Challenges for
Functional Renormalization

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}$$

'92

$$\left(\Gamma_k^{(2)}\right)_{ab}(q, q') = \frac{\delta^2 \Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$$

$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$

(fermions : STr)

important success in many areas

first rate tool if perturbative expansions or numerical simulations fail or are difficult

- models with fermions
- gravity
- models with largely different length scales
- non-perturbative renormalizability

different laws at different scales

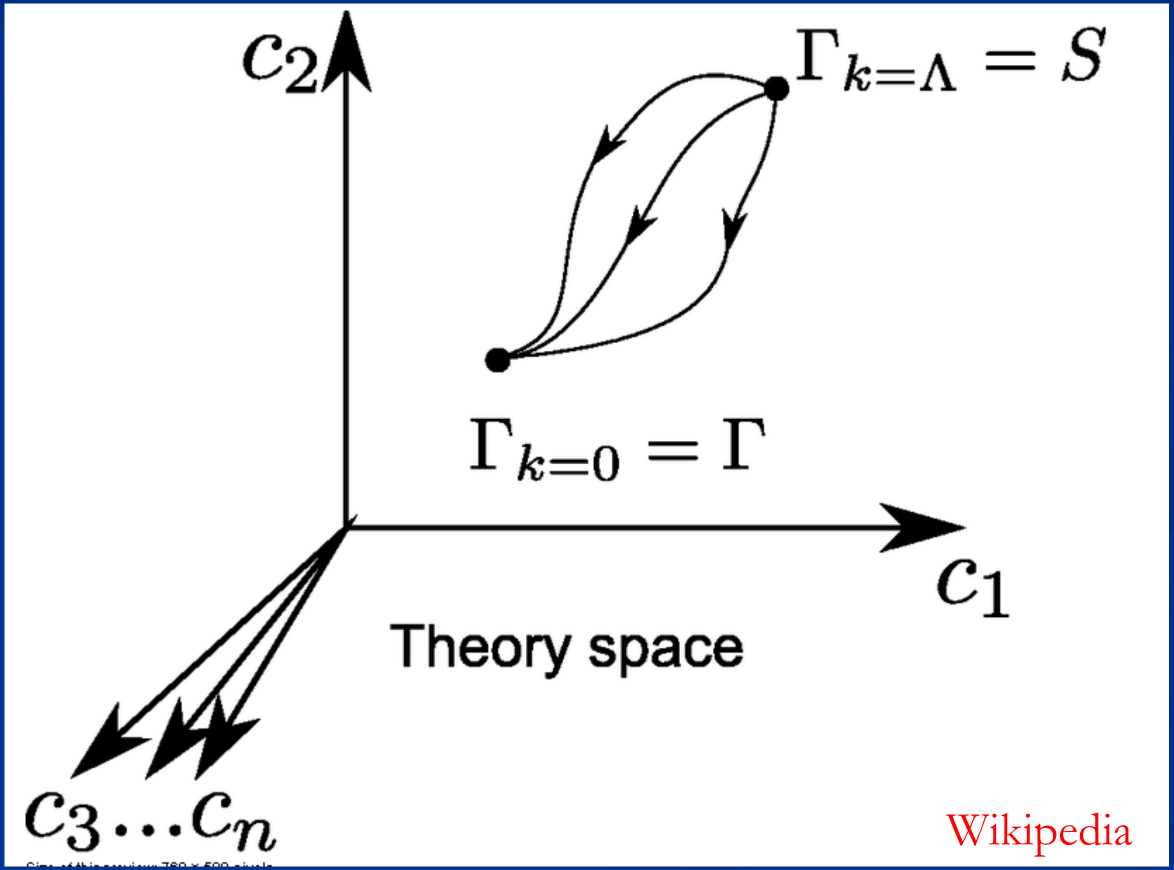
- fluctuations wash out many details of microscopic laws
- new structures as bound states or collective phenomena emerge

key problem in Physics !

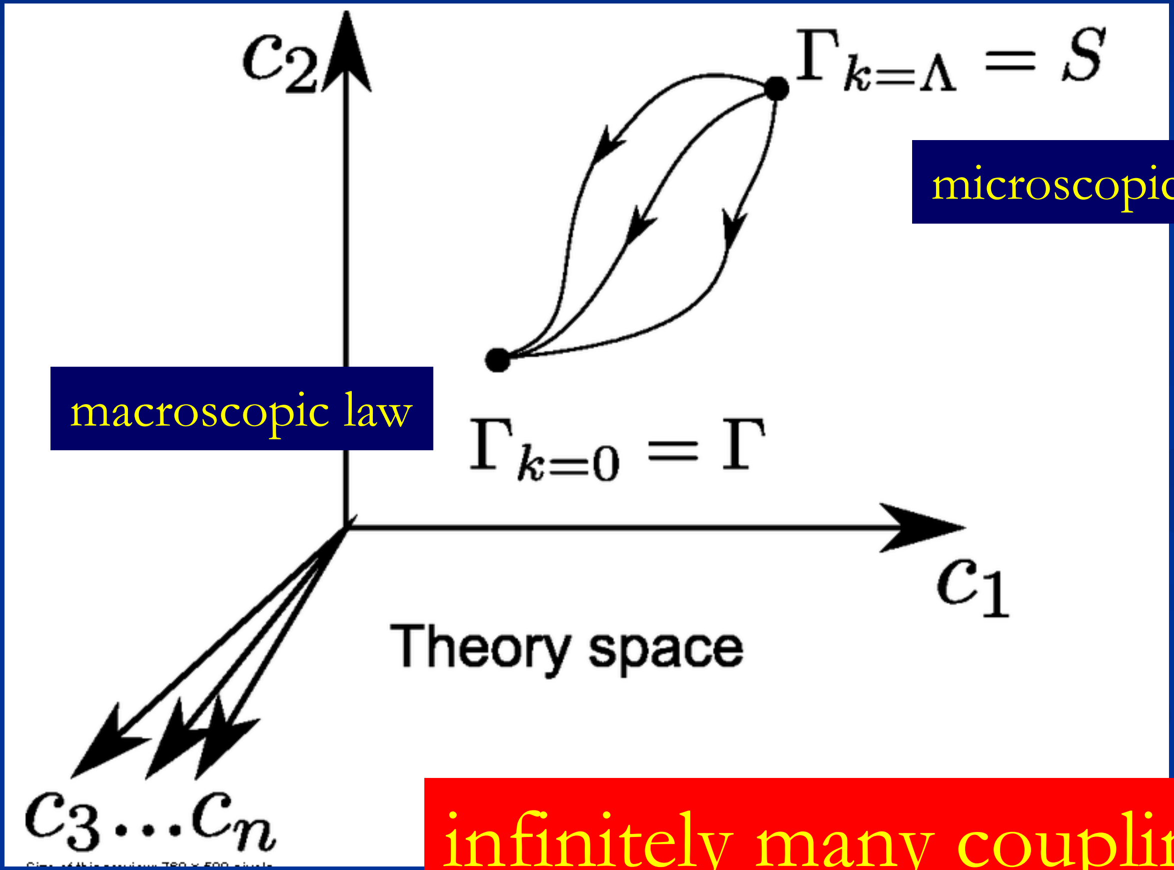
scale dependent laws

- scale dependent (running or flowing) couplings
- flowing functions
- flowing functionals

flowing action



flowing action



microscopic law

macroscopic law

infinitely many couplings

functional renormalization

- transition from microscopic to effective theory is made continuous
- effective laws depend on scale k
- flow in space of theories
- flow from simplicity to complexity if theory is simple for large k
- or opposite , if theory gets simple for small k

challenges for functional renormalization

- reliability and error estimates
- accessibility
- exploration of new terrain
- precision and benchmarking

truncation error

structure of truncated flow equation

$$\partial_t g = \varsigma$$

g : flowing data

ς: flow generators

flowing data

typically, g can be viewed as functions

effective potential $U(\rho)$

inverse relativistic propagator $P(p^2)$

inverse non-relativistic propagator $P(\omega, p^2)$

momentum dependent four-fermion vertex

truncation

finite number of functions

functions parameterized by finite set of data

e.g.

polynomial expansion

function values at given arguments

(can be single coupling)

truncation : limitation to restricted set of data

(finite set for numerical purposes)

exact flow equations

for given set of data :

flow generators ς can be computed **exactly** as formal expressions

O(N) – scalar model

first order derivative expansion

$$\Gamma_k[\phi] = \int d^dx \left\{ U_k(\rho) + \frac{1}{2} Z_k(\rho) \partial_\mu \phi^a \partial^\mu \phi_a + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial^\mu \rho + \mathcal{O}(\partial^4) \right\}$$

$$\rho \equiv \tfrac{1}{2} \phi_a \phi^a$$

flowing data g : U(ρ), Z(ρ), Y(ρ)

exact generator for U:

$$\partial_t U_k(\rho) = \frac{1}{2} \int \frac{d^dq}{(2\pi)^d} \frac{\partial R_k}{\partial t} \left(\frac{N-1}{M_0} + \frac{1}{M_1} \right)$$

$$\begin{aligned} M_0(\rho, q^2) &= Z_k(\rho, q^2) q^2 + R_k(q) + U_k'(\rho) \\ M_1(\rho, q^2) &= \tilde{Z}_k(\rho, q^2) q^2 + R_k(q) + U_k'(\rho) + 2\rho U_k''(\rho) \end{aligned}$$

momentum integration

one loop form of exact flow equation :

$$\zeta = \int_q \sigma(q)$$

$\sigma(q)$: input functions

one d-dimensional momentum integration necessary,
(sometimes analytical integration possible)

specification

For finite set of data the system of flow equations is not closed !

$\sigma(q)$ cannot be computed uniquely from g
one needs **prescription** how $\sigma(q)$ is determined in terms of g

$$g \rightarrow \sigma(q)$$

specification parameters

specification typically involves

specification parameters s

$$(g, s) \longrightarrow \sigma(q)$$

Lowest order derivative expansion for scalar O(N) model

- $Z(\rho) = Z$
- $Y(\rho) = 0$
- flowing data : $U(\rho)$, Z

Flow equation for average potential

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

R_k : IR-cutoff

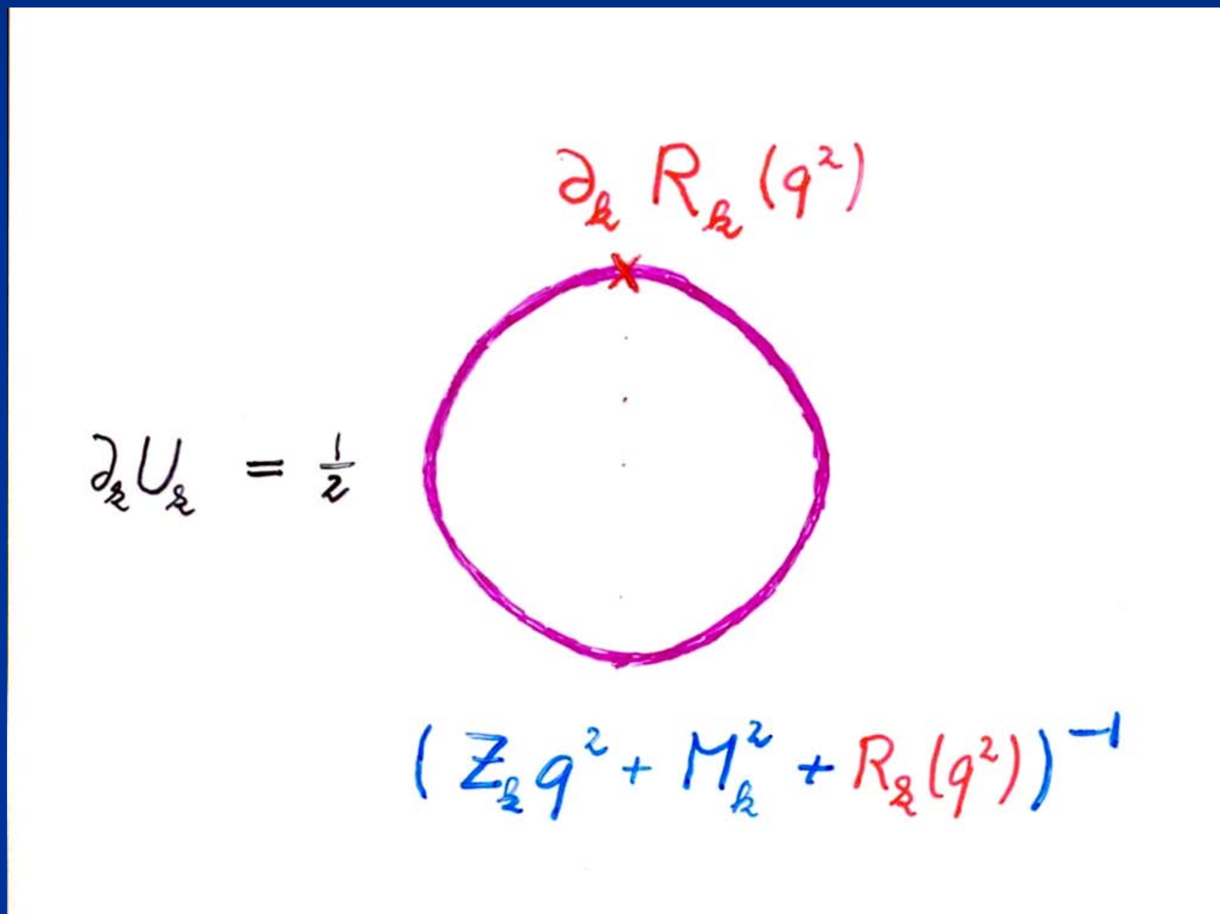
e.g $\quad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $\quad R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2) \quad$ (Litim)

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Simple one loop structure – nevertheless (almost) exact

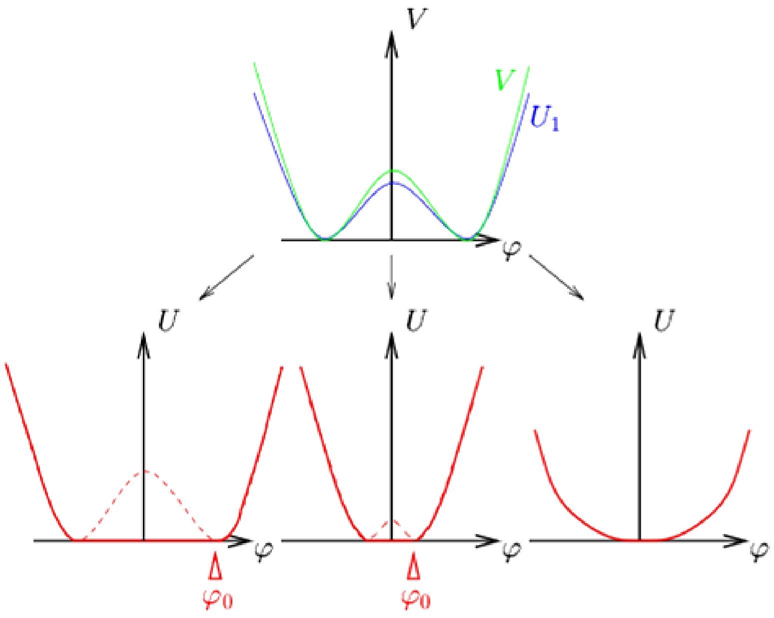


Scalar field theory

$\varphi_a(x)$: magnetization, density, chemical concentration, Higgs field, meson field, inflaton, cosmon

$O(N)$-symmetry:

$$S = \int d^d x \left\{ \frac{1}{2}\partial_\mu \varphi_a \partial_\mu \varphi_a + V(\rho) \right\}; \quad \rho = \frac{1}{2}\varphi_a \varphi_a$$



specification (1)

one has to specify the exact definition of Z from inverse propagator P(q)

P: second functional derivative at minimum of effective potential - Goldstone mode or radial mode

- $Z = \partial P / \partial q^2$ at $q^2 = 0$ or
- $Z = (P(q^2 = ck^2) - P(0)) / ck^2$

  c is one of the specification parameters

Which forms of effective action are compatible with given data g ?

specification (2)

different forms of inverse propagator are compatible with a given definition of Z

$$P = M^2(\rho) + Z\, f(q^2)$$

$$\frac{\partial f}{\partial q^2}\Big|_{q^2=0} = 1$$

choice of inverse propagator

example

$$f(q^2) = \begin{cases} q^2 + b q^4 & \text{for } q^2 < ck^2 \\ A(q^2 + ck^2)^{1-\frac{\eta}{2}} & \text{for } q^2 > ck^2 \end{cases}$$

$$A = \frac{1 + bck^2}{2} (2ck^2)^{\frac{\eta}{2}}$$

$$P = M^2(\rho) + Z f(q^2)$$

$$\frac{\partial f}{\partial q^2}\Big|_{q^2=0} = 1$$

Z_k: wave function renormalization

$$k \partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

physics knowledge can be put into choice of general form of input functions !

flow parameters w

- specification parameters
- cutoff parameters
- bosonization parameters

$$(g, w) \longrightarrow \xi$$

$$\partial_t g = \xi$$

error estimate

vary w within certain priors

accessibility

public program with structure

$$(g, w) \longrightarrow \zeta$$

$$\partial_t g = \zeta$$

for first step :
individual routines from users / library

numerical momentum integration

①

Flowing data g ↘
Flow parameters w ↗ → Input functions $\sigma(q)$

②

Input functions $\sigma(q)$ ⟶ Flow generators ς

momentum integration $\varsigma = \int_q \sigma(q)$

update of flow

(3)

Flow update of g

$$\partial_t g = \varphi$$

precision and benchmarking

benchmarks (1)

- universal critical physics:

  O(N)-models : exponents, amplitude ratios, equation of state (including non-perturbative physics as Kosterlitz -Thouless transition)
- non-abelian non-linear sigma-models in d=2 : mass gap, correlation functions
- Ising model on lattice and other exactly solvable models

benchmarks (2)

- quantum mechanics

  a) as d=1 functional integral

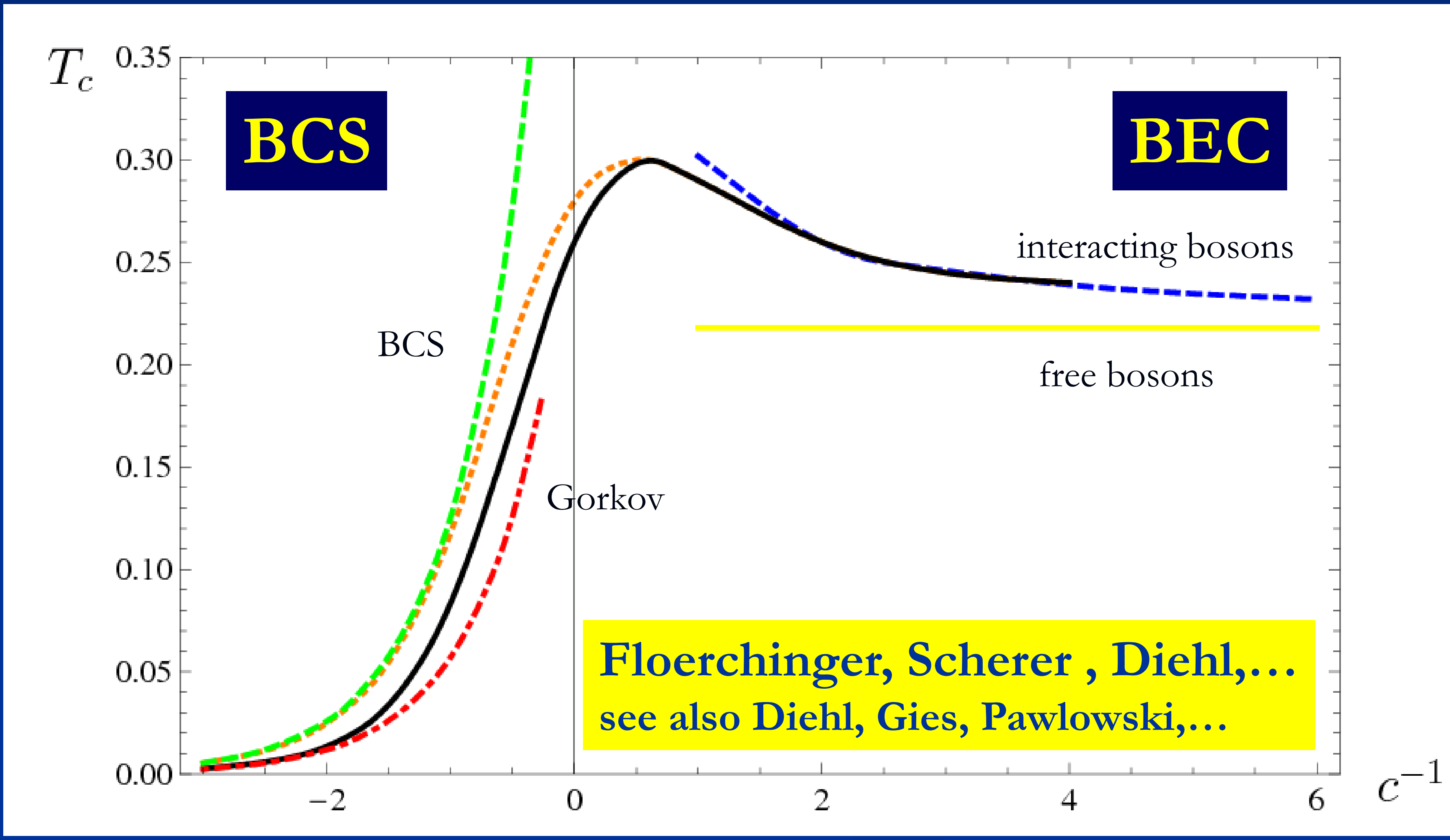
  b) limit $n \to 0$, $T \to 0$ of many body system,

  d=3+1

  scattering length for atoms, dimers,
  Efimov effect with Efimov parameter
- BCS-BEC crossover for ultracold atoms :

  all T, n, a

BCS – BEC crossover



explore new terrain

up to you !

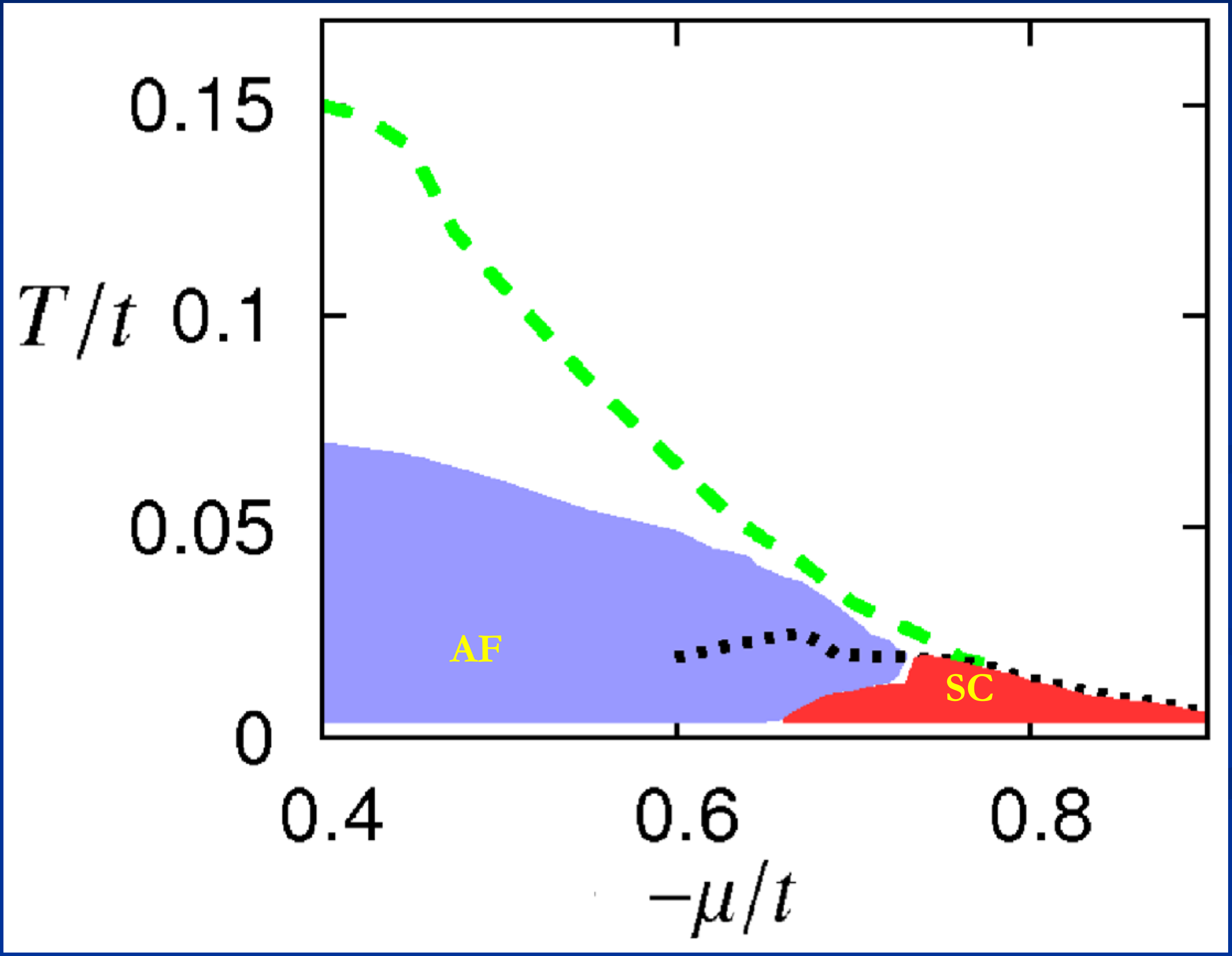
Phase transitions in Hubbard Model

Anti-ferromagnetic and superconducting order in the Hubbard model

A functional renormalization group study

T.Baier, E.Bick, …
C.Krahl, J.Mueller, S.Friederich

Phase diagram

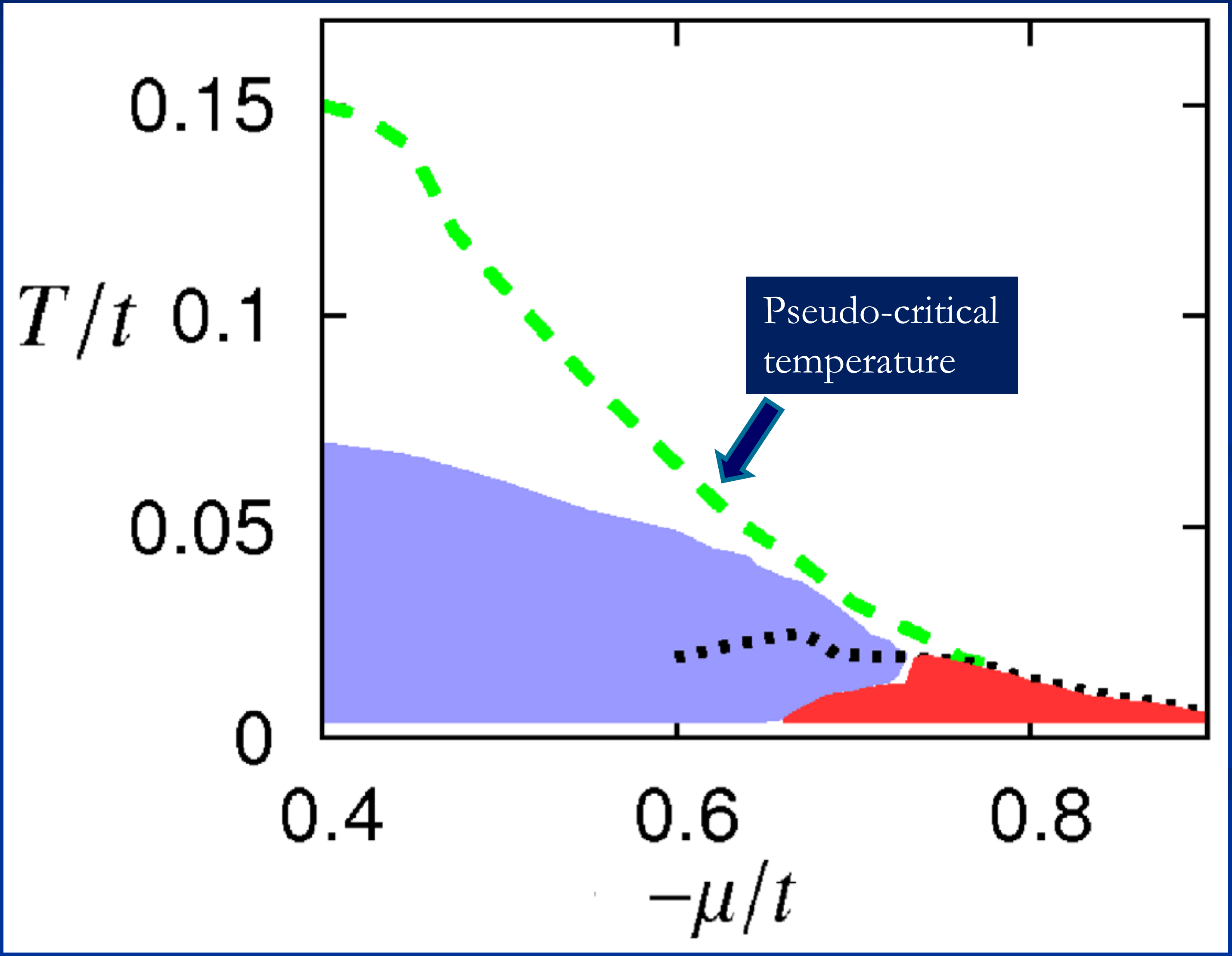


Mermin-Wagner theorem ?

No spontaneous symmetry breaking of continuous symmetry in $d=2$!

not valid in practice !

Phase diagram



challenges for functional renormalization

- reliability and error estimates
- accessibility
- exploration of new terrain
- precision and benchmarking

conclusions

functional renormalization :

bright future

substantial work ahead

end