The 2D $\mathcal{N}=2$ Wess-Zumino Model on the Lattice

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30.01.2009 / Heidelberg

seit 1558

Studienstiftung
des deutschen Volkes

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2. The model
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Motivation (Physics)

- The lattice breaks supersymmetry explicitly.
- No spontaneous supersymmetry breaking of the continuum model expected.
$\Rightarrow$ Supersymmetry restoration in continuum limit can be analyzed.
- In former works (M. Beccaria et al. (1998), S. Catterall and S. Karamov (2003)) only Wilson fermions with Nicolai improved action were used. Problems at stronger couplings.
- Effects of Nicolai improvement?

Introduction

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2. Motivation (Algorithms)

- Explicit investigation and improvement of the used algorithms, cf. e.g. Bergner et al. (2007) for WZ model in $1d$ with different discretizations.
- High precision measurements available in lower dimensions.

- The continuum action

$$S_{\text{cont}} = \int d^2x \left(2\bar{\partial}\bar{\varphi}\partial\varphi + \tfrac{1}{2}|W'(\varphi)|^2 + \bar{\psi}M\psi\right),$$

$$M = \gamma^z\partial + \gamma^{\bar{z}}\bar{\partial} + W''P_+ + \overline{W}''P_-$$

allows for 4 real supersymmetries, $\varphi = \varphi_1 + i\varphi_2$.

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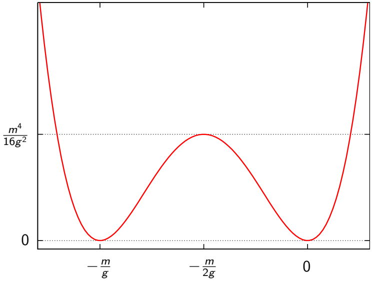
allows for 4 real supersymmetries, $\varphi = \varphi_1 + i\varphi_2$.

- We use $W(\varphi) = \frac{1}{2}m\varphi^2 + \frac{1}{3}g\varphi^3$ with dimensionless coupling $\lambda = \frac{g}{m}$.

Classical potential $|W'(\varphi_1, \varphi_2 = 0)|^2$:



- $\lambda = 0$ corresponds to free theory
  $\Rightarrow$ perturbative expansion in λ possible.

The action

$$S_{\text{cont}} = \int d^2x \left(2\bar{\partial}\bar{\varphi}\partial\varphi + \tfrac{1}{2}|W'(\varphi)|^2 + \bar{\psi}M\psi\right)$$

allows for discrete symmetries

$$\mathbb{Z}_2^{\mathrm{R}}\colon \varphi \mapsto -\frac{m}{g} - \varphi \quad \text{and} \quad \mathbb{Z}_2^{\mathrm{C}}\colon \varphi \to \bar{\varphi} \quad \text{due to chosen W}$$

$$\mathbb{Z}_2^{\mathrm{T}}\colon (z,\bar{z}) \mapsto (-\bar{z},-z) \quad \text{and} \quad \mathbb{Z}_2^{\mathrm{P}}\colon (z,\bar{z}) \mapsto (\bar{z},z) \quad \text{independent of W}$$

These should be recovered in the continuum limit of the lattice theory.
At least $\mathbb{Z}_2^{\mathrm{R}}$ and $\mathbb{Z}_2^{\mathrm{C}}$ are worth to keep because they correspond to the two classical minima of the action.

Using the Nicolai variable $\xi_x = 2(\bar{\partial}\bar{\varphi})_x + W_x$ an action on the lattice preserving one supersymmetry is given by

$$S = \frac{1}{2}\sum_x \bar{\xi}_x \xi_x + \sum_{xy} \bar{\psi}_x M_{xy} \psi_y$$

with $W_x = W'(\varphi_x)$, $W_{xy} := \partial W_x / \partial \varphi_y$ and

$$M_{xy} = \begin{pmatrix} W_{xy} & 2\bar{\partial}_{xy} \\ 2\partial_{xy} & \overline{W}_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi_x}{\partial \varphi_y} & \frac{\partial \xi_x}{\partial \bar{\varphi}_y} \\ \frac{\partial \bar{\xi}_x}{\partial \varphi_y} & \frac{\partial \bar{\xi}_x}{\partial \bar{\varphi}_y} \end{pmatrix}.$$

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In terms of the original fields the action reads

$$S = \sum_x \left(2\left(\bar{\partial}\bar{\varphi}\right)_x (\partial\varphi)_x + \frac{1}{2}\left|W_x\right|^2 + W_x(\partial\varphi)_x + \overline{W}_x(\bar{\partial}\bar{\varphi})_x \right) + \sum_{xy} \bar{\psi}_x M_{xy} \psi_y .$$

The difference to a straightforward discretization is given by surface terms

$$\Delta S = \sum_x \left(W_x(\partial\varphi)_x + \overline{W}_x(\bar{\partial}\bar{\varphi})_x \right).$$

We use different lattice derivatives *(the same for bosonic and fermionic degrees of freedom)*:

- Symmetric derivative $\left(\partial^S_\mu\right)_{xy} = \frac{1}{2}(\delta_{x+\hat{\mu},y} - \delta_{x-\hat{\mu},y})$ with standard Wilson term $W_x = W'(\varphi_x) - \frac{r}{2}(\Delta\varphi)_x$ using $(r = 1)$.

$$M_{xy} = \begin{pmatrix} W''(\phi_x)\delta_{xy} & 2\bar{\partial}_{xy} \\ 2\partial_{xy} & W''(\phi_x)\delta_{xy} \end{pmatrix} - \frac{r}{2}\Delta_{xy}$$

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- Symmetric derivative ∂^S with twisted Wilson term $W_x = W'(\varphi_x) + \frac{ir}{2}(\Delta\varphi)_x$.

$$M_{xy} = \begin{pmatrix} W''(\phi_x)\delta_{xy} & 2\bar{\partial}_{xy} \\ 2\partial_{xy} & W''(\phi_x)\delta_{xy} \end{pmatrix} + \gamma_3\frac{r}{2}\Delta_{xy}$$

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  The choice $r = 2/\sqrt{3}$ renders the mass of the free theory exact up to $\mathcal{O}(a^4)$.
- SLAC derivative $\partial_{x \neq y} = (-1)^{x-y} \frac{\pi/N}{\sin(\pi(x-y)/N)}$, $\partial_{xx} = 0$ with

  M_{xy} unchanged.

$\Rightarrow$ Simulate the (un)improved model with these different discretizations!
We use a combination of fourier acc. (DR)HMC with higher-order integrators.

Preserved discrete symmetries on the lattice:
For the improved model with SLAC fermions the symmetries are reduced:

$$\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{P}} \times \mathbb{Z}_2^{\mathrm{R}} \times \mathbb{Z}_2^{\mathrm{C}} \quad \longrightarrow \quad \mathbb{Z}_2^{\mathrm{TPR}} \times \mathbb{Z}_2^{\mathrm{PC}} := \mathrm{diag}(\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{P}} \times \mathbb{Z}_2^{\mathrm{R}}) \times \mathrm{diag}(\mathbb{Z}_2^{\mathrm{P}} \times \mathbb{Z}_2^{\mathrm{C}})$$

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	W. impr.	W. unimpr.	tw. W. impr.	SLAC impr.	SLAC unimpr.
lattice derivative	local	local	local	non-local	non-local
lattice artifacts	$\mathcal{O}(a)$	$\mathcal{O}(a)$	$\mathcal{O}(a)$	'perfect'	'perfect'
mod. superpot.	yes	yes	yes	no	no
discrete symmetries	$\mathbb{Z}_2^{\mathrm{PC}}$	$\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{P}} \times \mathbb{Z}_2^{\mathrm{C}}$	$\mathbb{Z}_2^{\mathrm{TR}}$	$\mathbb{Z}_2^{\mathrm{TPR}} \times \mathbb{Z}_2^{\mathrm{PC}}$	$\mathbb{Z}_2^{\mathrm{T}} \times \mathbb{Z}_2^{\mathrm{P}} \times \mathbb{Z}_2^{\mathrm{R}} \times \mathbb{Z}_2^{\mathrm{C}}$
supersymmetries	one	none	one	one	none

Limitations of improvement

For dynamical simulations of the improved model the bosonic action is fixed to $\langle S_\mathrm{B}\rangle = N = \#$ lattice points.

With SLAC fermions at different coupling strenghts we measure the improvement term $\Delta S = \sum_x \left(W_x(\partial\varphi)_x + \overline{W}_x(\bar{\partial}\bar{\varphi})_x\right)$:

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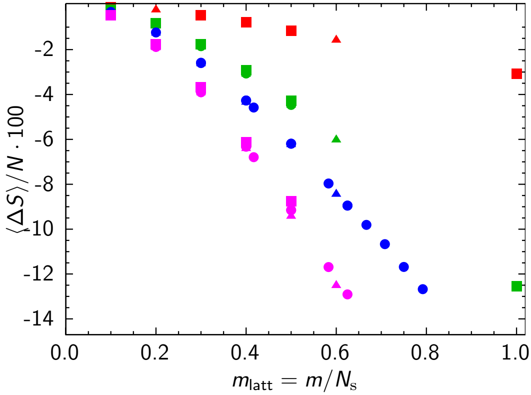
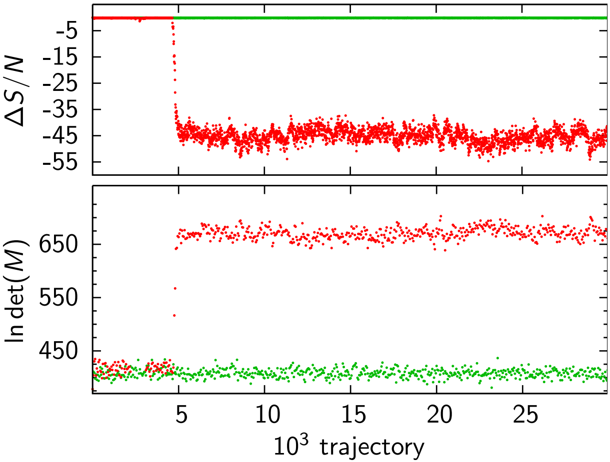


$N \in \{9 \times 9, 15 \times 15, 25 \times 25\}$

$\lambda = 0.8, 1.0, 1.2, 1.5$

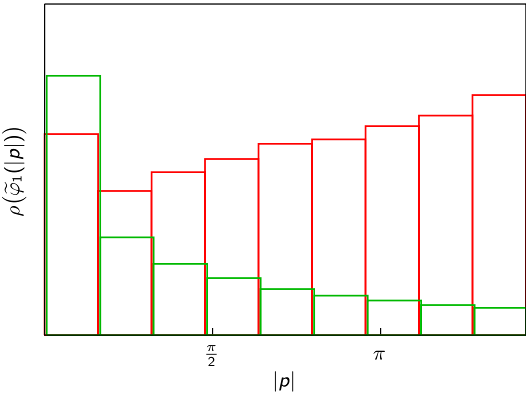
Simulations break down when $\langle \Delta S \rangle / N$ exceeds 14%.

MC history of the improvement term and the fermion determinant at $\lambda = 1.4$ and $\lambda = 1.7$ ($m_{\text{latt}} = 0.6$, $N = 15 \times 15$), $\langle S_B \rangle \approx N$ in each run:



Limitations of improvement

Analyzing the distribution of the fields in momentum space at $\lambda = 1.4$ and $\lambda = 1.7$:



$\Rightarrow$ For too large couplings λ (or lattice masses m_{latt}) the simulation samples only unphysical UV dominated configurations.

$\Rightarrow$ At larger couplings a careful analysis of the improvement term during the simulation must be ensured.

The sign problem

Positiveness of the fermion determinant cannot be guaranteed!
$\Rightarrow$ We need to check explicitely on the sign of the determinant.

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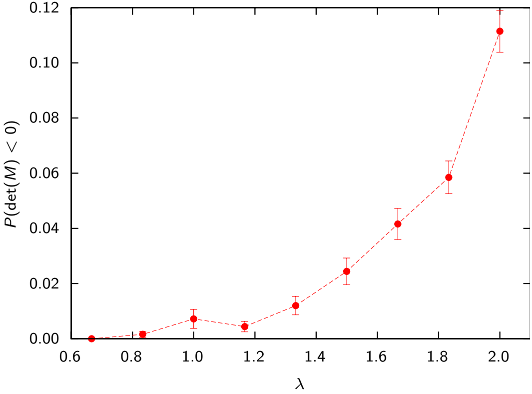
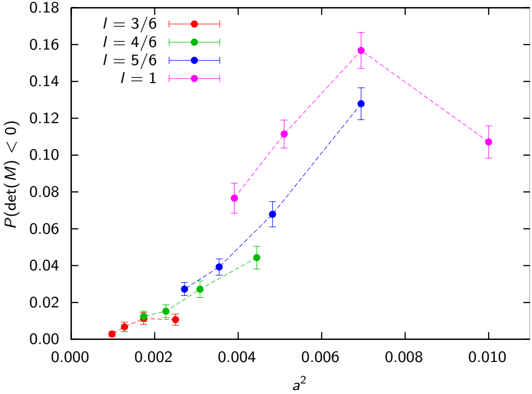


Wilson unimproved, $N = 14 \times 14$, $m_{\text{latt}} = 0.43$

Finite size scaling and continuum limit of the sign problem:



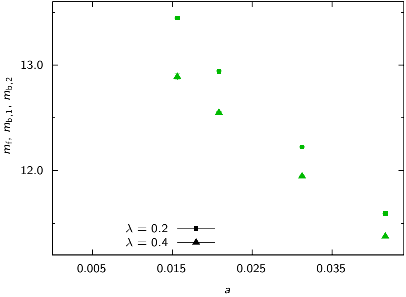
Wilson unimproved, $m = 6$, $\lambda = 2.0$

$\Rightarrow$ In the continuum limit at fixed box size the sign problem vanishes!

With Wilson fermions we test for supersymmetry breaking effects on the lattice at different lattice spacings for $\lambda \in \{0.2, 0.4\}$, $m = 15$.

Masses for bosons (φ_1, φ_2, statistics 10^6–10^7 configs) and fermions (statistics 10^4 configs)

improved:



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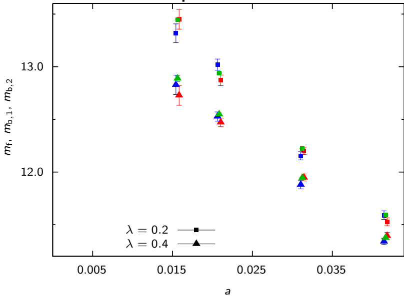

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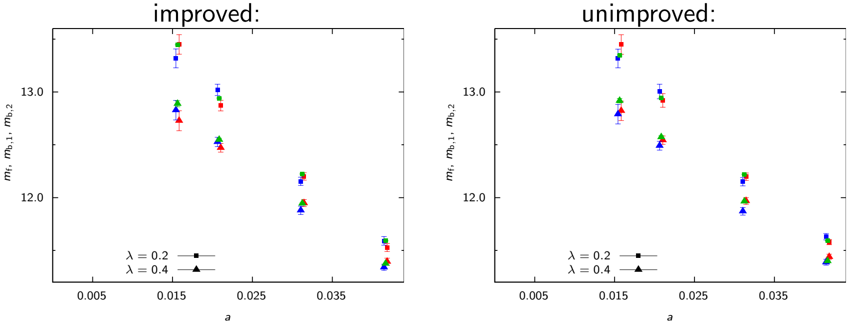
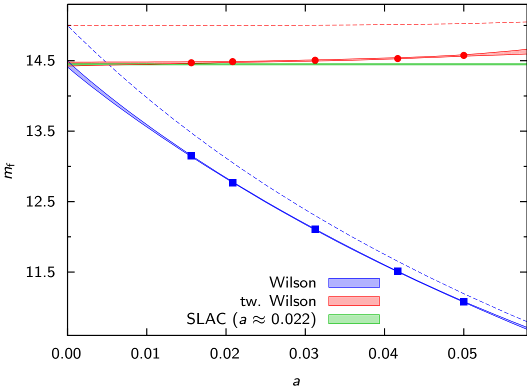


⇒ Improved and unimproved model can not be distinguished even with that high statistics.

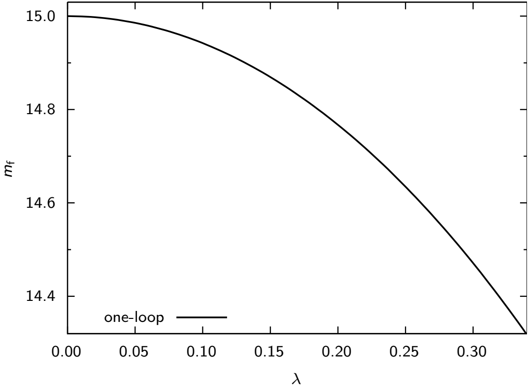
⇒ Bosonic and fermionic masses coincide.

Extrapolation from finite lattice spacing to the continuum using Wilson and twisted Wilson fermions for the improved model ($m = 15$, $\lambda = 0.3$):



$\Rightarrow$ All formulations yield the same continuum result.

The perturbative one-loop result $m_{\text{ren}}^2 = m^2\left(1 - \frac{4\lambda^2}{3\sqrt{3}}\right) + \mathcal{O}(\lambda^4)$ can be compared to the continuum extrapolation of the lattice data:



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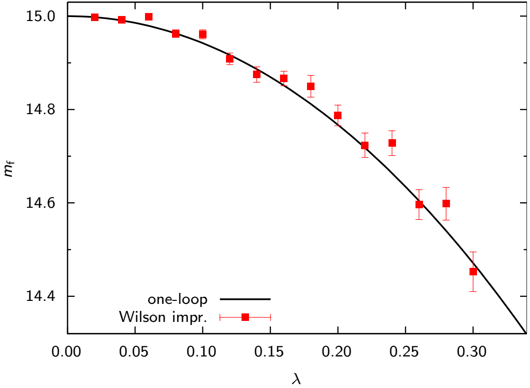

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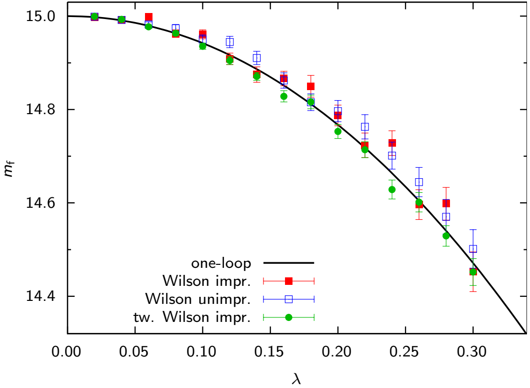
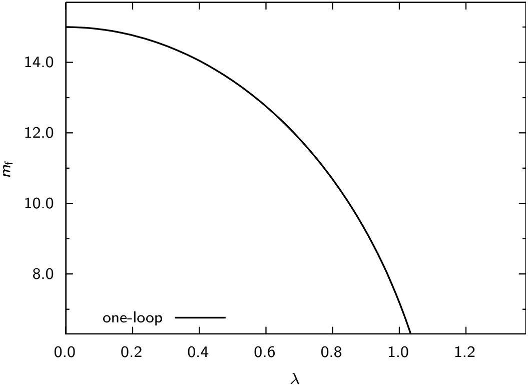


⇒ All different formulations coincide with perturbation theory.

⇒ The supersymmetric continuum limit is reached.

We checked for the limitations of the one-loop calculation using $\lambda \in [0, 1]$:



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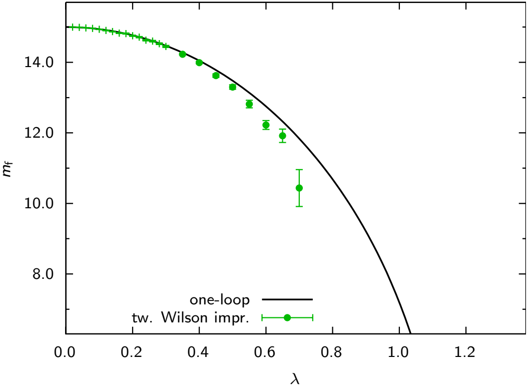

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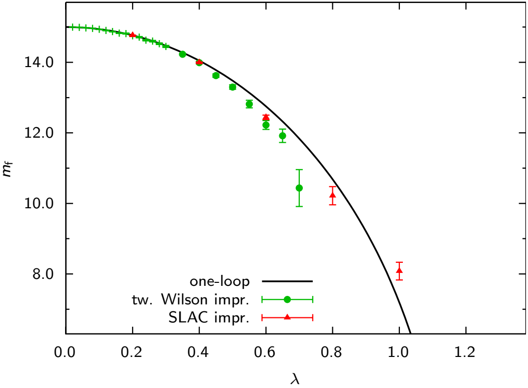

impr. tw. Wilson (cont.)
impr. SLAC ($N = 45 \times 45$)

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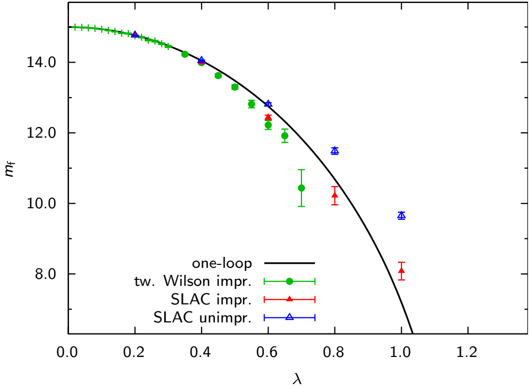

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$\Rightarrow$ For SLAC fermions with $\lambda \geq 0.6$ improved and unimproved models differ.
$\Rightarrow$ Larger lattice for $\lambda = 0.8$!

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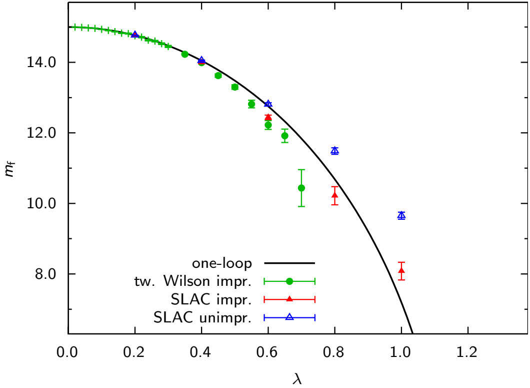

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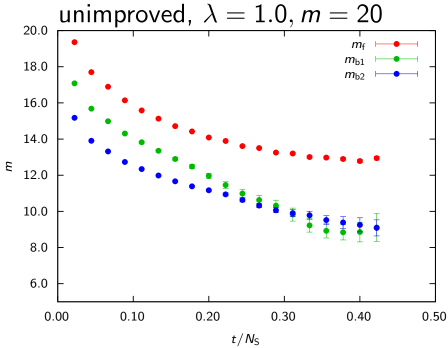
N_s	improved	unimproved
45	10.22(26)	11.49(9)
63	10.54(15)	10.70(19)

$\Rightarrow$ The correct continuum limit is reached for both models, where the improved SLAC model is closer to the continuum limit.

For smaller couplings the bosonic and fermionic masses coincide.
$\Rightarrow$ Check this at larger couplings $\lambda \gtrsim 1.0$ with SLAC fermions (45×45).



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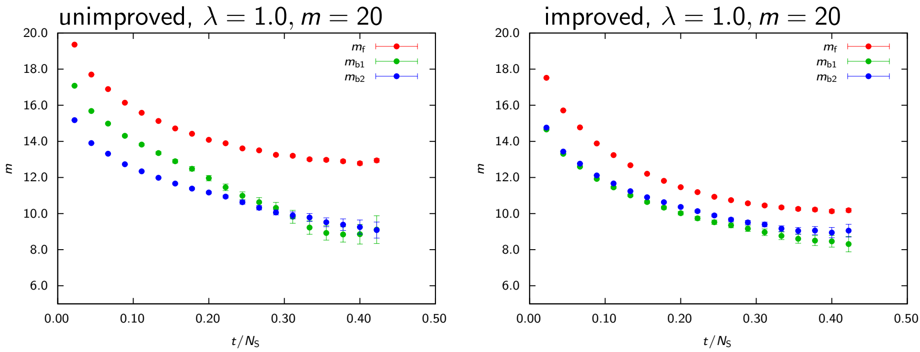
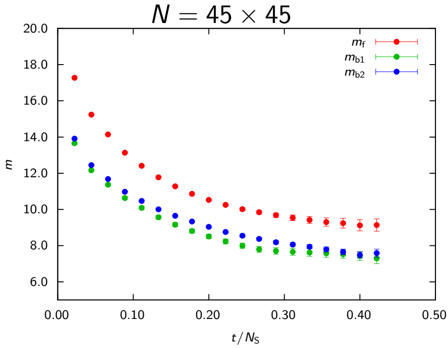


$\Rightarrow$ The mass ratio for the improved model is much closer to one.
Perhaps we are not close enough to the continuum?

Probing the continuum limit of the improved model at $\lambda = 1.1, m = 20$:



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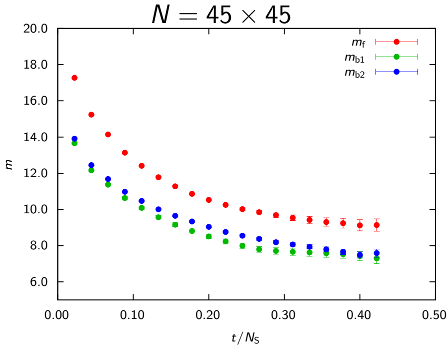

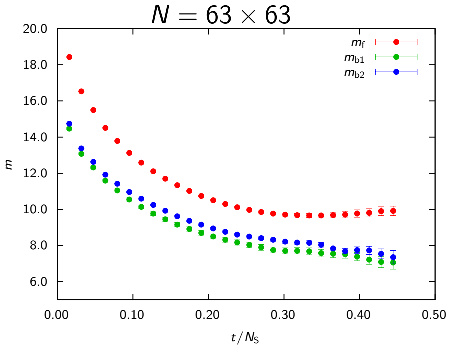

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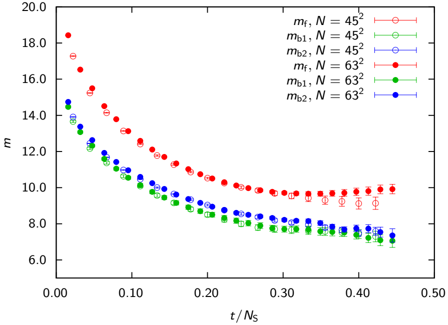

$\Rightarrow$ The masses are already in the scaling regime.

$\Rightarrow$ No discretization effect!

One exact supersymmertry in the improved model corresponds to one fulfilled Ward identity at finite lattice spacing.

$$\langle F(t)\rangle \equiv \left\langle \sum_{\alpha,x,x',t'} \psi_\alpha(t',x)\bar{\psi}_\alpha(t+t',x') \right\rangle = \left\langle \mathrm{Re} \sum_{x,x',t'} \bar{\varphi}(t',x)\xi(t+t',x') \right\rangle \equiv \langle B(t)\rangle$$

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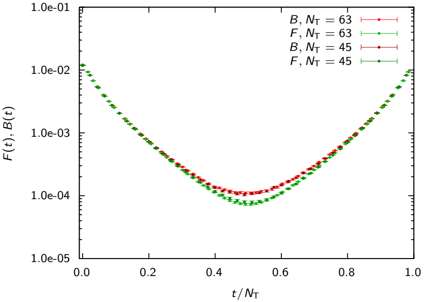

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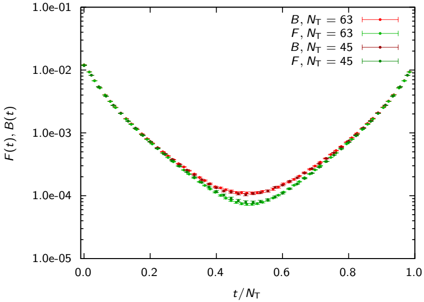
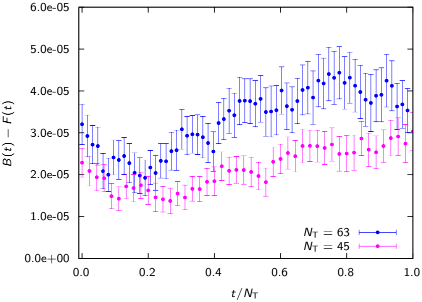
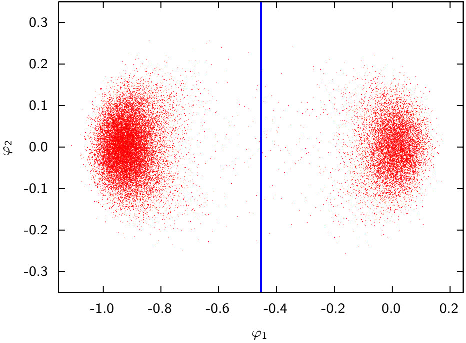




Not even the Ward identities are fulfilled!

On any finite lattice and $\lambda > 0$ there is a $\mathbb{Z}_2^{\mathrm{R}}$ symmetry.



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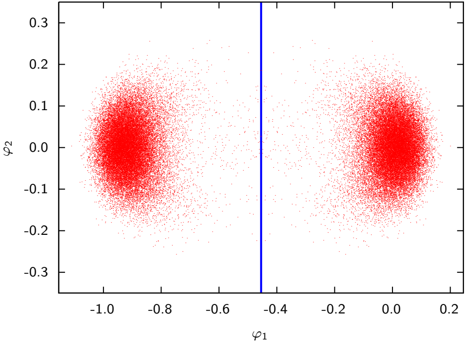
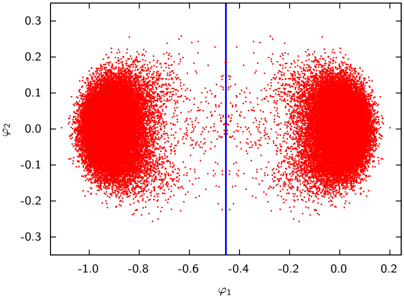


The Ward identities (with proper sampling) vanish for bosonic and fermionic channel seperately:

$$\langle B(t) \rangle = 0 = \langle F(t) \rangle$$

In the thermodynamic limit this $\mathbb{Z}_2^R$ is spontaneously broken.
$\Rightarrow$ We apply a projection to the classicle minimum around $\varphi = 0$.



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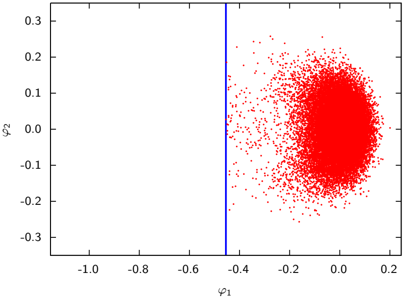

$$S_{\mathrm{B}} \to \lim_{\alpha\to\infty} \left(S_{\mathrm{B}} + \alpha \cdot \theta\big(N^{-1} \sum_x \varphi_1(x) + m/(2g)\big) \right)$$

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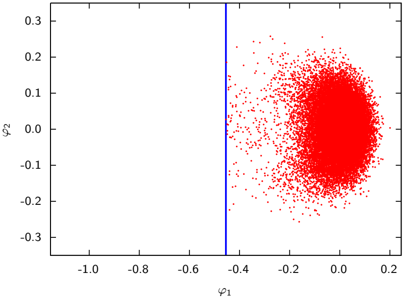

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$\Rightarrow$ Breaks supersymmetry explicitely!

- Not free of finite-size effects.
- Supersymmetry unbroken in finite volume, but we are not able to see degenerated masses of bosons and fermions.
- Even at $M \cdot l > 7$ tunneling events occur.

Summary

Results

- With very high statistics bosonic and fermionic masses can not be distinguished in the weak coupling region for both improved and unimproved formulation.
- For intermediate coupling the improved action in closer to the continuum limit (at least for SLAC fermions).
- The "Nicolai improvement" introduces new problems due to the sampling of unphysical (high-momentum) states. (no real improvement?)
- Even without improvement the correct continuum limit is reached.
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Outlook

- A detailled finite size study is in order to explore the strong coupling region ($\lambda > 1.0$).
- Use the elaborate algorithms to explore the $\mathcal{N} = 1$ WZ model in $d = 2$ (SUSY breaking expected).

Outlook: The $\mathcal{N} = 1$ WZ model

With $\varphi_2 \equiv 0$ and using Majorana fermions we end up with the $\mathcal{N} = 1$ WZ model:

$$S_{\text{cont}} = \int d^2x \left(\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2} V(\phi)^2 + \psi^T C M \psi \right),$$

$$M = \begin{pmatrix} \partial_0 + m + 2g\phi & -\partial_1 \\ -\partial_1 & -\partial_0 + m + 2g\phi \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

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We use the prepotential $V(\phi) = m\phi + g\phi^2$.
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Expectations

Beccaria, Feo et al. (2004):

- The broken SUSY comes together with an unbroken $\mathbb{Z}_2^R$.
- Even in the infinite volume the SUSY breaking survives.

On the lattice the path integral reduces to

$$\begin{aligned} Z_{\mathrm{PBC}} &= \int \mathcal{D}\phi\,\mathcal{D}\psi\ \exp\left(-\int d^2x\,\left(\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}V(\phi)^2+\psi^T CM\psi\right)\right) \\ &= \int \mathcal{D}\phi\,\mathrm{Pf}(CM)\ \exp\left(-\int d^2x\,\left(\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}V(\phi)^2\right)\right) \\ &= \int \mathcal{D}\phi\,\mathrm{sign}\,\mathrm{Pf}(CM)\,\sqrt{|\det(M)|}\ \exp\left(-\int d^2x\,\left(\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}V(\phi)^2\right)\right) \end{aligned}$$

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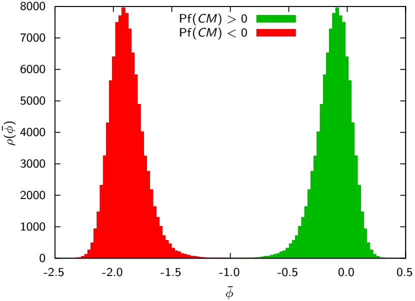
All but the sign of the Pfaffian can be handled by a standard DRHMC algorithm. Considering the symmetry $\mathbb{Z}_2^{\mathrm{R}} : \phi \to -\phi - m/g$,

$$\mathbb{Z}_2^{\mathrm{R}} : \mathrm{Pf}(CM) \to -\mathrm{Pf}(CM)$$

$$\Rightarrow \quad Z_{\mathrm{PBC}} = \mathrm{tr}[(-1)^F e^{-\beta H}] = 0 = \#\mathrm{bos.\ GS} - \#\mathrm{ferm.\ GS}$$

For every finite volume there may be tunneling processes:

$g = 4.5, m = 9, N = 9 \times 9$



$\mathbb{Z}_2^R$ broken
SUSY unbroken in infinite volume

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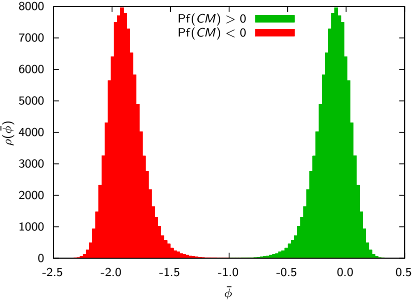

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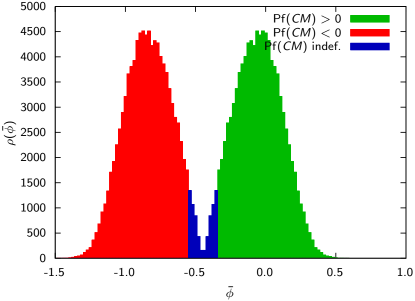
$g = 4.5, m = 4.02, N = 9 \times 9$



$\mathbb{Z}_2^R$ unbroken?
SUSY broken in infinite volume?

Perhaps we should use antiperiodic BCs? (Breaks SUSY explicitely!)
$\Rightarrow$ A lot of conceptual questions remaining!