Functional renormalization – concepts and prospects

physics at different length scales

- microscopic theories : where the laws are formulated
- effective theories : where observations are made
- effective theory may involve different degrees of freedom as compared to microscopic theory
- example: the motion of the earth around the sun does not need an understanding of nuclear burning in the sun

QCD :
Short and long distance
degrees of freedom are different !

Short distances : quarks and gluons
Long distances : baryons and mesons

How to make the transition?

confinement/chiral symmetry breaking

collective
degrees of freedom

Hubbard model

- Electrons on a cubic lattice

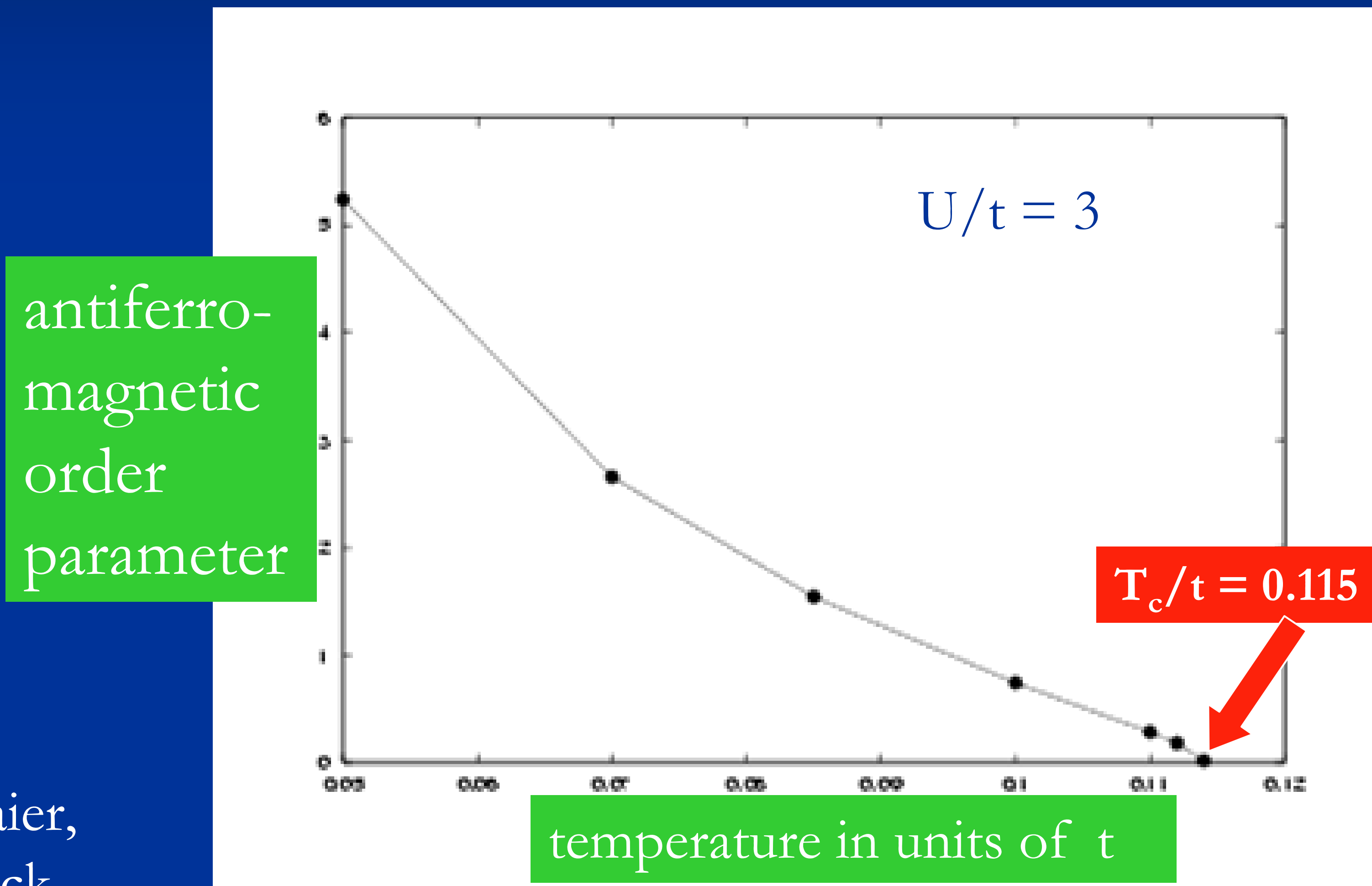
  here : on planes (d = 2)

- Repulsive local interaction if two electrons are on the same site
- Hopping interaction between two neighboring sites

In solid state physics :
“ model for everything “

- Antiferromagnetism
- High T_c superconductivity
- Metal-insulator transition
- Ferromagnetism

Antiferromagnetism in d=2 Hubbard model



T.Baier, E.Bick,…

Collective degrees of freedom are crucial !

for T < T_c

- nonvanishing order parameter

$$\hat{\vec{m}}(X) = \hat{\psi}^\dagger(X)\vec{\sigma}\hat{\psi}(X) \qquad \hat{\vec{m}}(Q) \rightarrow \vec{a}\delta(Q-\Pi)$$

- gap for fermions

- low energy excitations:
  antiferromagnetic spin waves

effective theory / microscopic theory

- sometimes only distinguished by different values of couplings
- sometimes different degrees of freedom

Functional Renormalization Group

describes flow of effective action from small to large length scales

perturbative renormalization : case where only couplings change , and couplings are small

How to come from quarks and gluons to baryons and mesons ?
How to come from electrons to spin waves ?

Find effective description where relevant degrees of freedom depend on momentum scale or resolution in space.

Microscope with variable resolution:

- High resolution , small piece of volume: quarks and gluons
- Low resolution, large volume : hadrons

From

Microscopic Laws

(Interactions, classical action)

to

Fluctuations!

Macroscopic Observation

(Free energy functional, effective action)

- block spins

  Kadanoff, Wilson

- exact renormalization group equations

  Wilson, Kogut

  Wegner, Houghton

  Weinberg

  Polchinski

  Hasenfratz^2

- Lattice finite size scaling

  Lüscher,...

- coarse grained free energy/average action

effective average action

Effective average potential : Unified picture for scalar field theories with symmetry O(N) in arbitrary dimension d and arbitrary N

linear or nonlinear sigma-model for
chiral symmetry breaking in QCD
or:
scalar model for antiferromagnetic spin waves
(linear O(3) – model)

fermions will be added later

Effective potential includes all fluctuations

Average potential U_k

$\equiv$ **scale dependent effective potential**
$\equiv$ **coarse grained free energy**

Only fluctuations with momenta $q^2 > k^2$ included

k: infrared cutoff for fluctuations, "average scale"

Λ: characteristic scale for microphysics

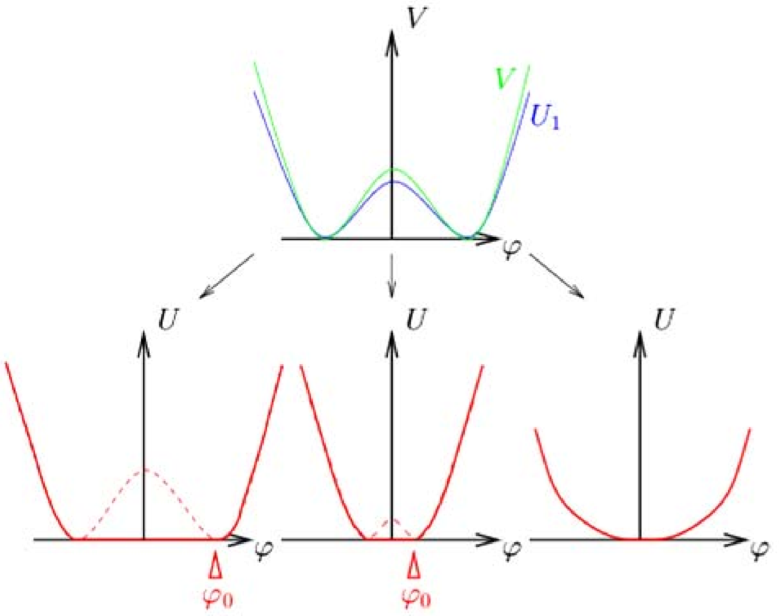
$$U_\Lambda \approx S \to U_0 \equiv U$$

Scalar field theory

$\varphi_a(x)$: magnetization, density, chemical concentration, Higgs field, meson field, inflaton, cosmon

$O(N)$-symmetry:

$$S = \int d^d x \left\{ \frac{1}{2} \partial_\mu \varphi_a \partial_\mu \varphi_a + V(\rho) \right\}; \quad \rho = \frac{1}{2} \varphi_a \varphi_a$$



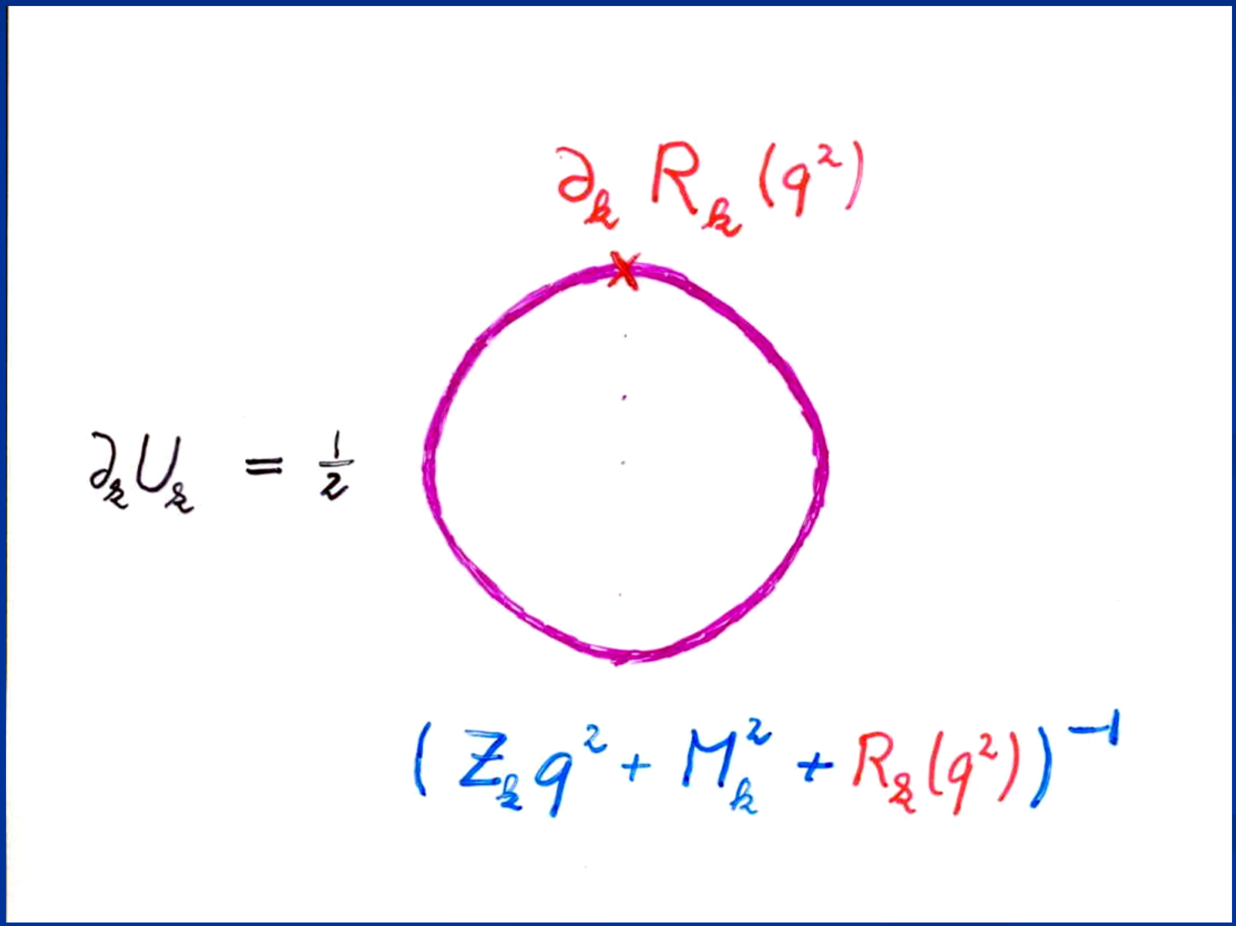
Flow equation for average potential

$$\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial \varphi_a \partial \varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

Simple one loop structure – nevertheless (almost) exact



Infrared cutoff

R_k : IR-cutoff

e.g $\quad R_k = \dfrac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $\quad R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2) \qquad$ (Litim)

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Flow equation for U_k

$$\boxed{\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}}$$

'91

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Partial differential equation for function U(k,φ) depending on two (or more) variables

$Z_k = c\, k^{-\eta}$

Regularisation

For suitable R_k :

$$R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$
$$R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$$

- Momentum integral is ultraviolet and infrared finite
- Numerical integration possible
- Flow equation defines a regularization scheme (ERGE –regularization)

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Integration by momentum shells

$$\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Momentum integral

is dominated by

$q^2 \sim k^2$.

Flow only sensitive to

physics at scale k

Wave function renormalization and anomalous dimension

Z_k: wave function renormalization

$$k\partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

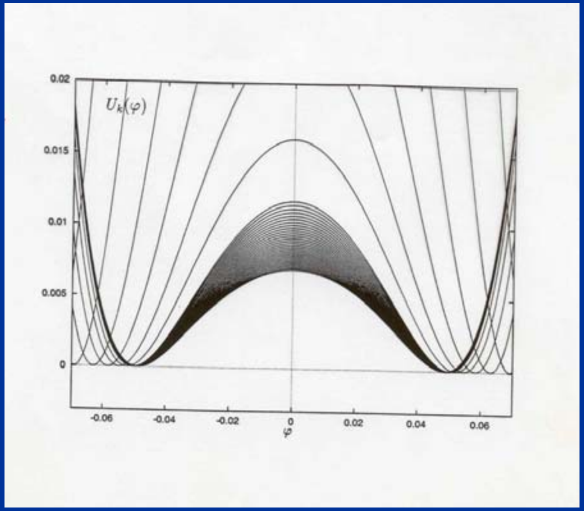
$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

for Z_k (φ,q^2) : flow equation is **exact !**

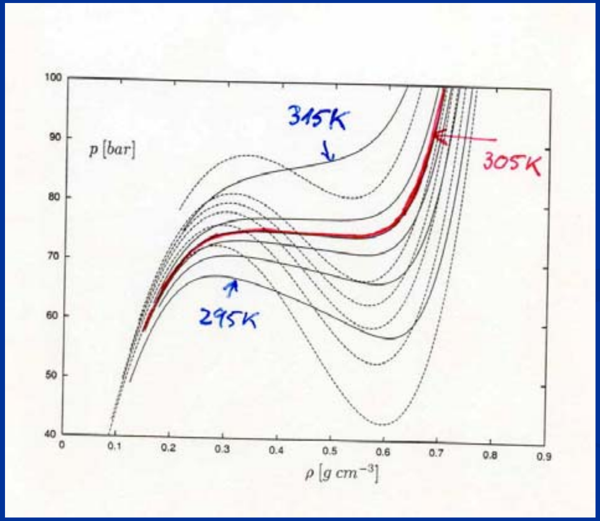
Flow of effective potential

Ising model



S.Seide …

CO_2



Experiment :

T_* =304.15 K
p_* =73.8.bar
ρ_* = 0.442 g cm-2

Critical exponents

$d = 3$

Critical exponents ν and η

N	ν		η	
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
10	0.881	0.886	0.028	0.025
100	0.990	0.980	0.0030	0.003
		↑		↑

"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

Critical exponents , d=3

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"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

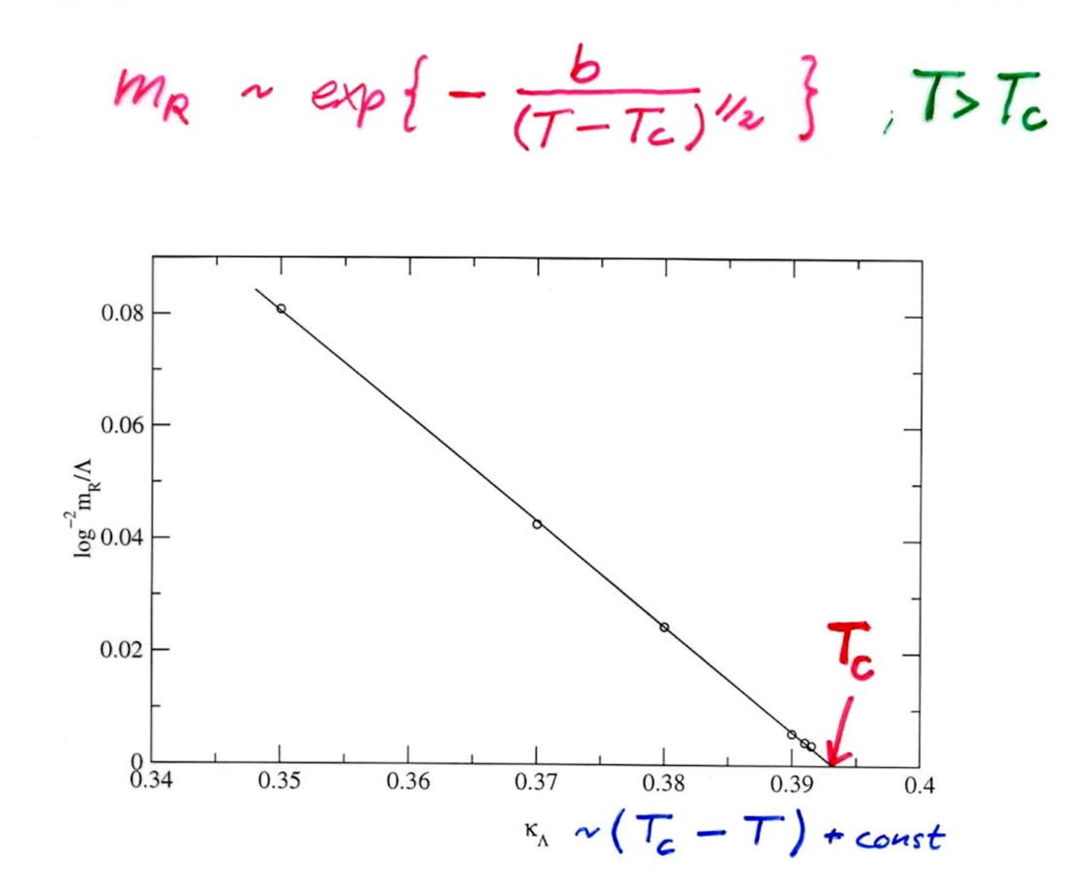
Solution of partial differential equation :

yields highly nontrivial non-perturbative results despite the one loop structure !

Example:

Kosterlitz-Thouless phase transition

Essential scaling : d=2,N=2



- Flow equation contains correctly the non-perturbative information !
- (essential scaling usually described by vortices)

Von Gersdorff …

Kosterlitz-Thouless phase transition (d=2,N=2)

Correct description of phase with

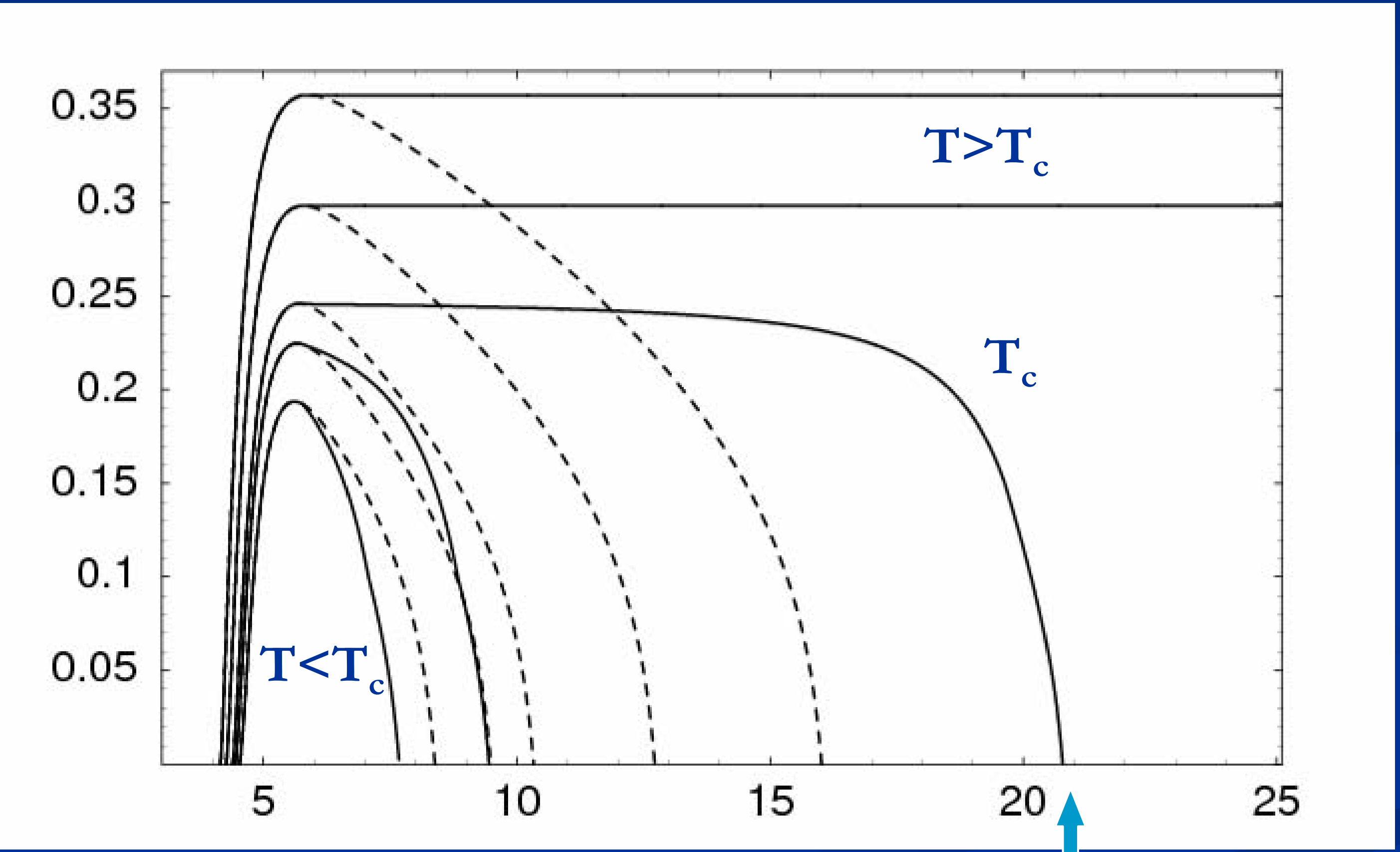
Goldstone boson

(infinite correlation length)

for $T<T_c$

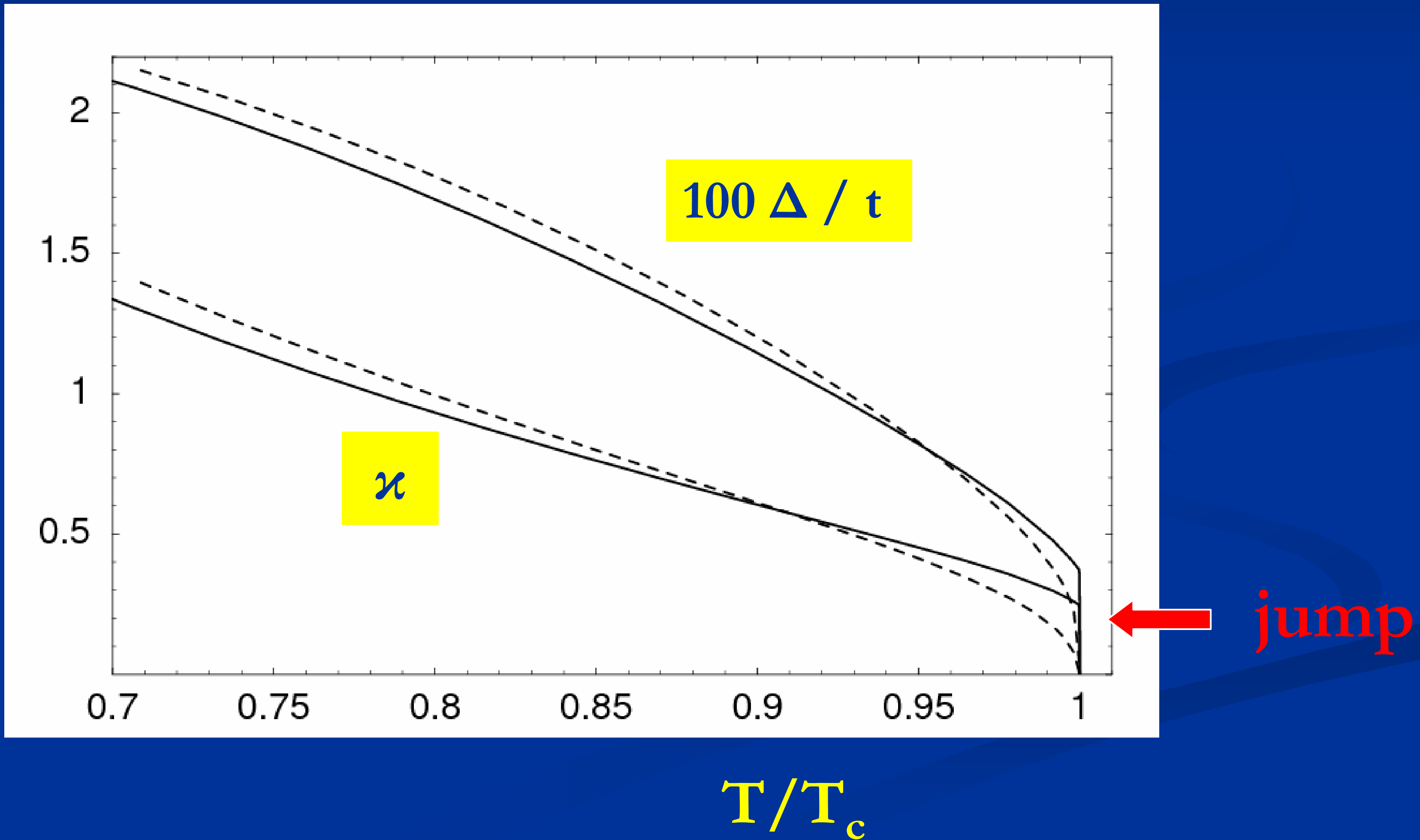
Running renormalized d-wave superconducting order parameter $\varkappa$ in Hubbard model

$\varkappa$

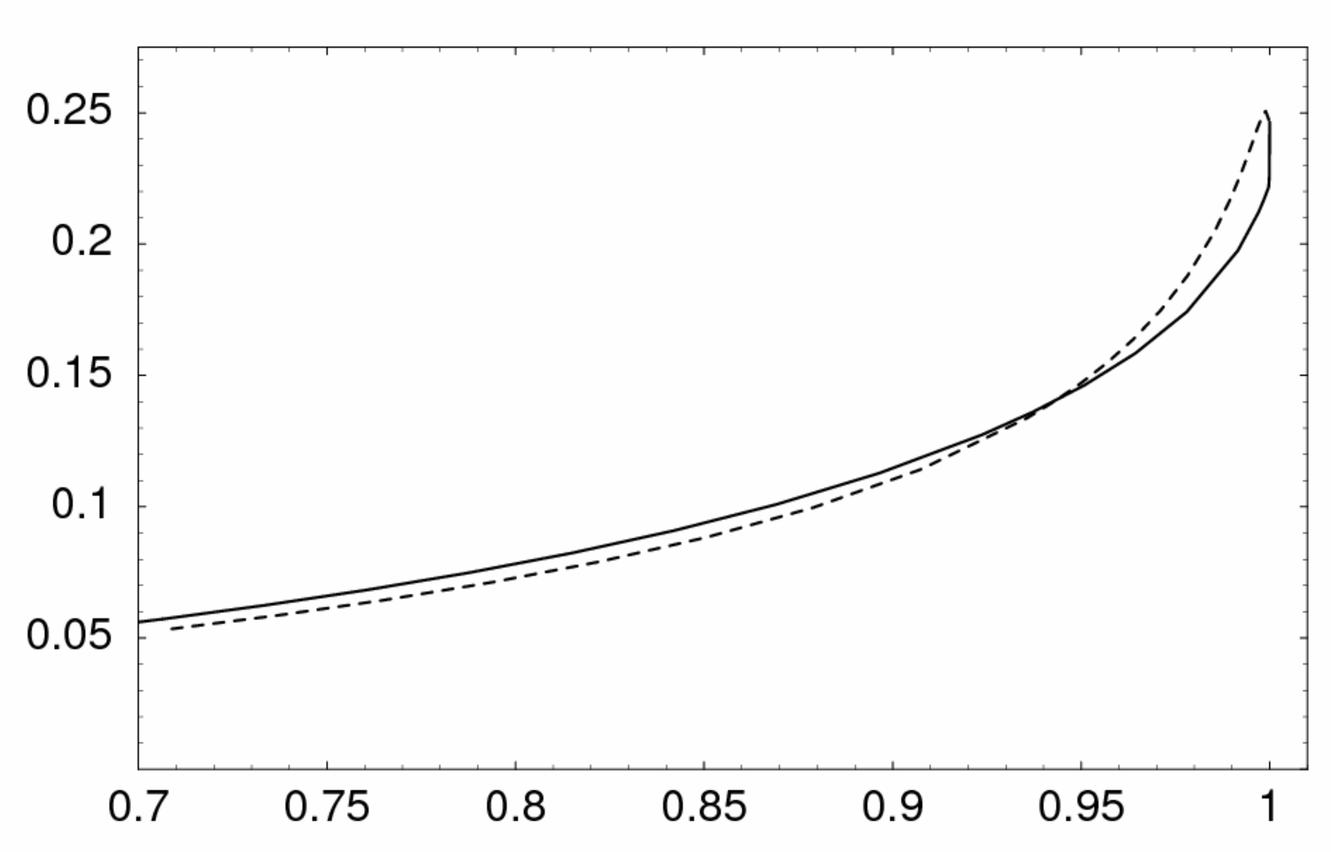


C.Krahl,… **- ln (k/Λ)** macroscopic scale 1 cm

Renormalized order parameter $\varkappa$ and gap in electron propagator Δ



Temperature dependent anomalous dimension η



η

T/T_c

Effective average action

and

exact renormalization group equation

Generating functional

generating functional for connected Green's functions in presence of quadratic infrared cutoff

$$W_k[j] = \ln \int \mathcal{D}\chi \exp\left(-S[\chi] - \Delta_k S[\chi] + \int d^d x\, j_a \chi_a\right)$$

$$\Delta_k S = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q^2) \chi_a(-q) \chi_a(q)$$

$$\text{e.g.} \quad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

$$\lim_{k \to 0} R_k = 0$$

$$R_{k \to \infty} \to \infty$$

Effective average action

$$\Gamma_k[\varphi] = -W_k[j] + \int d^dx\, j_a \varphi_a - \Delta_k S[\varphi]$$

$$\varphi_a = \langle \chi_a \rangle = \frac{\delta W_k}{\delta j_a}$$

$\Gamma_0[\varphi]$: quantum effective action

generates 1PI vertices

free energy: $F = \Gamma T + \mu n V$

Γ_k includes all fluctuations (quantum, thermal) with $q^2 > k^2$

Γ_Λ specifies microphysics

Loop expansion : perturbation theory with infrared cutoff in propagator

Quantum effective action

for $k \to 0$
all fluctuations (quantum + thermal)
are included

knowledge of $\Gamma_{k\to 0} \hat{=}$ solution of model

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}$$

'92

$$\left(\Gamma_k^{(2)}\right)_{ab}(q,q') = \frac{\delta^2 \Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$$

$$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$$

(fermions : STr)

Proof of exact flow equation

$$\partial_k \Gamma|_\phi = -\partial_k W|_j - \partial_k \Delta_k S[\varphi]$$
$$= \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k(\langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle)\right\}$$
$$= \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k W_k^{(2)}\right\}$$

$$W_k^{(2)}(\Gamma_k^{(2)} + R_k) = \mathbb{1}$$

$$(\Delta_k S^{(2)} \equiv R_k)$$

$$\Longrightarrow$$

$$\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k(\Gamma_k^{(2)} + R_k)^{-1}\right\}$$

Truncations

Functional differential equation –
cannot be solved exactly

Approximative solution by truncation of
most general form of effective action

derivative expansion

Tetradis,...; Morris

$O(N)$-model:

$$\Gamma_k = \int d^d x \{U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \varphi_a \partial_\mu \varphi_a + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial_\mu \rho + \cdots \}$$

$$(N = 1 : \quad Y_k \equiv 0)$$

field expansion
(flow eq. for 1PI vertices)

Weinberg; Ellwanger,...

$$\Gamma_k = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{j=0}^{n} d^d x_j \, \Gamma_k^{(n)}(x_1, x_2, \ldots, x_n) \prod_{j=0}^{n} (\phi(x_j) - \phi_0)$$

error estimate?

Expansion in canonical dimension of couplings

Lowest order:

$$d = 4: \quad \rho_0, \bar{\lambda}, Z$$

$$d = 3: \quad \rho_0, \bar{\lambda}, \bar{\gamma}, Z$$

$$U = \frac{1}{2}\bar{\lambda}(\rho - \rho_0)^2 + \frac{1}{6}\bar{\gamma}(\rho - \rho_0)^3$$

works well for $O(N)$ models
Tetradis,...; Tsypin

polynomial expansion of potential converges
if expanded around ρ_0
Tetradis,...; Aoki et al.

Exact flow equation for effective potential

- Evaluate exact flow equation for homogeneous field φ .
- R.h.s. involves exact propagator in homogeneous background field φ.

Flow equation for average potential

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

$$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b} \quad : \quad \text{Mass matrix}$$

$$\bar{M}^2_{k,i} \quad : \quad \text{Eigenvalues of mass matrix}$$

QCD :
Short and long distance
degrees of freedom are different !

Short distances : quarks and gluons
Long distances : baryons and mesons

How to make the transition?

confinement/chiral symmetry breaking

Nambu Jona-Lasinio model

$$S = \int d^4x \left\{ i \bar{\psi}^i_a \gamma^\mu \partial_\mu \psi^i_a + 2\lambda_\sigma (\bar{\psi}^i_{Lb} \psi^i_{Ra})(\bar{\psi}^j_{Ra} \psi^j_{Lb}) \right\}$$

$$\psi_{L,R} = \frac{1 \pm \gamma^5}{2} \psi$$

$i, j = 1 \ldots N_c$ color ($N_c = 3$)

$a, b = 1 \ldots N_F$ flavor ($N_F = 3, 2$)

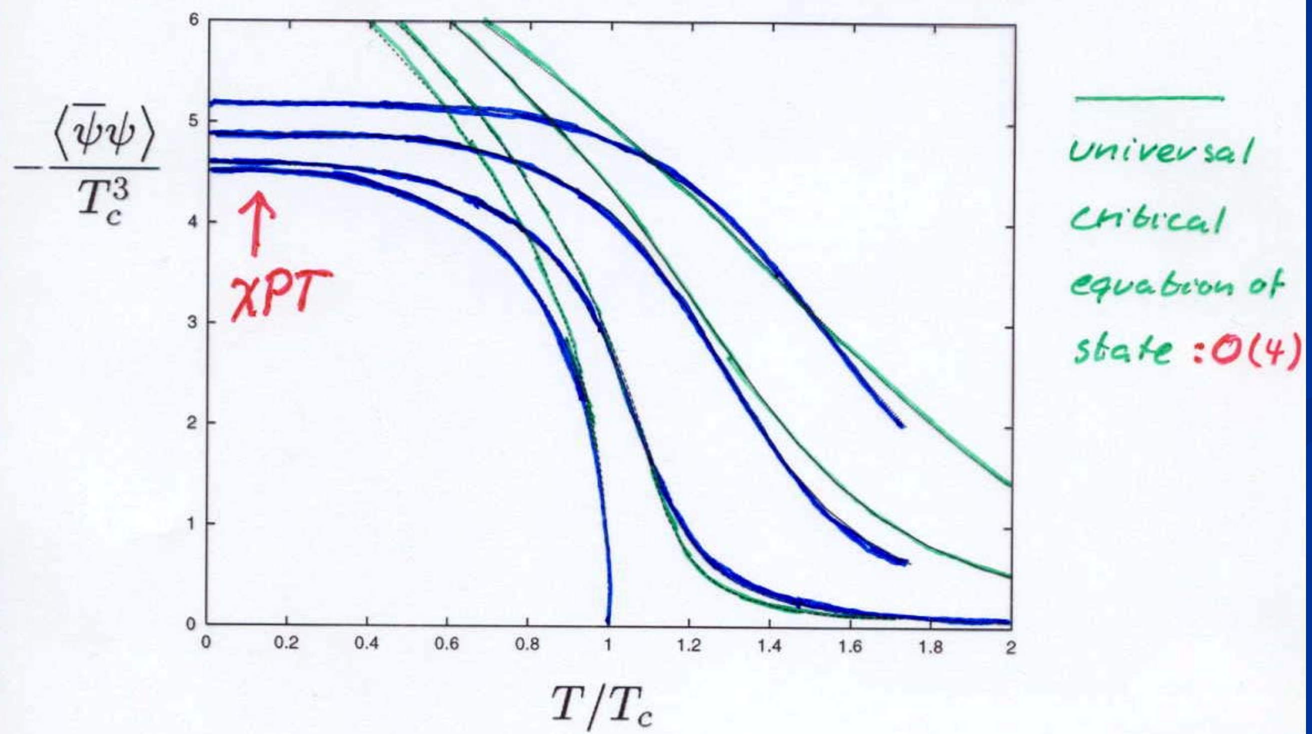
chiral flavor symmetry :

$$SU_L(N_F) \times SU_R(N_F)$$

...and more general quark meson models

Chiral condensate (N_f=2)

2nd order PT (expected for $O(4)$ Heisenberg model)



⟹ Explicit link between χPT domain of validity (4d) and critical (universal) domain near T_c (3d)

Berges, Jungnickel,…

Critical temperature , $N_f = 2$

$\frac{m_\pi}{\text{MeV}}$	0	45	135	230
$\frac{T_{pc}}{\text{MeV}}$	100.7	$\simeq 110$	$\simeq 130$	$\simeq 150$

for $f_\pi = 93$ MeV

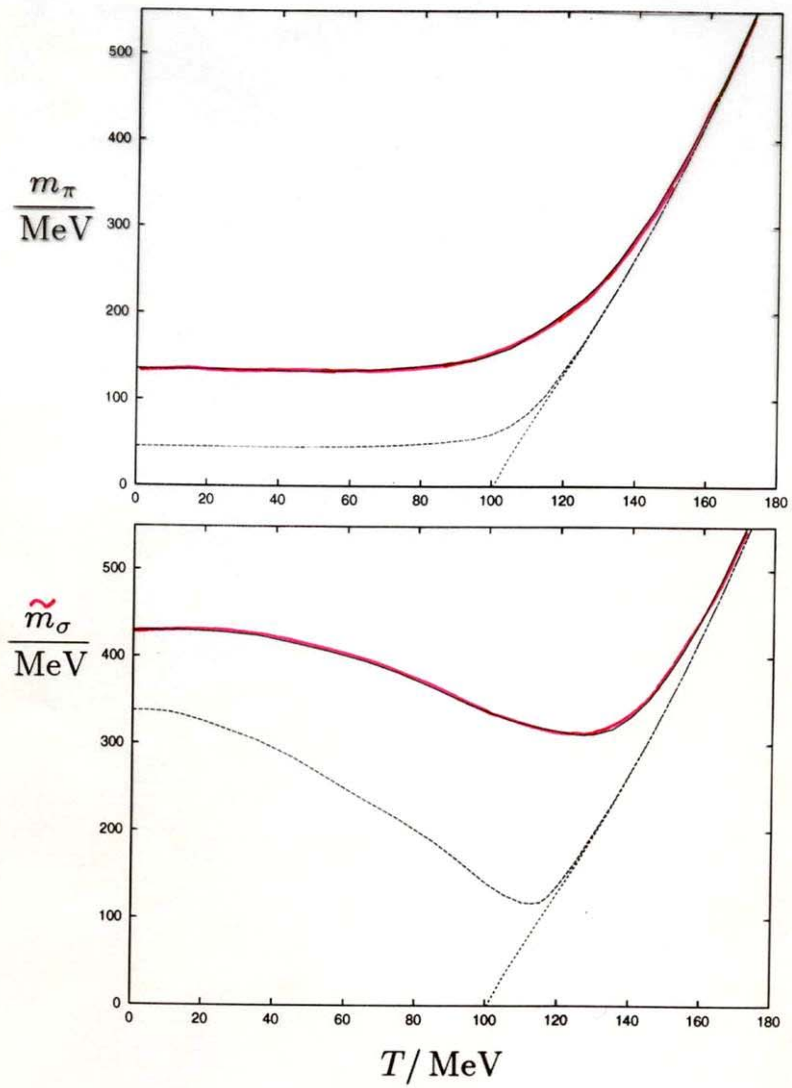
Lattice simulation

J.Berges,D.Jungnickel,…

temperature dependent masses

pion mass

sigma mass



? $m_\sigma < 2m_\pi$ for $T \gtrsim 100\,\text{MeV}$?

No long pion correlation length in thermal equilibrium!

Critical equation of state

Critical behavior for second order phase transitions :

correlation length $\xi = m_R^{-1}$

only relevant length scale

φ_R : renormalized field variable

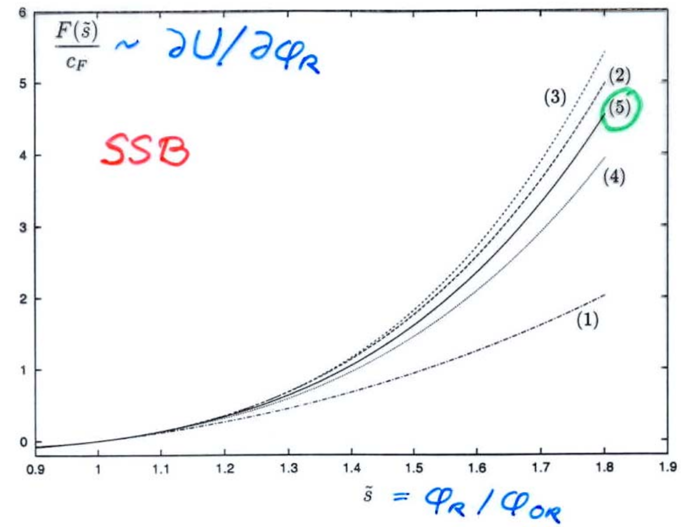
$U(\varphi_R)$ depends only on m_R

$$B \sim \frac{\partial U}{\partial \varphi_R} = F\left(\frac{\varphi_R}{\sqrt{m_R}}\right) m_R^{5/2}$$

↑

Widom scaling function

Scaling form of equation of state



critical equation of state

(2) ERGE, lowest order derivative exp.; Berges, Tetradis, ...

(5) ERGE, first order derivative exp.; Seide, ...

(1) mean field

(4) high-T-series, loop expansion, ε-expansion

(3) Monte Carlo

Berges, Tetradis,…

Universal critical equation of state is valid near critical temperature **if** the only light degrees of freedom are pions + sigma with O(4) – symmetry.

Not necessarily valid in QCD, even for two flavors !

conclusions

Flow equation for effective average action:

- Does it work?
- Why does it work?
- When does it work?
- How accurately does it work?

end