Gauge invariant flow equation

Gauge invariant effective action

- Variation yields gravitational field equations
- Generates Maxwell equations for QED

  (+ correction terms)

- Involves only one gauge field

Gauge invariant effective action

- Projected second functional derivative is inverse propagator for physical fluctuations
- Generates correlation function for physical fluctuations
- Example : power spectrum of cosmic fluctuations

- Involves only one gauge field

Background field formalism

- Background field formalism employs effective action with two gauge fields :
- expectation value of microscopic field : variation yields field equation and inverse propagator
- background field appears in gauge fixing

- effective action is invariant under simultaneous gauge transformation of both gauge fields

Field identification

Identify expectation value and background field :

- Effective action depends only on one gauge field
- Variation does not yield field equation and inverse propagator
- Second functional derivative has zero modes :

  gauge modes

Exact flow equation for effective average action

has been formulated in

background field formalism

M.Reuter, CW

Flow equation in background field formalism

Shortcomings:

- Either two gauge fields
- Or no gauge invariance

- Extended truncations necessary
- Or use of modified Slavnov Taylor identities or background field identities
- No closed gauge invariant flow equation with single gauge field

Aim : closed gauge invariant flow equation with single gauge field

problem :

- second functional derivative of gauge invariant effective action cannot generate inverse propagator
- second functional derivative is not invertible
- gauge modes !

Projection on physical fluctuations

$P = P^2$ projects on physical fluctuations

1-P projects on gauge fluctuations

$$\delta_\xi \bar{g} = (1 - P)\delta_\xi \bar{g}$$

Gauge fluctuations correspond to change of gauge field $\bar{g}$ under infinitesimal gauge fluctuation

Gauge invariant flow equation involves projection on physical fluctuations

$$k\partial_k \bar{\Gamma} = \zeta_k = \pi_k + \delta_k - \epsilon_k$$

$$\pi_k = \frac{1}{2} Str(k\partial_k \bar{R}_P G_P)$$

G_P : propagator for physical fluctuations

$$PG_P = G_P P^T = G_P$$

$$\delta_\xi \bar{g} = (1-P)\delta_\xi \bar{g}$$

measure contributions $\delta_k - \epsilon_k$ do not depend on effective action

Closed flow equation

projection on physical fluctuations makes second functional derivative invertible

$$\bar{\Gamma}_P^{(2)} = P^T \bar{\Gamma}^{(2)} P \qquad \bar{\Gamma}^{(2)ij} = \frac{\partial^2 \bar{\Gamma}}{\partial \bar{g}_i \partial \bar{g}_j}$$

$$\left(\bar{\Gamma}_P^{(2)} + \bar{R}_P \right) G_P = P^T$$

Projection in practice

- Not very complicated

  In flat space or Robertson Walker background :
- Metric fluctuations can be directly decomposed into physical fluctuations and gauge fluctuations
- Gauge invariant effective action involves physical fluctuations
- Physical gauge fixing involves only gauge fluctuations
- Second functional derivative is block diagonal
- Invert different blocks sepately

Measure terms in flow equation

$$k\partial_k \Gamma(g,\bar{g}) = \frac{1}{2} S\mathrm{tr}(k\partial_k \bar{R} G_P) + \delta_k - \epsilon_k$$

Gauge fixing contribution involves fixed differential operator Q used for gauge fixing

$$\delta_k = \frac{1}{2}\mathrm{tr}\{k\partial_k R_{gf}(1-P)(\bar{Q}+R_{gf})^{-1}(1-P^T)\}$$

does not involve effective action

Regularized Faddeev-Popov determinant (ghost contribution)

$$\epsilon_k = 2\delta_k .$$

Measure terms in flow equation

$$\delta_k - \epsilon_k = -\frac{1}{2}\mathrm{tr}\Big\{k\partial_k R_{gf}(\mathcal{D}_S)\big(\mathcal{D}_S + R_{gf}(\mathcal{D}_S)\big)^{-1}\Big\}$$

$$\begin{aligned}\delta_k &= \frac{1}{2}\,\mathrm{tr}\,\Big\{k\partial_k R_{\mathrm{gf}}(\bar{Q})\big(\bar{Q} + R_{\mathrm{gf}}(\bar{Q})\big)^{-1}\Big\}\\ &= \frac{1}{2}\,\mathrm{tr}\,\Big\{k\partial_k R_{\mathrm{gf}}(\mathcal{D}_S)\big(\mathcal{D}_S + R_{\mathrm{gf}}(\mathcal{D}_S)\big)^{-1}\Big\}\end{aligned}$$

$$\begin{aligned}M &= \det\big(-D^\mu(\bar{A})D_\mu(A')\big)\\ &= \det\big(\mathcal{D}_S + iD^\mu(A'_\mu - \bar{A}_\mu)\big)\end{aligned}$$

$$M_k = M(g', \bar{g})\, E_k(\bar{g})$$

$$E_k = \frac{\det\big(\mathcal{D}_S + R_{\mathrm{gf}}(\mathcal{D}_S)\big)}{\mathrm{Det}(\mathcal{D}_S)}.$$

$$\epsilon_k(\bar{g}) = \mathrm{tr}\,\big\{\ln k\partial_k E_k(\bar{g})\big\}$$

much simpler than the use of effective action for ghosts…

Gauge invariant metric

Fluctuations

$$h = g - \bar{g} = f + a$$

gauge fluctuations $a = \big(1 - P(\bar{g})\big)h$ $P(\bar{g})a = 0$

gauge transformation at fixed macroscopic gauge field

$$\hat{\delta}h = \bar{P}(\bar{g}+h)\delta_\xi(\bar{g}+h) = \bar{P}(\bar{g})\delta_\xi(\bar{g}) + \delta_h h$$

$$\bar{P} = 1 - P.$$

$$\delta_{\text{inh}}a = \bar{P}(\bar{g})\delta_\xi\bar{g}$$

physical fluctuations $f = P(\bar{g})h$

Yang – Mills theories

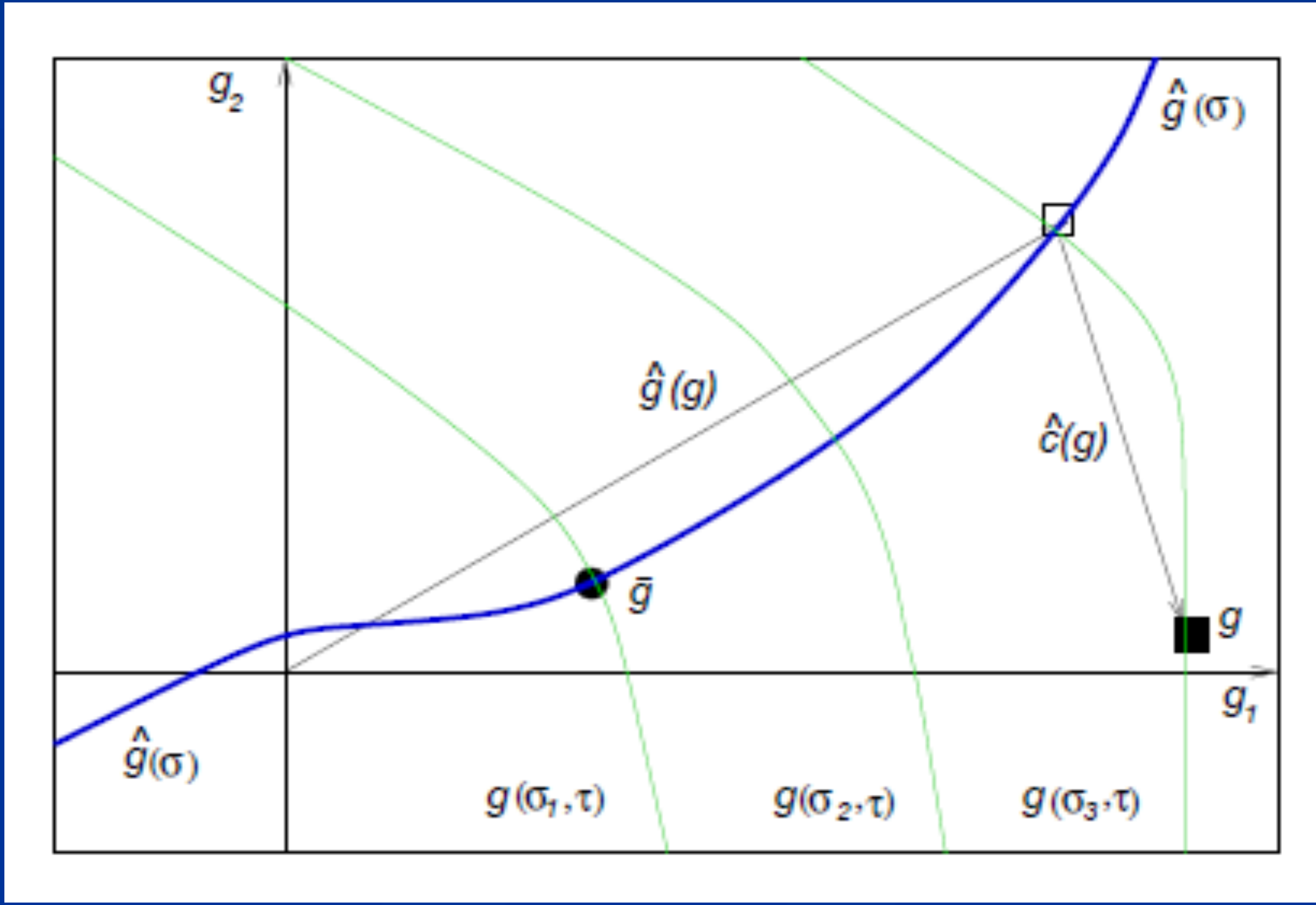
$$P_\mu{}^\nu = \delta_\mu^\nu - \bar{P}_\mu{}^\nu$$

$$\bar{P}_\mu{}^\nu = D_\mu D^{-2} D^\nu$$

projector depends on gauge field

gravity more complicated

Physical fluctuations generate gauge invariant fields no global projector



Gauge invariant fields and gauge invariant effective action

Physical gauge field $\hat{g}(g)$

Gauge invariant effective action $\bar{\Gamma}(g) = \bar{\Gamma}(\hat{g}(g))$

Projection properties :

$$\frac{\partial \hat{g}_k}{\partial g_i} P_i^{\ j}(g) = \frac{\partial \hat{g}_k}{\partial g_j}$$

$$\frac{\partial \bar{\Gamma}}{\partial g_k} = \frac{\partial \bar{\Gamma}}{\partial \hat{g}_j} \frac{\partial \hat{g}_j}{\partial g_k} = \frac{\partial \bar{\Gamma}}{\partial \hat{g}_j} \frac{\partial \hat{g}_j}{\partial g_i} P_i^k = \frac{\partial \bar{\Gamma}}{\partial g_i} P_i^k$$

Gauge invariance of effective action

$$\frac{\partial \bar{\Gamma}}{\partial \bar{g}} = \frac{\partial \bar{\Gamma}}{\partial \bar{g}} P$$

$$\delta_\xi \bar{\Gamma} = \frac{\partial \bar{\Gamma}}{\partial \bar{g}} P \delta_\xi \bar{g} = 0$$

$$\delta_\xi \bar{g} = (1 - P) \delta_\xi \bar{g}$$

Functional integral for flow equation

Functional integral

$$Z(L,\bar{g}) \;=\; \int \mathcal{D}g' \, M_k(g',\bar{g}) \exp\{-S(g') - S_{gf}(g',\bar{g}) - \Delta S_k(g',\bar{g}) + L^T g'\},$$

$$M_k = M(g',\bar{g}) E_k(\bar{g})$$

$$\epsilon_k(\bar{g}) = \mathrm{tr}\{\ln k\partial_k E_k(\bar{g})\}$$

$$\Delta S_k(g',\bar{g}) = \frac{1}{2}(g'-\bar{g})^T R_k(\bar{g})(g'-\bar{g})$$

$$W(L,\bar{g}) \;=\; \ln Z(L,\bar{g})$$

Effective action and exact flow equation

$$\tilde{\Gamma}(g,\bar{g}) = -W(L,\bar{g}) + L^T g$$

$$g = \frac{\partial W(L,\bar{g})}{\partial L} = \langle g' \rangle$$

$$L = \frac{\partial \tilde{\Gamma}(g,\bar{g})}{\partial g}$$

$$\Gamma_k(g,\bar{g}) = \tilde{\Gamma}_k(g,\bar{g}) - \frac{1}{2} h^T R_k(\bar{g}) h$$

$$\partial_k \Gamma = \frac{1}{2} S\mathrm{tr}(\partial_k R G) - \epsilon_k$$

$$G = (\Gamma^{(2)} + R)^{-1}$$

Reuter, CW

Physical gauge fixing for quantum gravity

$$S_{gf} = \frac{1}{2\alpha} \int_x \sqrt{\bar{g}} \left(D^\mu h'_{\mu\nu} \right)^2 \ , \ h'_{\mu\nu} = g'_{\mu\nu} - \bar{g}_{\mu\nu}$$

Projected propagator in flow equation

$$R_k = \bar{R}_k(\bar{g}) + \frac{1}{\alpha}\left(1 - P^T(\bar{g})\right)R_{k,gf}(\bar{g})\left(1 - P(\bar{g})\right)$$

$$\begin{aligned}\tilde{\Gamma}^{(2)} &= \hat{\Gamma}^{(2)} + \bar{R}_k \\ &\quad + \frac{1}{\alpha}(1 - P^T)(\bar{Q} + R_{gf})(1 - P)\end{aligned}$$

$$G = G_P + \alpha\Delta G \ , \ G_P = PGP^T$$

$$\tilde{\Gamma}^{(2)}_P G_P = P^T \ , \ \tilde{\Gamma}^{(2)}_P = P^T(\hat{\Gamma}^{(2)} + \bar{R})P$$

For α→0 flow equation decomposes into projected parts

Macroscopic gauge field instead of background field

- Projector , gauge fixing and infrared cutoff are all formulated with dynamical macroscopic gauge field $\bar{g}$
- Macroscopic gauge field replaces fixed background field
- Implicit definition of functional integral and effective action

Expectation value

- Macroscopic gauge field is not identical with expectation value of microscopic gauge field

- Expectation value of microscopic gauge field g can be expressed in terms of macroscopic gauge field $\bar{g}$

- Precise expression is not important

Implicit definition of effective action

$$\bar{\Gamma}[A] = -\ln\left(\int \mathcal{D}A' \prod_z \delta(G^z)\,\tilde{M} \times \exp\left\{-S[A'] + \int_x \frac{\partial\bar{\Gamma}}{\partial A^z_\mu}\,(A'^z_\mu - \hat{A}^z_\mu)\right\}\right)$$

$$\delta(G^z) = \lim_{\alpha\to 0} \exp\left\{-\frac{1}{2\alpha}\int_x (G^z)^2\right\}$$

Implicit definition of effective average action

$$\tilde{\Gamma}_k[A] = -\ln \int \mathcal{D}A'\, E_k(A) \qquad (28$$
$$\times \exp\left\{-(\tilde{S} + \Delta S_k)[A', A] + \int_x \frac{\partial \tilde{\Gamma}_k}{\partial A^z_\mu}\,(A'^z_\mu - A^z_\mu)\right\}$$

$$\bar{\Gamma}[A] = -\ln\Big(\int \mathcal{D}A' \prod_z \delta(G^z)\,\tilde{M}$$
$$\times \exp\Big\{-S[A'] + \int_x \frac{\partial \bar{\Gamma}}{\partial A^z_\mu}\,(A'^z_\mu - \hat{A}^z_\mu)\Big\}\Big)$$

$$\tilde{S} = S' + S_{\mathrm{gf}} - \ln M$$

$$A'_\mu = \hat{A}_\mu + b'_\mu + c'_\mu$$

$$\Delta S_k = \frac{1}{2}\int_x b'^\mu_z \,(R_k^{(\mathrm{ph})})^{z\nu}_{\mu y}\, b'^y_\nu + \frac{1}{2\alpha}\int_x c'^\mu_z\,(R_k^{(\mathrm{g})})^{z\nu}_{\mu y}\, c'^y_\nu$$

Subtract pieces such that

$$\partial_t \Gamma_k = \pi_k + \delta_k - \epsilon_k\,,$$
$$\pi_k = \frac{1}{2}\int_x \partial_t\,(R_k^{(\mathrm{ph})})^{z\nu}_{\mu y}\,\langle b'^y_\nu b'^\mu_z\rangle_c$$
$$\delta_k = \frac{1}{2\alpha}\int_x \partial_t\,(R_k^{(\mathrm{g})})^{z\nu}_{\mu y}\,\langle c'^y_\nu c'^\mu_z\rangle_c$$

$$\langle b'^y_\nu b'^\mu_z\rangle_c = (\tilde{G}_P)^{y\mu}_{\nu z}$$

Choice of macroscopic gauge field

Use freedom in precise definition of macroscopic gauge field in order to obtain simple expression of physical propagator in terms of gauge invariant effective action

$$\tilde{G}_P = G_P$$

$$\left(\bar{\Gamma}^{(2)}_P + \bar{R}_P\right) G_P = P^T$$

$$\bar{\Gamma}^{(2)}_P = P^T \bar{\Gamma}^{(2)} P$$

$$\bar{\Gamma}^{(2)ij} = \frac{\partial^2 \bar{\Gamma}}{\partial \bar{g}_i \partial \bar{g}_j}$$

Optimize choice of macroscopic field and effective action

$$\tilde{\Gamma}_k[A] = -\ln \int \mathcal{D}A' \, E_k(A) \qquad (28$$

$$\times \exp\left\{ -(\tilde{S} + \Delta S_k)[A', A] + \int_x \frac{\partial \tilde{\Gamma}_k}{\partial A^z_\mu} (A'^z_\mu - A^z_\mu) \right\}$$

modify

procedure

Logic : accept more complicated definition of

effective average action and

precise relation between macroscopic field and

expectation value of microscopic field

in order to obtain :

simple relations for physical correlation function

and flow equation

Gauge invariant flow equation involves projection on physical fluctuations

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$$\pi_k = \frac{1}{2} Str(k\partial_k \bar{R}_P G_P)$$

G_P : propagator for physical fluctuations

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measure contributions $\delta_k - \epsilon_k$ do not depend on effective action

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projection on physical fluctuations makes second functional derivative invertible

$$\bar{\Gamma}_P^{(2)} = P^T \bar{\Gamma}^{(2)} P \qquad \bar{\Gamma}^{(2)ij} = \frac{\partial^2 \bar{\Gamma}}{\partial \bar{g}_i \partial \bar{g}_j}$$

$$\left(\bar{\Gamma}_P^{(2)} + \bar{R}_P \right) G_P = P^T$$

Measure terms in flow equation

$$k\partial_k\Gamma(g,\bar{g}) = \frac{1}{2}S\mathrm{tr}(k\partial_k\bar{R}G_P) + \delta_k - \epsilon_k$$

$$\delta_k = \frac{1}{2}\mathrm{tr}\{k\partial_k R_{gf}(1-P)(\bar{Q}+R_{gf})^{-1}(1-P^T)\}$$

does not involve effective action

$$\epsilon_k = 2\delta_k.$$

Conclusion

- Closed gauge invariant flow equation involving only one gauge field exists.
- Is effective action local enough to admit simple truncations ?

Particular gauge fixing for quantum gravity

$$S_{gf} = \frac{1}{2\alpha} \int_x \sqrt{\bar{g}} \left(D^\mu h'_{\mu\nu} \right)^2 \ , \ h'_{\mu\nu} = g'_{\mu\nu} - \bar{g}_{\mu\nu}$$