The complex Langevin method: Successes and Difficulties

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Overview

1. Introduction
2. General discussion
3. Quadratic actions
4. Mathematical and Practical Problems
5. Some examples 6. Extension to manifolds?
7. Outlook

1. Introduction

Complex Langevin first (?) proposed:

Parisi, Phys. Lett. **131 B** (1983) 393; Klauder, Acta Phys. Austriaca Suppl. **XXXV** (1983) 251.

Many studies in 1980's and 1990's, e.g.

Hüffel&Rumpf 1984, Klauder&Petersen 1984, J. Ambjørn and S.-K. Yang 1985, Ambjørn, Flensburg&Peterson 1986, Nakazata&Yamanaka 1986, Gausterer&Klauder 1986, Söderberg 1988, Haymaker&Wosiek 1987, Söderberg 1988, Okamoto, Okano, Schülke and Tanaka 1989, Haymaker&Peng 1989, Gausterer 1993, L. L. Salcedo 1993, 1997, S. Lee 1994, Gausterer&Thaler 1998.

In principle

Complex Langevin solves sign problem.

Sign problem arises in

- QCD at finite density
- Quantum Field Theory in Minkowski Space
- Relativistic Bose Gas
- ...

Successes and Failures

In some simple cases good convergence to the right limit.

Example: $U(1)$ LGT in $2D$ (Ambjørn et al 1986).

Practical Problems:

- Runaways (divergence)
- convergence to wrong limit.

Mathematical questions unresolved:

Quotes: *. . . conspicuous absence of general spectral theorems . . .*

(Klauder&Petersen 1984)

. . . a rather experimental character: for some situations the method works, while it fails for other choices of the action . . .

(Haymaker&Wosiek 1988)

Resurrection

Berges&Stamatescu 2005: Simulation of Minkowski space QFT
(Hüffel&Rumpf 1984, Nakamoto&Yamanaka 1986)

Continuation: Berges et al 2007, Berges&Sexty 2007

Finite density: Aarts&Stamatescu 2008
Complex relativistic Bose gas: Aarts 2009
– Numerically impressive results
– approach appears again promising
– but problems lingering.
Guralnik&Pehlevan 2008-2009: Effective potential to resolve ambiguities

2. General discussion

'Flat' case: defined on $\mathcal{M} = \mathbb{R}^n$, analytically continued to $\mathcal{M}_c \equiv \mathbb{C}^n$.

Complex Langevin:

$$dz = -\nabla S dt + dw$$

dw increment of Wiener process on $\mathbb{R}^n$ (formally $dw = \eta(t)dt$, η white noise).

This is real stochastic process:

$$\begin{aligned} dx &= K_x dt + dw \\ dy &= K_y dt \,, \end{aligned} \tag{1}$$

$$
\begin{aligned}
K_x &= -\mathrm{Re}\nabla_x S(x+iy) \\
K_y &= -\mathrm{Im}\nabla_x S(x+iy)
\end{aligned}
\tag{2}
$$

$\Longrightarrow$ Real Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,y;t) = L_{FP} P(x,y;t)\,; \quad P(x,y;0) = \delta(x-x_0)\delta(y-y_0)\,,$$

P probability density in $\mathbb{R}^{2n}$,
Real Fokker-Planck operator:

$$L_{FP} \equiv \nabla_x[\nabla_x - K_x] - \nabla_y K_y$$

Complex Fokker-Planck Equation: Given y_0, define

$$\frac{\partial}{\partial t}\rho_{y_0}(x;t) = L^c_{y_0}\rho_{y_0}(x;t)\,,$$

where $\rho_{y_0}(x;t)$ is complex density defined on $\mathbb{R}^n + iy_0$,

$$L^c_{y_0} \equiv \nabla_x \left[\nabla_x + (\nabla_x S(x+iy_0))\right]\,.$$

Special case: $S(x)$ real for x real:

Complex FPE $\rightarrow$ standard FPE

Real FPE lives still in $\mathbb{R}^{2n}$, but has stationary solution

$$P(x,y) \propto \exp[-S(x)]\delta(y)\,.$$

FP Hamiltonian

$L^c_{y_0}$ operator on $\mathcal{H}_2 \equiv L^2(e^{Re\ S}dx)$.
Unitary map $U : L^2(dx) \to \mathcal{H}_2$:

$$U\psi = \exp(-\tfrac{1}{2}S)\psi\,,$$

$$H_{FP} \equiv -U^{-1}L^c_{y_0}U = -\left(\nabla - \tfrac{1}{2}(\nabla S)\right)\left(\nabla + \tfrac{1}{2}(\nabla S)\right)\;;$$

S real: H_{FP} manifestly positive.

Fact: spectrum and numerical range of $-H_{FP}$ and $L^c_{y_0}$ agree.

Goal and Questions

Goal: Produce expectation values of holomorphic observables O:

$$\langle O \rangle \equiv \frac{\int O(x+iy_0) e^{-S(x+iy_0)} d^n x}{\int e^{-S(x+iy)} d^n x}\,;$$

independent of y_0 by Cauchy's theorem.

Hope: obtainable as long time limit of

$$\langle O \rangle_{P,t} \equiv \frac{\int O(x+iy) P(x,y;t) d^n x d^n y}{\int P(x,y;t) d^n x d^n y}\,;$$

and by ergodicity as

$$\lim_{t\to\infty} \frac{1}{t} \int O(z(t) dt\,.$$

Question: Relation to 'ρ-expectations'

$$\langle O\rangle_{\rho,t} \equiv \frac{\int O(x+iy_0)\rho(x;t)d^n x}{\int \rho_{y_0}(x;t)d^n x} \ ?$$

Transpose operator:

$$(L^c_{y_0})^T \equiv \left[\nabla_x - (\nabla_x S(x+iy_0))\right]\nabla_x \,,$$

$$L^T_{FP} \equiv \left[\nabla_x - \mathrm{Re}(\nabla_x S(x+iy))\right]\nabla_x - \mathrm{Im}(\nabla_x S(x+iy))\nabla_y$$

defined such that

$$\partial_t \langle O\rangle_{\rho,t,y} = \langle (L^c_{y_0})^T O\rangle_{\rho,t} \text{ and } \quad \partial_t\langle O\rangle_{P,t} = \langle L^T_{FP} O\rangle_{P,t} \,.$$

Result

Assume

- $P(x,y;0) = \delta(y-y_0)\rho(x;0)$
- for all y_0 $\quad L^c_{y_0}$ generates quasibounded holomorphic semigroup (i.e. $\|e^{tL^c_{y_0}}\| \leq C_1 e^{C_2 t}$)
- L_{FP} generates quasibounded (strongly continuous) semigroup on $L^2(\mathbb{R}^n)$(i.e. $\|e^{tL_{FP}}\| \leq C_1 e^{C_2 t}$)
- for all y_0 $\quad O(x+iy_0) \in L^2(\mathbb{R}^n, d^n x)$.

Then

$$\boxed{\langle O\rangle_{\rho,t} = \langle O\rangle_{P,t} \quad \forall\, y_0,\, t\geq 0}$$

Proof

1. Initial conditions agree.

2. Let $O(x+iy_0;t) \equiv \exp\left[t(L^c_{y_0})^T\right] O(x+iy_0)$, the unique solution of DE

$$\partial_t O(x+iy_0;t) = (L^c_{y_0})^T O(x+iy_0;t) \quad (t \geq 0)\,;$$

$O(x+iy_0;t)$ still determines holomorphic $O(x+iy;t)$.

3. Consider $F(t,\tau) \equiv \int P(x,y;t-\tau) O(x+iy;\tau)$.

$$F(t,0) = \langle O \rangle_{P,t}; \quad F(t,t) = \langle O \rangle_{\rho,t}$$

(second equation: use integration by parts)

Claim: $F(t,\tau)$ independent of τ.

Reason:

$$\begin{aligned}\frac{\partial}{\partial\tau}F(t,\tau) &= -\int (L_{FP}P(x,y;t-\tau)O(x+iy;\tau)d^nxd^ny \\ &+ \int P(x,y;t-\tau)(L^c_{y_0})^T O(x+iy;\tau)d^nxd^ny \qquad (3)\end{aligned}$$

Second term: can replace $(L^c_{y_0})^T$ by L^T_{FP} (Cauchy-Riemann equations).

Integration by parts $\Rightarrow \frac{\partial}{\partial\tau}F(t,\tau) = 0$.

Generalization

Introduce $N_I, N_R > 0, \quad N_R = N_I + 1$

Complex Langevin:

$$dz = -\nabla S dt + N_R dw_R + N_I dw_I$$

w_R, w_I independent Wiener processes on $\mathbb{R}^{2n}$

Real FP operator:

$$L_{FP} \equiv N_R \nabla_x [\nabla_x - K_x] + N_I \nabla_y [\nabla_y - K_y]$$

Complex FP operator unchanged!

(Reason: Cauchy-Riemann equations)

Comments

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  Need only convergence of $\rho(x;t)$.

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- Convergence of $P(x,y;t)$ not necessary. Need only convergence of $\rho(x;t)$.
- Need: spectrum of $L^c_{y_0}$ in left half plane.
- $\mathrm{spec}(L^c_{y_0}) \subset \mathrm{spec}(L_{FP})$. Pseudospectrum?

Equilibrium distribution

Existence not proven!

Assume existence, $N_I > 0$. Stationary real FPE:

$$\left[N_R \Delta_x + N_I \Delta_y - \vec{K} \cdot \vec{\nabla} - (\mathrm{div}\,\vec{K})\right] P(\vec{x}, \vec{y}) = 0$$

Facts:

- P smooth
- $(\vec{x}_*, \vec{y}_*)$ stable fixed point $\Longrightarrow P(\vec{x}_*, \vec{y}_*) \geq \langle P \rangle_\epsilon$,
  $\langle \cdot \rangle_\epsilon$ average over a circle of radius ϵ (ϵ small enough)
- $P(\vec{x}_*, \vec{y}_*)$ local maximum $\Longrightarrow \mathrm{div}\,\vec{K}(\vec{x}_*, \vec{y}_*) < 0$.

Analogous for local minima.

Reasons:

- Elliptic regularity
- Rescale $(\vec{x}, \vec{y}) = S(\vec{\xi}, \vec{\eta})$ to obtain

$$\left[\Delta_\xi + \Delta_\eta - \vec{L} \cdot \vec{\nabla} - (\mathrm{div}\, \vec{L})\right] Q(\vec{\xi}, \vec{\eta}) = 0$$

  Fixed point structure unchanged;
  near (ξ_*, η_*) $\mathrm{div}\, \vec{L} < 0 \Longrightarrow Q$ superharmonic;
- Near fixed point $\vec{K} = A\vec{x} + O(\vec{x}^2) \quad \Longrightarrow$

$$\mathrm{div}\, \vec{L}(\vec{x}_*, \vec{y}_*) = \mathrm{div}\, \vec{K}(\vec{\xi}_*, \vec{\eta}_*)$$

3. Quadratic Actions

Almost trivial, but instructive. Complete analysis possible.
(cf. Ambjørn&Yang 1985, Haymaker&Peng 1989)

Setting:

$$S = -\tfrac{1}{2}(x, Ax), \quad x \in \mathbb{R}^n \,,$$

$A = A_r + iA_i$ complex symmetric matrix; A_r and A_i real symmetric matrices.

Assumptions:

- $-A$ *strictly dissipative*: $A_r = \frac{1}{2}(A + A^\dagger) > 0$.
- A diagonalizable by a complex orthogonal matrix O: $A = O^T DO$ with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Generic!

Fact: $\operatorname{Re}\lambda_1, \ldots \lambda_n > 0$ because $-A$ strictly dissipative.
Converse not true:

$$A = \begin{pmatrix} -1 & 2i \\ 2i & 3 \end{pmatrix}$$

has eigenvalues $\lambda_1 = \lambda_2 = 1$, but

$$\frac{1}{2}(A + A^\dagger) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

not positive definite, i.e. $-A$ not dissipative.

1D example

$$S = \frac{1}{2} a x^2, \quad a = a_r + i a_i, \quad a_r > 0$$

$$L_{FP} = \partial_x^2 + a_r(\partial_x x + \partial_y y) + a_i(-\partial_x y + \partial_y x)\,.$$

L_{FP} not dissipative:

$$\frac{1}{2}(L_{FP} + L_{FP}^\dagger) = \partial_x^2 + 2a_r\,.$$

But stationary solution:

$$P(x, y; \infty) = c \exp\left[-a_r x^2 - \frac{2a_r^2}{a_i} xy - \frac{a_r}{a_i^2}(2a_r^2 + a_i^2) y^2\right]\,.$$

Integrable for $a_r > 0$.

Remark: Level lines of $P(x, y; \infty)$ are tilted ellipses:

$$P(x, y; \infty) = c \exp[-Q(x, y)]$$

with

$$Q(x, y) = \frac{a_r}{2}\left[x + y(\alpha + \sqrt{1+\alpha^2})\right]^2 + \frac{a_r}{2}\frac{1+\alpha^2-\sqrt{1+\alpha^2}}{1+\alpha^2+\sqrt{1+\alpha^2}}\left[x(\alpha + \sqrt{1+\alpha^2}) - y\right]^2 . \quad (3)$$

where $\alpha = a_r/a_i$.

Time-dependent solution

(Haymaker&Peng 1989):

$$X = \begin{pmatrix} x \\ y \end{pmatrix} , X_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} , Z(t) = X - e^{-a_r t} \begin{pmatrix} \cos a_i t & \sin a_i t \\ -\sin a_i t & \cos a_i t \end{pmatrix} X$$

$$\boxed{P(x, y; t) = \exp\left[-\frac{1}{2} Z(t)^T \Sigma^{-1}(t) Z(t)\right]}$$

with $\Sigma(t) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$

$$\sigma_{11} = \frac{1}{a_r} + \frac{a_r}{2(a_r^2+a_i^2)} + e^{-2a_r t}\left[\frac{-a_r\cos(2a_i t)+a_i\sin(2a_i t)}{2(a_r^2+a_i^2)} - \frac{1}{2a_r}\right]$$

$$\sigma_{12} = -\frac{a_r}{2(a_i^2+a_i^2)} + e^{-2a_r t}\left[\frac{a_r\sin(2a_i t)+a_i\cos(2a_i t)}{2(a_r^2+a_i^2)}\right]$$

$$\sigma_{22} = \frac{1}{a_r} - \frac{a_r}{2(a_r^2+a_i^2)} + e^{-2a_r t}\left[\frac{a_r\cos(2a_i t)-a_i\sin(2a_i t)}{2(a_r^2+a_i^2)} - \frac{1}{2a_r}\right]$$

Complex FP equation

$$L^c_{y_0} = \partial_x^2 + a\partial_x(x + iy_0)\,;$$

not dissipative if $a_i \neq 0$.
FP Hamiltonian:

$$H_{FP} = -\partial_x^2 - \tfrac{1}{2}a + \tfrac{1}{4}a^2(x + iy_0)^2\,,$$

For $y_0 = 0$ and rescaled $x \mapsto x\sqrt{2}$: standard harmonic oscillator

$$H_{h.o.} = -\tfrac{1}{2}\tfrac{d^2}{dx^2} + \tfrac{1}{2}\omega^2 x^2 - \tfrac{\omega}{2}$$

Mehler formula

$$\exp(-tH_{h.o.}(x,x_0) \equiv Q_t(x,x_0)\,,$$

with

$$Q_t^\omega(x,x_0) = \sqrt{\frac{\omega}{\pi(1-e^{-2\omega t})}}\exp\left[-\frac{\omega(x^2+{x_0}^2)}{2\tanh(\omega t)} - \frac{\omega x x_0}{\sinh(\omega t)}\right]\,.$$

Using unitary map U:

$$\exp(tL_0^c)(x,x_0) = e^{-ax^2/4}Q_t^\omega\left(\frac{x}{\sqrt{2}},\frac{x_0}{\sqrt{2}}\right)e^{a{x_0}^2/4}\,.$$

Reintroduce y_0:

$$\exp(tL_{y_0}^c)(x,x_0) = \exp(tL_0^c)(x+iy_0,x_0+iy_0)\,.$$

Higher dimensions

$$L_{FP} = \Delta_x + \nabla_x \cdot A_r x + \nabla_y \cdot A_r y - \nabla_x \cdot A_i y + \nabla_y \cdot A_i x \,,$$

$$L_{FP}^{\dagger} = \Delta_x - (A_r x) \cdot \nabla_x - (A_r y) \cdot \nabla_y + \nabla_x \cdot A_i y - \nabla_y \cdot A_i x \,.$$

$$\tfrac{1}{2}(L_{FP} + L_{FP}^{\dagger}) = \Delta_x + 2\,\mathrm{tr}\, A \,,$$

so L_{FP} is again not dissipative.

Solution by Mehler kernel

First $A_i = 0$: $\exists O$ (orthogonal)

$$A = O^T D$$

with $D = diag(\lambda_1, \dots , \lambda_n)$.
Put $Ox = x'$, $Ox_0 = x'_0$:

$$\exp(-tH_{FP})(x, x_0) = \prod_{i=1}^{n} Q_t^{\lambda_i} \left(\frac{(Ox)_i}{\sqrt{2}}, \frac{(Ox_0)_i}{\sqrt{2}} \right) .$$

$$e^{L_{y_0} t}(x, x_0) = \exp(-\frac{S(x+iy_0))}{2} \prod_{i=1}^{n} Q_t^{\lambda_i} \left(\frac{(Ox)_i}{\sqrt{2}}, \frac{(Ox_0)_i}{\sqrt{2}} \right) \exp\left[\frac{S(x_0+iy_0)}{2} \right]$$

Remarks:

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- Relaxation to equilibrium if $\mathrm{Re}\,\lambda_i > 0$, $i = 1, \ldots, n$.
- Moral reason: all classical trajectories attracted to origin.

4. Problems

Mathematical and practical difficulties:

- *Existence* of the semigroup generated by L_{FP}.
  Not known: L_{FP} never manifestly dissipative.
  Hope: with new scalar product L_{FP} dissipative.

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Mathematical and practical difficulties:

- *Existence* of the semigroup generated by L_{FP}.
  Not known: L_{FP} never manifestly dissipative.
  Hope: with new scalar product L_{FP} dissipative.
- *Runaways:* In typical cases deterministic motion can go to ∞ in finite time.
  Reason: Drift ∇S grows in some directions. $1D$:

$$\dot{z} = -S' \Longrightarrow t - t_0 = -\int \frac{dz}{S'}$$

  (integration on curve with dz real multiple of S').

- Pseudospectrum (see below)

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- Convergence to wrong limit

  Noticed by Klauder&Petersen 1985, Ambjørn et al 1986:

  "Quantum mechanical desasters of the first degree":

$$S = -\beta \cos \theta - i\theta$$

  works for large β, fails for small β.

  "Non-abelian desasters of the third degree":

$$S = -\beta \,\mathrm{tr}\, U - \log \mathrm{tr}\, U, \quad U \in SU(2), SU(3)\,,$$

  works for large β, fails for small β.

– Haymaker&Wosiek 1987:

$$S = -\beta cos\theta - \log\cos\theta$$

Simulates restricted range $[-\pi/2, \pi/2]$.
Reason: zero of $\cos\theta$.

– Gausterer 1993: criterion for correctness.
(1) 1D, S poynomial, $e^{-S} \in \mathcal{S}$
(2) $\int_{\mathbb{R}} e^{-S(x)} dx \neq 0$
(3) $\forall k \in \mathbb{R} \quad lim_{t\to\infty} \langle e^{ikz} \rangle_{P,t}$ exists and is $\in \mathcal{S}(\mathbb{R})$.
Not really practical.

5. Examples

Example 1 (Aarts& Stamatescu 2008)

$$S = -\beta \cos x - \kappa \cos(x - i\mu)$$

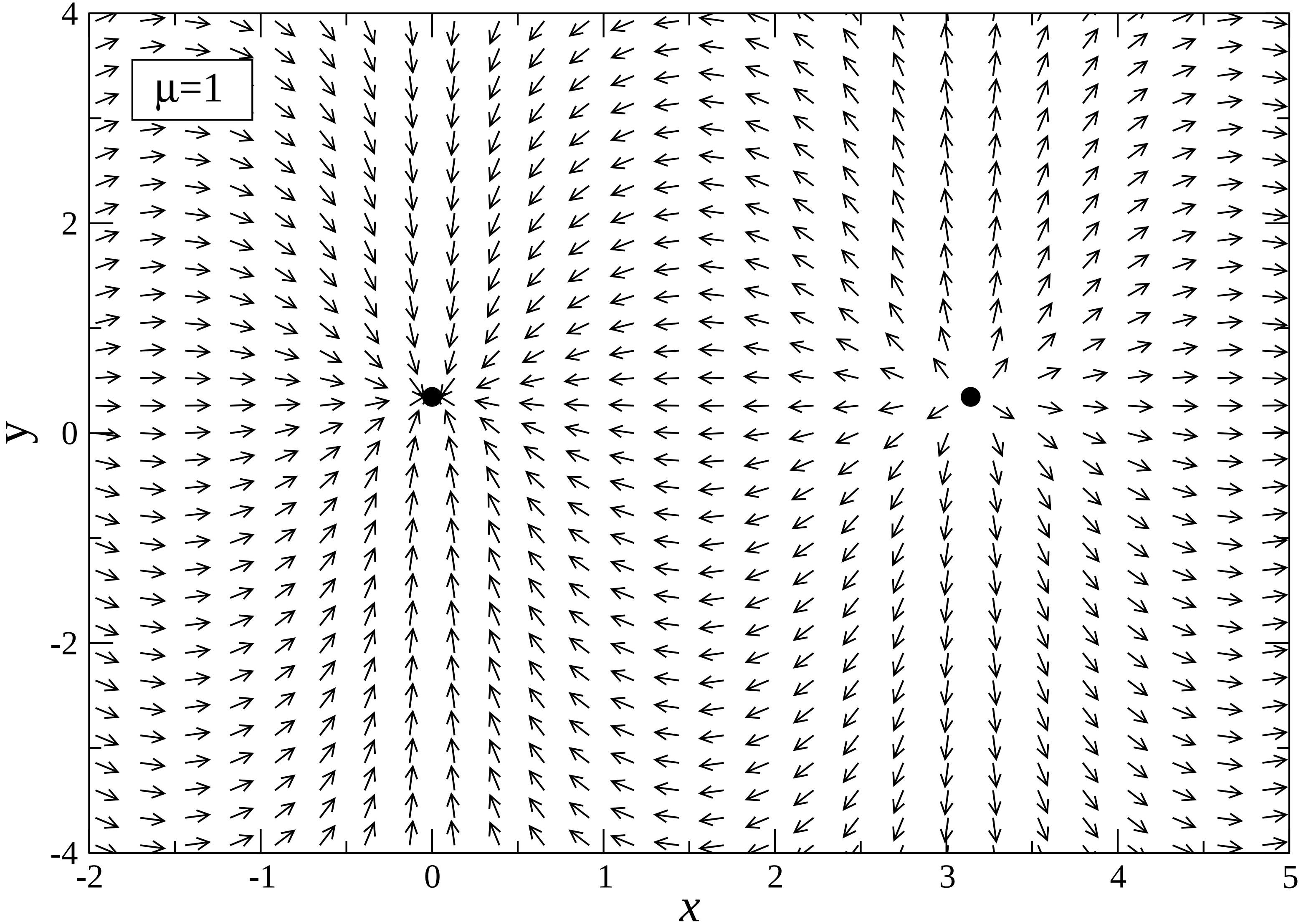
Complex Langevin equation

$$dx = K_x dt + dw, \qquad dy = K_y dt$$

with

$$\begin{aligned} K_x &= -\sin x \left[\beta \cosh y + \kappa \cosh(y - \mu)\right] \\ K_y &= -\cos x \left[\beta \sinh y + \kappa \sinh(y - \mu)\right] \end{aligned} \tag{2}$$

From (Aarts& Stamatescu 2008): Drift pattern



Real FP operator:

$$L_{FP} = \partial_x[\partial_x - K_x] - \partial_y K_y]$$

Complex FP operator:

$$L^c_{y_0} = \partial_x[\partial_x + \beta\sin(x + iy_0) + \kappa\sin(x + iy_0 - i\mu)]$$

Drift K_x, K_y parallel to gradient of

$$G(x, y) = \exp\left[-\frac{\cos x}{\beta\cosh y + \kappa\cosh(y - \mu)}\right] .$$

G is candidate Lyapunov function:

$$\frac{d}{dt}G(x(t),y(t)) = (K_x\partial_x + K_y\partial_y)G(x,y) =$$

$$-\left[\sin^2 x + \cos^2 x \left(\frac{\beta \sinh y + \kappa \sinh(y-\mu)}{\beta \cosh y + \kappa \cosh(y-\mu)}\right)^2\right] G \leq 0\,,$$

Vanishes only on fixed points $(0, y_*)$, (π, y_*);
$\Rightarrow$ all points with $x \neq \pi$ attracted to (O, y_*).

G also candidate stochastic Lyapunov function:

$$L_{FP}^T G < 0$$

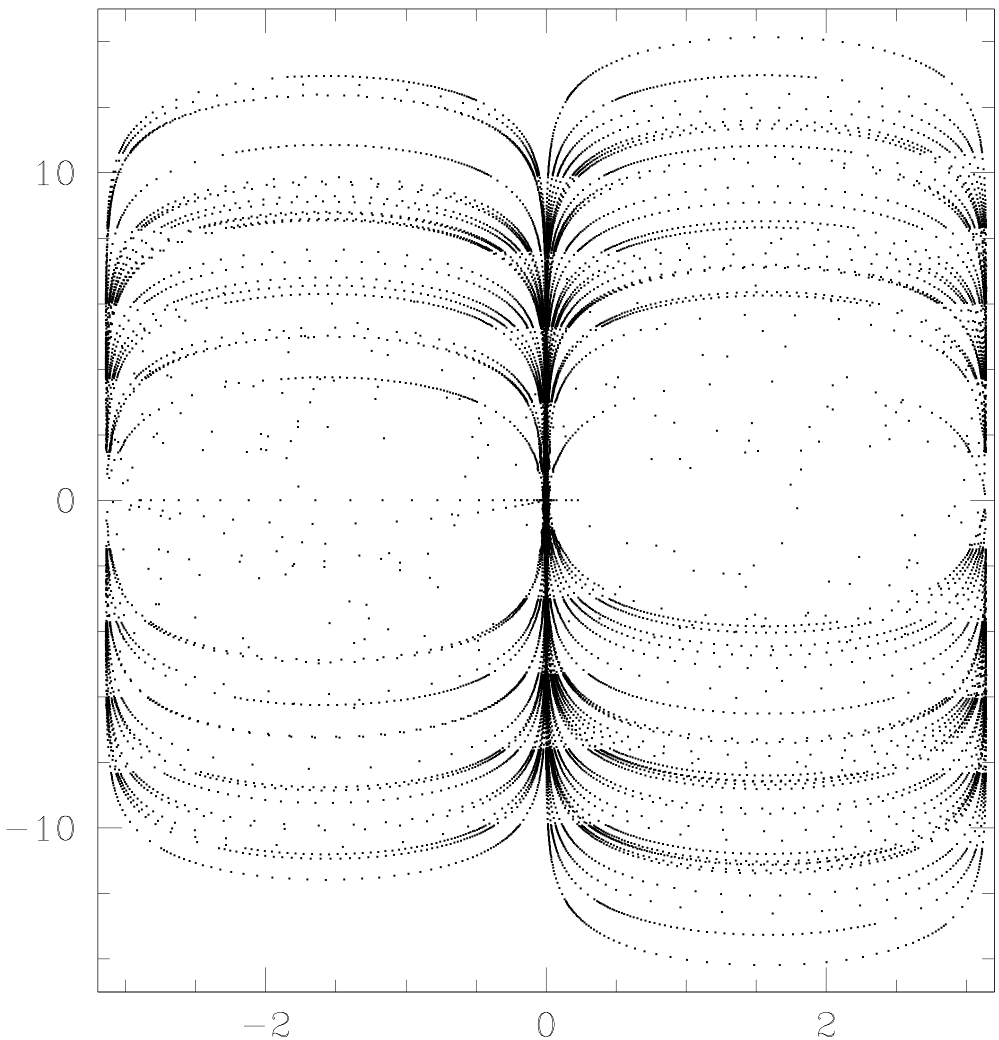
for $|y|$ large enough.
Need (Khasminskii 1980):

$$L_{FP}^T G \to -\infty \quad \text{for } |y| \to \infty \, .$$

Open problem.

Practically large excursions cause problems even if stationary $P(x,y)$ exists.

Simulation



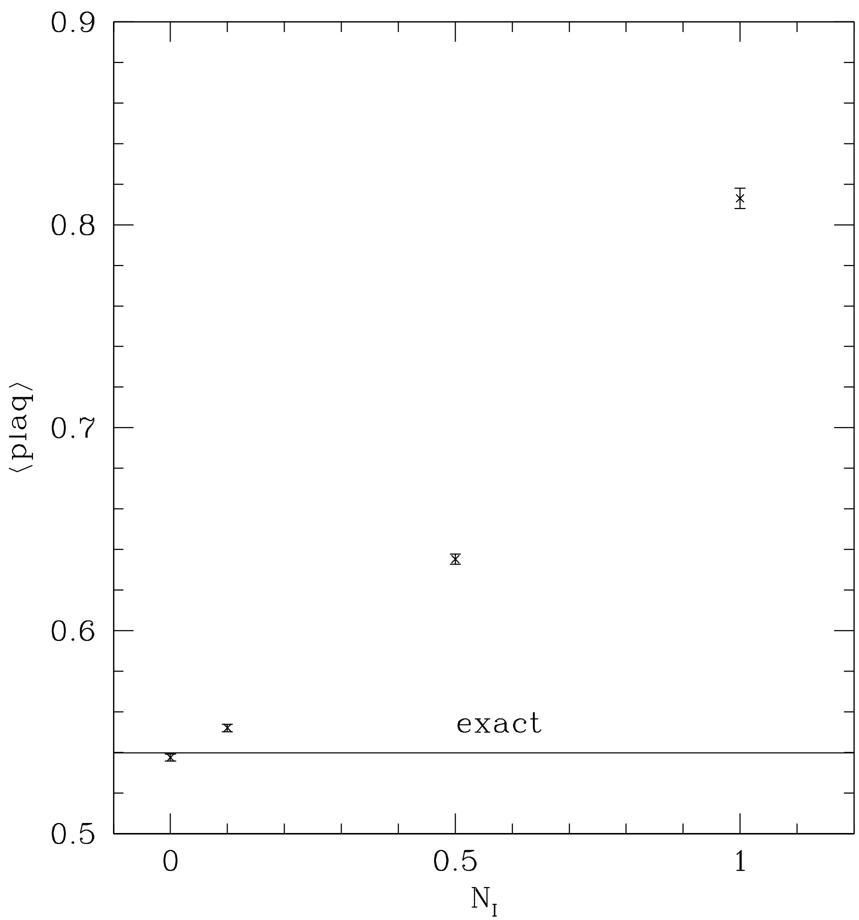
$\beta = 100.$

$\kappa = 0.0$

$N_I = 1.0$

Problem

Convergence to wrong limit for $N_I > 0$: What is going on?

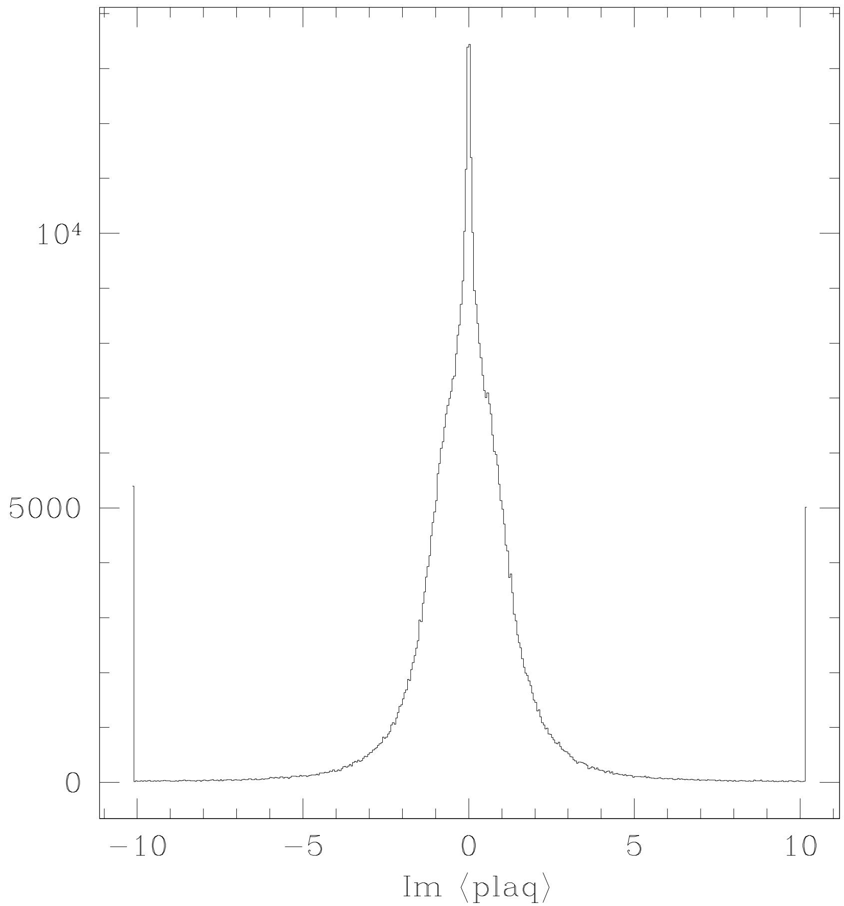


$\beta = 1.$

$\kappa = 0.5$

$\mu = 1.0$

Possible reason: Non-Gaussian large fluctuations?



$\beta = 1.$

$\kappa = 0.5$

$\mu = 1.0$

$N_I = 1.0$

Example 2 (Guralnik&Pehlevan 2009)

$$S = -\beta(iz + \tfrac{i}{3}z^3)$$

Attractive fixed point: $z = i$

Repulsive fixed point: $z = -i$

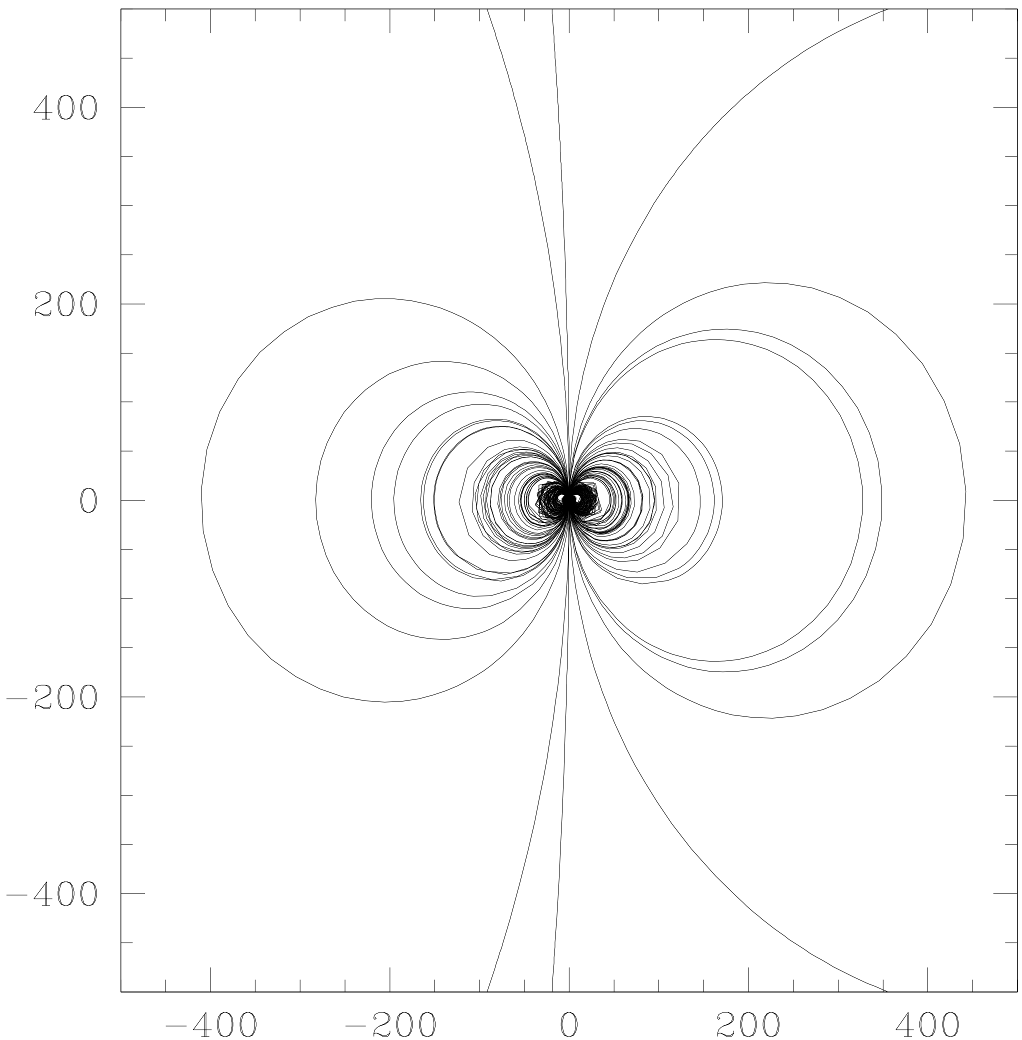
Classical orbits: Circles

$$z(t) = \frac{z_o + i \tanh t}{1 - i z_o \tanh t}$$

Möbius transformation $w \equiv \tanh t \mapsto z(t)$,

$z(0) = z_o, \quad z(\infty) = i$

Simulation



$N_I = 1.0, \beta = 1.0$

Exact results

$$\beta = 1.0 \ : \qquad Z(j) = \int dx \exp[ix + \tfrac{i}{3}x^3 + jx]$$

Schwinger-Dyson eq.:

$$-iZ'' - iZ = jZ$$

leads to

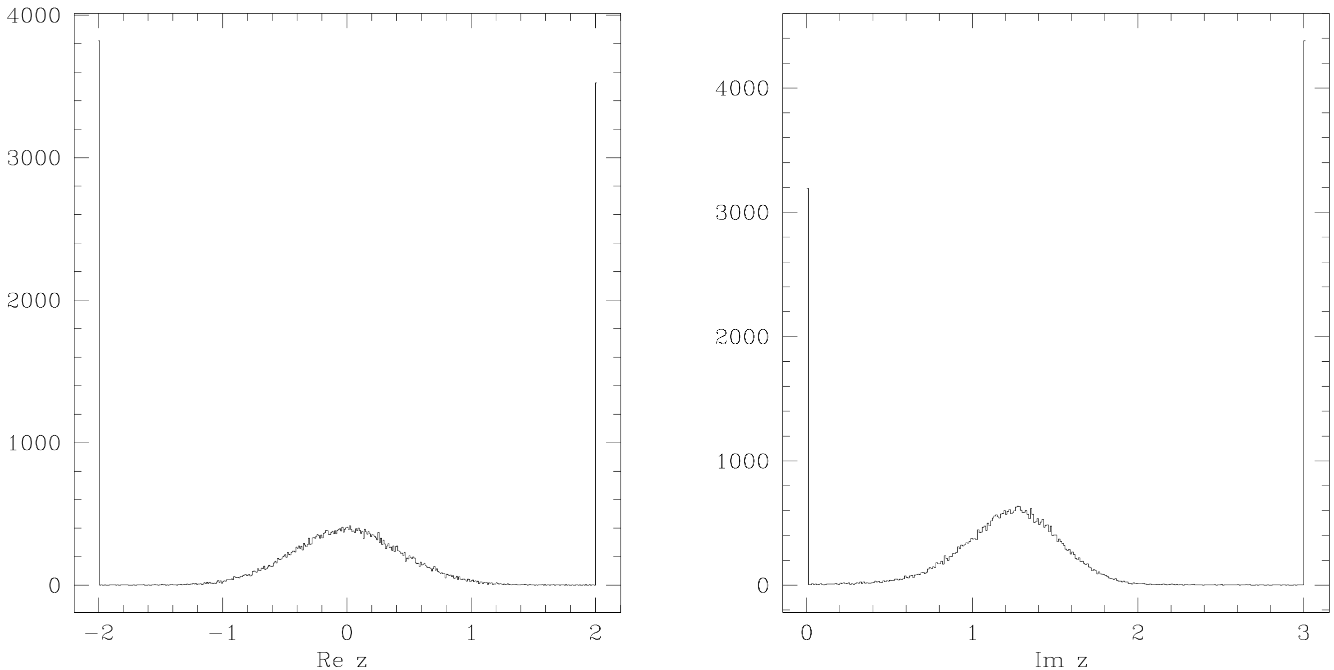
$$\langle z \rangle = -i\frac{Ai'(1)}{Ai(1)} \approx 1.17632i$$

$$\langle z^2 \rangle = -1.0, \quad \langle z^3 \rangle = i - \langle z \rangle \approx -0.17632i$$

Problem

Again convergence to wrong limit for $N_I > 0$.

Reason: Non-Gaussian large fluctuations?

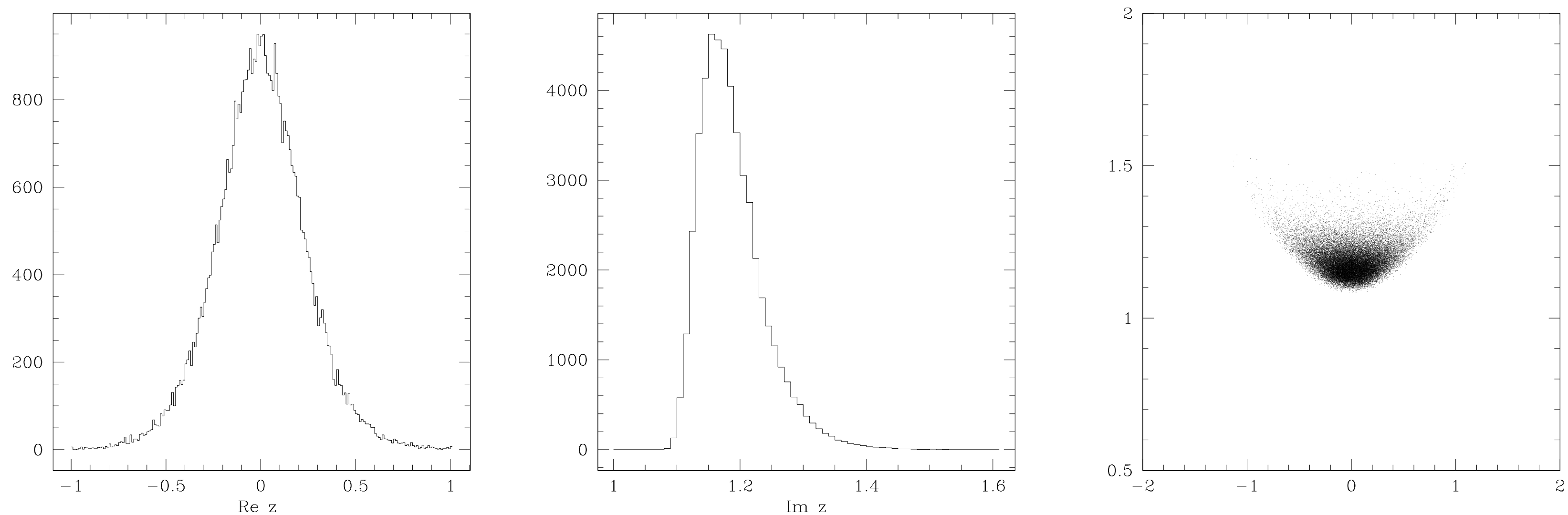


$$\beta = 1.0, \quad N_I = 1.0$$

Problem

But convergence to right limit for $N_I << 1$..

In spite of: Small non-Gaussian fluctuations



$$\beta = 1.0, \quad N_I = 0.0$$

Pseudospectrum

Typically:

Spectrum of L_{FP}^c and $-H_{FP}$ in left half plane, but
not dissipative: $Re\,(\psi, H_{FP}\psi) < 0$ for some ψ

Price to pay: Pseudospectrum

Definition: $\mathrm{spec}_\epsilon(A) \equiv \{z \in \mathbb{C}|\, \|(A-z)^{-1}\| > \epsilon^{-1}\}$

Signifies instability:

$$\mathrm{spec}_\epsilon(A) = \bigcup_B \{\mathrm{spec}(A+B), \|B\| < \epsilon\}$$

Tiny perturbation can eliminate “pseudo”

Example 3 (Davies&Kuijlaars, 2004): Spectral projections P_n of complex harmonic oscillator grow:

$$\|P_n\| \geq a\,C^{2n+1}, \quad C > 1\,;$$

poor convergence of eigenfunction expansions:

$$e^{-Ht}\psi = \sum_n e^{-\omega(n+1/2)t} P_n \psi$$

– Eigenfunctions do not form Riesz basis
– e^{-Ht} not bounded semigroup
– $\exists$ pseudospectrum far from spectrum!
(Davies 1999)

[*Riesz basis* $(\phi_n)_{n=1}^{\infty}$:

$\exists$ bounded operator S with S^{-1} bounded such that

$$S\phi_n = e_n \quad n = 1, \ldots \infty\,,$$

where $(e_n)_{n=1}^{\infty}$ orthonormal basis.]

6. Extension to manifolds

Gausterer&Thaler 1998, Aarts&Stamatescu 2008:
Compact connected Lie groups.

Examples:
$U(1)$ complexified to $U(1) \times \mathbb{R}$
$SU(N)$ complexified to $SL(N, \mathbb{C})$

More generally:
– $\mathcal{M}$ Riemannian manifold $\Rightarrow \exists$ Wiener process $\Rightarrow$ noise in real directions well defined
– Real manifold $\mathcal{M}$ has to have complexification $\mathcal{M}_C$.

Formal arguments carry over; problems remain.

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- Validation necessary: check with analytic or otherwise known result.
- More general procedures to represent complex measures by positive ones (Salcedo 1997-2007, Bender et al 1998-2008, Weingarten 2002, Bernard& Savage 2001)
- Hope for the best, be prepared for the worst