Infrared Solution for Yang-Mills Theory in the Maximally Abelian Gauge

Markus Q. Huber[1] K. Schwenzer[2] R. Alkofer[1]

[1]Institut für Physik, Karl-Franzens-Universität Graz
[2]Department of Physics, Washington University St. Louis

April 15, 2009

arxiv:0904.1873

Seminar: Simulational methods in physics

Der Wissenschaftsfonds.

SIC!QFT

Table of contents

Phenomenology of the Strong Interaction

Particles under the influence of the strong force: hadrons, e. g. π, K, η, proton, neutron, Λ, Σ, ...

- High energy experiments: point-like particles inside the hadrons (quarks).
- Quarks only exist in bound states, never as free particles (confinement).
- Mediator of the strong force: gluons (also confined).
- Theory: Quantum Chromodynamics (QCD).
- At high energies QCD is asymptotically free, i. e. the coupling gets small and we can "observe" quarks (Nobel prize 2004).
- At lower energies non-perturbative methods are needed.

In this talk we will focus on the low energy behavior of Yang-Mills theory (gluonic part of QCD).

Confinement of quarks and gluons

- Confinement is a long-range $\leftrightarrow$ IR phenomenon: We do not see individual $\sim$ infinitely separated quarks or gluons.
- One expects that the property of being confined is encoded in the particles' propagators.
- Different confinement criteria for the propagators:
  - Positivity violations: negative norm contributions $\rightarrow$ not a particle of the physical state space
  - Gribov-Zwanziger (Landau gauge, Coulomb gauge): IR suppression of the gluon propagator $\rightarrow$ no long-distance propagation
  - Kugo-Ojima: quartet mechanism, e. g. Gupta-Bleuler formalism in QED: timelike and longitudinal photon cancel each other

Functional methods employ

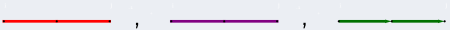
correlation functions/Green fcts./n-point fcts./propagators and vertices.

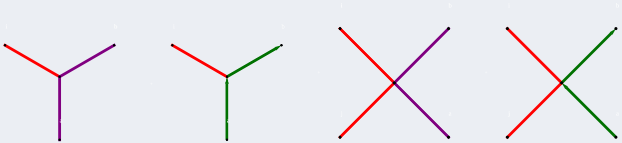
The equations of motion of these are the Dyson-Schwinger equations.

Propagators and vertices

The theory is encoded in the Green functions: "building blocks" for functional equations.
They describe propagation and interactions of fields.

Graphical notation (in anticipation of the MAG)

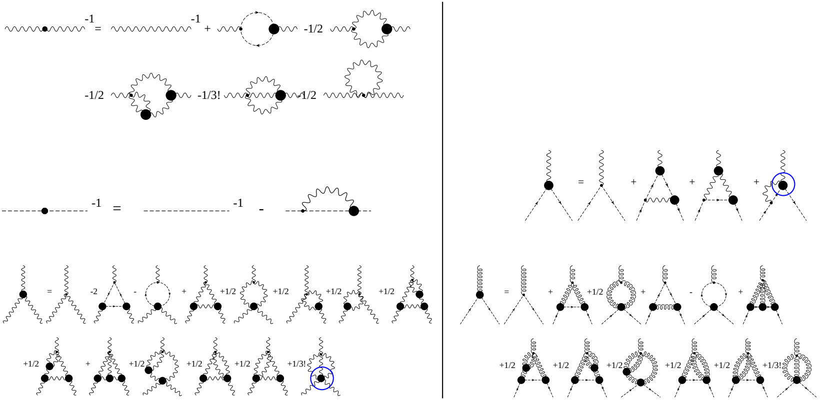
Propagators: 

Vertices:  , ...

The propagators and interactions are given by the Lagrangian of the theory.
Shorthand notation: propagator of field A is AA, quartic interaction is $AAAA$ etc.

The tower of DSEs

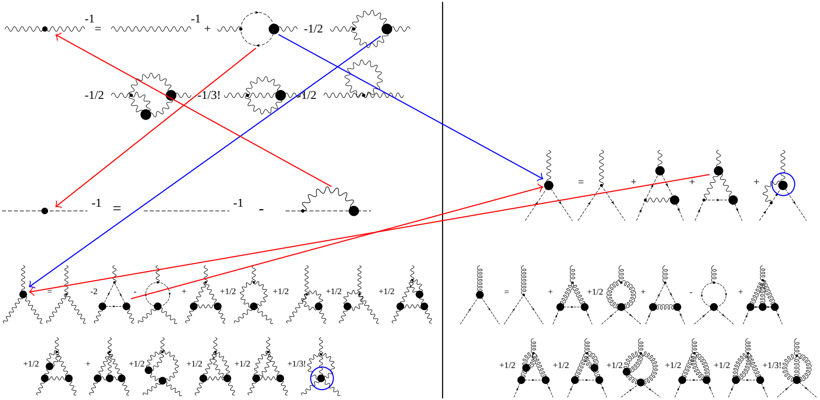
DSE describe non-perturbatively how particles propagate and interact.



n-point functions couple to n-point, (n+1)- and (n+2)-point functions.

The tower of DSEs

DSE describe non-perturbatively how particles propagate and interact.

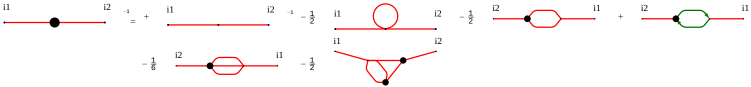


n-point functions couple to n-point, (n+1)- and (n+2)-point functions.

Landau Gauge: Propagators

Colored propagators: Output of *DoDSE* [Alkofer, MQH, Schwenzer, CPC (2009)].

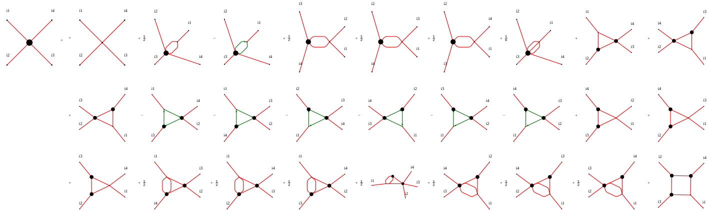
Gluon propagator:

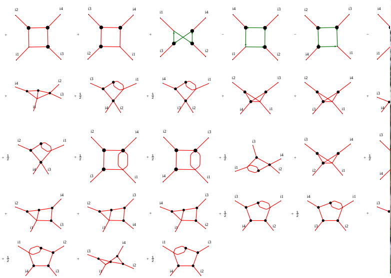


Ghost propagator:

Landau Gauge: Four-Gluon Vertex

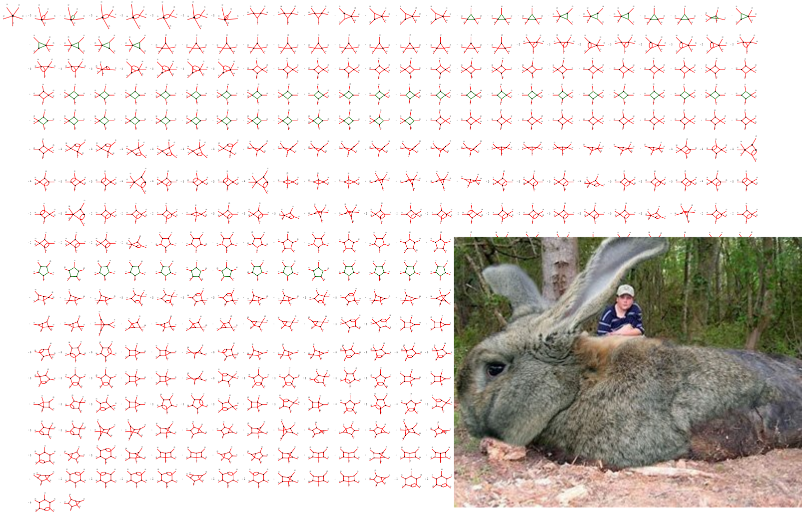
66 terms





Landau Gauge: Five-Gluon Vertex

434 terms



Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Pros:

- Exact equations
  $\rightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\rightarrow$ complement lattice method

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Pros:

- Exact equations
  $\rightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\rightarrow$ complement lattice method

Cons?:

- Truncations
- Gauge-dependent

Dyson-Schwinger equations (DSEs) for investigating QCD

Infinitely large tower of equations

Equations of motion of Green functions

Pros:

- Exact equations
  $\rightarrow$ non-perturbative regime accessible
- Continuum, different scales accessible
  $\rightarrow$ complement lattice method

Cons?:

- Truncations (not for all tasks)
- Gauge-dependent
  $\rightarrow$ Exploit advantages of different gauges

The path integral

Cf. statistical physics: The partition function allows to compute expectation values and correlation functions, e. g. Ising model:

$$Z[H] = \sum_{\text{all conf.}} e^{-K\sum_{\langle i,j\rangle} S_i S_j - H\sum_i S_i}.$$

Expectation value of the magnetization:

$$\left\langle \sum_k S_k \right\rangle = \frac{1}{Z} \sum_{\text{all conf.}} \left(\sum_k S_k \right) e^{-J\sum_{\langle i,j\rangle} S_i S_J - H\sum_i S_i} = -\frac{\partial}{\partial H} log(Z[H])$$

Similar in quantum field theory:

- Path integral is an integral over all possible field configurations.
- "Boltzmann factor" is given by the exponentiated action.
- External sources allow to derive correlation functions $\langle \phi_i \phi_j \ldots \rangle$.

$$Z[J] = \int [d\phi]\, e^{-\int dx \left(\mathcal{L} + \phi(x) J(x) \right)}$$

The path integral for Yang-Mills theory

Yang-Mills theory: Assume quarks to be infinitely heavy $\rightarrow$ only gluons.

$$Z[J] = \int [dA] e^{-\int dx (\mathcal{L}_{\boldsymbol{YM}} + A_\mu(x) J_\mu(x))}$$

The path integral for Yang-Mills theory

Yang-Mills theory: Assume quarks to be infinitely heavy → only gluons.

$$Z[J] = \int [dA] e^{-\int dx (\mathcal{L}_{\boldsymbol{YM}} + A_\mu(x) J_\mu(x))}$$

- Essential symmetry requirement of any gauge theory is invariance of the action under gauge transformations.
- In QCD: gauge transformations = rotations in color space.

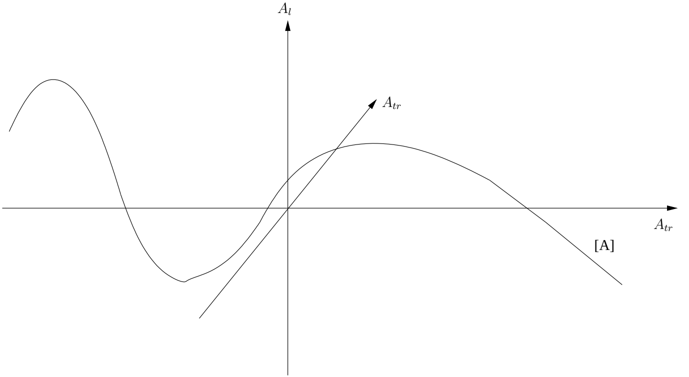
⇒ Equivalent configurations, $\mathcal{L}(A) = \mathcal{L}(A')$, exist (gauge copies).

- But functional integration $\int [dA]$ should only count one representative of a set of gauge copies ⇒ gauge fixing.

[Other problems without gauge fixing: propagator not properly defined, commutation relations of the field operators cannot be obeyed.]

Pictorial sketch of gauge fixing

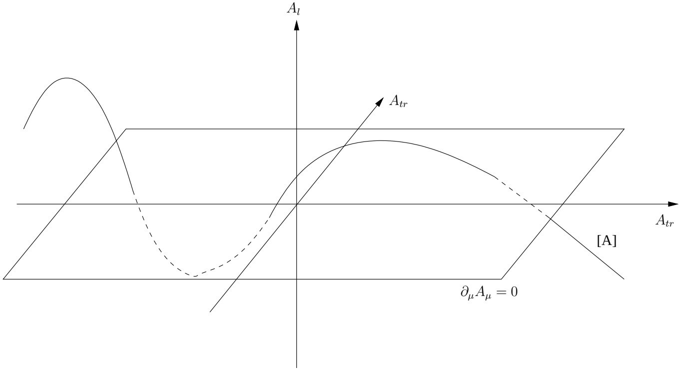
Sketch of field configuration space:



Configurations connected by a gauge transformation lie on a gauge orbit $[A]$.

Pictorial sketch of gauge fixing

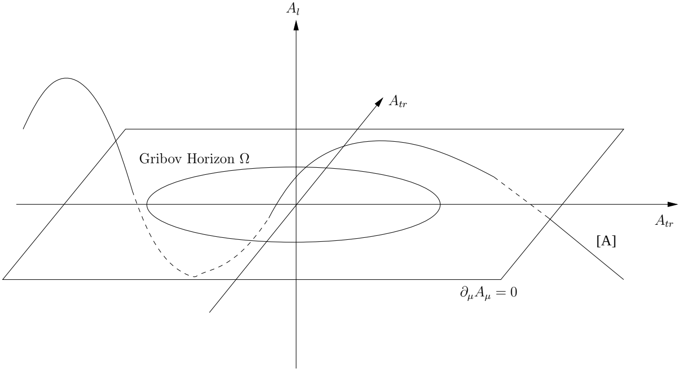
Sketch of field configuration space:



Restrict integration the hyperplane $\partial_\mu A_\mu = 0$ (Landau gauge).

Pictorial sketch of gauge fixing

Sketch of field configuration space:



Minimize some functional to get only one gauge configuration per orbit $\rightarrow$ Gribov region, but still copies (Gribov copies).

Gauge fixing

Task: Functional integration $\int [dA]$ should only count one representative of a set of gauge copies.

$\Rightarrow$ Restriction to a hyperplane in configuration space, e. g. $\partial_\mu A_\mu = 0$.

Gauge fixing

Task: Functional integration $\int[dA]$ should only count one representative of a set of gauge copies.

$\Rightarrow$ Restriction to a hyperplane in configuration space, e. g. $\partial_\mu A_\mu = 0$.

Convenient to introduce new Grassmann fields c and $\bar{c}$ to account for this restriction:

$$\Rightarrow Z[J, J_c, J_{\bar{c}}] = \int [dA\bar{c}c] e^{-\int dx \left(\mathcal{L}_{YM} + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \boxed{\bar{c}(-\partial_\mu D_\mu)c} + AJ + \bar{c}J_{\bar{c}} + J_c c\right)}$$

The maximally Abelian gauge (MAG)

Ezawa, Iwazaki, PRD 25 (1981): Hypothesis of Abelian dominance
(Abelian part should dominate in the infrared part of the theory)

Gauge field: $A_\mu = A^r_\mu T^r, \quad r = 1, \ldots, N^2 - 1$
T^r is the generator of the gauge group $SU(N)$

Abelian subalgebra: $[T^i, T^j] = 0$, can be written as diagonal matrices
Split the gauge field: Abelian/Diagonal and non-Abelian/off-diagonal fields

$$A_\mu = \boldsymbol{A^i_\mu T^i} + \boldsymbol{B^a_\mu T^a}, \quad i = 1, \ldots, N-1, \quad a = N, \ldots, N^2 - 1$$

Fix gauge of off-diagonal field $\boldsymbol{B}$ by $D_\mu \boldsymbol{B_\mu} = 0$.
Fix gauge of diagonal gluon field $\boldsymbol{A}$ by Landau gauge condition: $\partial_\mu \boldsymbol{A_\mu} = 0$.

Peculiarities of the maximally Abelian gauge for $SU(2)$

- Yang-Mills vertices split: ***ABB***, ***AABB***, ***BBBB***.
- Non-linear gauge fixing condition (depends on ***A***) $\rightarrow$ ***Acc***, ***AAcc***, ***BBcc***.
- Renormalizability requires an additional quartic ghost interaction $\rightarrow$ ***cccc***.
- Ghosts also split into diagonal and off-diagonal parts, but diagonal ghosts decouple (diagonal ghost equation).
- Two gauge fixing parameters: $\alpha_A = 0$ (Landau gauge), α_B.

Note: For $SU(N)$ there are four interactions more.

Landau gauge and maximally Abelian gauge

	Landau gauge	MAG ($SU(2)$)
propagators	***A***, ***c***	***A***, ***B***, ***c***
interactions	***AAA***, ***Acc***; ***AAAA***	***ABB***, ***Acc***; ***AABB***, ***AAcc***, ***BBcc***, ***BBBB***, ***cccc***
Gribov region	bounded in all directions	bounded in off-diagonal and unbounded in diagonal direction [Capri et al, PRD 79 (2009)]
decoupling sol.	lattice, extended Gribov-Zwanziger framework and functional equations	lattice [Mendes et al., arxiv:0809.3741], ext. Gribov-Zwanziger framework [Capri et al., PRD 77 (2008)]
scaling solution	[von Smekal, Alkofer, Hauck, PRL 79 (1997)]	this talk ($SU(N)$) [M. Q. H., Schwenzer, Alkofer, arxiv:0904.1873]

Loop integrals for low external momenta

We want to know how a vertex function behaves, when the external momenta approach 0 simultaneously:

$$\Gamma(p_1, p_2, \ldots) \text{ for } p_i \rightarrow 0$$

Generic propagator

$$L_{(\mu\nu)} \cdot \frac{D(p)}{p^2},$$

assume power law behavior at low p

$$D^{IR}(p) = A \cdot (p^2)^{\;\delta}$$

Example: Ghost propagator

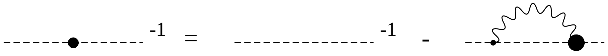
$$\int \frac{d^d q}{(2\pi)^d} P_{\mu\nu} \frac{D^{AA}(q)}{q^2} \Gamma^{Acc,0}(p,q) \frac{D^{cc}(p-q)}{(p-q)^2} \Gamma^{Acc}(p,q)$$

Integrals are dominated by $1/(p-q)^2 \rightarrow$ use IR expressions for all quantities.

Vertices also assume power law behavior [Alkofer, Fischer, Llanes-Estrada, PLB 611 (2005)] (skeletion expansion).

Power counting

- The ghost propagator DSE:



- Plug in power law ansätze for dressing functions in the IR (In Landau gauge the ghost-gluon vertex has an IR constant dressing.):

$$\left(\frac{B\cdot(p^2)^\beta}{p^2}\right)^{-1}\sim\int\frac{d^dq}{(2\pi)^d}\,P_{\mu\nu}\,\frac{A\cdot(q^2)^\alpha}{q^2}\;\frac{B\cdot((p-q)^2)^\beta}{(p-q)^2}\;(p-q)_\mu q_\nu$$

Power counting

- The ghost propagator DSE:

- Plug in power law ansätze for dressing functions in the IR (In Landau gauge the ghost-gluon vertex has an IR constant dressing.):

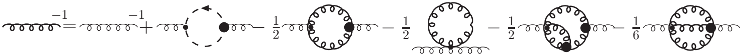
$$\left(\frac{B\cdot(p^2)^\beta}{p^2}\right)^{-1}\sim\int\frac{d^dq}{(2\pi)^d}\,P_{\mu\nu}\,\frac{A\cdot(q^2)^\alpha}{q^2}\,\frac{B\cdot((p-q)^2)^\beta}{(p-q)^2}\,(p-q)_\mu q_\nu$$

- Only one momentum scale
  $\rightarrow$ simple power counting is possible:
- $1-\beta=\frac{d}{2}+\alpha-1+\beta-1+\frac{1}{2}+\frac{1}{2}\Longrightarrow -2\beta=\alpha+\frac{d}{2}-2$

Systems of inequalities

- For every diagram such expressions can be written down.
- At least the IRE of one diagram must equal the IRE of the vertex function on the lhs.
- No diagram can be more IR divergent than the vertex function on the lhs $\rightarrow \delta_{lhs} \leq \delta_{rhs}$.
- Not knowing which diagram is leading on the rhs, we can write inequalities from all diagrams.

$$-\delta_{gl} = \min(\underbrace{0}_{\text{bare prop.}}, \underbrace{2\delta_{gl} + \delta 3g}_{\text{gh loop}}, \underbrace{2\delta_{gh} + \delta_{gg}}_{\text{gl loop}}, \underbrace{\delta_{gl}}_{\text{tadpole}}, \underbrace{4\delta_{gl} + 2\delta_{3g}}_{\text{squint}}, \underbrace{3\delta_{gl} + \delta_{4g}}_{\text{sunset}})$$



Infrared exponent for an arbitrary diagram

Having so many diagrams, isn't there a shorter way than writing all expressions down explicitly?

Arbitrary Diagram v

Numbers of vertices and propagators related $\Rightarrow$ possible to get a formula for the IR exponent by pure combinatorics.

Function of:

- propagator IR exponents δ_{x_i}
- number of external legs m^{x_i}
- number of vertices.

$$\delta_v = \boxed{-\frac{1}{2}\sum_i m^{x_i}\delta_{x_i}} + \\ + \sum_i (\#\text{ of dressed vertices})_i C_1^i + \sum_i (\#\text{ of bare vertices})_i C_2^i$$

Only depends on the external legs $\rightarrow$ equal for all diagrams in a DSE/RGE.

[Similar formula from physical arguments: Fischer, Pawlowski, 0903.2193 (2009).]

Relevant inequalities

dressed vertices	$C_1^i = \delta_{vertex} + \frac{1}{2} \sum_{\substack{\text{legs } j \text{ of} \\ \text{vertex}}} \delta_j \geq 0$	from RGEs
prim. divergent vertices	$C_2^i = \frac{1}{2} \sum_{\substack{\text{legs } j \text{ of} \\ \text{prim. div.} \\ \text{vertex}}} \delta_j \geq 0$	from DSEs/RGEs

Here we included inequalities from renormalization group equations (RGEs) [Fischer, Pawlowski, PRD 75 (2007)].
Only some inequalities are restrictive.

Some inqualities are contained within others.
E. g. in MAG: $\delta_B \geq 0$ and $\delta_c \geq 0$ render $\delta_B + \delta_c \geq 0$ useless.

Maximally infrared divergent solution

The inequalities derived from DSEs and RGEs allow to derive a lower bound on the IREs.

$$C_1^i \geq 0, \qquad C_2^i \geq 0.$$

IR solution:

$$\delta_v = -\frac{1}{2}\sum_i m^{x_i}\delta_{x_i} + \sum_i (\#\ \text{dr. vert.})_i C_1^i + \sum_i (\#\ \text{bare vert.})_i C_2^i.$$

Maximally infrared divergent solution

The inequalities derived from DSEs and RGEs allow to derive a lower bound on the IREs.

$$C_1^i \geq 0, \qquad C_2^i \geq 0.$$

$\Rightarrow$ Maximally IR divergent solution:

$$\boxed{\delta_{v,max} = -\frac{1}{2}\sum_i m^{x_i}\delta_{x_i}} + \cancel{\sum_i (\#\text{ dr. vert.})_i C_1^i} + \cancel{\sum_i (\#\text{ bare vert.})_i C_2^i}\,.$$

IR scaling solutions

A general analysis of propagator DSEs yields that at least one inequality from a prim. divergent vertex has to be saturated. Can be traced back to the one bare vertex. Consistency condition between DSEs and RGEs [Fischer, Pawlowski, PRD 75 (2007)].

Landau gauge	MAG
$\delta_{gl} \geq 0$	$\delta_B \geq 0,\ \delta_c \geq 0$
$\frac{1}{2}\delta_{gl} + \delta_{gh} \geq 0$	$\delta_A + \delta_B \geq 0,\ \delta_A + \delta_c \geq 0$

- Saturation in first row corresponds to trivial solution: $\delta_i = 0$ ($\rightarrow$ perturbation theory)
- Second row yields scaling relations: $\delta_{gl} = -2\delta_{gh} = 2\kappa_{LG}$ and $\delta_B = \delta_c = -\delta_A = \kappa_{MAG}$
- Known IR scaling solution of Landau gauge [von Smekal, Hauck, Alkofer, PRL (1997)].
- New scaling solution for MAG [M.Q.H., Schwenzer, Alkofer, arxiv:0904.1873].

IR Scaling solutions for other gauges

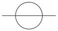
Linear covariant gauges	ghost-antighost symmetric gauges
scaling solution only, if the longitudinal part of the gluon propagator gets dressed	quartic ghost interaction $\rightarrow \delta_{gh} \geq 0 \rightarrow$ with non-negative IREs only the trivial solution can be realized

$\Longrightarrow$

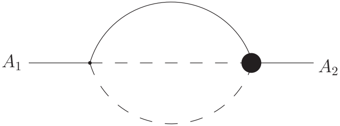
- Either the existence of a scaling solution is something special or
- a more refined analysis is needed.

Leading diagrams

Leading diagrams are determined by bare ***AABB*** or ***AAcc*** vertices:

sunset 	squint 
leading	possibly leading

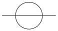
n-point functions (n even): Successively add pairs of fields:



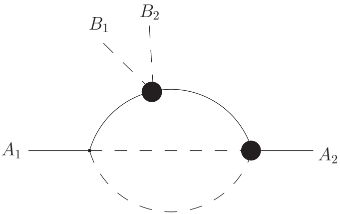
n odd: At least one vertex with an odd number of legs, cannot be determined uniquely.

Leading diagrams

Leading diagrams are determined by bare ***AABB*** or ***AAcc*** vertices:

sunset 	squint 
leading	possibly leading

n-point functions (n even): Successively add pairs of fields:



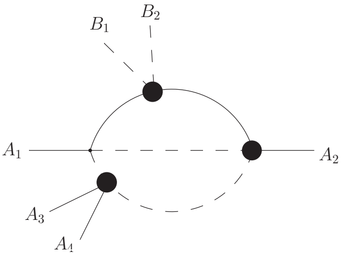
n odd: At least one vertex with an odd number of legs,
cannot be determined uniquely.

Leading diagrams

Leading diagrams are determined by bare $AABB$ or $AAcc$ vertices:

sunset	squint
leading	possibly leading

n-point functions (n even): Successively add pairs of fields:



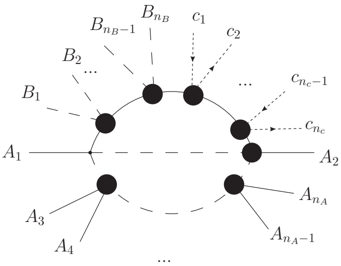
n odd: At least one vertex with an odd number of legs,
cannot be determined uniquely.

Leading diagrams

Leading diagrams are determined by bare ***AABB*** or ***AAcc*** vertices:

sunset	squint
leading	possibly leading

n-point functions (n even): Successively add pairs of fields:



n odd: At least one vertex with an odd number of legs, cannot be determined uniquely.

Scaling solution for the MAG

Scaling relations

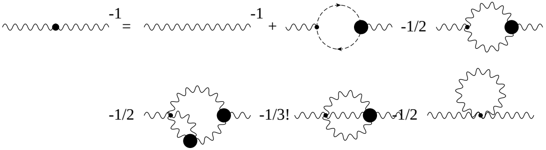
- $-\delta_A = \delta_B = \delta_c := \kappa \geq 0$,
- $\delta_{A^{n_A} B^{n_b} c^{n_c}} = \frac{1}{2}(n_A - n_B - n_c)\kappa \qquad (n_A \text{ even}),$
- $\delta_{A^{n_A} B^{n_B} c^{n_c}} = \frac{1}{2}(n_A - n_B - n_c + \eta)\kappa \qquad (n_A \text{ odd})$

η determines the behavior of the vertices with an odd number of legs.

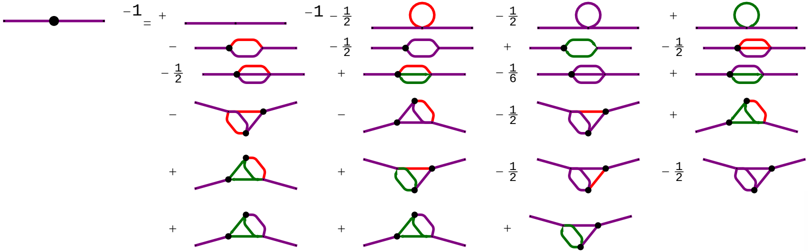
- Diagonal gluon propagator is IR enhanced ($\delta_A \leq 0$). $\Rightarrow$ Supports hypothesis of Abelian dominance.
- Off-diagonal propagators are IR suppressed.
- Two-loop terms are leading.

Numerical solution

In Landau gauge trunction "straight forward": keep one-loop terms (consistent UV behavior, contain IR leading term)

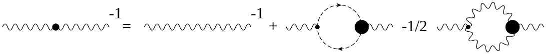


In MAG: two-loop terms leading $\longrightarrow$ for consistent UV behavior keep ALL two-loop terms = no truncation

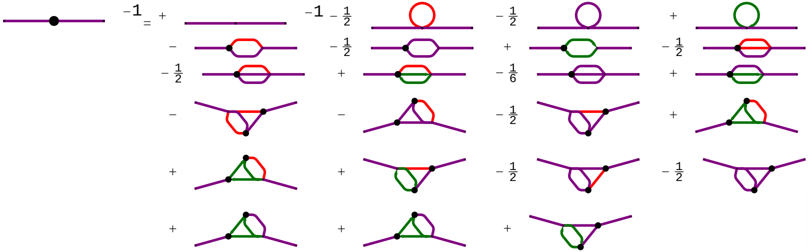


Numerical solution

In Landau gauge trunction "straight forward": keep one-loop terms (consistent UV behavior, contain IR leading term)



In MAG: two-loop terms leading $\longrightarrow$ for consistent UV behavior keep ALL two-loop terms = no truncation



The MAG in $SU(3)$

In general there are more interactions than included above.
$\rightarrow$ Different solution for "physical system", i. e. $SU(3)$?

4 additional vertices: ***BBB***, ***Bcc***, ***ABBB***, ***ABcc***
Constraints:

$$\frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_B + \delta_c \geq 0,$$

$$\frac{1}{2}\delta_A + \frac{3}{2}\delta_B \geq 0, \qquad \frac{1}{2}\delta_A + \frac{1}{2}\delta_B + \delta_c \geq 0$$

Already contained in "old" system $\rightarrow$ nothing new, solution still valid.
No new solutions possible $\rightarrow$ unique solution.

Conclusions

- The MAG may possess an IR scaling solution.
- This solution is in support of the hypothesis of Abelian dominance, because the diagonal gluon propagator is IR enhanced.
- Although the DSEs are more complicated for general $SU(N > 2)$, the qualitative behavior is the same as in $SU(2)$.
- Even if this solution turns out not to exist, we learned about IR scaling solutions and how to retrieve the corresponding scaling relations.

The existence of the IR scaling solution in the MAG has to be verified by a numerical solution of the DSEs, which is more involved than in Landau gauge.