Functional Renormalization

Lecture 1: Setting the stage. Functional integral for scalar quantum field theory

a) partition function (elementary magnets)
b) bosonic atoms
c) Higgs scalar
d) generating functional for Greens functions
e) effective action
f) thermodynamics

Lecture 2: Perturbation theory. Infrared flow. Functional renormalization

a) background identity for effective action
b) one-loop perturbation theory for effective potential
c) infrared divergences
d) infrared regulator
e) flow equation
f) renormalization group improvement
g) functional renormalization

Lecture 3: Exact functional renormalization group equation. Truncations

a) effective average action or flowing action
b) derivation of exact flow equation
c) renormalization group improved one loop equation
d) properties of flow equation
e) derivative expansion
f) flow of effective potential for N-component scalar theory in arbitrary dimension
g) other truncations

Lecture 4: Results and generalizations

a) unified picture of scalar theories in arbitrary dimensions
b) critical phenomena
c) fermionic models

1. Functional integral for scalar quantum field theory

a) partition function

lattice points x ;

elementary magnets $\chi_a(x)$, $a = 1 \ldots N$

here $\chi_a(x) \in \mathbb{R}$

special cases classical spins :

$N = 3$, $\chi_a(x)\chi_a(x) = 1$: Heisenberg model

$N = 1$, $\chi_a(x) = \pm 1$: Ising model

microstates of classical statistical ensemble :

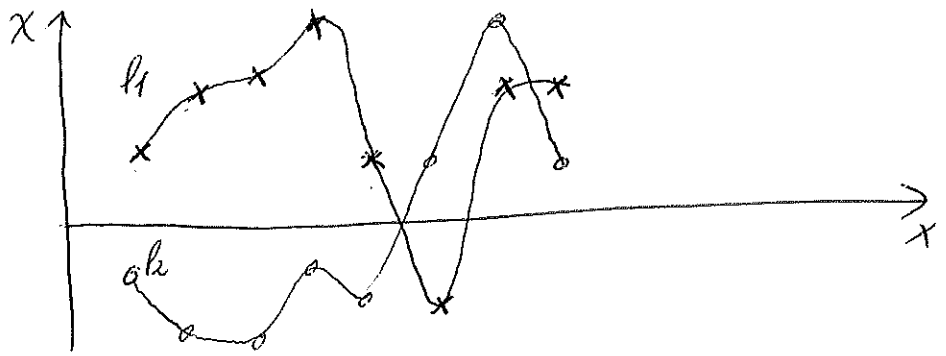
configurations $\{\chi_a(x)\}$

partition function :

$$Z = \sum_{\{\chi_a(x)\}} e^{-\beta H} = \sum_{\{\chi_a(x)\}} e^{-S}$$

$$= \int \mathcal{D}\chi \, e^{-S}$$

functional measure $\int \mathcal{D}\chi$: integral over all functions $\chi_a(x)$



Continuum limit: lattice distance $\Delta \to 0$

φ^4- theory: $S = \int d^D x \left\{ \frac{1}{2} \vec{\nabla}\chi_a \vec{\nabla}\chi_a + m^2 \tilde{\rho} + \frac{1}{2}\lambda \tilde{\rho}^2 \right\}$

$$\tilde{\rho} = \frac{1}{2} \chi_a \chi_a$$

D : space - dimension

classical potential: $U(\tilde{\rho}) = m^2 \tilde{\rho} + \frac{1}{2}\lambda \tilde{\rho}^2$

for $m^2 < 0$: minimum at $\tilde{\rho}_0 \neq 0$, $\tilde{\rho}_0 = -m^2/\lambda$

non- linear σ- models (Heisenberg, Ising):

$$\rho_0 = 1, \quad \lambda \to \infty$$

b) <u>bosonic atoms</u> ~~ultracold~~

quantum statistics, quantum field theory description

$\chi_a(x) = \chi_a(\tau, \vec{x})$, $d = D+1$, τ: euclidean time

$N = 2$:

complex formulation $\chi(x) = \frac{1}{\sqrt{2}}(\chi_1(x) + i\chi_2(x))$

$$\tilde{\rho}(x) = \chi^*(x)\chi(x)$$

* ground state ($T=0$):

$$Z = \int D\chi \, e^{-S} \quad , \quad S = \beta(H - \mu N)$$

$$S = \int d^d x \, \{ \chi^* \partial_\tau \chi + \vec{\nabla} \chi^* \vec{\nabla} \chi - \mu \chi^* \chi$$

chemical potential

$$+ m^2 \chi^* \chi + \frac{1}{2} \lambda (\chi^* \chi)^2 \}$$

energy shift, detuning

interaction, repulsive for $\lambda > 0$

units : $\hbar = 1$, atom mass M, $2M = 1$

* non-zero temperature $T \neq 0$:

momentum space : $(q_0, \vec{q}) \hat{=} (\tau, \vec{x})$

$$q_0 = 2\pi T n, \quad n \in \mathbb{N}$$

$$\int dq_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq_0 = T \sum_{n=-\infty}^{\infty}$$

Matsubara formalism

<u>expectation value of χ</u>

$$\varphi_0 = \langle \chi \rangle = Z^{-1} \int \mathcal{D}\chi \, \chi \, e^{-S}$$

$\varphi_0 \neq 0$: BEC

in principle, φ_0 can depend on x

c) Higgs field

$$S = \int d^4x \left\{ \frac{1}{2} \partial_\mu \chi_a \partial_\mu \chi_a + m^2 \tilde{\rho} + \frac{1}{2} \lambda \tilde{\rho}^2 \right\}$$

$N = 4$, $x^\mu = (\tau, \vec{x})$, $\partial_\mu = \left(\frac{\partial}{\partial \tau}, \vec{\nabla}\right)$, $c = 1$

$\varphi_0 \neq 0$: spontaneous symmetry breaking of $SO(4)$ symmetry

Coupling to gauge fields $\to$ massive gauge fields, Higgs phenomenon

$T \neq 0$: Matsubara formalism as for non-relativistic bosons

d) generating functional for Greens functions

sources $j_a(x)$

$$Z[j] = \int \mathcal{D}\chi \; \exp\left\{ -S + \int_x j_a(x) \chi_a(x) \right\}$$

$$W[j] = \ln Z[j]$$

$$\varphi_a(x) = Z^{-1} \int \mathcal{D}\chi \; \chi_a(x) \, e^{-S + \int j \chi}$$

$$= \frac{\delta W}{\delta j_a(x)}$$

solution $\varphi[j]$

invert $j[\varphi]$

e) <u>effective action</u>

$$\Gamma[\varphi] = -W + \int_x \varphi_a(x)\, j_a(x)$$

insert $j[\varphi]$; Legendre transform

exact quantum field equation:

$$\boxed{\frac{\delta \Gamma}{\delta \varphi_a(x)} = j_a(x)}$$

proof

$$\frac{\delta \Gamma}{\delta \varphi_a(x)} = -\int_y \frac{\delta W}{\delta j_b(y)} \frac{\delta j_b(y)}{\delta \varphi_a(x)} + j_a(x)$$

$$+ \int_y \varphi_b(y) \frac{\delta j_b(y)}{\delta \varphi_a(x)}$$

includes all quantum fluctuations

gravitational field eq., Maxwell eq. are of this type

($j \mathrel{\hat{=}}$ energy momentum tensor, current)

Γ is free energy functional ; Landau theory absence of sources, $j_a(x) \to 0$

$\varphi_{0,a}(x)$ is location of extremum of Γ

(stability: minimum)

<u>symmetries</u> : Γ has same symmetries as S,

e.g. $SO(N)$ symmetry

expansion of Γ in derivatives

$$\Gamma = \int d^d x \left\{ U(\rho) + \frac{1}{2} Z(\rho)\, \partial_\mu \varphi_a \partial_\mu \varphi_a + \dots \right\}$$

$U(\rho)$: effective potential

$\rho = \frac{1}{2} \varphi_a \varphi_a$

typically : minimum of Γ for $\partial_\mu \varphi_a = 0$

$\Rightarrow$ derivative terms do not contribute for minimum

$\Rightarrow$ $\boxed{\varphi_0 \text{ given by minimum of } U}$

(situations with $\partial_\mu \varphi_0 \neq 0$ also possible)

computation of BEC or expectation value of Higgs field :

compute $U(\rho)$!

* $\Gamma[\varphi]$ generating functional for 1PI-Greens functions

Γ known $\Leftrightarrow$ model solved

f) thermodynamics

evaluate Γ at minimum

$$\Gamma_{min}(\mu, T) = \Gamma[\varphi_{0,a}(x); \mu, T]$$

$Z(\mu, T)$: grand canonical partition function

$$\Gamma_{min}(\mu, T) = T^{-1}\Phi_G(\mu, T)$$

Gibbs free energy

$(k_B = 1)$

thermodynamics can be computed from Γ_{min}

2. Perturbation theory. Infrared flow. Functional renormalization

a) background identity for effective action

$$\exp\{-\Gamma[\varphi]\} = e^{W} \cdot \exp\{-\int_x j\varphi\}$$

$$= Z \exp\{-\int_x j\varphi\}$$

$$= \int D\chi \exp\{-S + \int_x j\chi - \int_x j\varphi\}$$

$$\chi = \underset{\substack{\uparrow \\ \text{background field}}}{\varphi} + \underset{\substack{\uparrow \\ \text{fluctuation}}}{\chi'}$$

$$\exp\{-\Gamma[\varphi]\} = \int D\chi' \exp\{-S[\varphi+\chi'] + \frac{\delta\Gamma}{\delta\varphi}\chi'\}$$

* saddle point expansion

$$S[\varphi+\chi'] = S[\varphi] + \int_x \frac{\delta S}{\delta\varphi_a(x)}\chi'_a(x)$$

$$+ \frac{1}{2}\int_{x,y} S^{(2)}_{ab}(x,y)\,\chi'_a(x)\,\chi'_b(y) + \dots$$

$$S^{(2)}_{ab}(x,y) = \frac{\delta^2 S}{\delta\varphi_a(x)\,\delta\varphi_b(y)}$$: matrix of second functional derivatives

b) <u>one loop perturbation theory for effective potential</u>

neglect $S^{(3)}$ etc. , neglect $\int_x \left(\frac{\delta \Gamma}{\delta \varphi} - \frac{\delta S}{\delta \varphi} \right) \chi'$

$\Rightarrow$

$$\Gamma[\varphi] = \underbrace{S[\varphi]}_{\uparrow} - \underbrace{\ln \int \mathcal{D}\chi' \exp\left\{ -\int_{xy} \frac{1}{2} S^{(2)}_{ab}(x,y)\, \chi'_a(x) \chi'_b(y) \right\}}_{\uparrow} \qquad (1)$$

tree contribution = classical action — one loop contribution

$$= S[\varphi] + \Gamma^{(1)}[\varphi]$$

Gauss integral $\int \prod_i dx_i \exp\{-A_{ij} x_i x_j\} = (\det A)^{-1/2} \cdot \text{const.}$

$$\Gamma^{(1)}[\varphi] = -\ln\left(\det S^{(2)}\right)^{-1/2} + \text{const}$$

$$= \frac{1}{2} \ln \det S^{(2)} + \text{const} \qquad ; \qquad \ln \det A = \operatorname{tr} \ln A$$

$$= \frac{1}{2} \operatorname{tr} \underline{\ln S^{(2)}} + \text{const.}$$

- for effective potential: evaluate for constant φ !
- momentum basis

$$\boxed{\cancel{\Gamma^{(1)}[\varphi] = \frac{1}{2} \underbrace{\int \frac{d^d q}{(2\pi)^d}}_{\int_q} \sum_a \left(\ln S^{(2)} \right)_{aa}(q,q)}}$$

$$\operatorname{tr} \to \int_q \sum_a \ , \qquad \int_q = \int \frac{d^d q}{(2\pi)^d}$$

$$S^{(2)}_{ab}(q,q') = \left(q^2 \delta_{ab} + \frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b} \right) (2\pi)^d \delta^d(q-q')$$

$$V = m^2 \rho + \frac{1}{2} \lambda \rho^2$$

$$M^2_{ab} = \frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b} \quad : \text{depends on } \varphi$$

* one loop effective potential

$$U = V + U^{(1)}$$

$$\boxed{U^{(1)} = \frac{1}{2} \int_q \operatorname{Tr} \ln (q^2 + M^2)}$$; Tr : internal trace over index a

$m_a^2(\varphi)$: eigenvalues of M^2

$$U^{(1)} = \frac{1}{2} \sum_a \int_q \ln (q^2 + m_a^2(\varphi))$$

* Computation mass eigenvalues (depends on specific model)

$$\frac{\partial V}{\partial \varphi_a} = V' \varphi_a \quad , \quad V' = \frac{\partial V}{\partial \rho}$$

$$\frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b} = V' \delta_{ab} + V'' \varphi_a \varphi_b$$

take $\varphi_a = \varphi\, \delta_{a1}$, $\rho = \frac{1}{2} \varphi_1^2$

$\Rightarrow M_1^2 = V' + 2\rho V''$, $M_a^2 = V'$ for $a \neq 1$

at minimum of V:

a) $\varphi_0 = 0$: $V' = m^2$

b) $\varphi_0 \neq 0$: $V' = 0$:

$\Rightarrow$ $N-1$ Goldstone Bosons

$$U^{(1)} = \frac{1}{2}\int_q \ln(q^2 + V' + 2\rho V'') + \frac{N-1}{2}\int_q \ln(q^2 + V')$$

$$V' = m^2 + \lambda\rho \quad , \quad V'' = \lambda \quad , \quad V''' = 0$$

c) <u>infrared divergences</u>

derivatives of U

$$U' = m^2 + \lambda\rho + \frac{1}{2}\int_q \frac{1}{q^2 + V' + 2\rho V''} \cdot 3V''$$

$$+ \frac{N-1}{2}\int_q \frac{1}{q^2+V'} \cdot V''$$

$$U'' = \lambda - \frac{1}{2}\int_q (q^2 + V' + 2\rho V'')^{-2}\, 9(V'')^2$$

$$- \frac{N-1}{2}\int_q (q^2+V')^{-2}(V'')^2$$

$$U''' = \int_q (q^2+V'+2\rho V'')^{-3}\, 27(V'')^3 + (N-1)\int_q (q^2+V')^{-3}(V'')^3$$

infrared divergence for Goldstone contribution

$$d=3 : \; U'' \sim \int \frac{d^3q}{q^4} \quad , \quad d=4 : \; U''' \sim \int \frac{d^4q}{q^6}$$

* phase transition as function of m^2

~~SB~~ symmetric phase (SYM) : $\rho_0 = 0$

SSB phase : $\rho_0 > 0$

transition : $\rho_0 = 0$, ~~[illegible]~~ $U'(\rho_0) = 0$

classical approximation $m^2 = 0$, $V = \frac{1}{2}\lambda\rho^2$, $V' = \lambda\rho$

at phase transition: also contribution from „radial mode" φ_1 is infrared divergent

universal critical behavior, fixed points etc. are related to this infrared divergence

d) <u>infrared regulator</u>

strategy: approach the problem stepwise

(can be generalized to many similar problems)

modify ~~a~~ classical action by introducing infrared (IR) regulator

$$\Delta S_k = \frac{1}{2}\int_q \chi_a(q) R_k(q) \chi_a(q)$$

$$q^2 \to q^2 + R_k(q^2)$$

e.g. $q^2 \to 0$: $R_k \sim k^2$

$k^2 \to 0$: $R_k \to 0$

loop contributions are regulated : $(q^2+V')^{-n} \to (q^2+R_k+V')^{-n}$

$$U_k = V + \frac{1}{2} \int_q \ln\left(q^2 + R_k + V' + 2\rho V''\right)$$

$$+ \frac{N-1}{2} \int_q \ln\left(q^2 + R_k + V'\right)$$

d) <u>flow equation</u>

at the end, one wants to remove cutoff, $k \to 0$

flow equation

$$\frac{\partial U_k(\rho)}{\partial k} = \partial_k U_k = \frac{1}{2} \int_q \partial_k R_k \left(q^2 + R_k + V' + 2\rho V''\right)^{-1}$$

$$+ \frac{N-1}{2} \int_q \partial_k R_k \left(q^2 + R_k + V'\right)^{-1}$$

* finite momentum integrals if $\partial_k R_k$ falls off sufficiently fast for large q^2

  e.g. $R_k \sim \exp\left(-\frac{q^2}{k^2}\right)$

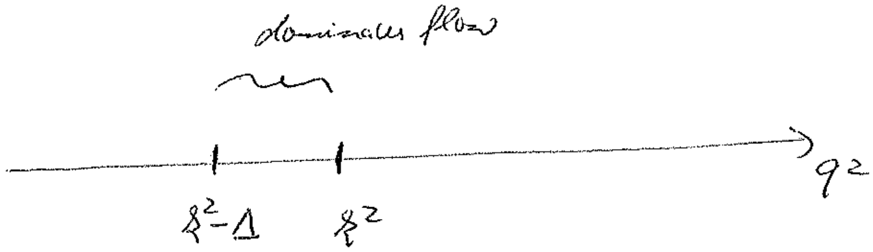
  or $R_k = (k^2 - q^2)\,\Theta(k^2 - q^2)$

  such that

$$q^2 + R_k = \begin{cases} k^2 & \text{for } q^2 < k^2 \\ q^2 & \text{for } q^2 > k^2 \end{cases}$$

$$\partial_k R_k = 0 \quad \text{for } q^2 > k^2$$

momentum integrals dominated by $q^2 \approx k^2$



no ultraviolet problems in flow equation!

e) <u>renormalization group improvement</u>

U_k : effective potential with quantum fluctuations $q^2 > k^2$ included

plays role of classical action for computation of $U_{k'}$, $k' < k$

RG - improvement: replace on r.h.s. of flow equation $V \to U_k$

$$\boxed{\partial_k U_k = \frac{1}{2} \int_q \partial_k R_k \left(q^2 + R_k + U_k' + 2\rho U_k'' \right)^{-1} + \frac{N-1}{2} \int_q \partial_k R_k \left(q^2 + R_k + U_k' \right)^{-1}}$$

* non-linear ~~partial~~ differential equation for function $U_k(\rho)$ (function of two variables k, ρ)

* no dependence of flow equation on V: loss of memory of short distance physics, i.e. $q^2 = \Lambda^2$

* short distance physics influences flow
  only through initial conditions $U_\Lambda = V$

* solution of flow equation does not
  encounter infrared problems;
  can account for universal critical behavior

f) functional renormalization

* flow deals with arbitrary functions $U_k(\rho)$
  "function renormalization"
* can be generalized to flow of effective action $\Gamma_k[\varphi]$.
  "functional renormalization"

3. Exact functional renormalization group equation. Truncations.

a) effective average action or flowing action

* add to action infrared (IR) cutoff piece

$$\Delta_k S = \frac{1}{2} \int_q \chi_a(-q) R_k(q) \chi_a(q)$$

family of theories, one for every value of k

$$W_k[j] = \ln \int \mathcal{D}\chi \exp\left\{ -S - \Delta_k S + \int j\chi \right\}$$

same as before, but with modified action

* Legendre transform

$$\tilde{\Gamma}_k[\varphi] = -W_k + \int j\varphi$$

$$\varphi_a(x) = \frac{\delta W_k}{\delta j_a(x)} \quad ; \quad \text{relation between } \varphi \text{ and } j \text{ depends on } k$$

* effective average action

$$\Gamma_k[\varphi] = \tilde{\Gamma}_k[\varphi] - \frac{1}{2} \int_q \varphi_a(-q) R_k(q) \varphi_a(q)$$

subtract cutoff piece

* matrix notation

$\varphi_a(x)$: vector

$R_k^{ab}(x,y)$: matrix

$$\Delta_k S[\varphi] = \frac{1}{2}\int_{x,y} \varphi_a(x)\, R_k^{ab}(x,y)\, \varphi_b(x)$$

$$\equiv \frac{1}{2}\, \varphi\, R_k\, \varphi$$

$R_k^{ab}(x,y)$: Fourier transform of $R_k^{ab}(q,q') = R_k(q)\,\delta^{ab}\,\delta(q,q')$

$$R_k^{ab}(x,y) = \int_q e^{iq(x-y)}\, R_k(q)\, \delta^{ab}$$

matrix relations hold in arbitrary basis, e.g. position space or momentum space

$$\frac{\delta \Gamma_k}{\delta \varphi} = J - R_k \varphi$$

background identity

$$\exp\{-\Gamma_k[\varphi]\} = \int \mathcal{D}\chi' \exp\left\{-S[\varphi+\chi'] + \frac{\delta\Gamma}{\delta\varphi}\chi' - \frac{1}{2}\chi' R_k \chi'\right\}$$

only IR-cutoff for fluctuations!

* limits

(i) assume $\lim\limits_{k\to\infty} R_k(q) \to \infty$

$$\Rightarrow \lim_{k\to\infty} \Gamma_k[\varphi] = S[\varphi]$$

corrections in saddle point approximation around fluctuations are cut off

(ii) assume $\lim\limits_{k\to 0} R_k(q) = 0$

$$\Rightarrow \lim_{k\to 0} \Gamma_k[\varphi] = \Gamma[\varphi]$$

no IR cutoff, all fluctuations included in quantum effective action

$$\boxed{\text{Flowing action } \Gamma_k[\varphi] \text{ interpolates between microscopic action } S[\varphi] \text{ and macroscopic quantum effective action } \Gamma[\varphi].}$$

In practice: microscopic (UV) scale Λ

instead $k \to \infty$: $k \to \Lambda$

Γ_Λ can be associated with microscopic action

(in principle: first step computes Γ_Λ from S)

* Symmetries of Γ_k:

all symmetries of $S + \Delta_k S$ (in absence of anomalies)
(or Γ_Λ and $\Delta_k S$)
IR-cutoff can violate symmetries (e.g. gauge interactions)

* effective laws

Γ_k encodes effective laws at momentum scale k,
length scale k^{-1}

microscope with variable resolution

fluctuations with $q^2 < k^2$ not yet included

analogue to experiment with finite size of probe $\sim k^{-1}$

"flowing action"

* can also be interpreted as averaging of fields

"effective average action"

b) exact flow equation

$$\boxed{\partial_k \Gamma_k = \frac{1}{2} \operatorname{tr}\{(\Gamma^{(2)} + R_k)^{-1} \partial_k R_k\}}$$

derivation in several steps

i) $\partial_k \tilde{\Gamma}_k [\varphi] = - \partial_k W_k [j]$

proof:

$$\partial_k \tilde{\Gamma}_k|_\varphi = - \partial_k W_k|_j - \int_x \frac{\delta W_k}{\delta j_a(x)} \partial_k j_a(x)|_\varphi + \int_x \varphi_a(x)\, \partial_k j_a(x)|_\varphi$$

ii) $\partial_k \tilde{\Gamma}_k = \frac{1}{2} \langle \chi \partial_k R \chi \rangle = \frac{1}{2} \int_{x,y} \langle \chi_a(x) \partial_k R_k^{ab}(x,y)\, \chi_b(y) \rangle$

proof:

$$- \partial_k W_k [j] = - \partial_k \ln \int D\chi \exp\{-S - \Delta_k S + j\chi\}$$

$$= - \frac{1}{Z} \int D\chi \exp\{-S - \Delta_k S + j\chi\} (-\partial_k \Delta_k S)$$

$$= \partial_k \langle \Delta_k S \rangle$$

only $\Delta_k S$ depends on k!

iii) propagator

$$G_{ab}(x,y) = \langle \chi_a(x)\chi_b(y)\rangle - \langle \chi_a(x)\rangle\langle \chi_b(y)\rangle ,$$

$$\langle \chi_a(x)\chi_b(y)\rangle = G_{ab}(x,y) + \varphi_a(x)\varphi_b(y)$$

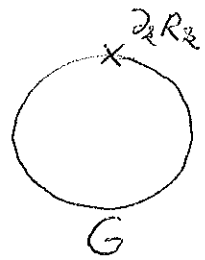
$$G_{ab}(x,y) = G_{ba}(y,x)$$

$$\partial_k \tilde{\Gamma}_k = \frac{1}{2}\int_{xy} \partial_k R_k^{ab}(x,y) G_{ab}(x,y)$$
$$+ \frac{1}{2}\int_{xy} \varphi_a(x)\, \partial_k R_k^{ab}(x,y)\varphi_b(y)$$

$\Longrightarrow$

$$\partial_k \Gamma_k = \frac{1}{2}\int_{x,y} G_{ab}(x,y)\,\partial_k R_k^{ba}(y,x)$$

$$= \frac{1}{2}\,\mathrm{tr}\left\{ G\, \partial_k R_k \right\}$$



iv) propagator and $\Gamma^{(2)}$

$$G\,\tilde{\Gamma}^{(2)}_k = 1$$

$$G\,(\Gamma^{(2)}_k + R_k) = 1$$

$$G = (\Gamma^{(2)}_k + R_k)^{-1}$$

proof

$$G_{ab}(x,y) = \frac{\delta^2 W_k}{\delta J_a(x)\,\delta J_b(y)}$$

$$= \frac{\delta}{\delta J_a(x)}\left(\frac{1}{Z}\frac{\delta Z}{\delta J_b(y)}\right)$$

$$= \frac{1}{Z}\frac{\delta^2 Z}{\delta J_a(x)\delta J_b(y)} - \frac{1}{Z^2}\frac{\delta Z}{\delta J_a(x)}\frac{\delta Z}{\delta J_b(y)}$$

$$= \langle \chi_a(x)\,\chi_b(y)\rangle - \langle \chi_a(x)\rangle\langle \chi_b(y)\rangle$$

recall $Z = \int \mathcal{D}\chi \exp\{-S - \Delta_k S + J\chi\}$

$$G\,\tilde{\Gamma}_k^{(2)} = \int_y \frac{\delta^2 W}{\delta j_a(x)\,\delta j_b(y)} \; \frac{\delta^2 \tilde{\Gamma}_k}{\delta \varphi_b(y)\,\delta \varphi_c(z)}$$

$$= \int_y \frac{\delta \varphi_b(y)}{\delta j_a(x)} \; \frac{\delta j_c(z)}{\delta \varphi_b(y)} = \frac{\delta j_c(z)}{\delta j_a(x)} = \delta_{ac}\,\delta(x-z)$$

$$= 1$$

This concludes derivation of exact flow equation.

No approximation.

All non-perturbative effects included.

Particular form of matrix R_k not important:

only *quadratic* form of IR cutoff matters

Generalization to many other problems!

c) renormalization group improved one loop equation

formal expression

$$\partial_k \Gamma_k = \frac{1}{2} \tilde{\partial}_k \operatorname{tr} \ln \left(\Gamma^{(2)} + R_k \right)$$

$$\tilde{\partial}_k = \partial_k R_k \frac{\partial}{\partial R_k}$$

recall one loop formula

$$\Gamma^{(1)} = \frac{1}{2} \operatorname{tr} \ln S^{(2)}$$

close correspondence of Feynman graphs

more vertices, since Γ has richer structure than S

two important advantages :

- momentum integrals in loop expression are finite
- no higher loops needed

(higher loop effects can be reproduced by iterative solution of flow equation)

e) derivative expansion

* expand $\Gamma_k[\varphi]$ in number of derivatives (~~...~~ theory with $SO(N)$ sym.)

$$\Gamma_k = \int_x \Big\{ U_k(\rho) + \frac{1}{2} Z_k(\rho)\partial_\mu \varphi_a(x)\partial_\mu \varphi_a(x)$$

$$+ \frac{1}{4} Y_k(\rho)\partial_\mu \rho\, \partial_\mu \rho + O(\partial^4) \Big\}$$

$$\rho = \frac{1}{2}\varphi_a\varphi_a$$

first order derivative expansion: three functions U, Z, Y

simplify further : $Y_k = 0$, Z_k independent of ρ

" leading potential approximation "

* flow equation for effective potential $U_k(\rho)$

evaluate Γ_k for φ independent of x (derivative terms vanish)

needed $\Gamma_k^{(2)}$;

momentum space $\left(\Gamma_k^{(2)}\right)_{ab}(q,q') = \left(Zq^2\delta_{ab} + \frac{\partial^2 U}{\partial\varphi_a\partial\varphi_b}\right)(2\pi)^d\delta^d(q-q')$

(similar to $S^{(2)}$ in Lecture 2)

$$\boxed{\partial_k U = \frac{1}{2}\int_q \partial_k R_k \left(Zq^2 + R_k + U' + 2\rho U''\right)^{-1} + \frac{N-1}{2}\int_q \partial_k R_k \left(Zq^2 + R_k + U'\right)^{-1}}$$

Compare with exact flow equation for U

$$\partial_k U = \frac{1}{2}\int_q \partial_k R_k \left(\tilde{Z}(\rho, q^2) q^2 + R_k + U' + 2\rho U'' \right)^{-1}$$
$$+ \frac{N-1}{2}\int_q \partial_k R_k \left(Z(\rho, q^2) q^2 + R_k + U' \right)^{-1}$$

k③

* choose cutoff function

$$R_k = Z_k (k^2 - q^2)\, \Theta(k^2 - q^2)$$

$$Z_k q^2 + R_k = Z_k \begin{cases} q^2 & \text{for } q^2 > k^2 \\ k^2 & \text{for } q^2 < k^2 \end{cases}$$

$\eta = - k\, \partial_k \ln Z_k$: anomalous dimension (small)

neglect η in $\partial_k R_k$:

$$\partial_k R_k = 2 Z_k k\, \Theta(k^2 - q^2) \quad ; \quad (\partial_k \Theta(k^2 - q^2)\text{-term vanishes})$$

$\Rightarrow$

$$k\partial_k U = \partial_t U = \frac{1}{2}\int_{q^2 < k^2} \frac{2 Z_k k^2}{Z_k k^2 + U' + 2\rho U''} + \frac{N-1}{2} \dots$$

$$w_1 = \frac{U' + 2\rho U''}{Z_k k^2} \;,\quad w_2 = \frac{U'}{Z_k k^2} \;,\quad \int_{q^2 < k^2} = \alpha_d k^d$$

$$\boxed{\partial_t U = \alpha_d k^d \left\{ \frac{1}{1 + w_1} + \frac{N-1}{1 + w_2} \right\}}$$

$$\alpha_2 = \frac{1}{4\pi} \;,\quad \alpha_3 = \frac{1}{6\pi^2} \;,\quad \alpha_4 = \frac{1}{32\pi^2} \;,\quad \left(v_d = \frac{d}{4}\alpha_d \right)$$

f) flow of effective potential ~~in~~ for N-component scale theory in arbitrary dimension

* scaling form of flow equation

eliminate explicite dependence ~~of~~ on scale k, and on Z_k

$$u_k = \frac{U_k}{k^d}$$

renormalized dimensionless fields: $\tilde{\rho} = Z_k k^{2-d} \rho$

$$\kappa = Z_k k^{2-d} \rho_0$$

$$u' = \frac{\partial u}{\partial \tilde{\rho}} = k^{-d} \frac{\partial U}{\partial \rho} \frac{\partial \rho}{\partial \tilde{\rho}} = \frac{1}{Z_k k^2} U' = w_2$$

$$u'' = \frac{1}{Z_k k^2} \frac{k^{d-2}}{Z_k} U''$$

$$\tilde{\rho} u'' = \frac{1}{Z_k k^2} \rho U'' \quad \Rightarrow \quad w_1 = u' + 2\tilde{\rho} u''$$

$$\partial_t u_{|\rho} = \alpha_d \left(\frac{N-1}{1+u'} + \frac{1}{1+u'+2\tilde{\rho}u''} \right) - d u$$

wanted: k-dependence at fixed $\tilde{\rho}$

$$\partial_t u_{|\tilde{\rho}} = \partial_t u_{|\rho} + \frac{\partial u}{\partial \rho} \partial_t \rho_{|\tilde{\rho}}$$

$$= \partial_t u_{|\rho} + u' Z_k k^{2-d} (d-2+\eta) \rho$$

$$\boxed{\begin{aligned}\partial_t u_{|\tilde\rho} &= -du + (d-2+\eta)\tilde\rho u' \\ &\quad + \alpha_d \left(\frac{N-1}{1+u'} + \frac{1}{1+u'+2\tilde\rho u''} \right)\end{aligned}}$$

scaling form: no k, no Z_k

* scaling solution

$$\partial_t u_{|\tilde\rho} = 0$$

simultaneous fixed point for κ, $u''(\kappa)$, $u'''(\kappa)$ etc

fixed point for infinitely many couplings

no scale present, all dimensionful couplings scale with appropriate powers of k

example for approximative scaling solution

$$u_* = \frac{1}{2}\lambda_* (\tilde\rho - \kappa_*)^2$$

(i) fixed κ_*

$$\rho_0 = \frac{k^{d-2}\kappa_*}{Z_k} \quad ; \quad d=3 : \rho_0 \sim k$$

(ii) fixed λ_*

$$\bar\lambda = U_k''(\rho_0) = Z_k^2 k^{4-d} \lambda_* \quad ; \quad d=3 : \bar\lambda \sim k$$

solution of IR-divergence for $\bar\lambda$ in $d=3$!

fixed point for λ_k in $d=3$

Wilson-Fisher fixed point

no fixed point or scaling solution in $d=4$ (triviality)

4. Results and generalizations

a) unified picture of scalar theories in arbitrary dimensions

consider simple truncation for u

$$u = \frac{1}{2}\lambda(\tilde{\rho}-\kappa)^2$$

$$u' = \lambda(\tilde{\rho}-\kappa), \quad u'' = \lambda, \quad u''' = 0$$

$$\partial_t u = -du + (d-2+\eta)\tilde{\rho}u' + \alpha_d\left(\frac{N-1}{1+u'} + \frac{1}{1+u'+2\tilde{\rho}u''}\right)$$

evolution of κ | $\kappa = Z_k k^{2-d}\rho_0$

$u'(\kappa) = 0$ for all t

$$\partial_t u' + u''\partial_t\kappa = 0, \quad \partial_t\kappa = -\frac{1}{\lambda}\partial_t u'(\kappa)$$

evolution of $\lambda = u''(\kappa)$

$$\partial_t\lambda = \partial_t u''(\kappa)$$

needed: $\partial_t u'$, $\partial_t u''$: derivatives of flow equation

$$\partial_t u' = (-2+\eta)u' + (d-2+\eta)\tilde{\rho}u''$$

$$-\alpha_d u''\left(\frac{N-1}{(1+u')^2} + \frac{3}{(1+u'+2\tilde{\rho}u'')^2}\right)$$

$$\partial_t u'' = (d-4+2\eta)u'' + 2\alpha_d (u'')^2 \left(\frac{N-1}{(1+u')^3} + \frac{9}{(1+u'+2\tilde{\rho}u'')^3} \right)$$

$u' \to 0$, $u'' \to \lambda$ for $\partial_t \kappa$, $\partial_t \lambda$

$$\partial_t \kappa = -\frac{1}{\lambda} \left\{ (d-2+\eta)\lambda\kappa - \alpha_d \lambda \left(N-1 + \frac{3}{(1+2\lambda\kappa)^2} \right) \right\}$$

$$\boxed{\begin{aligned} \partial_t \kappa &= -(d-2+\eta)\kappa + \alpha_d \left(N-1 + \frac{3}{(1+2\lambda\kappa)} \right) \\ \partial_t \lambda &= (d-4+2\eta)\lambda + 2\alpha_d \lambda^2 \left(N-1 + \frac{9}{(1+2\lambda\kappa)^3} \right) \end{aligned}}$$

one needs to compute η from flow of kinetic term

this computation is omitted in this lecture

for given $\eta(\lambda\kappa)$: two coupled nonlinear

differential equations for λ and κ

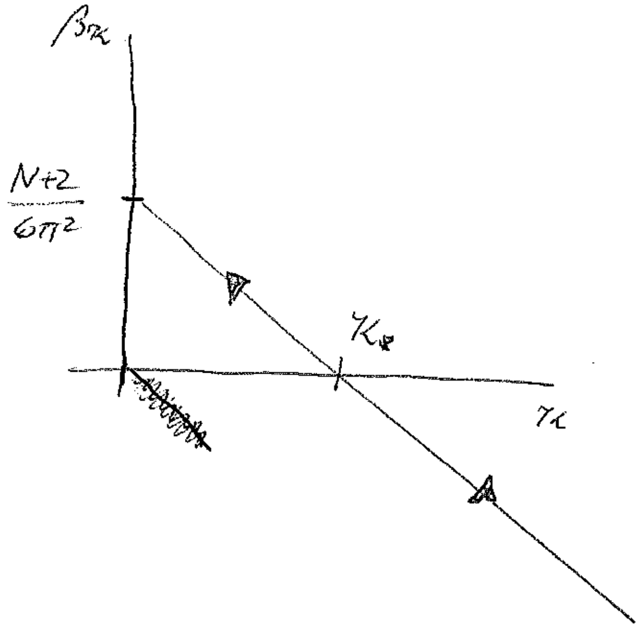
$$\partial_t \kappa = \beta_\kappa (\kappa, \lambda)$$

$$\partial_t \lambda = \beta_\lambda (\kappa, \lambda)$$

$\boxed{d=3}$

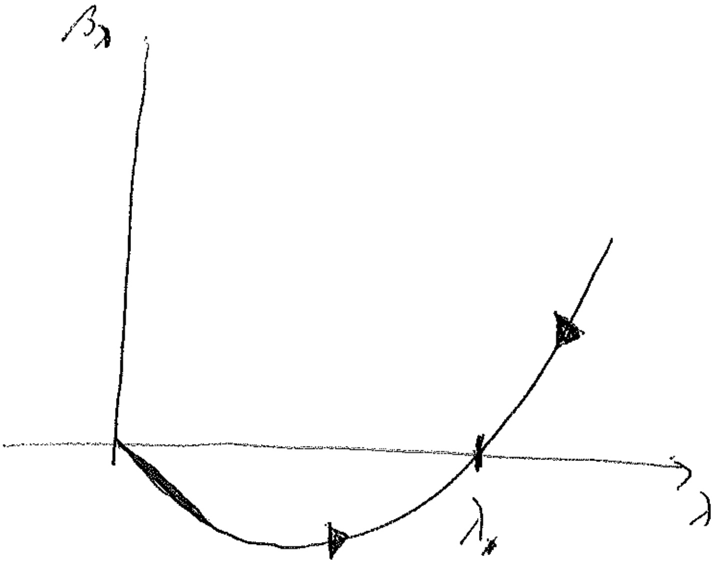
$$\partial_t \kappa = -(1+\eta)\kappa + \frac{1}{6\pi^2}\left(N-1+\frac{3}{(1+2\lambda\kappa)^2}\right)$$

$$\partial_t \lambda = -(1-2\eta)\lambda + \frac{\lambda^2}{3\pi^2}\left(N-1+\frac{9}{(1+2\lambda\kappa)^3}\right)$$



arrows: flow towards smaller k

IR - unstable



IR - stable

Fixed point (κ_*, λ_*): Wilson-Fisher fixed point

★ Small deviations from fixed point ($\lambda_\Lambda = \lambda_*$ for simplicity)

a) $\kappa_\Lambda > \kappa_*$

κ gets large

$$\partial_t \kappa = -(1+\eta)\kappa$$

$$\kappa = \kappa_0 \, k^{-(1+\eta)}$$

$$\rho_0 = \frac{k}{Z_k}\kappa \quad , \quad Z_k \sim k^{-\eta} \qquad (\eta = -\partial_t \ln Z_k)$$

$\Rightarrow$ $\rho_0 \to$ const ; spontaneous symmetry breaking

SSB

b) $\kappa_\Lambda < \kappa_*$

κ runs to zero (then $u = \tilde{m}^2 \tilde{\rho} + \frac{1}{2}\lambda \tilde{\rho}^2$)

minimum of U at $\rho_0 = 0$

symmetric phase, SYM

c) $\kappa_\Lambda = \kappa_*$: $\rho_0 \sim k^{(1+\eta)} \kappa_* \to 0$

phase transition for $\kappa_\Lambda = \kappa_*$ ($\lambda_\Lambda = \lambda_*$)

$$\kappa_\Lambda = \kappa_* + \delta\kappa_\Lambda$$

$\delta\kappa_\Lambda \to 0 \;\Rightarrow\; \rho_0 \to 0 \;\Rightarrow$ second order phase transition

b) critical phenomena

Classical statistics in thermal equilibrium

$$\delta\chi_\Lambda = a(T_c - T)$$

Compute $\rho_0(\delta\chi_\Lambda)$ (for SSB)

find $\rho_0 \sim (T_c - T)^{\beta/2}$

critical exponent β

* other critical exponents

$$G(q^2) = (q^2)^{-1+\frac{\eta}{2}}$$: correlation fct. for $T = T_c$

$$\xi \sim |T - T_c|^{-\nu}$$: correlation length

$$\xi = \frac{1}{m_R^2} = \frac{Z}{\bar{m}^2}$$ (SYM), $U = \bar{m}^2\rho + \dots$

Computation

η : from scaling solution

β, ν : plot $\frac{\partial \ln \rho_0}{\partial \ln \delta\chi_\Lambda}$, $\frac{\partial \ln m_R^2}{\partial \ln |\delta\chi_\Lambda|}$ for $k = 0$

$\boxed{d=4}$

$\eta \sim \lambda^2$ for small λ : neglect

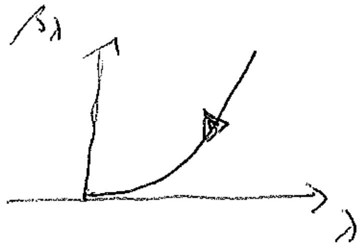
$$\partial_t \kappa = -2\kappa + \frac{1}{32\pi^2}\left(N-1 + \frac{3}{(1+2\lambda\kappa)^2}\right)$$

$$\partial_t \lambda = \frac{\lambda^2}{16\pi^2}\left(N-1 + \frac{9}{(1+2\lambda\kappa)^3}\right)$$

only fixed point for $\lambda_* = 0$: Gaussian fixed point

„φ^4-theory is trivial"

one loop perturbation theory : $\partial_t \lambda = \dfrac{(N+8)\lambda^2}{16\pi^2}$



strong coupling λ gets small very fast

$$\partial_t \lambda^{-1} = -\frac{1}{16\pi^2}\left(N-1+\dots\right)$$

---

phase transition with trivial critical exponents

(up to tiny corrections since $\lambda \neq 0$, small runs very slowly)

1812.4447

standard model of particle physics close
to phase transition; small value ρ_0 / M_p^2 needs
explanation; but no fine tuning problem
in renormalization group improved perturbation theory
(additional symmetry at phase transition: dilatation symmetry —
approximative since still slow running)

in presence of gauge and Yukawa couplings:
infrared interval instead of fixed point

lower end of IR-interval favored: $M_H = 126$ GeV

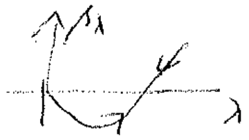
$\boxed{d=2}$

<u>for large κ:</u> $\eta = \frac{1}{4\pi\kappa} + O(\kappa^{-2})$, $\alpha_2 = \frac{1}{4\pi}$

$$\partial_t \kappa = \beta_\kappa = \frac{N-2}{4\pi} + O(\kappa^{-1})$$

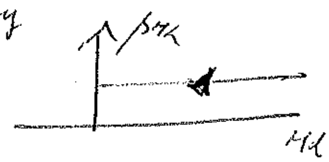
$$\partial_t \lambda = -2\lambda + \frac{\lambda^2}{2\pi}(N-1) + O(\kappa^{-1})$$

IR-fixed point λ_*



no fixed point for κ for $N > 2$

no phase transition in $d=2$ for $SO(N)$ symmetric scalar theories; no spontaneous symmetry breaking; Mermin-Wagner theorem



non-linear σ-model: $g^2 = \frac{1}{\kappa}$ $(N \geq 3)$

$$\partial_t \kappa^{-1} = -\frac{N-2}{4\pi}\kappa^{-2} \mathrel{\widehat{=}} \partial_t g^2 = -\frac{N-2}{4\pi} g^4$$

asymptotic freedom!

scale where g^2 gets large $\widehat{=}$ κ gets small

symmetric phase of $SO(N)$-theory

k_{SSB} given by $\varkappa(k_{SSB}) = 0$

$$m_R^2 = C\, k_{SSB}^2$$

$$\varkappa(k) = \varkappa(\Lambda) - \frac{N-2}{4\pi} \ln \frac{\Lambda}{k}$$

$$\frac{N-2}{4\pi} \ln \frac{\Lambda}{k_{SSB}} = \varkappa_\Lambda = \frac{1}{g_\Lambda^2}$$

$$\ln \frac{\Lambda}{k_{SSB}} = \frac{4\pi}{(N-2)\, g_\Lambda^2} \quad , \quad k_{SSB} = \Lambda \exp\left\{ - \frac{4\pi}{(N-2)\, g_\Lambda^2} \right\}$$

$$m_R^2 = C^2 \Lambda^2 \exp\left\{ - \frac{8\pi}{(N-2)\, g_\Lambda^2} \right\}$$

↑ non-perturbative effect in non-linear σ-model

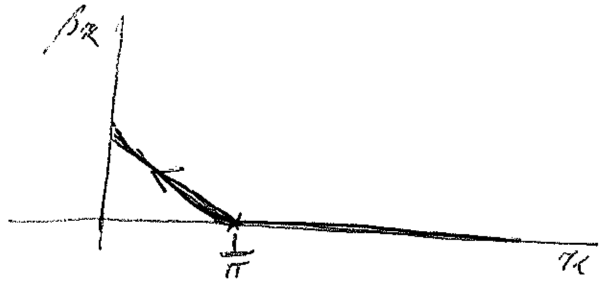
Non-perturbative region of non-linear non-abelian σ-model finds simple explanation of symmetric phase of linear σ-model

(duality : large $g^2 \Leftrightarrow$ small $\varkappa$

small $g^2 \Leftrightarrow$ large $\varkappa$)

* Kosterlitz-Thouless phase transition (BEC)

$N=2, \; d=2$



$$\rho_\kappa \sim \frac{1}{\nu}(\kappa_0 - \kappa)^{3/2}$$

phase transition!

SSB - phase $\quad \kappa(k=0) \neq 0$, 1 Goldstone boson

$$\rho_0 = Z^{-1}\kappa_0$$

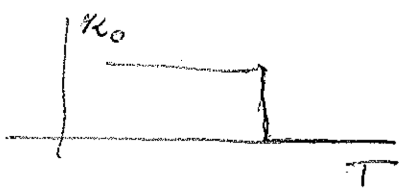
$$Z \sim k^{-\eta}, \quad \eta = \frac{1}{4\pi\kappa} > 0$$

$$Z(k \to 0) \to \infty \Rightarrow \rho_0 \to 0$$

(in accordance with Mermin-Wagner theorem)

* anomalous dimension depends on $T_c - T$: $\eta \sim \frac{1}{4\pi\kappa_\Lambda}$

* jump in κ at phase transition (jump in superfluid density)



* essential scaling in SYM - phase

$$m_R \sim \exp\left\{-\frac{b}{\sqrt{T-T_c}}\right\}$$