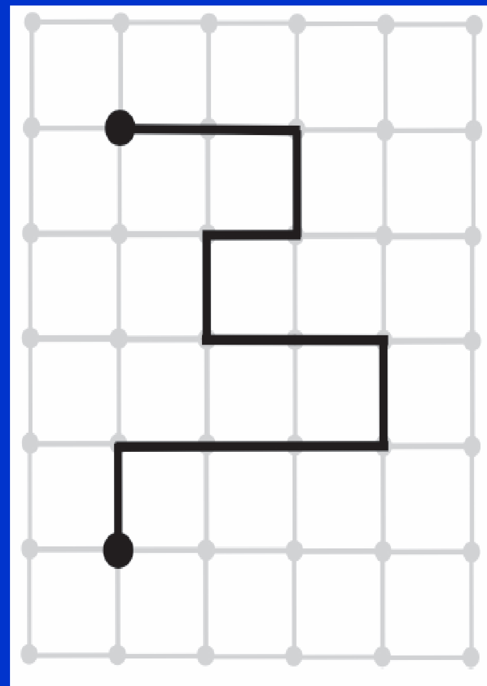
$$x_B\, p_- = \frac{2\pi}{N_- a_-} n \qquad x_B = \frac{2n}{N_- - 2}\ , \quad n = 0, \ldots, N_-/2 - 1$$

Color-Dipole Model



- Consider scalar QCD $\Phi(x) = \Phi_+(x) + \Phi_-(x)$
- Interpolating color-dipole field of two heavy quarks with separation $\vec{d}$ in config. space:

  $\Longrightarrow$ Gluons represented by Schwinger string
- Equip with total momentum $(p_-,\vec{p}_\perp) = (p_-,\vec{0}_\perp)$



$$\left|d(p_-,\vec{d}_\perp)\right\rangle = \frac{1}{(2\pi)^3}\int\prod_{j=1}^{n}\frac{dp_-^{l\,j}}{(2\pi)}\int dx^-\,d^2x_\perp\,e^{-\mathrm{i}p_-x^-}\,\Phi_+^\dagger(x^-,\vec{x}_\perp)$$
$$\cdot\left[\int\prod_{j=1}^{n}dy_j^-\,e^{-\mathrm{i}\sum_{j=1}^{n}p_-^{l\,j}y_j^-}\,S_{q\,\bar{q}}(x^-,\{x^-+y_j^-\},\vec{x}_\perp)\right]\Phi_-(x^-,\vec{x}_\perp+\vec{d}_\perp)\,|\Omega\rangle$$

$$p_-^S = p_- - p_-^q - p_-^{\bar{q}}$$

Gluon distribution function of the color dipole

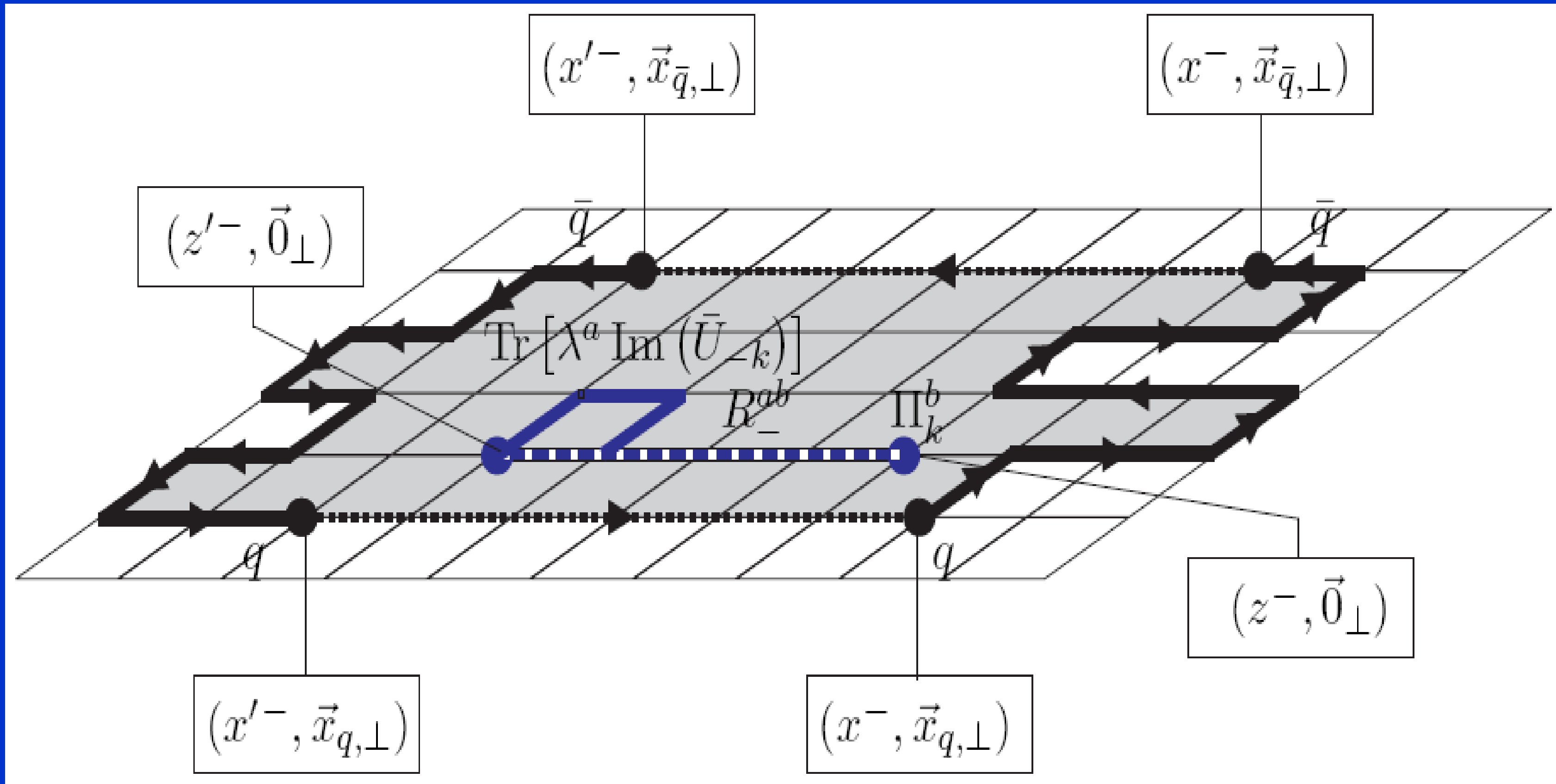
- Eliminate quark operators with the eikonal approximation of the propagator

$$g_n(x_B; p_-^S; p_-) = \lim_{\eta \to 0} \frac{1}{x_B\, p_-} \frac{1}{N_-^2} \sum_{z^-, z'^-} e^{-\mathrm{i} x_B\, p_-(z^- - z'^-)} \langle d(p_-, n \cdot \vec{e}_\perp) | \, G(z^-, z'^-) \, | d(p_-, n \cdot \vec{e}_\perp) \rangle_c$$

$$\begin{aligned}
&\langle d(p_-, n \cdot \widehat{e}_\perp | \, G(z^-, z^{-\prime}) \, | d(p_-, n \cdot \widehat{e}_\perp) \rangle \\
&= \frac{2\, p_- V}{N} \sum_{\vec{x}_\perp} \sum_{\{y_j^-\}, \{y_j'^-\}} \rho_n(p_-^S, \{y_j^-\}, \{y_j'^-\}) \\
&\qquad \cdot \langle \Psi_0 | \left[S_{q\bar{q}}(0, \{y_j'^-\}, \vec{x}_\perp)^\dagger \right]_{ab} G(z^-, z^{-\prime}) \left[S_{q\bar{q}}(0, \{y_j^-\}, \vec{x}_\perp) \right]_{ba} | \Psi_0 \rangle
\end{aligned}$$

$$\begin{aligned}
\rho_n(p_-^S, \{y_j^-\}, \{y_j'^-\}) &= \sum_{\{p_-^{l\,j}\}, \{p_-^{l\,j\,\prime}\}} \delta(p_-^S - \sum_{j=1}^n p_-^{l\,j}) \, \delta(p_-^S - \sum_{j=1}^n p_-^{l\,j\,\prime}) \, e^{-\mathrm{i} \sum_{j=1}^n \left(p_-^{l\,j}\, y_j^- - p_-^{l\,j\,\prime}\, y_j'^- \right)} \\
p_-^S &= p_- - p_-^q - p_-^{\bar{q}}
\end{aligned}$$

Visualization of matrix elements



Recursion relation

- Increase of number of transversal links corresponds to an increase of resolution Q^2 if the physical dipole size is fixed
- Gluon distribution function obeys DGLAP type of evolution equation

$$g_n(x_B;p_-^S;p_-)=\sum_{p_-^{S'}=0}^{p_-^S} g_{n-1}(x_B;p_-^{S'};p_-)\,P_{n\to n-1}(p_-^{S'};p_-^S)$$

$$P_{n\to n-1}(p_-^{S'};p_-^S)=\frac{F_{n-1}(p_-^{S'})F_1(p_-^S-p_-^{S'})}{\sum_{p_-^{S''}=0}^{p_-^S} F_{n-1}(p_-^{S''})F_1(p_-^S-p_-^{S''})}\frac{n}{n-1}$$

$$F_n(p_-^S)=\sum_{\{y_j^-\},\{y_j^{'-}\}}\rho_n(p_-^S;\{y_j^-\},\{y_j^{'-}\})\prod_{j=1}^{n}\left\langle 1/2\,Tr\left[U_{-k}\right]\right\rangle^{|y_j^{'-}-y_j^-|}$$

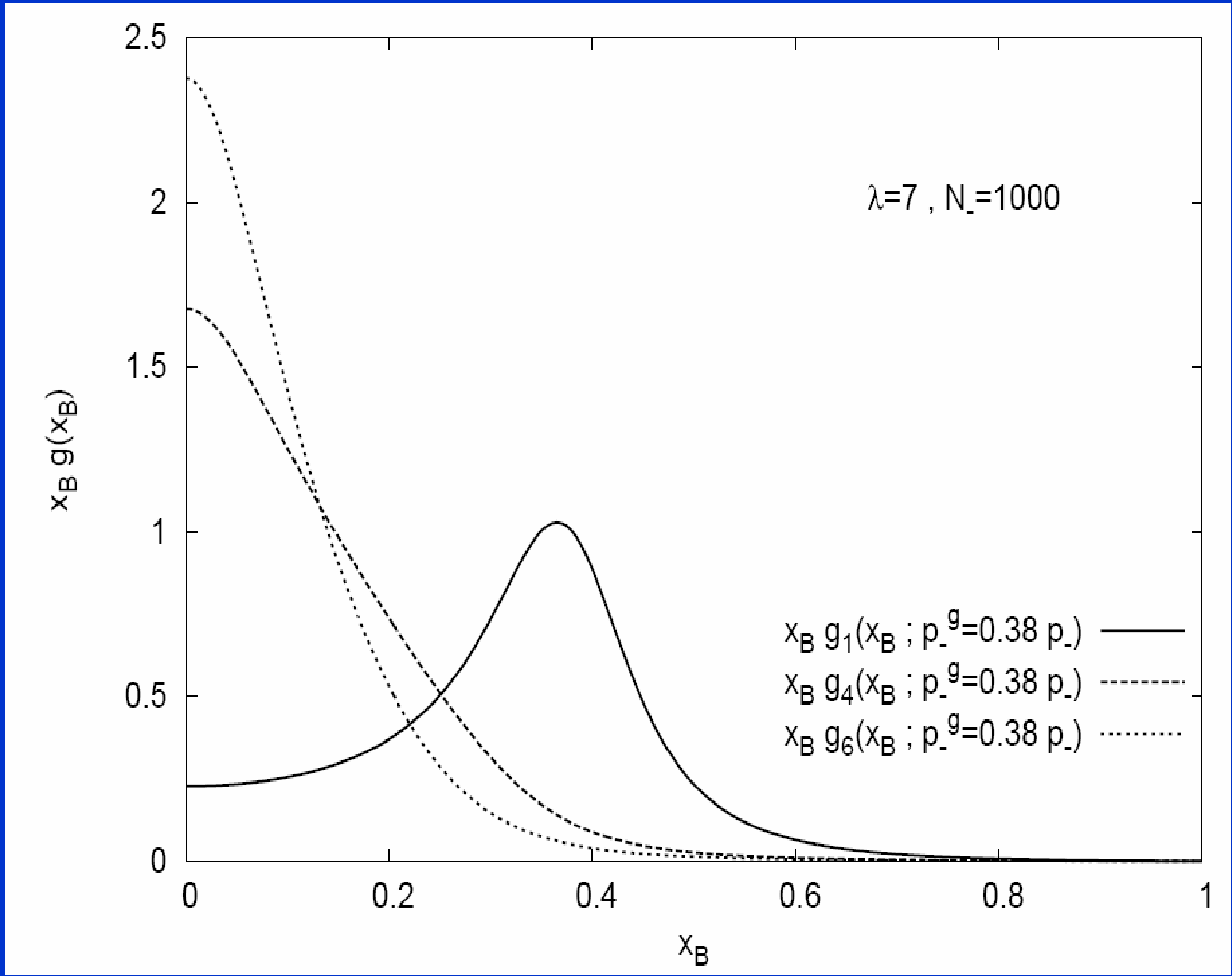
- Indeed, if one assumes

$$p_-^S=p_-\ ,\ p_-\to\infty\ ,\ g_n(x_B;p_-;p_-)=g_n(x_B)\ \Rightarrow$$

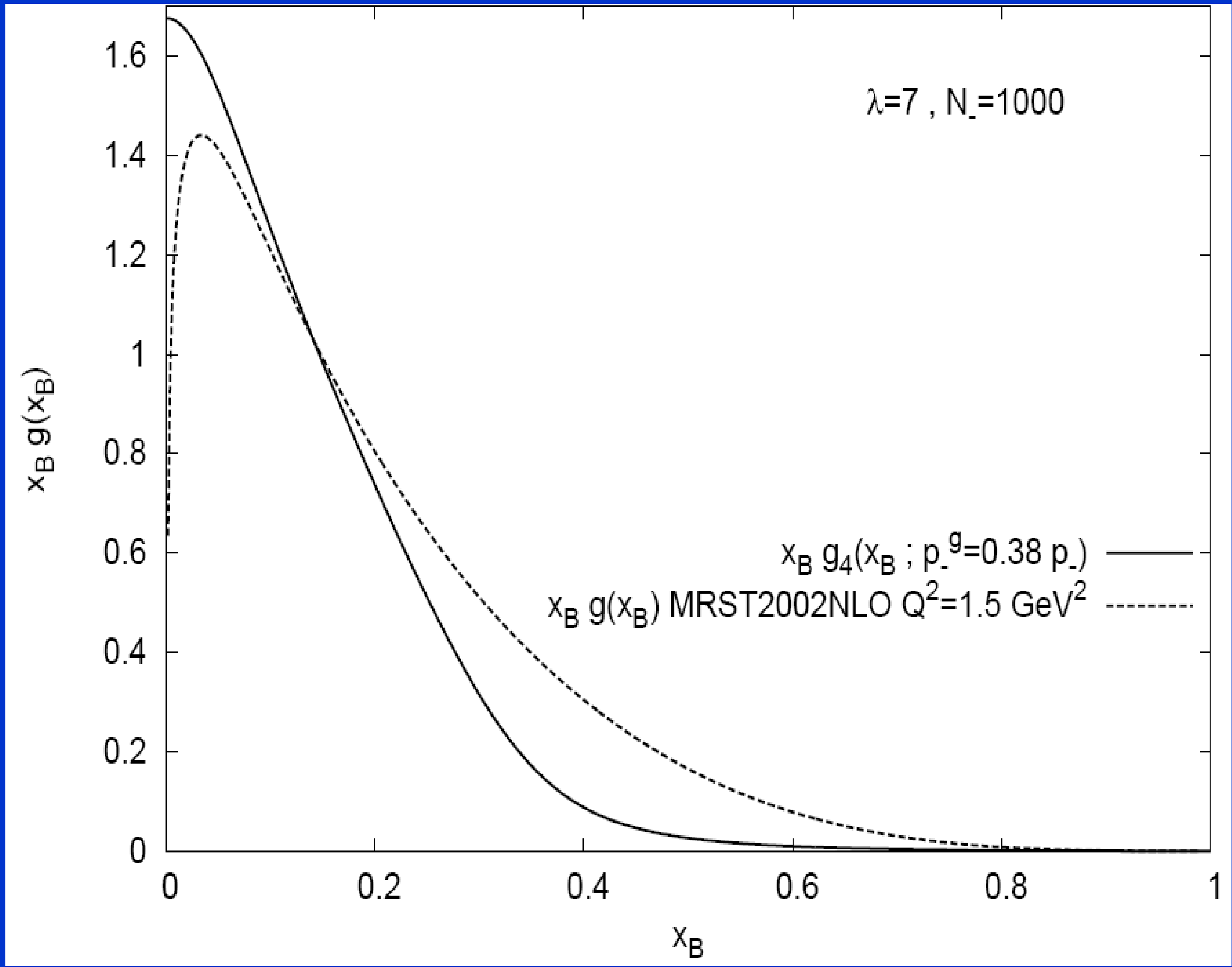
$$g_n(x_B)=\int_{x_B}^{1} dz_B\, g_{n-1}\left(x_B/z_B\right)P_{n\to n-1}(z_B)$$

Results as a function of the dipole size

- Set p_-^S to $p_-^S = \left\langle x_B^g \right\rangle_{NLO\,MRST\,2002}\Big|_{Q^2=1.5GeV^2} p_-$ (Eur.Phys.J. C28 (2003) 455-473)



Comparison with „experiment“



Summary and conclusions:

- Near light cone coordinates are well suited to describe high energy scattering on the lattice. In particular, they allow in principle the determination of entire parton distribution functions
- Euclidean path integral treatments of the theory are not possible due to complex phases during the update process
- An effective lattice Hamiltonian avoiding this problem can be derived
- Ground state wave functionals have been constructed for strong and weak coupling which motivate a variational Ansatz valid over the whole coupling regime
- We model a color dipole state equipped with longitudinal momentum on top of the variational ground state

- We find the full gluon distribution function g(xB) for this state
  - It obeys a DGLAP type of evolution
  - Nice agreement with „experimental“ data
- Outlook:
  - Use improved ground state wave functional in the gluonic sector

Thank you for your attention…

Backup Slides

Problems of LGT near the LC

• Euclidean gluonic Lagrange density

$$x^+ = -\mathrm{i}\, x_E^+ \qquad S = \mathrm{i}\int d^4x_E \mathcal{L}_E \equiv \mathrm{i}S_E \qquad Z = \int DAe^{-S_E}$$

$$\mathcal{L}_E \equiv \frac{1}{2}F^a_{+-}F^a_{+-} + \sum_k \left(\frac{\eta^2}{2}F^a_{+k}F^a_{+k} - \mathrm{i}F^a_{+k}F^a_{-k}\right) + \frac{1}{2}F^a_{12}F^a_{12}$$

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g\, f^{abc} A_\mu{}^b A_\nu{}^c$$

• a complex action remains (similar to finite baryonic density) -> sign problem

• Possible way out: Hamiltonian formulation

$\Longrightarrow$ Sampling of the ground state wavefunctional with guided diffusion quantum Monte-Carlo

$$\begin{aligned} |\Psi_0\rangle &= \lim_{t\to\infty} \exp\left[-t\left(\widehat{H}_0 - E\right)\right]|\Phi\rangle \\ &= \lim_{\substack{\Delta t\to 0\\ N\Delta t\to\infty}} \prod_{n=1}^{N}\left\{\exp\left[-\Delta t\left(\widehat{H}_0 - E\right)\right]\right\}|\Phi\rangle \end{aligned}$$

Analytic asymptotic solutions

- Strong coupling wavefunctional (perturbation theory)

$$\lambda = \frac{4}{g^4}$$

$$\left|\Psi_0\right\rangle = \prod_{\vec{x}} \exp\left\{\frac{1}{3}\,\lambda\,\tilde{\eta}^2 \mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right] \right.$$
$$\left. \frac{1}{16}\,\frac{\lambda}{1+\tilde{\eta}^2}\sum_k \Big(\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right]\Big)^2\right\}\left|\Psi_0^{(0)}\right\rangle + \mathcal{O}(\lambda^2)$$

  - Product state of single plaquette wavefunctionals

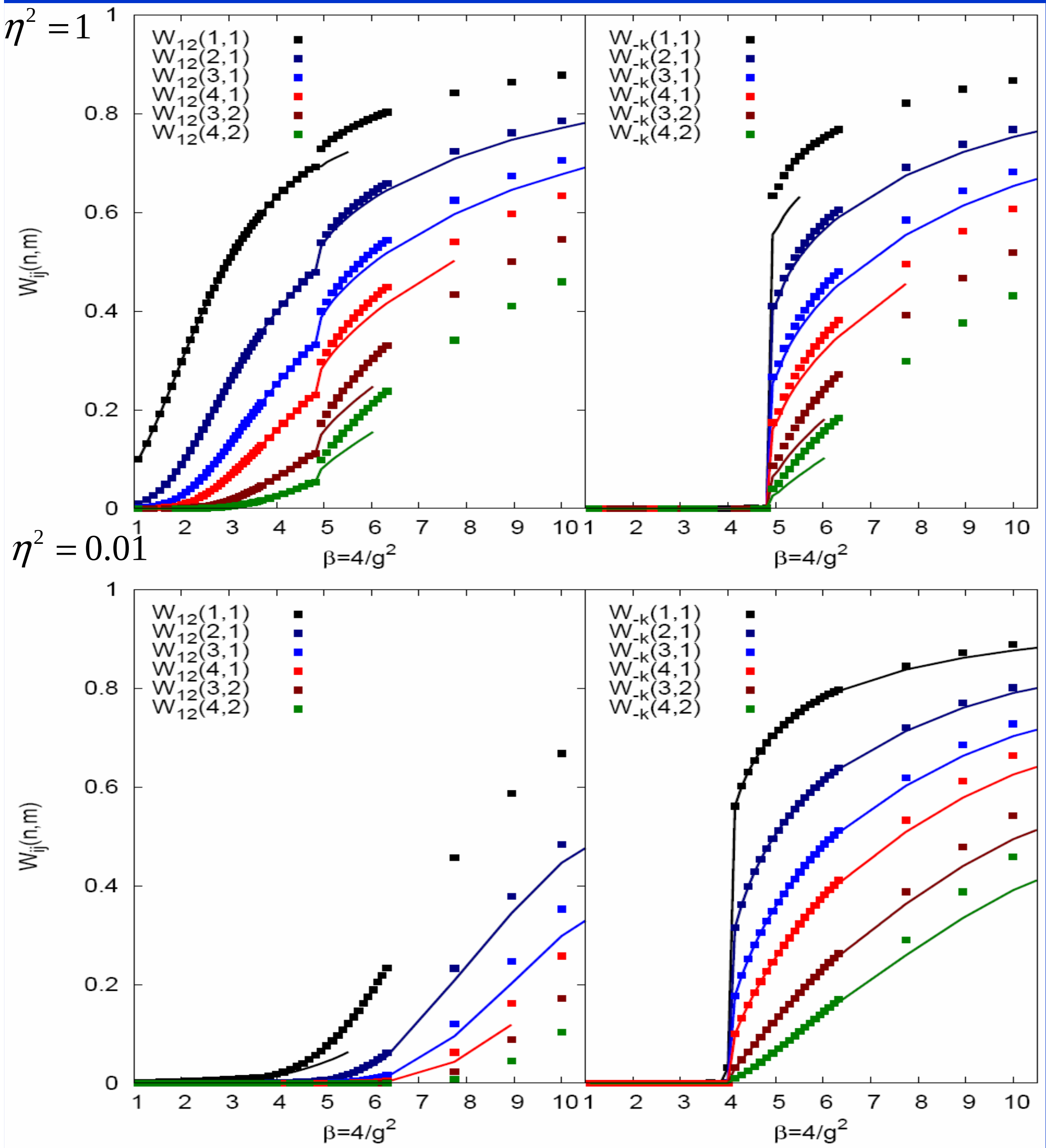
- Weak coupling wavefunctional

$$\Psi_0 = \exp\left\{-\sqrt{\lambda}\sum_{\vec{x},\vec{x}'}\sum_a \frac{1}{2}\vec{B}^a(\vec{x})\boldsymbol{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}')\frac{1}{2}\vec{B}^a(\vec{x}')\right\}$$

$$\boldsymbol{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}') \equiv \begin{pmatrix} \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 & 0 \\ 0 & \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 \\ 0 & 0 & \tilde{\eta}^2\gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') \end{pmatrix}$$

$$\begin{aligned} U_{ij}(\vec{x}) &= \exp\left(\mathrm{i}F^a_{ij}(\vec{x})\lambda^a\right) \\ F^a_{ij}(\vec{x}) &= \epsilon_{ijk}B^a_k(\vec{x}) + g\,f^{abc}A^b_i(\vec{x})A^c_j(\vec{x}) \\ B^a_k(\vec{x}) &= \epsilon_{klm}\left[A^a_m(\vec{x}) - A^a_m(\vec{x}-\vec{e}_l)\right] \end{aligned}$$

  - Multivariate Gaussian wavefunctional



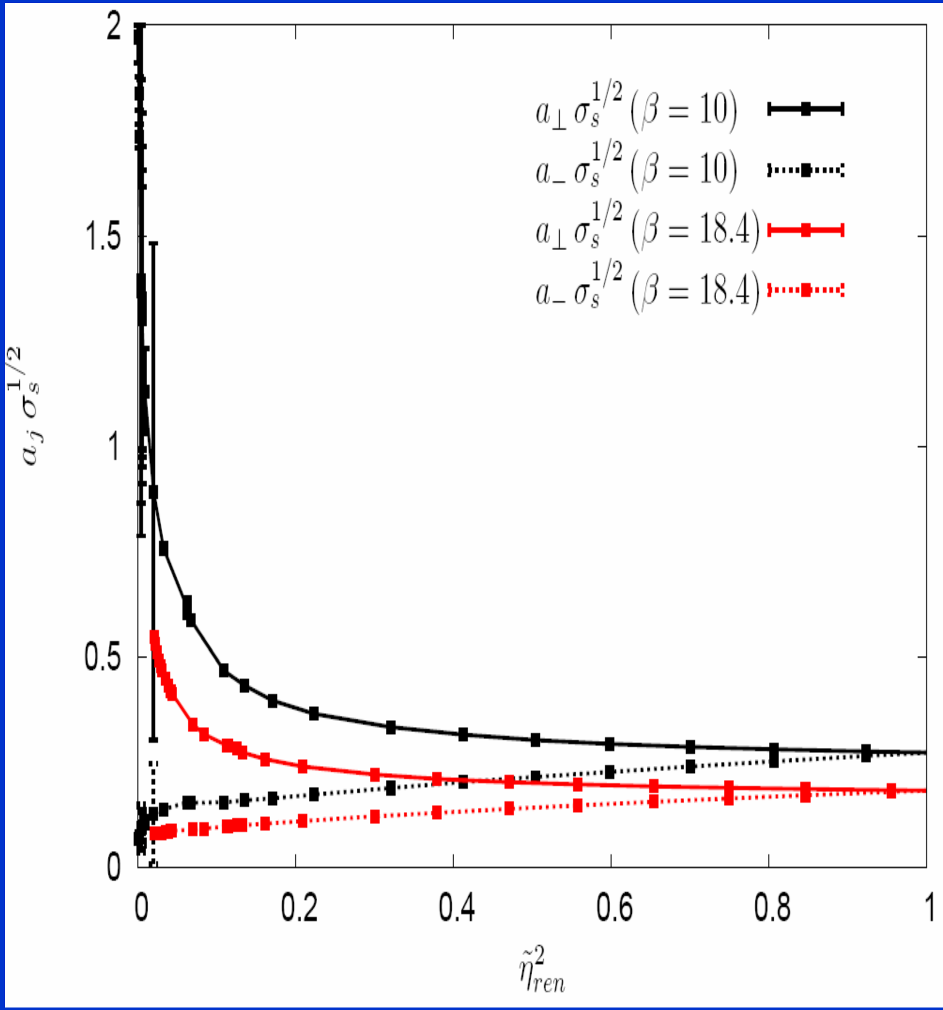
Wilson loop expectation values

$$\left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{-k}\right)\right]\right\rangle_{\Psi_0(\rho_0,\delta_0)} = \frac{I_2(4\,\rho_0)}{I_1(4\,\rho_0)}$$

$$\left\langle W_{ij}(n,m)\right\rangle_{\Psi_0(\rho_0,\delta_0)} = \left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{ij}\right)\right]\right\rangle^{n\cdot m}_{\Psi_0(\rho_0,\delta_0)}$$

- Nice strong coupling behavior
- Better agreement to strong coupling for smaller values of η

Lattice spacings



- $a_\perp = a_\perp(\beta, \eta) \quad \Rightarrow$
  the transversal lattice constant $a_\perp$ is varying with the boost parameter η

  $\Rightarrow$ UNPHYSICAL !

- Introduce two different couplings λ_- and $\lambda_\perp$ for the longitudinal and transversal part of the Hamiltonian
- $\Rightarrow$ three couplings $\lambda_-, \lambda_\perp, \eta$ which can be tuned in such a way that
  $a_\perp$ is independent of η_{ren}
  a_- is $a_- = \eta_{ren}\, a_\perp$
- Work in progress