Spectral Sums, Polyakov Loops and the Banks-Casher Relation

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1. Chiral Symmetry and Breaking
2. Center Transformations and Loops on Lattice
3. Dirac Operator and Spectral Sums
4. Numerical investigations for SU(2)
5. Spectral Sums for Continuum Theory

Finite temperature gauge theories

- confinement
  order parameter: Polyakov loop $\langle \mathrm{Tr}\,\mathcal{P}\rangle$, $\mathcal{P} = \mathcal{P}\exp\left(i\int A_0 d\tau\right)$
  induced by topological defects (monopoles,...)
- chiral symmetry breaking
  order parameter: chiral condensate $\langle\bar\psi\psi\rangle$
  induced by instantons
- deconfining and chiral phase transitions at same T_c determined by low lying eigenvalues of $\not{D}$ (Banks-Casher; Gattringer)
- spectrum of $\not{D} \leftrightarrow$ Polyakov loop
  Gattringer, Bilgici, Bruckmann, Hagen, Soldner, ...

Chiral Symmetry and Breaking

- chiral boundary conditions or $\mathbb{T}^4, S^4, \dots$

$$i\not{D}_A \psi_p = \lambda_p \psi_p$$

- massive quark propagator

$$\langle q(x)\bar{q}(y)\rangle_A = \langle x|\frac{1}{i\not{D}_A + im}|y\rangle = \sum_p \frac{\psi_p(x)\psi_p^\dagger(y)}{\lambda_p + im}$$

current quark mass m, external gauge potential A

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- quark condensate

$$\langle\bar{q}q\rangle = \lim_{V\to\infty}\frac{1}{V}\Big\langle \int d^4x \big\langle\bar{q}(x)q(x)\big\rangle_A\Big\rangle$$

relation to spectral density $\rho(\lambda)$ of Dirac operator

$$\langle\bar{q}q\rangle = 2im\int d\lambda\,\frac{\langle\rho_A(\lambda)\rangle}{\lambda^2+m^2}\quad \rho_A(\lambda)=\frac{1}{V}\sum_{\lambda_p>0}\delta\big(\lambda-\lambda_p\big),$$

- main contribution $\lambda \le m \ll \Lambda_{\rm QCD}$
  first $V\to\infty$ then chiral limit $m\to 0$

$$\Sigma = \lim_{m\to 0}|\langle\bar{q}q\rangle| = \pi\rho(0),\quad \rho(\lambda)=\langle\rho_A(\lambda)\rangle,\quad \text{Banks-Casher}$$

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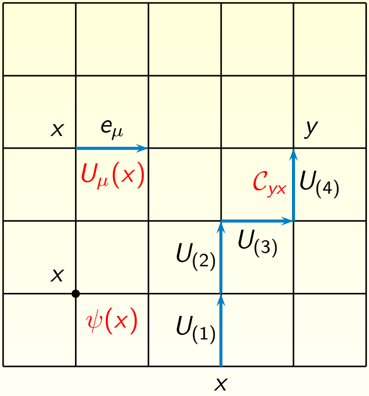
Lattice Formulation



matter fields $x \to \psi(x)$

gauge fields (x, e_μ)
$(x, e_\mu) \to U_\mu(x) \in G$ (unitary)

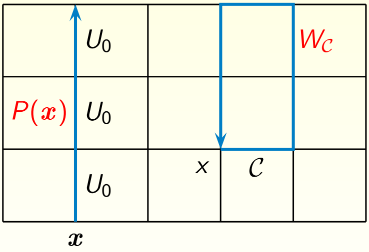
$\mathcal{W}_{\mathcal{C}_{yx}}$ parallel transport along $\mathcal{C}_{yx}$

gauge transformation

$$\psi(x) \to V_x \psi(x), \quad V_x \in G$$
$$\mathcal{W}_{\mathcal{C}_{yx}} \to V_y \mathcal{W}_{\mathcal{C}_{yx}} V_x^{-1}$$

Wilson and Polyakov loops

- finite temperature:
  asymmetric $N_\tau \times N_s^3$-lattice, $N_\tau \ll N_s$, $V = N_\tau \cdot N_s^3$
- parallel transport along loop $\mathcal{C} \Rightarrow$ gauge invariant Wilson loop $W_\mathcal{C}$
- Polyakov loops



loops winding around periodic time direction

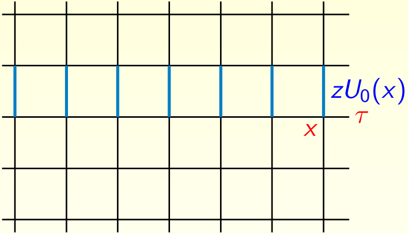
$$P_{\boldsymbol{x}} = \mathrm{Tr}\,\mathcal{P}_{\boldsymbol{x}}$$

$$\mathcal{P}_{\boldsymbol{x}} = \prod_{\tau=1}^{N_\tau} U_0(\tau, \boldsymbol{x})$$

Center transformations

multiply all $U_0(\tau, \boldsymbol{x})$ in one time-slice with center element z

$$\text{configuration } \{U_\mu(x)\} \longrightarrow \{{}^zU_\mu(x)\} \text{ twisted configuration}$$



$\mathcal{C}$ contractable

$$\mathcal{W}_\mathcal{C} \longrightarrow \mathcal{W}_\mathcal{C} \Rightarrow S_\mathrm{w} \text{ invariant}$$

non-contractable Polyakov loop

$$\mathcal{P}_{\boldsymbol{x}} \longrightarrow z\mathcal{P}_{\boldsymbol{x}}$$

$\mathcal{C}$ winds n-times around periodic time direction:

$$\mathcal{W}_\mathcal{C} \longrightarrow z^n \mathcal{W}_\mathcal{C}$$

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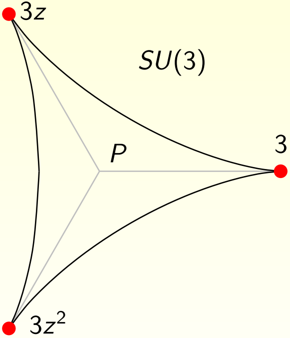
probability distribution for order parameter $P_{\boldsymbol{x}}$

$$e^{-S_{\rm eff}[L]} \propto \int \mathcal{D}U\,\delta\big(L_{\boldsymbol{x}},\mathrm{Tr}\,\mathcal{P}_{\boldsymbol{x}}\big)\,e^{-S[U]}$$



center symmetry of $S \Rightarrow$

$$S_{\rm eff}[L] = S_{\rm eff}[z\cdot L]$$

high temperatures: $\langle P\rangle \neq 0 \Rightarrow$
breaking of center symmetry

strong coupling $\longrightarrow$
generalized Potts models
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Dirac Operator

- forward/backward covariant derivatives: $D^{\mathrm{f}}_\mu = -(D^{\mathrm{b}}_\mu)^\dagger$:

$$\begin{aligned}(D^{\mathrm{b}}_\mu\psi)(x) &= \psi(x) - U_\mu(x-e_\mu)\psi(x-e_\mu)\\ (D^{\mathrm{f}}_\mu\psi)(x) &= U_{-\mu}(x+e_\mu)\psi(x+e_\mu) - \psi(x)\end{aligned}$$

- naive anti-hermitean Dirac operator

$$\not{D} = \tfrac{1}{2}\gamma^\mu\big(D^{\mathrm{f}}_\mu + D^{\mathrm{b}}_\mu\big), \quad \gamma_\mu = \gamma_\mu^\dagger$$

- doublers $\Rightarrow$ add hermitian covariant Laplacian

$$D^2 = \sum\nolimits_\mu D^{\mathrm{b}}_\mu D^{\mathrm{f}}_\mu = \sum\nolimits_\mu (D^{\mathrm{f}}_\mu - D^{\mathrm{b}}_\mu)$$

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- Wilson operator

$$\mathcal{D} = -\not{D} + m - \tfrac{1}{2} D^2, \quad \gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}^\dagger$$

- nearest neighbor interaction

$$\mathcal{D}_{xy} = (m+d)\delta_{xy} - \tfrac{1}{2} \sum_\mu \Big((1+\gamma^\mu) U_{-\mu}(y) \delta_{x,y-e_\mu} + (1-\gamma^\mu) U_\mu(y) \delta_{x,y+e_\mu} \Big)$$

- hop from site x to $x \pm e_\mu$: factor $\propto (1 \mp \gamma^\mu) U_\mu(x)$
  staying at x: factor $(m+d)$
  $\mathcal{D}^\ell$: chains of ℓ or less hops on lattice

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Spectral Sums

- $\langle x|\mathcal{D}^\ell|x\rangle$: Wilson loops $\mathcal{W}_{\mathcal{C}_x}$ with $|\mathcal{C}_x| \le \ell$
- twisting:

$$U \to {}^zU, \quad \mathcal{D} \to {}^z\mathcal{D}, \quad \lambda_p \to {}^z\lambda_p$$

$\ell < N_\tau \Rightarrow$ loops contractable $\Longrightarrow {}^z\mathcal{W}_\mathcal{C} = \mathcal{W}_\mathcal{C} \Longrightarrow$

$$\langle x|\,{}^z\mathcal{D}^\ell|x\rangle = \langle x|\mathcal{D}^\ell|x\rangle \Longrightarrow \sum_{\text{center}} \bar{z}_k \big\langle x\big|\,{}^{z_k}\mathcal{D}^\ell\big|x\big\rangle = 0, \quad \ell < N_\tau$$

trace: $$\sum_{\text{center}} \bar{z}_k \Big(\sum\nolimits_p {}^{z_k}\lambda_p^\ell\Big) = 0, \quad \ell < N_\tau$$

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- $\ell = N_\tau$: $\langle x|\mathcal{D}^{N_\tau}|x\rangle$ contains Polyakov loops, use $\sum \bar{z}_k z_k = |\mathcal{Z}| \Rightarrow$

$$\sum_k \bar{z}_k \, \langle x|^{z_k}\mathcal{D}^{N_\tau}|x\rangle \quad \propto \quad \mathcal{P}(\boldsymbol{x})$$

$$\sum_k \bar{z}_k \, \mathrm{Tr} \left({}^{z_k}\mathcal{D}\right)^{N_\tau} \quad = \quad \kappa \, L, \quad L = \frac{1}{V_s}\sum_{\boldsymbol{x}} P(\boldsymbol{x})$$

- first relation Polyakov loop $\leftrightarrow$ spectral data of $\mathcal{D}$
- Problem: in continuum limit $N_\tau \to \infty$,

$$L = \frac{1}{\kappa}\sum_k \bar{z}_k \Big(\sum_p {}^{z_k}\lambda_p^{N_\tau}\Big), \quad \kappa = (-1)^{N_\tau} 2^{[d/2]-1} V|\mathcal{Z}|$$

$\Rightarrow$ need generalized spectral sums

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twisting $\lambda_p \to {}^z\lambda_p$: consider

$$\Delta_p \equiv \tfrac{1}{3}\big(|\lambda_p - {}^z\lambda_p| + |{}^z\lambda_p - {}^{\bar z}\lambda_p| + |{}^{\bar z}\lambda_p - \lambda_p|\big)$$

observations:

- Δ_p maximal for low-lying eigenvalues
- spectral sum: main contribution from large eigenvalues:

$$\Sigma_n = \frac{1}{\kappa}\sum_k \bar z_k \sum_{p=1}^{n} {}^{z_k}\lambda_p^{N_\tau}$$

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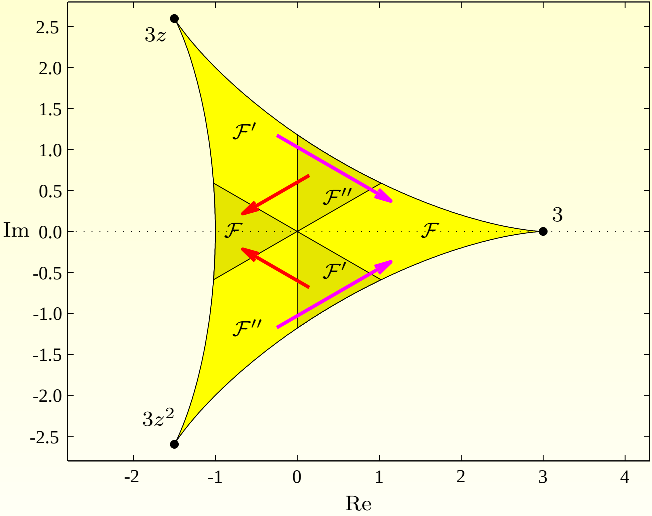

fundamental domain for P, definition of $L^{\rm rot}$, $\Sigma^{\rm rot}$

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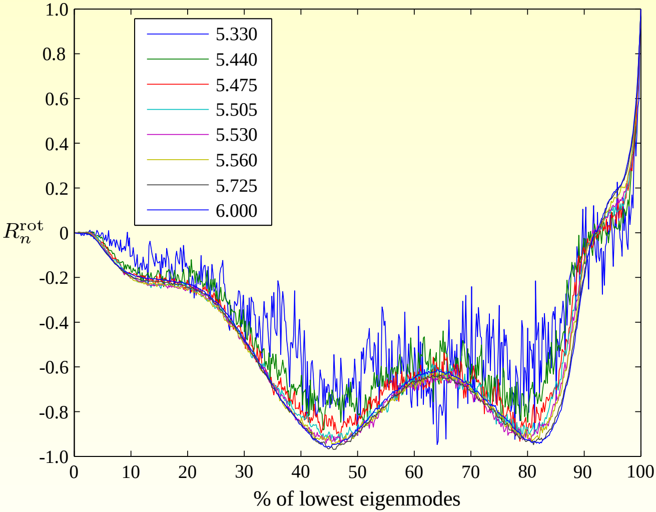

$R_n^{\rm rot} = \frac{\langle \Sigma_n^{\rm rot} \rangle}{\langle L^{\rm rot} \rangle}$, % of lowest eigenvalues ($\beta_c = 5.49$)

universal behavior for $\langle L \rangle > 0.6$, $\beta > 5.5$

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Generalized spectral sums

$$\mathcal{S}_{f,n} = \frac{1}{\kappa}\sum_k \bar{z}_k \sum_{p=1}^{n} f\left({}^{z_k}\lambda_p\right) \qquad f \text{ support in IR}$$

- sum over all eigenvalues $\Rightarrow$ traces

$$\mathcal{S}_f = \frac{1}{\kappa}\sum_k \bar{z}_k \sum_{\text{all } p} f\left({}^{z_k}\lambda_p\right) = \frac{1}{\kappa}\sum_k \bar{z}_k \,\mathrm{Tr}\, f\left({}^{z_k}\mathcal{D}\right)$$

- $S(f)$ is order parameter for center symmetry

$$S_f \xrightarrow{z} \frac{1}{\kappa}\sum_k \bar{z}_k \,\mathrm{Tr}\, f\left({}^{z_k z}\mathcal{D}\right) = z S_f$$

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- Gattringer: $\Sigma_n = \mathcal{S}_{f,n}$ for $f(\lambda) = \lambda^{N_t}$
- propagator sum

$$\Sigma^{(-1)} = \frac{1}{\kappa} \sum_k \mathrm{Tr}\left(\frac{\bar{z}_k}{z_k \mathcal{D}}\right)$$

- enters Banks-Casher relation
- hopping parameter expansion
  $\mathcal{D} = (m+d)\mathbb{1} - V$: expansion in powers of V,

$$\mathcal{D}^{-1} = \frac{1}{m+d} \sum_k \frac{V^k}{(m+d)^k} \Longrightarrow$$

$$\Sigma^{(-1)} = \frac{(-1)^{N_\tau}}{(m+d)^{N_\tau+1}} L + \dots$$

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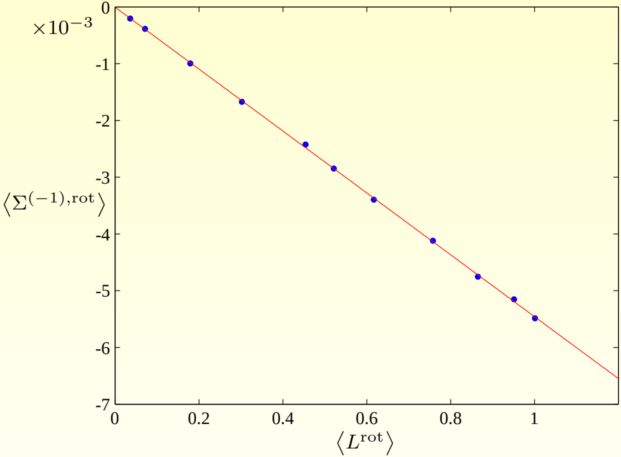

fit: $\langle \Sigma^{(-1),\mathrm{rot}} \rangle = -0.00545 \cdot \langle L^{\mathrm{rot}} \rangle - 4.379 \cdot 10^{-6}$

hopping-expansion on $4^3 \times 3 : \langle \Sigma^{(-1)} \rangle = -0.004 \cdot \langle L \rangle$

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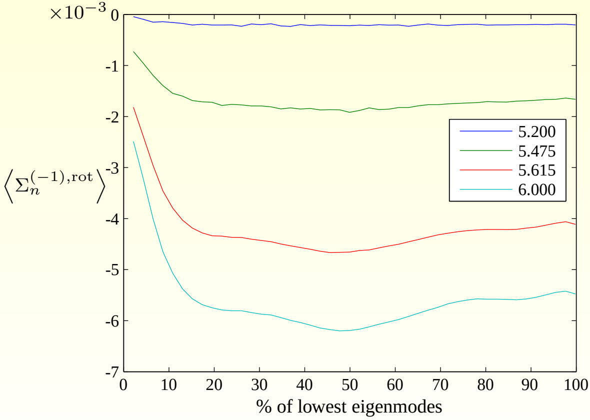
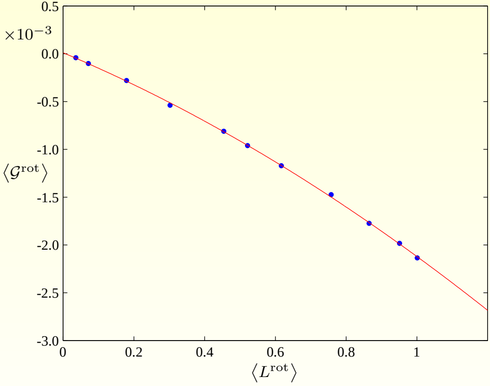
ultraviolet vs. infrared contributions: partial sums

$$\Sigma_n^{(-1)} = \frac{1}{\kappa}\sum_k \bar{z}_k \sum_{p=1}^{n} \frac{1}{z_k \lambda_p} \Rightarrow \text{lowest } 10\%$$



- Gaussian spectral sums (zeta-function, proper-time)

$$\mathcal{G} = \sum_k \bar{z}_k \,\mathrm{Tr}\exp\left(-{}^{z_k}\mathcal{D}^{\dagger}\,{}^{z_k}\mathcal{D}\right)$$



The expectation value of $\mathcal{G}^{\mathrm{rot}}$ as function of L^{rot}

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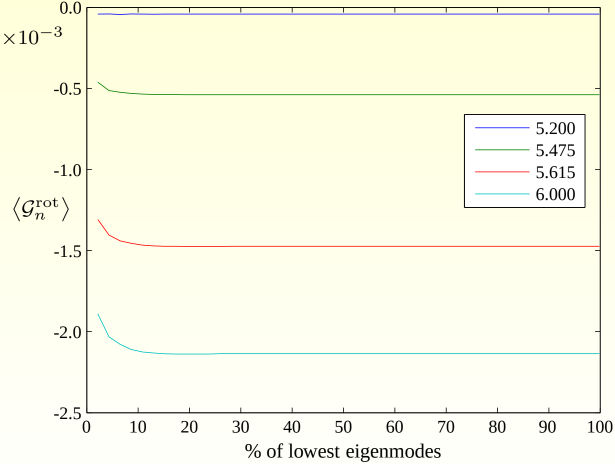
$$\mathcal{G}_{n} = \frac{1}{\kappa}\sum_k \bar{z}_k \sum_{p=1}^{n} e^{-|z_k\lambda_p|^2} \Rightarrow \text{lowest } 3\%$$



Numerical investigations for SU(2)

- static quark potential

$$V(r) = -T \log C(r), \quad C(r) = \langle P(\boldsymbol{x}) P(\boldsymbol{x} + r\boldsymbol{e}_3) \rangle$$

  $P(\boldsymbol{x}) = \Sigma(\boldsymbol{x}) \Rightarrow$ cancellation of huge contributions
  $\Rightarrow$ must include UR eigenfunction

- use IR-dominated spectral sum, e.g. $\mathcal{G}_n$

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- simulations: staggered fermions, SU(2)

$$\mathcal{G}_n(x) := \frac{1}{8}\sum_{p=1}^{n}\Big(|\psi_p(x)|^2\, e^{-\lambda_p^2/\mu^2} - |{}^z\psi_p(x)|^2\, e^{-{}^z\lambda_p^2/\mu^2}\Big)$$

- improved action (rotational invariance, scaling)

$$S = \beta \sum_{\mu>\nu,x}\Big[\gamma_1 P_{\mu\nu}(x) + \gamma_2 P^{(2)}_{\mu\nu}(x)\Big]$$

$$\beta = 1.35,\ \gamma_1 = .0348,\ \gamma_2 = -0.10121,\ \sigma a^2 = 0.1244(7)$$

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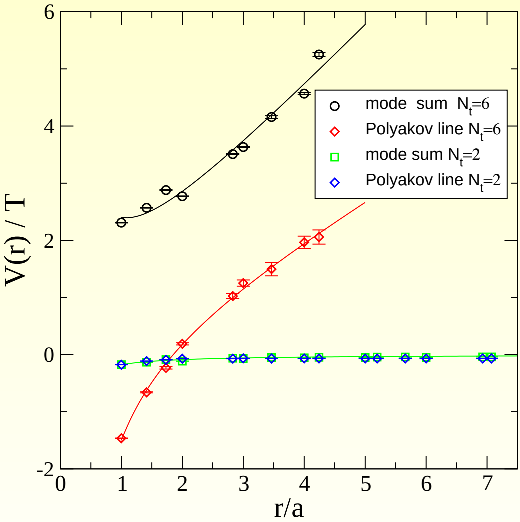
- compare $V(r)$ with

$$V_n^{\mathcal{G}}(r) = -T \log C_n^{\mathcal{G}}(r), \quad C_n^{\mathcal{G}}(r) = \langle \mathcal{G}_n(\boldsymbol{x}) \mathcal{G}_n(\boldsymbol{x} + r\boldsymbol{e}_3) \rangle$$

- simulation parameters

β	σa^2	lattice	T/T_c	configurations
1.35	0.1244(7)	$12^3 6$	0.7	8658
1.35	0.1244(7)	$12^3 4$	1.0	12000
1.35	0.1244(7)	$12^3 2$	2.1	12000



potential from P and $\mathcal{G}_n$, $N_t = 6$ confined, $N_t = 2$ deconfined

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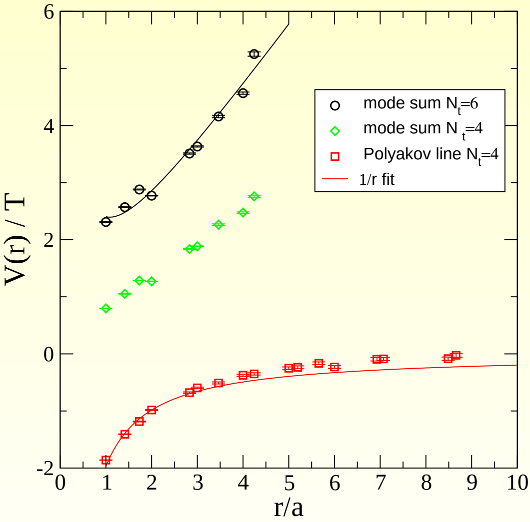

potential from P and $\mathcal{G}_n$ for $T \approx T_c(N_t = 4)$

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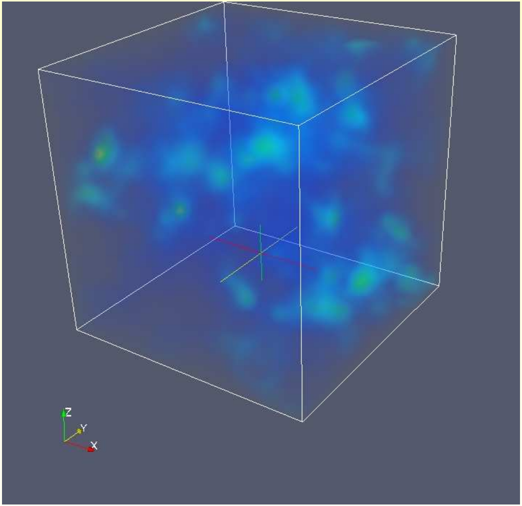

mode sum $|\mathcal{G}_n(\boldsymbol{x})|$ in a 20^3 spatial hypercube. $L = 3.2$ fm
smooth texture at scale 0.3 fm

Spectral Sums for Continuum Theory

- Euclidean Dirac operator (on T^4)

$$\mathcal{D} = i\gamma^\mu D_\mu + im,$$

- finite temperature spectral problem

$$\mathcal{D}\psi_p = \lambda_p \psi_p, \quad \psi_p(x_0+\beta, \boldsymbol{x}) = -\psi_p(x_0, \boldsymbol{x}) \text{ normalized}$$

- gauge invariant spectral data

$$|\psi_p(x)|^2 \quad \text{and} \quad \lambda_p$$

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- center transformation = 'gauge-transformation' with non-periodic

$$g(x_0+\beta,\boldsymbol{x}) = z\,g(x_0,\boldsymbol{x}),\quad z\in\mathcal{Z}$$

transformed fields ${}^gA$, ${}^g\psi$:

$${}^gA(x_0+\beta,\boldsymbol{x}) = {}^gA(x_0,\boldsymbol{x}),\quad {}^g\psi(x_0+\beta,\boldsymbol{x}) = -z\,{}^g\psi(x_0,\boldsymbol{x})$$

- twisted Diracoperator

$${}^z\mathcal{D}_A \equiv \mathcal{D}_{{}^gA} = g\mathcal{D}_A g^{-1}$$

${}^g\psi$ not eigenfunctions of ${}^z\mathcal{D}$ for $z\neq\mathbb{1}$

- twisting: $A\to {}^gA \Longrightarrow$

$$\mathcal{D}\to{}^z\mathcal{D},\quad \lambda_p\to{}^z\lambda_p,\quad \psi_p(x)\to{}^z\psi_p(x)\quad ({}^z\psi_p\neq{}^g\psi_p)$$

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- spectral sums in continuum: weighted center average of $\langle x|f(\mathcal{D})|x\rangle$

$$\mathcal{S}_f(x) = \lim_{n\to\infty}\sum_{p=1}^{n}\sum_k z_k^* \left|{}^{z_k}\psi_p(x)\right|^2 f({}^{z_k}\lambda_p)$$

$$\mathcal{S}_f = \int d^dx\,\mathcal{S}_f(x) = \sum_k z_k^*\,\mathrm{Tr}\,f\left({}^{z_k}\mathcal{D}\right)$$

- order parameters for center symmetry

$$\mathcal{S}_f(x) \stackrel{A\to{}^gA}{\longrightarrow} z\,\mathcal{S}_f(x)$$

- hopping parameter expansion, symmetry arguments, constant $F_{\mu\nu}$

$$\mathcal{S}_f(\boldsymbol{x}) \stackrel{L\gg\beta}{\longrightarrow} \text{const}\cdot P(\boldsymbol{x})$$

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Constant field strength configurations

- Here: Schwinger model
- 'boundary conditions'

$$\psi(x_0+\beta, x_1) = -\psi(x_0, x_1) \quad , \quad \psi(x_0, x_1+L) = e^{i\gamma(x)}\psi(x_0, x_1)$$
$$\gamma = -2\pi q\, x_0/\beta \quad , \quad q \text{ instanton number}$$

- instanton configuration

$$A_0 = -Bx_1 + \frac{2\pi}{\beta}h \Longrightarrow F_{01} \equiv B, \quad P(\boldsymbol{x}) = e^{2\pi i h - i\Phi\, x_1/L}$$

- $U(1)$ twist with $g = e^{2\pi i \alpha x_0/\beta} \Rightarrow$

$$z = e^{2\pi i\alpha}, \quad {}^{g}A_0 = -Bx_1 + \frac{2\pi}{\beta}(h+\alpha)$$

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- eigenvalues $\mu_p = \lambda_p^2 = 2pB$ center-invariant
- center-average

$$z = e^{2\pi i\alpha}, \quad \sum_k z_k^* \ldots \longrightarrow \int_0^1 d\alpha e^{-2\pi i\alpha} \ldots$$

- all spectral sums are proportional to $P(\boldsymbol{x})$

$$\mathcal{S}_f(\boldsymbol{x}) = -\frac{q}{L} P(\boldsymbol{x}) \sum_{p=0}^{\infty} f(\mu_p) \left\{ L_p(\pi q\tau) + L_{p-1}(\pi q\tau) \right\} e^{-\pi q\tau/2}$$

similar result for constant field strength on T^4

- Gaussian spectral sum for $f(\mathcal{D}) = \exp(t\mathcal{D}^2)$:

$$\mathcal{G}_t(\boldsymbol{x}) = -\frac{q}{L}\coth(tB)\,\exp\left(-\frac{\pi q\tau}{2}\coth(tB)\right)P(\boldsymbol{x})$$

- small-t and large-t asymptotics

$$\mathcal{G}_t(\boldsymbol{x}) \;\longrightarrow\; \begin{array}{ll} -\frac{1}{L}\frac{q}{tB}e^{-\pi q\tau/(2tB)}\,P(\boldsymbol{x}) & \text{for} \quad t\to 0 \\ -\frac{1}{L}(q-2qe^{-2tB})e^{-\pi q\tau/2}\,P(\boldsymbol{x}) & \text{for} \quad t\to\infty \end{array}$$

- asymptotic expansion for small t: all $a_n = 0$

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On the convergence of spectral sums: general case

- zero-mode subtracted heat kernel

$$K'(t,x) = K(t,x) - \varrho_0(x), \quad K(t,x) = \langle x | e^{t\mathcal{D}^2} | x \rangle$$

- large and small-t asymptotics

$$K'(t,x) \;\rightarrow\; e^{-t\mu_1} \big|\psi_1(x)\big|^2 \qquad \text{for} \quad t \to \infty$$

$$K(t,x) \;\rightarrow\; t^{-d/2} \Big\{ \sum_{n=0}^{N} a_n(x) t^n + \mathcal{O}(t^{N+1}) \Big\} \qquad \text{for} \quad t \to 0$$

- a_n gauge-invariant function of $F_{\mu\nu}$ and its covariant derivatives
  $\Rightarrow$ a_n center-invariant,

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- Mellin transform

$$\zeta'_{\mathcal{D}}(s,x) = \langle x|\frac{1}{(\mathcal{D}^2)^s}|x\rangle = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}K'(t,x)$$

d even: simple poles at $s=\frac{d}{2},\dots,1$, residues $a_0,\dots,a_{\frac{d}{2}-1}$

- a_n center-invariant $\Rightarrow$

$$\Sigma^{(-2s)}(x) = \frac{1}{\Gamma(s)}\int dt\, t^{s-1}\mathcal{G}'(t,x) = \sum_k z_k^*\,\zeta'_{z_k\mathcal{D}}(s,x)$$

no poles in $s \Rightarrow$ entire function

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Conclusions, remarks

- all spectral sums define order parameters for center symmetry
- spectral sum approach can be formulated for continuum theories
- first sum over center elements and afterwards over the eigenvalues
  $\Rightarrow$ spectral sums well-defined for almost all $f(\mathcal{D})$
- reconstructed Polyakov loop locally from spectral sums on lattice
- same construction for continuum theory and constant $F_{\mu\nu}$
- beyond constant field strength?
  relation to Banks-Casher in continuum (CSB $\leftrightarrow$ confinement)

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