Emergence of macroscopic laws with Functional Renormalization

Macroscopic understanding does not need all details of underlying microscopic physics

1) motion of planets : m_i

Newtonian mechanics of point particles

probabilistic atoms → deterministic planets

2) thermodynamics : T, μ, Gibbs free energy $J(T,\mu)$

3) antiferromagnetic waves for correlated electrons

$\Gamma[\, s_i(x)]$

How to get from microphysics to macrophysics ?

1) motion of planets : m_i

compute or measure mass of objects

(second order more complicated : tides etc.)

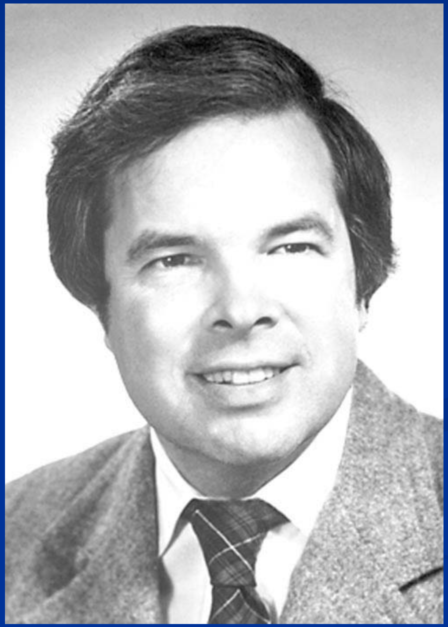
2) thermodynamics : $J(T, \mu)$

integrate out degrees of freedom

3) antiferromagnetic waves for correlated electrons

$\Gamma[s_i(x)]$ change degrees of freedom

Do it stepwise : functional renormalization


Leo Kadanoff Kenneth Wilson Franz Wegner

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}$$

'92

$$\left(\Gamma_k^{(2)}\right)_{ab}(q,q') = \frac{\delta^2\Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$$

$\mathrm{Tr} : \sum_a \int \frac{d^dq}{(2\pi)^d}$

(fermions : STr)

From

Microscopic Laws

(Interactions, classical action)

to

Fluctuations!

Macroscopic Observation

(Free energy functional, effective action)

different laws at different scales

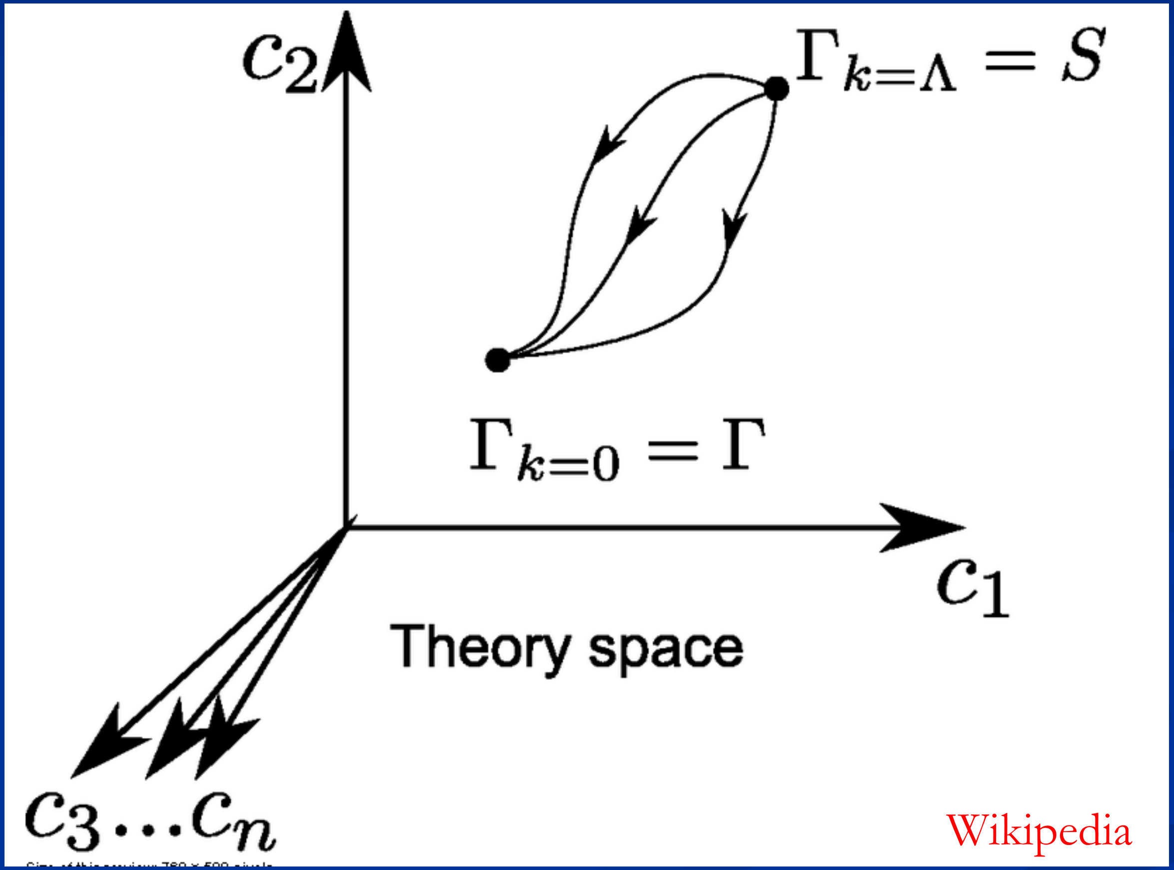
- fluctuations wash out many details of microscopic laws
- new structures as bound states or collective phenomena emerge
- elementary particles – earth – Universe :

key problem in Physics !

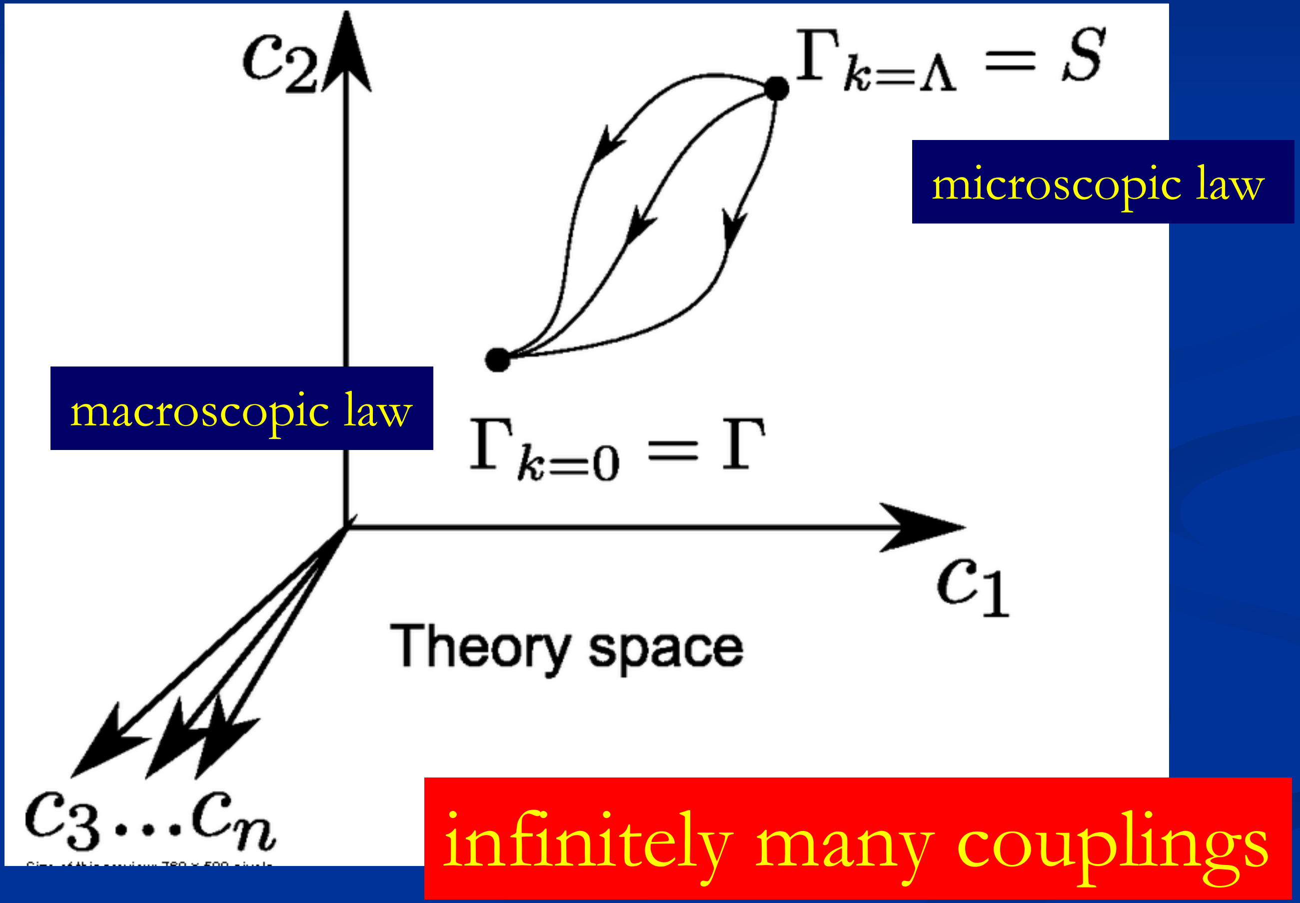
scale dependent laws

- scale dependent (running or flowing) couplings
- flowing functions
- flowing functionals

flowing action



flowing action



flow of functions

Effective potential includes all fluctuations

Average potential U_k

$\equiv$ **scale dependent effective potential**
$\equiv$ **coarse grained free energy**

Only fluctuations with momenta $q^2 > k^2$ included

k: infrared cutoff for fluctuations, "average scale"

Λ: characteristic scale for microphysics

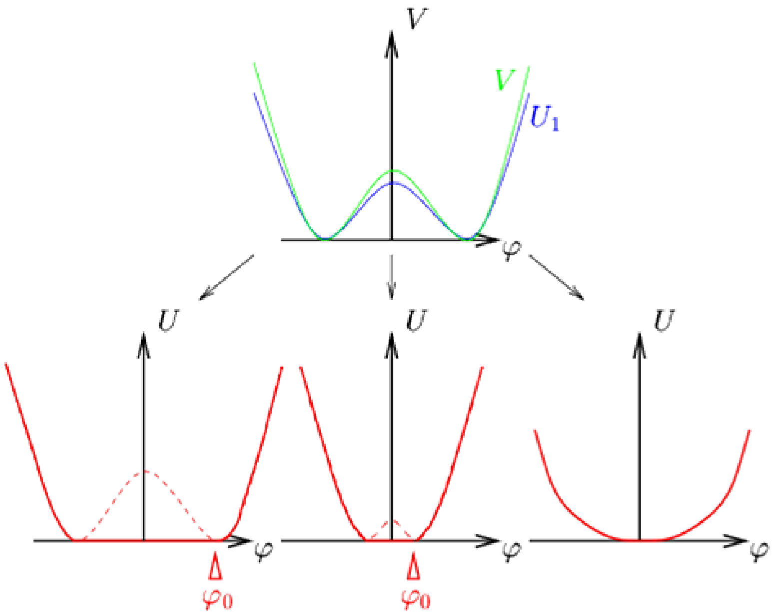
$$U_\Lambda \approx S \to U_0 \equiv U$$

Scalar field theory

$\varphi_a(x)$: magnetization, density, chemical concentration, Higgs field, meson field, inflaton, cosmon

$O(N)$-symmetry:

$$S = \int d^d x \left\{ \frac{1}{2} \partial_\mu \varphi_a \partial_\mu \varphi_a + V(\rho) \right\}; \quad \rho = \frac{1}{2} \varphi_a \varphi_a$$



Flow equation for average potential

cutoff

$$\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

propagator with cutoff

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial \varphi_a \partial \varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

R_k : IR-cutoff

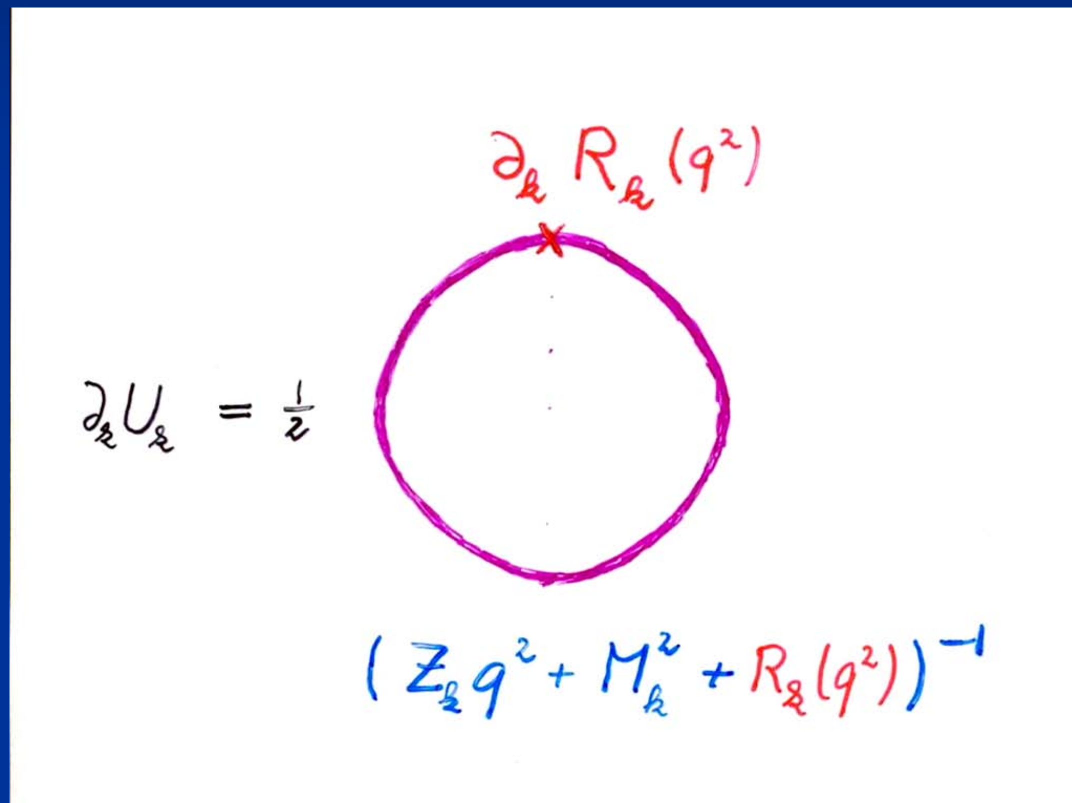
e.g $R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$ (Litim)

$\lim_{k \to 0} R_k = 0$

$\lim_{k \to \infty} R_k \to \infty$

Simple one loop structure – nevertheless (almost) exact



$$\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Simple differential equation for O(N) – models , dimension d

$$\partial_t u|_{\tilde{\rho}} = -du + (d-2+\eta)\tilde{\rho}u' + 2v_d\{l_0^d(u'+2\tilde{\rho}u'';\eta) + (N-1)\, l_0^d(u';\eta)\}$$

$$u = \frac{U_k}{k^d}$$

$$\tilde{\rho} = Z_k k^{2-d}\rho$$

$$u' = \frac{\partial u}{\partial \tilde{\rho}} \quad \text{etc.}$$

linear cutoff:

$$l_0^d(w;\eta) = \frac{2}{d}\left(1-\frac{\eta}{d+2}\right)\frac{1}{1+w}$$

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)$$

t = ln(k)

Wave function renormalization and anomalous dimension

Z_k: wave function renormalization

$$k\partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

for $Z_k(\varphi, q^2)$: flow equation is **exact !**

Scaling form of evolution equation

$$u = \frac{U_k}{k^d}$$
$$\tilde{\rho} = Z_k k^{2-d} \rho$$
$$u' = \frac{\partial u}{\partial \tilde{\rho}} \quad \text{etc.}$$

$$\partial_t u|_{\tilde{\rho}} = -du + (d - 2 + \eta)\tilde{\rho} u' + 2v_d \{ l_0^d(u' + 2\tilde{\rho}u''; \eta) + (N-1)\, l_0^d(u'; \eta) \}$$

$$v_d^{-1} = 2^{d+1} \pi^{d/2} \Gamma\left(\frac{d}{2}\right)$$

linear cutoff:

$$l_0^d(w;\eta) = \frac{2}{d}\left(1 - \frac{\eta}{d+2}\right)\frac{1}{1+w}$$

On r.h.s. :
neither the scale k
nor the wave function
renormalization Z
appear explicitly.

Scaling solution:
no dependence on t;
corresponds
to second order
phase transition.

Tetradis …

unified approach

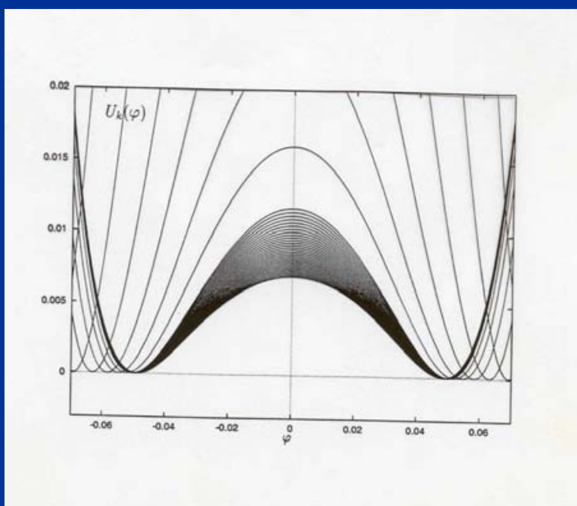
- choose N
- choose d
- choose initial form of potential
- run !

(quantitative results : systematic derivative expansion in second order in derivatives)

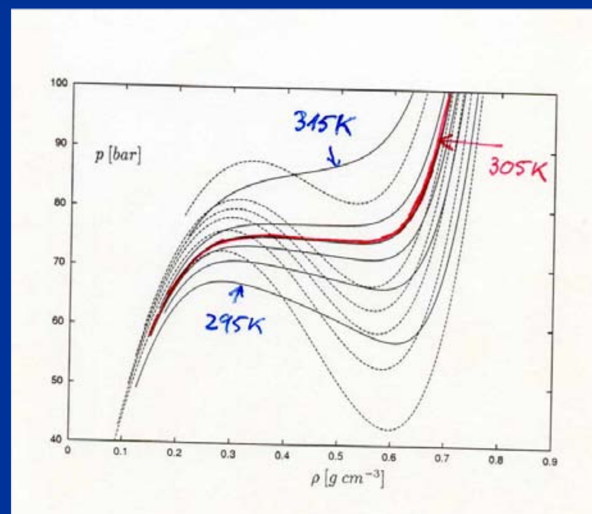
unified description of scalar models for all d and N

Flow of effective potential

Ising model



CO_2



Experiment :

T_* =304.15 K
p_* =73.8.bar
ρ_* = 0.442 g cm-2

Critical exponents

$d = 3$

Critical exponents ν and η

N	ν		η	
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
10	0.881	0.886	0.028	0.025
100	0.990	0.980	0.0030	0.003
		↑		↑

"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

S.Seide …

Critical exponents , d=3

N	ν (ERGE)	ν (world)	η (ERGE)	η (world)
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
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100	0.990	0.980	0.0030	0.003

ERGE world ERGE world

"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

critical exponents , BMW approximation

N	η	η (other)	ν	ν (other)	ω (prelim.)	ω (other)
0	0.033(3)	0.028(3) [1]	0.588	0.588(1) [1]	0.80	
1	0.039(3)	0.0364(2) [2] 0.0368(2) [3] 0.033(3) [1]	0.6298(4)	0.6301(2) [2] 0.6302(1) [3] 0.630(1) [1]	0.78	0.79(1) [1]
2	0.041(3)	0.0381(2) [4] 0.035(3) [1]	0.6719(4)	0.6717(1) [4] 0.670(2) [1]	0.78	0.79(1) [1]
3	0.040(3)	0.0375(5) [5] 0.036(3) [1]	0.709	0.7112(5) [5] 0.707(4) [1]	0.73	
4	0.038(3)	0.035(5)[1] 0.037(1) [6]	0.738	0.741(6) [1] 0.749(2) [6]	0.74	0.77(2) [1]
5	0.035(3)	0.031(3) [8] 0.034(1) [7]	0.768	0.764(4) [8] 0.779(3) [7]	0.73	0.77(2) [1]
10	0.022(2)	0.024 [9]	0.860	0.859 [9]	0.81	
20	0.012(1)	0.014 [9]	0.929	0.930 [9]	0.94	
100	0.0023(2)	0.0027 [10]		0.989 [10]	0.99	

[1] R. Guida and J. Zinn-Justin '98. [2] M. Campostrini, A. Pelissetto, P. Rossi, E. Vicari '02.
[3] Y. Deng and H. W. J. Blote '03. [4] M. Campostrini, M. Hasenbusch, A. Pelissetto, E. Vicari '06.
[5] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, E. Vicari '02. [6] M. Hasenbusch '01.
[7] M. Hasenbusch, A. Pelissetto, E. Vicari '05. [8] A. Butti and F. Parisen Toldin '05.
[9] S. A. Antonenko and A. I. Sokolov '95. [10] M. Moshe and J. Zinn-Justin '03.

Blaizot, Benitez , … , Wschebor

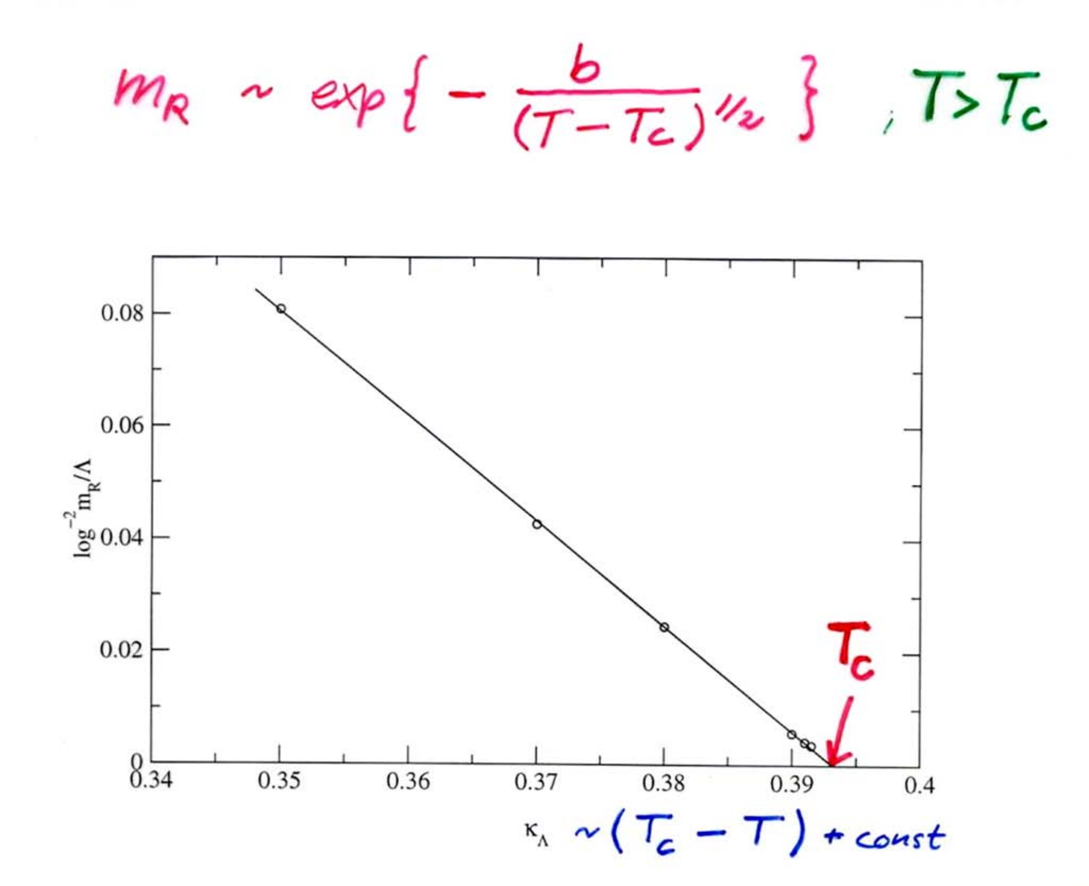
Solution of partial differential equation :

yields highly nontrivial non-perturbative results despite the one loop structure !

Example:

Kosterlitz-Thouless phase transition

Essential scaling : d=2,N=2



- Flow equation contains correctly the non-perturbative information !
- (essential scaling usually described by vortices)

Von Gersdorff …

Kosterlitz-Thouless phase transition (d=2,N=2)

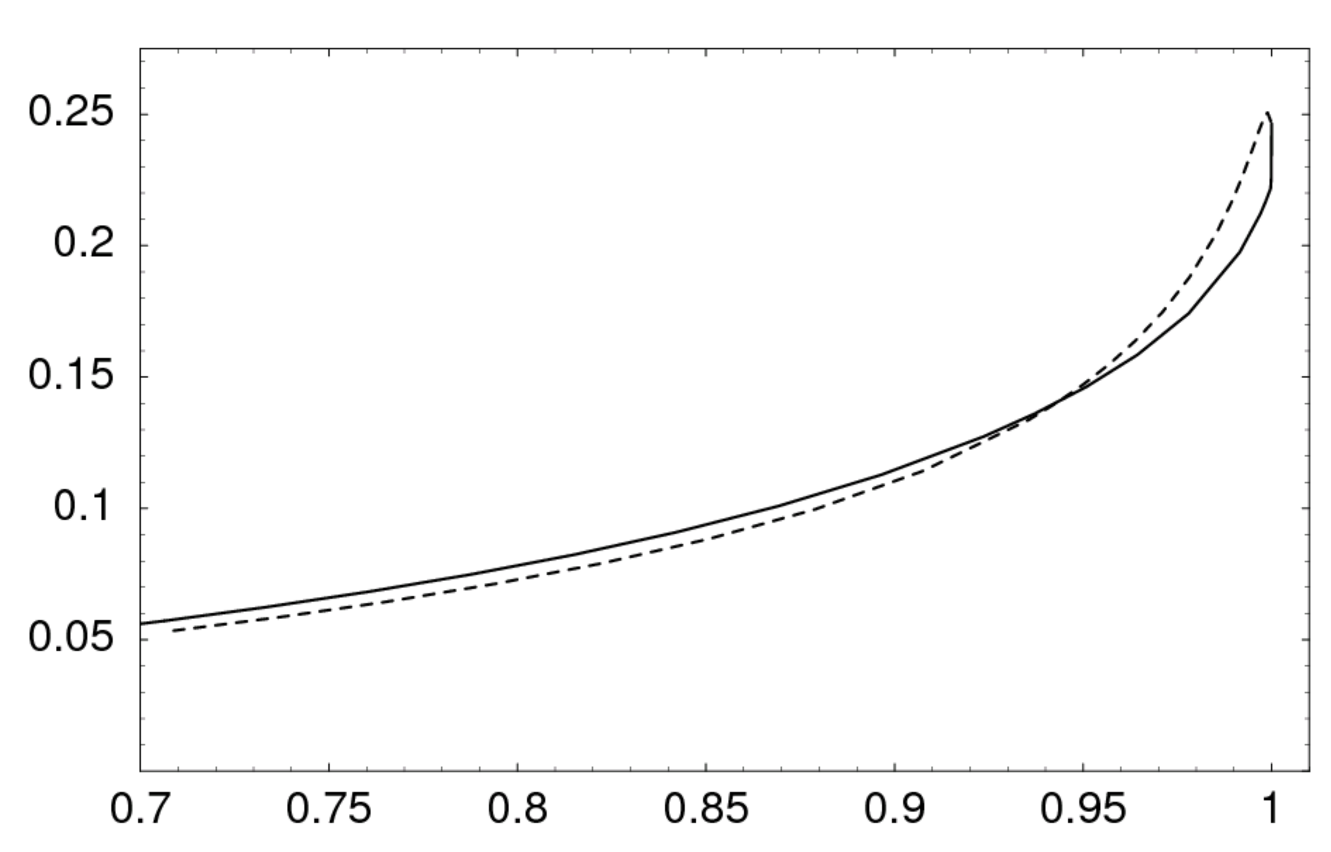
Correct description of phase with

Goldstone boson

(infinite correlation length)

for $T<T_c$

Temperature dependent anomalous dimension η



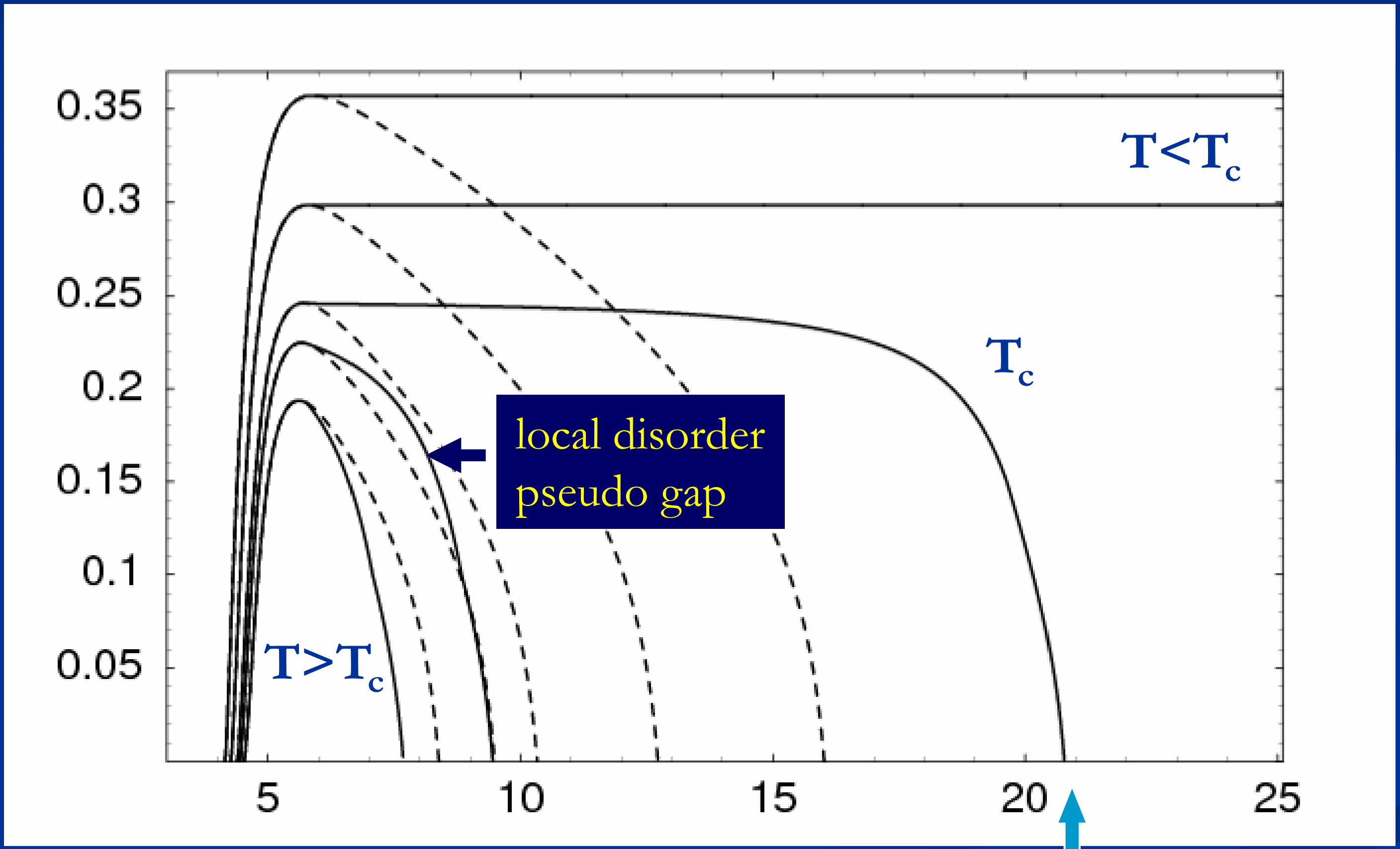
η

T/T_c

Running renormalized d-wave superconducting order parameter $\varkappa$ in doped Hubbard (-type) model

$\varkappa$

location of minimum of u

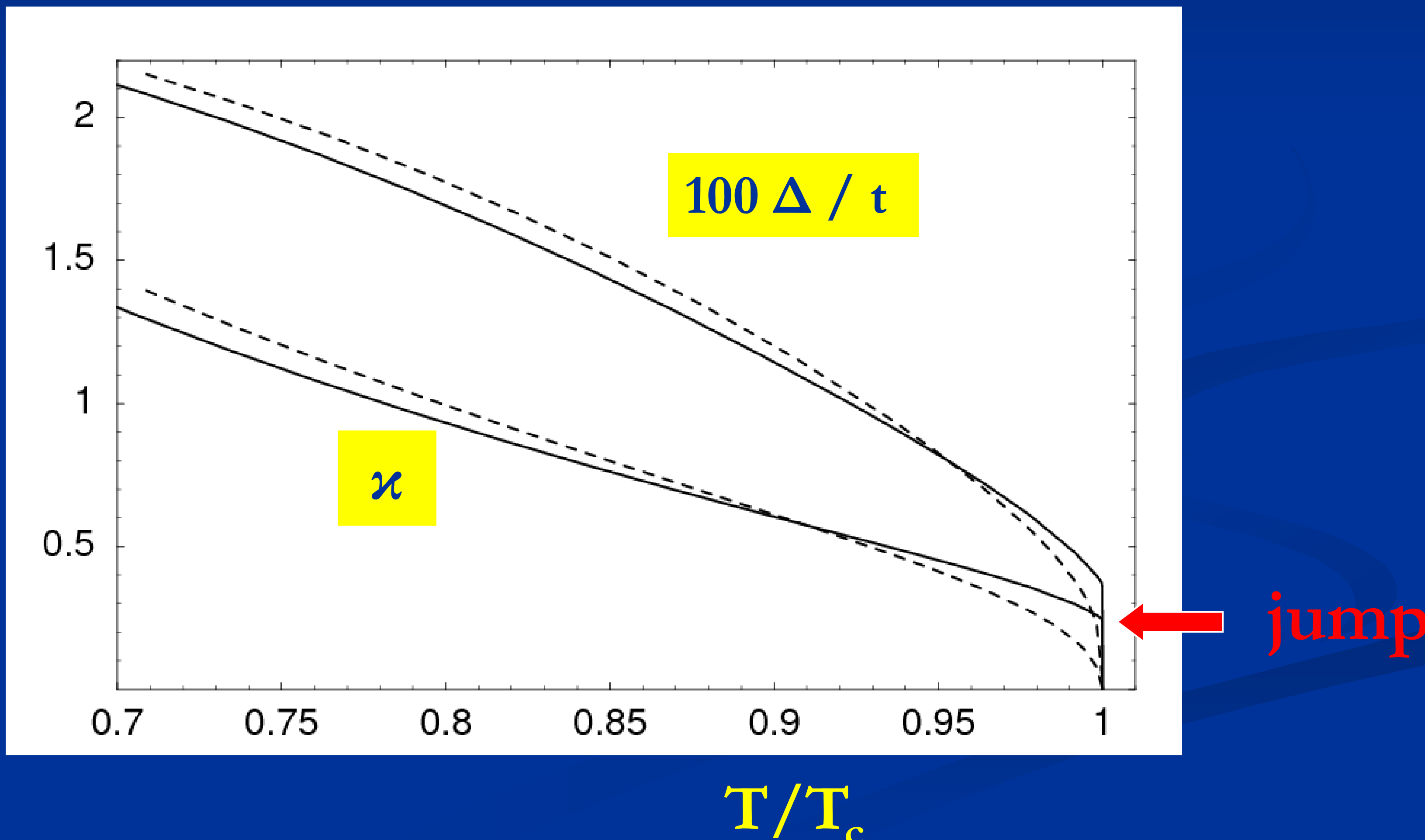


C.Krahl,…

- ln (k/Λ)

macroscopic scale 1 cm

Renormalized order parameter ϰ and gap in electron propagator Δ in doped Hubbard model



unification

abstract laws

quantum gravity

grand unification

standard model

electro-magnetism

gravity

Landau theory | universal critical physics | functional renormalization

complexity

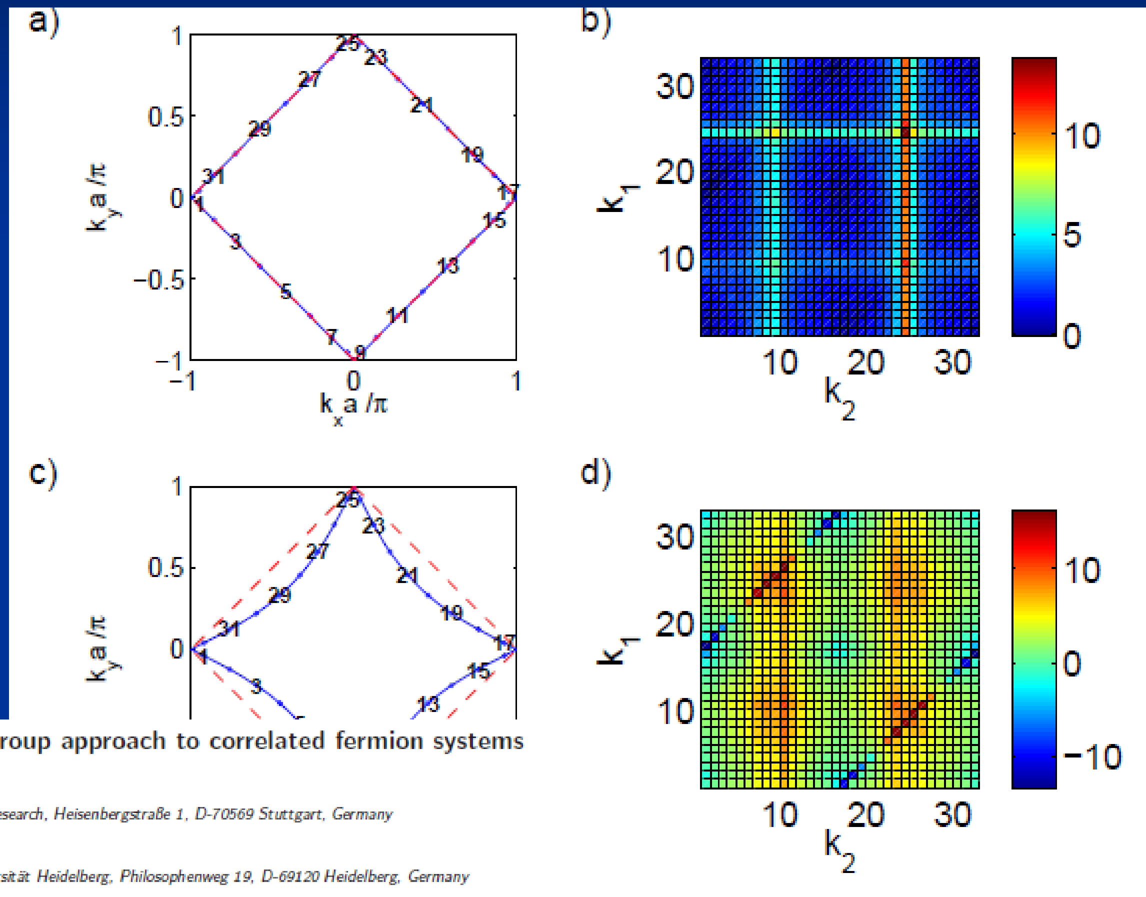
unification: functional integral / flow equation

- simplicity of average action
- explicit presence of scale
- differentiating is easier than integrating…

Unification from Functional Renormalization

- fluctuations in d=0,1,2,3,...
- linear and non-linear sigma models
- vortices and perturbation theory
- bosonic and fermionic models
- relativistic and non-relativistic physics
- classical and quantum statistics
- non-universal and universal aspects
- homogenous systems and local disorder
- equilibrium and out of equilibrium

Flow of four point function
Hubbard model



Functional renormalization group approach to correlated fermion systems

Walter Metzner

Max-Planck-Institute for Solid State Research, Heisenbergstraße 1, D-70569 Stuttgart, Germany

Manfred Salmhofer

Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, D-69120 Heidelberg, Germany

Carsten Honerkamp

Institut für Theoretische Festkörperphysik and JARA-Fundamentals of Future Information Technology, RWTH Aachen University, D-52056 Aachen, Germany

Volker Meden

Institut für Theorie der Statistischen Physik and JARA-Fundamentals of Future Information Technology, RWTH Aachen University, D-52056 Aachen, Germany

Kurt Schönhammer

Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1, D-37077 Göttingen, Germany

FRG for disordered systems

flow of functionals

$$f(\varphi) \longrightarrow f\,[\varphi(x)]$$

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left\{ \left(\Gamma_k^{(2)}[\varphi] + R_k \right)^{-1} \partial_k R_k \right\}$$

'92

$$\left(\Gamma_k^{(2)} \right)_{ab} (q, q') = \frac{\delta^2 \Gamma_k}{\delta \varphi_a(-q) \delta \varphi_b(q')}$$

exact inverse propagator, depends on field configuration φ(x) or φ(q)

$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$

(fermions : STr)

some history … (the parents)

- **exact RG equations :**

  Symanzik eq. , Wilson eq. , Wegner-Houghton eq. , Polchinski eq. ,

  mathematical physics

- **1PI :** RG for 1PI-four-point function and hierarchy

  Weinberg

  formal Legendre transform of Wilson eq.

  Nicoll, Chang

- **non-perturbative flow :**

  d=3 : sharp cutoff ,

  no wave function renormalization or momentum dependence

  Hasenfratz^2

functional renormalization

- transition from microscopic to effective theory is made continuous
- effective laws depend on scale k
- flow in space of theories
- flow from simplicity to complexity if theory is simple for large k
- or opposite , if theory gets simple for small k

Truncation

- equation is exact

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}$$

- solution needs truncation

- systematic expansions sometimes , not always possible
- that's where experience , knowledge or intuition enter
- not a black box !

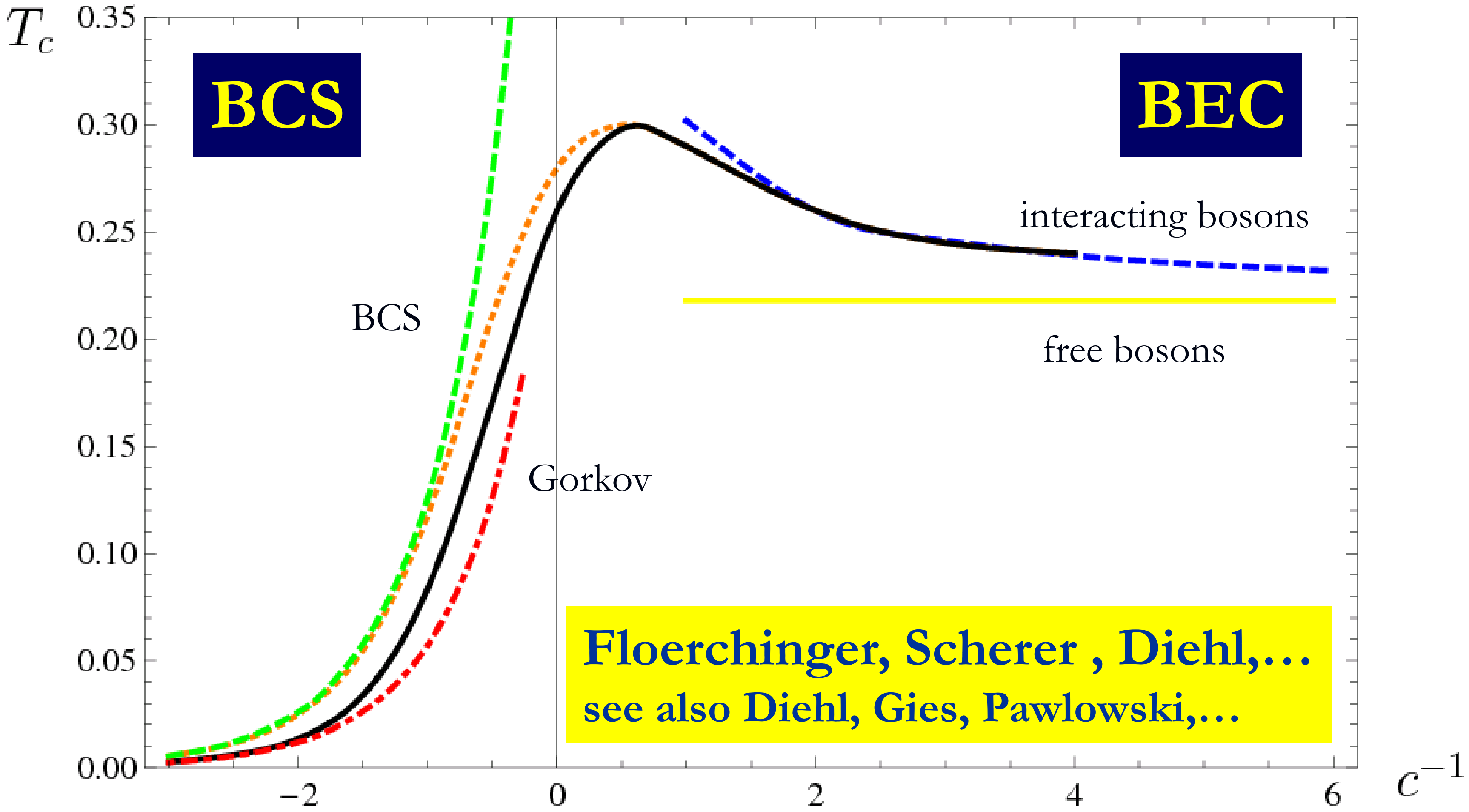
QCD :
Short and long distance degrees of freedom are different !

Short distances : quarks and gluons
Long distances : baryons and mesons

How to make the transition?

confinement/chiral symmetry breaking

BCS – BEC crossover in ultracold Fermi gases



Floerchinger, Scherer , Diehl,…
see also Diehl, Gies, Pawlowski,…

how to change continuously degrees of freedom ?

add in and remove !

Anti-ferromagnetic order in the Hubbard model

A functional renormalization group study

T.Baier, E.Bick, …
C.Krahl, J.Mueller, S.Friederich

Fermion bilinears

$$\tilde{\rho}(X) = \hat{\psi}^{\dagger}(X)\hat{\psi}(X),$$
$$\tilde{\vec{m}}(X) = \hat{\psi}^{\dagger}(X)\vec{\sigma}\hat{\psi}(X)$$

are described by “ composite “ bosons

fermion interactions can be
partially accounted for
by exchange of
(composite) bosons

Initial fermion – boson action

fermion kinetic term + local interaction ~ U

$$S_{F,\mathrm{kin}} = \sum_Q \hat{\psi}^\dagger(Q)(i\omega_F - \mu - 2t(\cos q_1 + \cos q_2))\hat{\psi}(Q),$$

boson quadratic term (“classical propagator”)

$$S_B = \frac{1}{2}\sum_Q \Big(U_\rho \hat{\rho}(Q)\hat{\rho}(-Q) + U_m \hat{\vec{m}}(Q)\hat{\vec{m}}(-Q)\Big)$$

Yukawa coupling

$$S_Y = -\sum_{QQ'Q''} \delta(Q - Q' + Q'') \times \\ (U_\rho \hat{\rho}(Q)\hat{\psi}^\dagger(Q')\hat{\psi}(Q'') + U_m \hat{\vec{m}}(Q)\hat{\psi}^\dagger(Q')\vec{\sigma}\hat{\psi}(Q''))$$

no boson dynamics in absence of Yukawa coupling , typical initial situation , bosons only auxiliary fields

Flowing fermion – boson action

$$
\begin{aligned}
\Gamma_k[\psi,\psi^*,\phi] = & \sum_Q \psi^*(Q) P_{\psi,k} \psi(Q) \\
& + \frac{1}{2} \sum_Q \phi(-Q) P_{\phi,k}(Q) \phi(Q) \\
& - \sum_Q h_k(Q) \phi(Q) \tilde{\phi}(-Q) \\
& + \sum_Q \lambda_{\psi,k}(Q) \tilde{\phi}(Q) \tilde{\phi}(-Q)
\end{aligned}
$$

what generates Yukawa coupling ?

Flowing bosonisation

k-dependent
field redefinition
(variable change)

$$\phi_k(Q) = \phi_{\bar{k}}(Q) + \Delta\alpha_k(Q)\tilde{\phi}(Q)$$

$$\partial_k \phi_k(Q) = -\,\partial_k \alpha_k(Q)\tilde{\phi}(Q)$$

shuffles parts of four – fermion interaction
generated by the flow into
boson exchange interaction
exact formalism !

H.Gies , …

Flowing bosonisation

Evolution with k-dependent field variables

$$\partial_k \Gamma_k[\psi,\psi^*,\phi_k] = \partial_k \Gamma_k[\psi,\psi^*,\phi_k]|_{\phi_k} + \sum_Q \left(\frac{\delta}{\delta\phi_k}\Gamma[\psi,\psi^*,\phi_k]\right)\partial_k\phi_k$$
$$= \partial_k \Gamma_k[\psi,\psi^*,\phi_k]|_{\phi_k} + \sum_Q \big(-\partial_k\alpha_k(Q) P_{\phi,k}(Q)\phi_k(Q)\tilde{\phi}(-Q) + h_k(Q)\partial_k\alpha_k(Q)\tilde{\phi}(Q)\tilde{\phi}(-Q)\big)$$

modified flow of couplings

$$\partial_k h_k(Q) = \partial_k h_k(Q)|_{\phi_k} + \partial_k \alpha_k(Q) P_{\phi,k}(Q),$$
$$\partial_k \lambda_{\psi,k}(Q) = \partial_k \lambda_{\psi,k}(Q)|_{\phi_k} + h_k(Q)\,\partial_k\alpha_k(Q).$$

Choose α_k in order to absorb the four fermion coupling in corresponding channel ➡

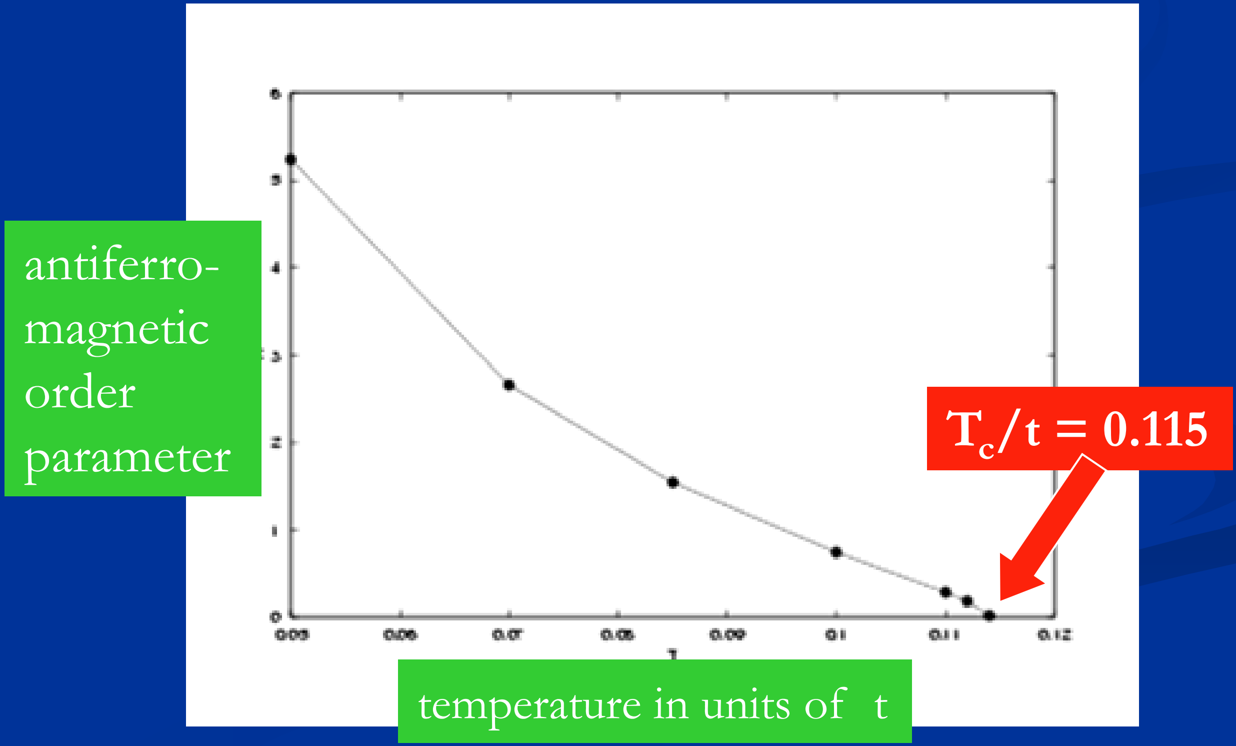
$$\partial_k h_k(Q) = \partial_k h_k(Q)|_{\phi_k} - \frac{P_{\phi,k}(Q)}{h_k(Q)}\partial_k\lambda_{\psi,k}(Q)|_{\phi_k}$$

variable change
exploits the
freedom of functionals

Below the critical temperature :

Infinite-volume-correlation-length becomes larger than sample size

finite sample ≈ finite k : order remains effectively

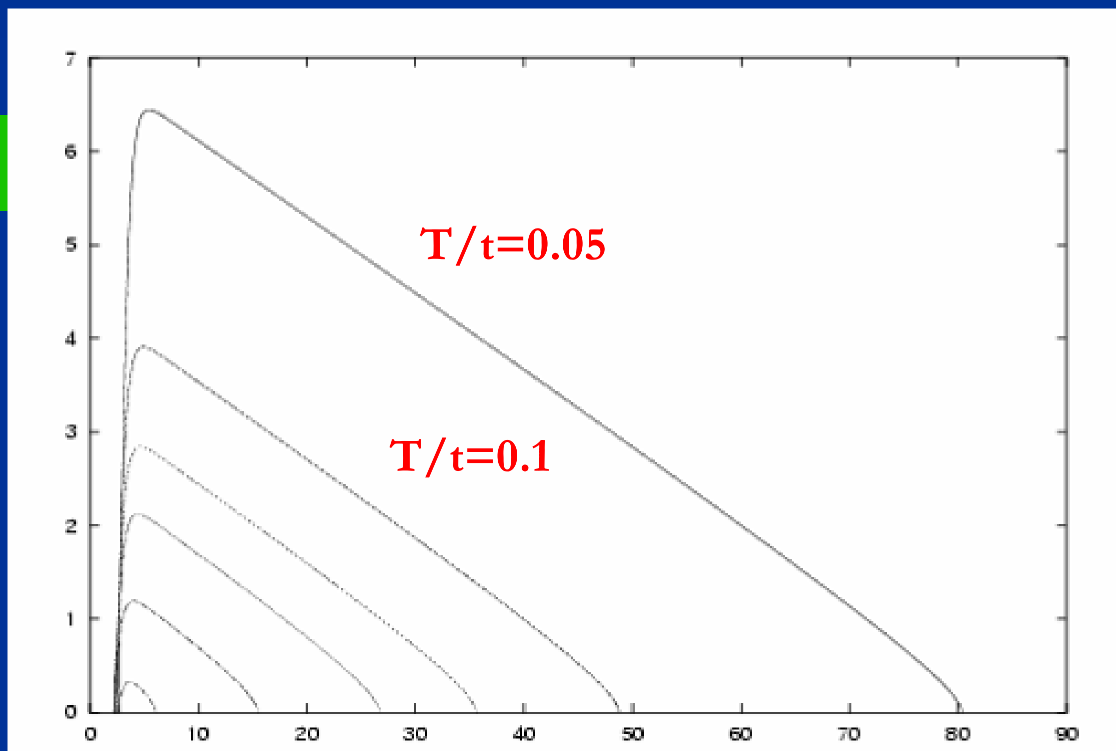


Critical temperature

For $T<T_c$: $\varkappa$ remains positive for $k/t > 10^{-9}$

size of probe > 1 cm

$\varkappa$



local disorder
pseudo gap

SSB

-ln(k/t)

$$\kappa_a = \frac{Z_a t^2}{T} \alpha_0$$

T_c=0.115

Pseudo-critical temperature T_{pc}

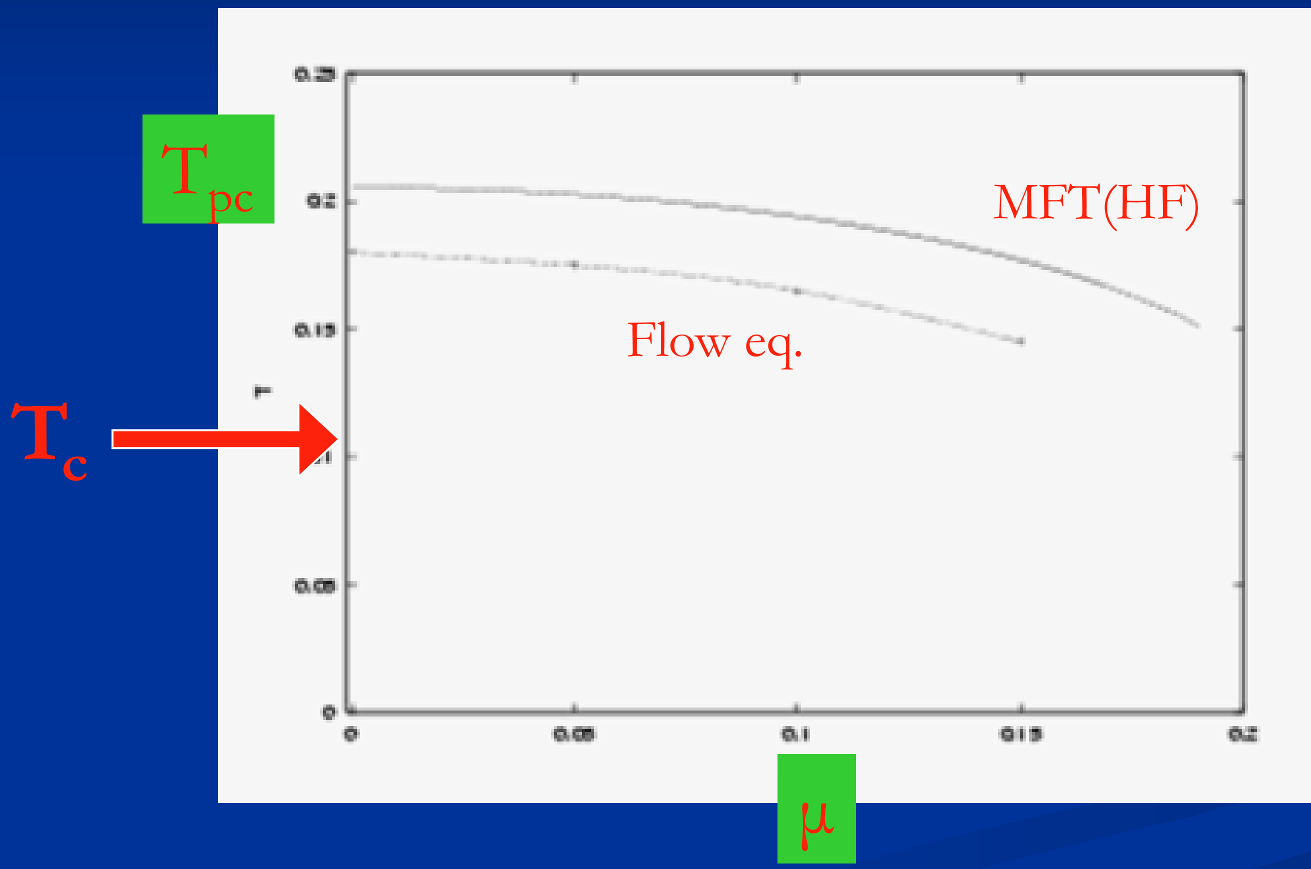
Limiting temperature at which bosonic mass term vanishes ($\varkappa$ becomes nonvanishing)

It corresponds to a diverging four-fermion coupling

This is the "critical temperature" computed in MFT !

Pseudo-gap behavior below this temperature

Pseudocritical temperature



Below the pseudocritical temperature

the reign of the goldstone bosons

effective nonlinear O(3) – σ - model

critical behavior

for interval $T_c < T < T_{pc}$
evolution as for classical Heisenberg model

cf. Chakravarty,Halperin,Nelson

$$k\partial_k \kappa = \frac{1}{4\pi} + \frac{1}{16\pi^2\kappa} + 0(\kappa^{-2})$$

critical correlation length

$$\xi t = c(T)\exp\left\{20.7\beta(T)\frac{T_c}{T}\right\}$$

c,β : slowly varying functions

exponential growth of correlation length
compatible with observation !

at T_c : correlation length reaches sample size !

Mermin-Wagner theorem ?

No spontaneous symmetry breaking of continuous symmetry in d=2 !

not valid in practice !

change of degrees of freedom
is crucial for
simple picture

qualitative changes that make non-perturbative physics accessible :

(1) basic object is simple

average action ~ classical action

~ generalized Landau theory

direct connection to thermodynamics

(coarse grained free energy)

qualitative changes that make non-perturbative physics accessible :

(2) Infrared scale k instead of Ultraviolet cutoff Λ

short distance memory not lost

no modes are integrated out , but only part of the fluctuations is included

simple one-loop form of flow

simple comparison with perturbation theory

infrared cutoff k

cutoff on momentum resolution

or frequency resolution

e.g. distance from pure anti-ferromagnetic momentum or from Fermi surface

intuitive interpretation of k by association with physical IR-cutoff , i.e. finite size of system :

arbitrarily small momentum differences cannot be resolved !

qualitative changes that make non-perturbative physics accessible :

(3) only physics in small momentum range around k matters for the flow

ERGE regularization

simple implementation on lattice

artificial non-analyticities can be avoided

qualitative changes that make non-perturbative physics accessible :

(4) flexibility

change of fields

microscopic or composite variables

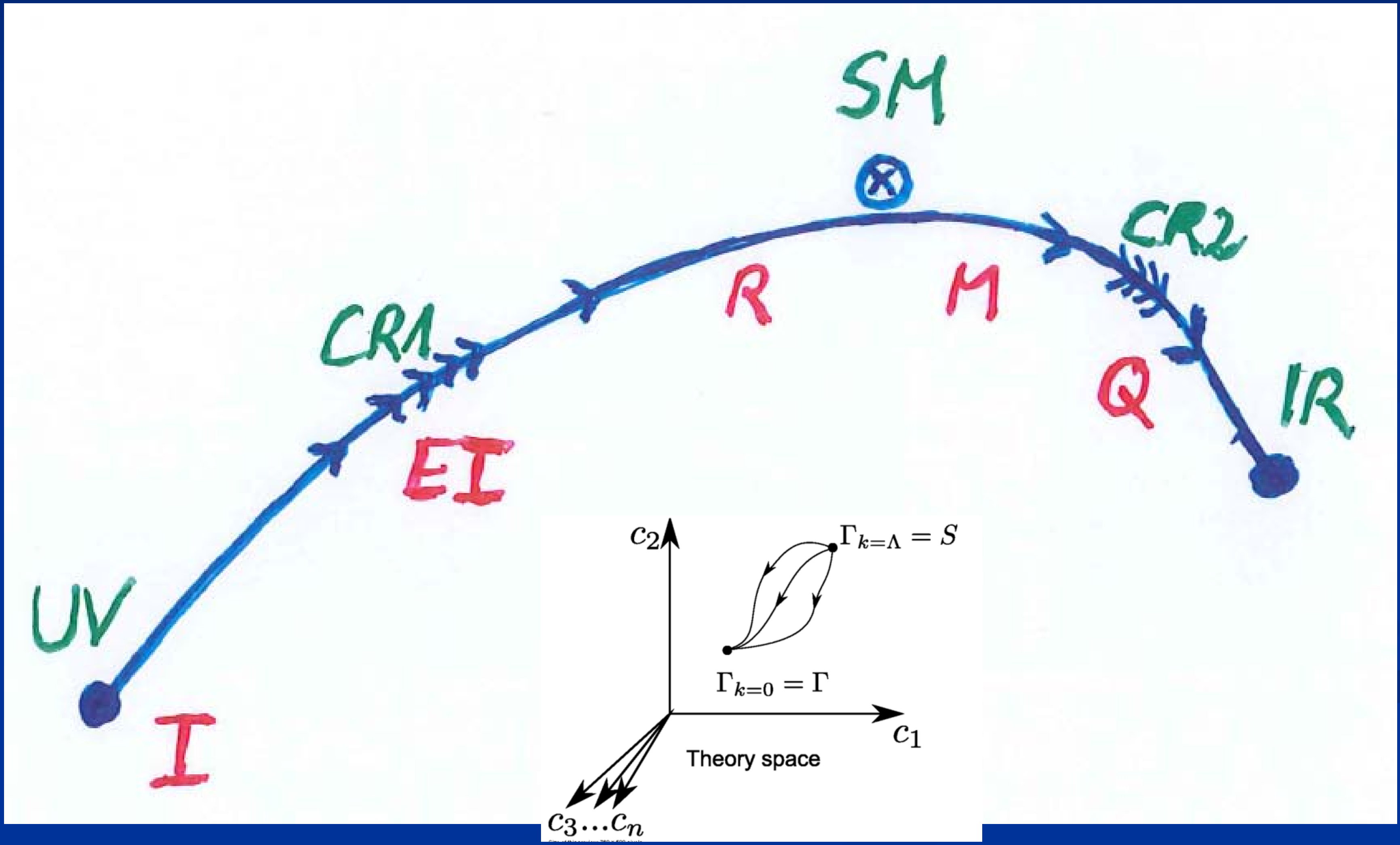
simple description of collective degrees of freedom and bound states

many possible choices of "cutoffs"

The more you want , the more complicated it gets

- works well and conceptually simple for scalars and fermions, including chiral fermions, in equilibrium
- more complex for local gauge theories, gravity
- non- equilibrium : first successes
- disorder : a challenge
- biology, finance ….?

Crossover in quantum gravity



Variable Gravity

$$\Gamma = \int_x \sqrt{g}\left\{-\frac{1}{2}\chi^2 R + \mu^2\chi^2 + \frac{1}{2}\big(B(\chi/\mu) - 6\big)\partial^\mu\chi\partial_\mu\chi\right\}$$

quantum effective action,
variation yields field equations

Einstein gravity : $\Gamma = \int_x \sqrt{g}\left\{-\frac{1}{2} M^2 R\right\}$

Variable Gravity

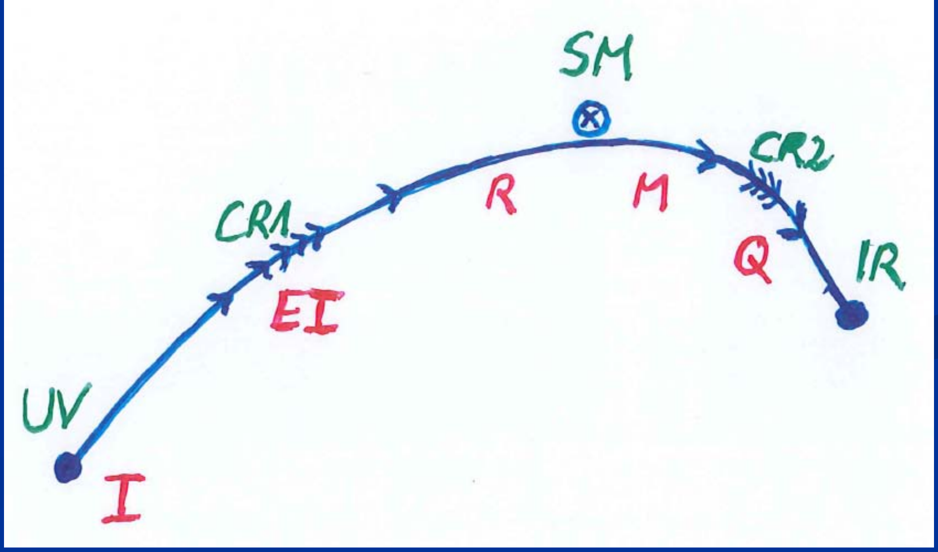
- Scalar field coupled to gravity
- Effective Planck mass depends on scalar field
- Simple quadratic scalar potential involves intrinsic mass μ
- Nucleon and electron mass proportional to dynamical Planck mass

$$\Gamma = \int_x \sqrt{g} \left\{ -\frac{1}{2}\chi^2 R + \mu^2\chi^2 + \frac{1}{2}\left(B(\chi/\mu) - 6\right)\partial^\mu\chi\partial_\mu\chi \right\}$$

Cosmological solution : crossover from UV to IR fixed point

- Dimensionless functions as B depend only on ratio μ/χ .
- IR: $\mu\to 0$, $\chi\to\infty$
- UV: $\mu\to\infty$, $\chi\to 0$

Cosmology makes crossover between fixed points by variation of χ .



renormalization flow and cosmological evolution

- renormalization flow as function of μ

  is mapped by dimensionless functions to

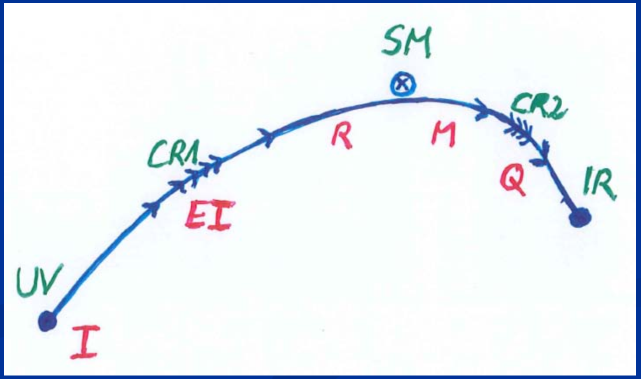
- field dependence of effective action on scalar field χ

  translates by solution of field equation to

- dependence of cosmology an time t or η

Origin of mass

- UV fixed point : scale symmetry unbroken
  all particles are massless



- IR fixed point :
  scale symmetry spontaneously broken,
  massive particles , massless dilaton

- crossover : explicit mass scale μ important

- approximate SM fixed point : approximate scale symmetry spontaneously broken, massive particles , almost massless cosmon, tiny cosmon potential

Spontaneous breaking of scale symmetry

- expectation value of scalar field breaks scale symmetry spontaneously
- massive particles are compatible with scale symmetry
- in presence of massive particles : sign of exact scale symmetry is exactly massless Goldstone boson – the dilaton

Approximate scale symmetry near fixed points

- UV : approximate scale invariance of primordial fluctuation spectrum from inflation

- IR : cosmon is pseudo Goldstone boson of
  spontaneously broken scale symmetry,
  tiny mass,
  responsible for dynamical Dark Energy

Simplicity

simple description of all cosmological epochs

natural incorporation of Dark Energy :

- inflation
- Early Dark Energy
- present Dark Energy dominated epoch

Asymptotic safety

if UV fixed point exists :

quantum gravity is

non-perturbatively renormalizable !

S. Weinberg , M. Reuter

Fundamental setting

several relevant directions at UV – fixed point

crossover to IR fixed points in several steps

- 1) decoupling of gravity , end of inflation
- 2) Fermi scale , decoupling of weak interactions
- 3) QCD- scale , decoupling of hadrons
- 4) Beyond standard model physics :

  crossover in neutrino sector →

  onset of dark energy domination

end