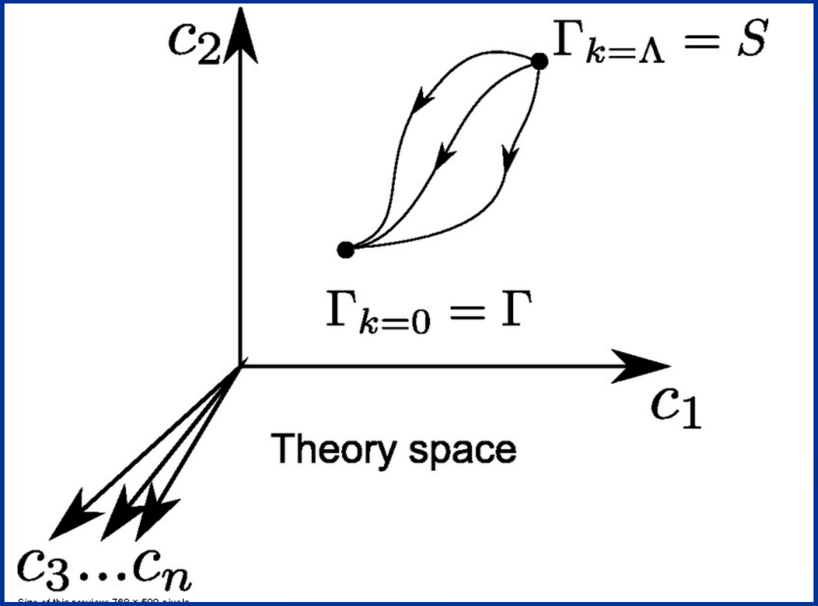
Functional renormalisation: a bridge from microphysics to macrophysics



Functional renormalisation: from microphysical laws to macrophysical complexity

Functional integral

Wide applications:

- partition function in statistical physics
- whenever you deal with a probability distribution
- also more general complex weight distributions

Microphysics

Formulated as partition function or functional integral

Microphysical laws are encoded in classical action S (microphysical action, related to Hamiltonian)

weight factor in probability distribution e^{-S}

atomic interactions, quantum gravity,
standard model of particle physics, …

Macrophysics

Landau type theories for relevant degrees of freedom

extract properties from variation of effective action :

field equations, correlation function, 1PI-vertices

superconductors, superfluidity …

Macroscopic understanding does not need all details of underlying microscopic physics

1) motion of planets : $\mathbf{m_i}$

Newtonian mechanics of point particles

probabilistic atoms → deterministic planets

2) thermodynamics : T, μ, Gibbs free energy $\mathbf{J(T,\mu)}$

3) antiferromagnetic waves for correlated electrons

$\mathbf{\Gamma[\, s_i(x)]}$

How to get from microphysics to macrophysics ?

1) motion of planets : m_i

compute or measure mass of objects

(second order more complicated : tides etc.)

2) thermodynamics : $J(T, \mu)$

integrate out degrees of freedom

3) antiferromagnetic waves for correlated electrons

$\Gamma[s_i(x)]$ change degrees of freedom *central role of fluctuations*

Classical and effective action

- classical action : microscopic laws

- quantum effective action : macroscopic laws

  includes all fluctuation effects

  (quantum, thermal, whatsoever…)

  field equations are exact

  Landau type theory

  generates 1PI- correlation functions

Effective action

$$\exp(-\Gamma[\tilde{\varphi}]) = \int D\tilde{\chi} \exp\left\{-S[\tilde{\varphi}+\tilde{\chi}] + \int_x \frac{\partial \Gamma}{\partial \tilde{\varphi}} \tilde{\chi}\right\}$$

Field equations

- The field equations we use for electromagnetism, gravity, or superfluidity are macroscopic equations.
- They obtain by variation of the effective action, not the microscopic action.
- "Classical field theory" is exact, but only with macroscopic field equations

Emergence of macroscopic laws
with
Functional Renormalisation

Do it stepwise : functional renormalisation

Leo Kadanoff   Kenneth Wilson   Franz Wegner

Scale dependent effective action

- average effective action, flowing effective action
- introduces momentum scale k by an
  infrared cutoff
- all fluctuations with momenta larger k are included
- fluctuations with momenta smaller k are not yet included

effective laws at scale k

From

Microscopic Laws

(Interactions, classical action)

to

Fluctuations!

Macroscopic Observation

(Free energy functional, effective action)

Exact renormalisation group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}$$

'92

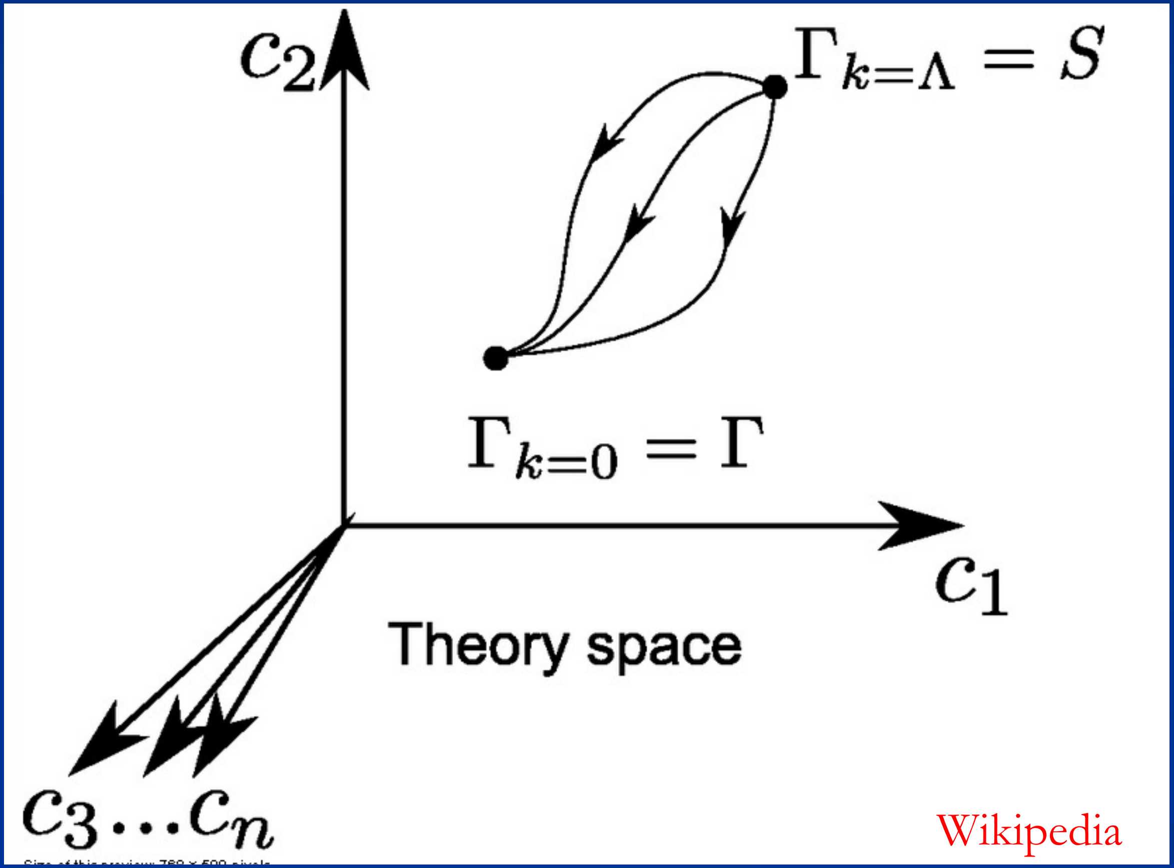
$$\left(\Gamma_k^{(2)}\right)_{ab}(q,q') = \frac{\delta^2\Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$$

$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$

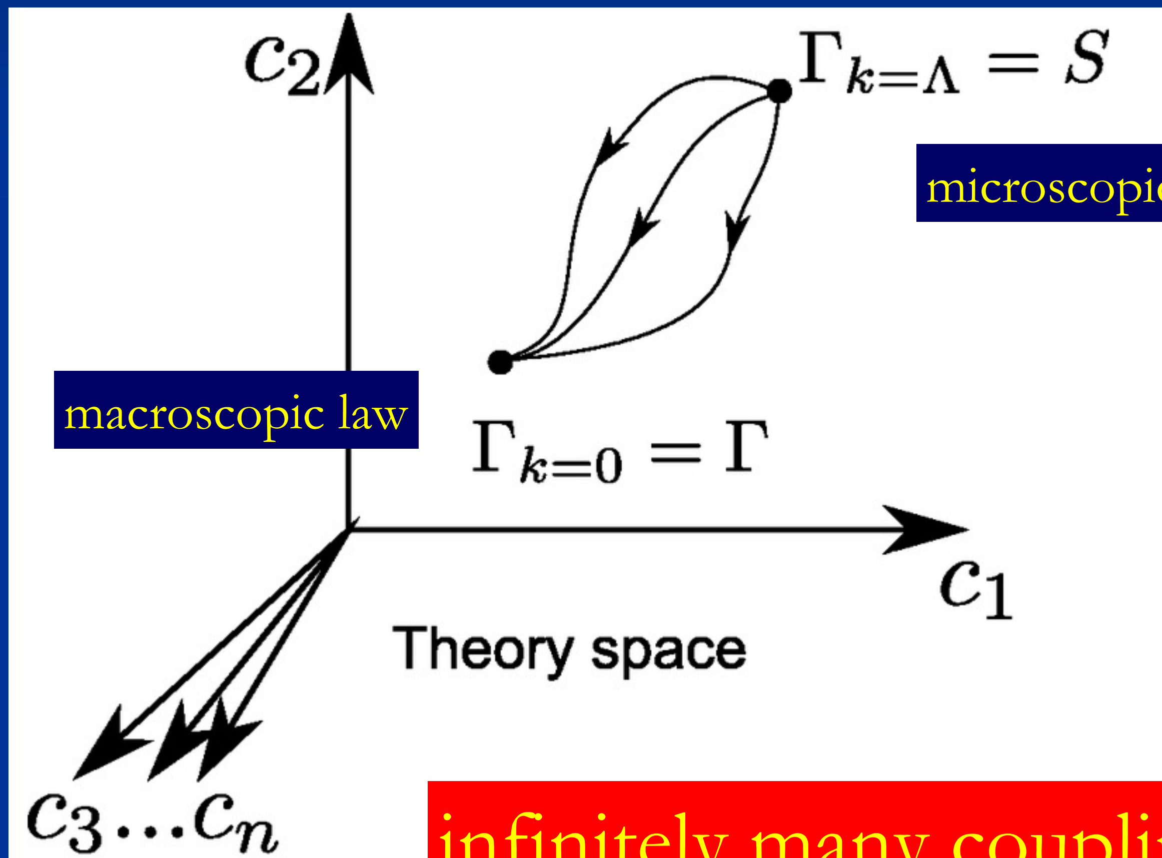
(fermions : STr)

R_k : cutoff function
does not affect
high momentum fluctuations
cuts off
"infrared fluctuations"

Flowing action



Flowing action



microscopic law

macroscopic law

infinitely many couplings

Effective potential

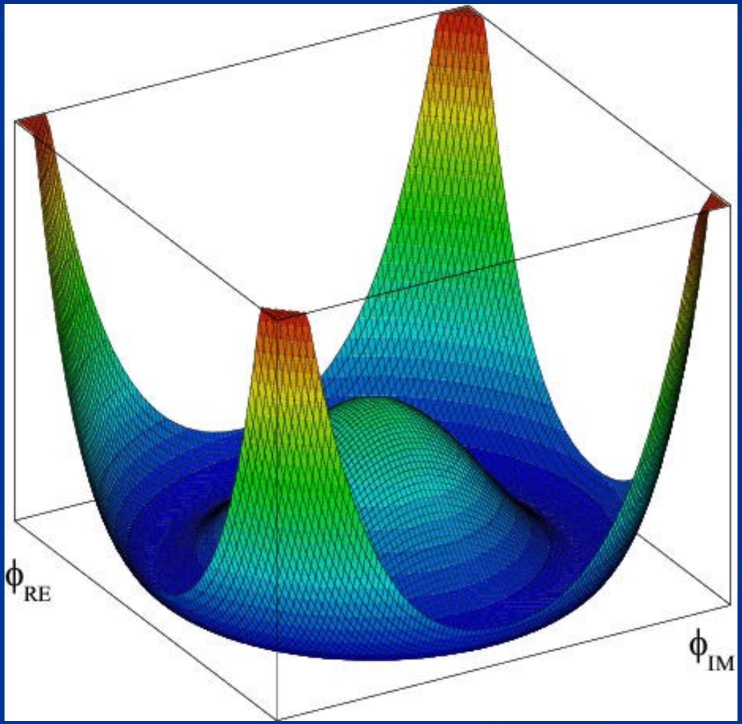
Effective potential

=

non – derivative

part of

effective action



Effective potential includes all fluctuations

Average potential U_k

$\equiv$ **scale dependent effective potential**
$\equiv$ **coarse grained free energy**

Only fluctuations with momenta $q^2 > k^2$ included

k: infrared cutoff for fluctuations, "average scale"

Λ: characteristic scale for microphysics

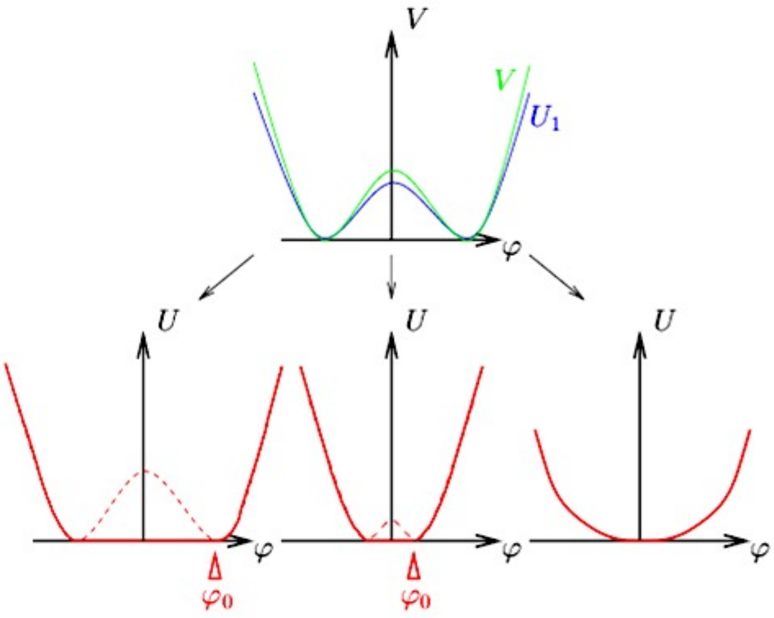
$$U_\Lambda \approx S \to U_0 \equiv U$$

Scalar field theory

$\varphi_a(x)$: magnetization, density, chemical concentration, Higgs field, meson field, inflaton, cosmon

$O(N)$-symmetry:

$$S = \int d^d x \left\{ \frac{1}{2} \partial_\mu \varphi_a \partial_\mu \varphi_a + V(\rho) \right\}; \quad \rho = \frac{1}{2} \varphi_a \varphi_a$$



Flow equation for average potential

cutoff

Flow equation for U_k

$$\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

propagator with cutoff

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

R_k : IR-cutoff

e.g $R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$

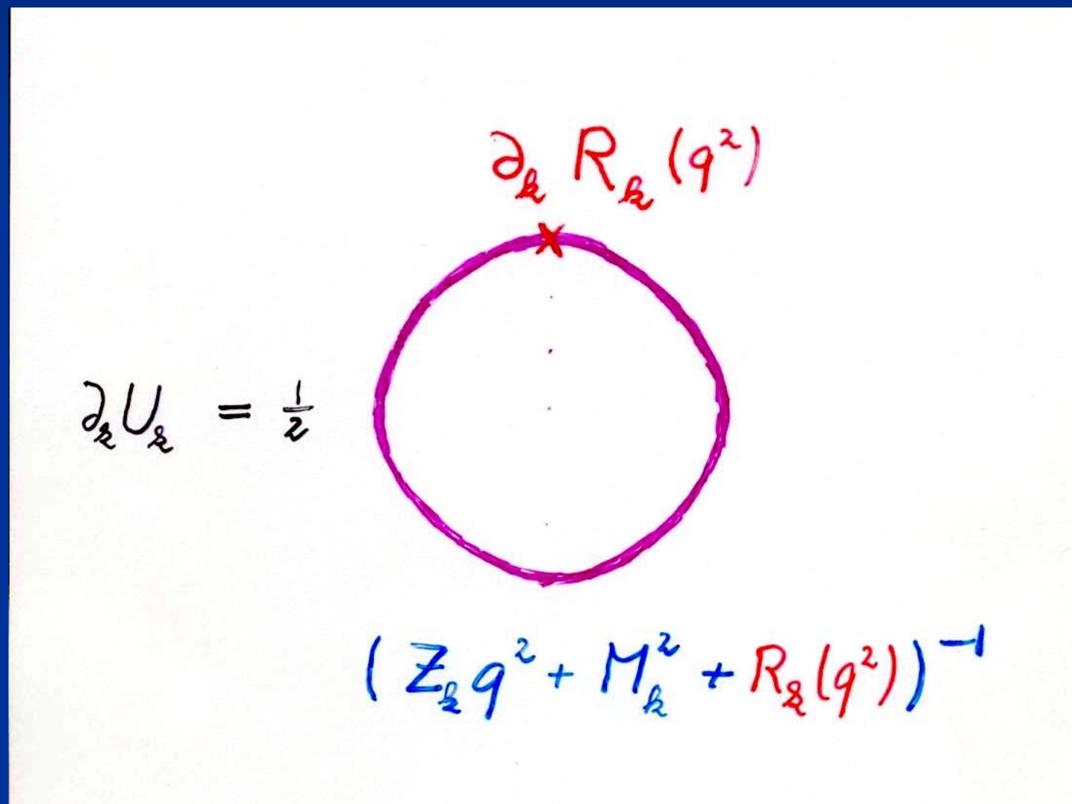
or $R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$ (Litim)

$\lim_{k\text{-}}$ $\lim_{k\to 0} R_k = 0$

$\lim_{k\text{-}}$ $\lim_{k\to\infty} R_k \to \infty$

2

Simple one loop structure –nevertheless (almost) exact



$$\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Simple differential equation for O(N) – models , dimension d

$$\partial_t u|_{\tilde\rho} = -du + (d-2+\eta)\tilde\rho u' + 2v_d\{l_0^d(u' + 2\tilde\rho u'';\eta) + (N-1)\, l_0^d(u';\eta)\}$$

$$u = \frac{U_k}{k^d}$$
$$\tilde\rho = Z_k k^{2-d}\rho$$
$$u' = \frac{\partial u}{\partial \tilde\rho} \quad \text{etc.}$$

linear cutoff:

$$l_0^d(w;\eta) = \frac{2}{d}\left(1 - \frac{\eta}{d+2}\right)\frac{1}{1+w}$$

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)$$

t = ln(k)

Wave function renormalization and anomalous dimension

Z_k: wave function renormalization

$$k\partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

for $Z_k(\varphi, q^2)$: flow equation is **exact !**

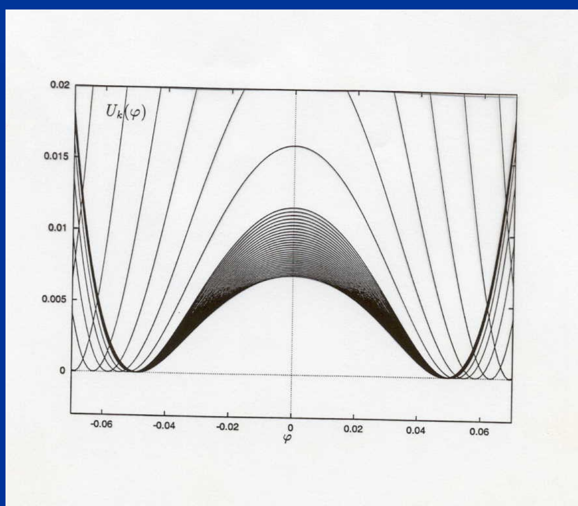
unified approach

- choose N
- choose d
- choose initial form of potential
- run !

unified description of scalar models for all d and N

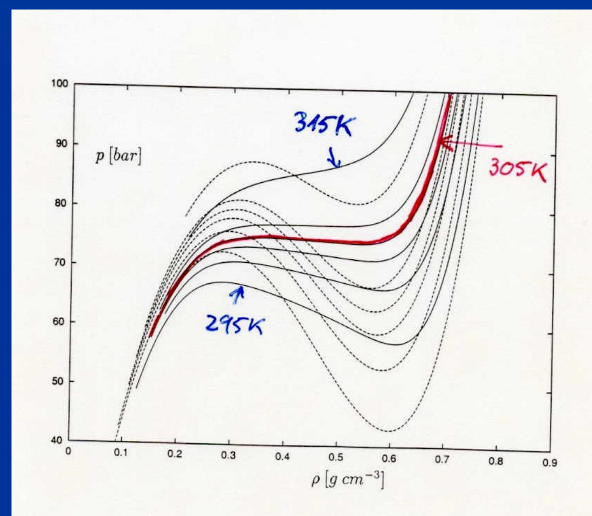
Flow of effective potential

Ising model



S.Seide …

CO_2



Experiment :

T_* =304.15 K
p_* =73.8.bar
ρ_* = 0.442 g cm-2

Critical exponents

$d = 3$

Critical exponents ν and η

N	ν		η	
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
10	0.881	0.886	0.028	0.025
100	0.990	0.980	0.0030	0.003
		↑		↑

"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

Critical exponents , d=3

N	ν ERGE	ν world	η ERGE	η world
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
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"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

More sophisticated approximations

Correlation-length exponent ν										
N	LPA	DE_2	DE_4	DE_6	LPA″	BMW	MC	PT	ϵ-exp	CB
0	0.5925	0.5879(13)	0.5876(2)	–	–	0.589	0.58759700(40)	0.5882(11)	0.5874(3)	0.5876(12)
1	0.650	0.6308(27)	0.62989(25)	0.63012(16)	0.631	0.632	0.63002(10)	0.6304(13)	0.6292(5)	0.629971(4)
2	0.7090	0.6725(52)	0.6716(6)	–	0.679	0.674	0.67169(7)	0.6703(15)	0.6690(10)	0.6718(1)
3	0.7620	0.7125(71)	0.7114(9)	–	0.725	0.715	0.7112(5)	0.7073(35)	0.7059(20)	0.7120(23)
4	0.805	0.749(8)	0.7478(9)	–	0.765	0.754	0.7477(8)	0.741(6)	0.7397(35)	0.7472(87)

Anomalous dimension η									
N	DE_2	DE_4	DE_6	LPA″	BMW	MC	PT	ϵ-exp	CB
0	0.0326(47)	0.0312(9)	–	–	0.034	0.0310434(30)	0.0284(25)	0.0310(7)	0.0282(4)
1	0.0387(55)	0.0362(12)	0.0361(11)	0.0506	0.039	0.03627(10)	0.0335(25)	0.0362(6)	0.0362978(20)
2	0.0410(59)	0.0380(13)	–	0.0491	0.041	0.03810(8)	0.0354(25)	0.0380(6)	0.03818(4)
3	0.0408(58)	0.0376(13)	–	0.0459	0.040	0.0375(5)	0.0355(25)	0.0378(5)	0.0385(13)
4	0.0389(56)	0.0360(12)	–	0.0420	0.038	0.0360(4)	0.0350(45)	0.0366(4)	0.0378(32)

Correction-to-scaling exponent ω								
N	LPA	DE_2	DE_4	BMW	MC	PT	ϵ-exp	CB
0	0.66	1.00(19)	0.901(24)	0.83	0.899(14)	0.812(16)	0.841(13)	–
1	0.654	0.870(55)	0.832(14)	0.78	0.832(6)	0.799(11)	0.820(7)	0.82968(23)
2	0.672	0.798(34)	0.791(8)	0.75	0.789(4)	0.789(11)	0.804(3)	0.794(8)
3	0.702	0.754(34)	0.769(11)	0.73	0.773	0.782(13)	0.795(7)	0.791(22)
4	0.737	0.731(34)	0.761(12)	0.72	0.765	0.774(20)	0.794(9)	0.817(30)

The nonperturbative functional renormalization group and its applications

N. Dupuis[a], L. Canet[b], A. Eichhorn[c,d], W. Metzner[e], J. M. Pawlowski[d,f], M. Tissier[a], N. Wschebor[g]

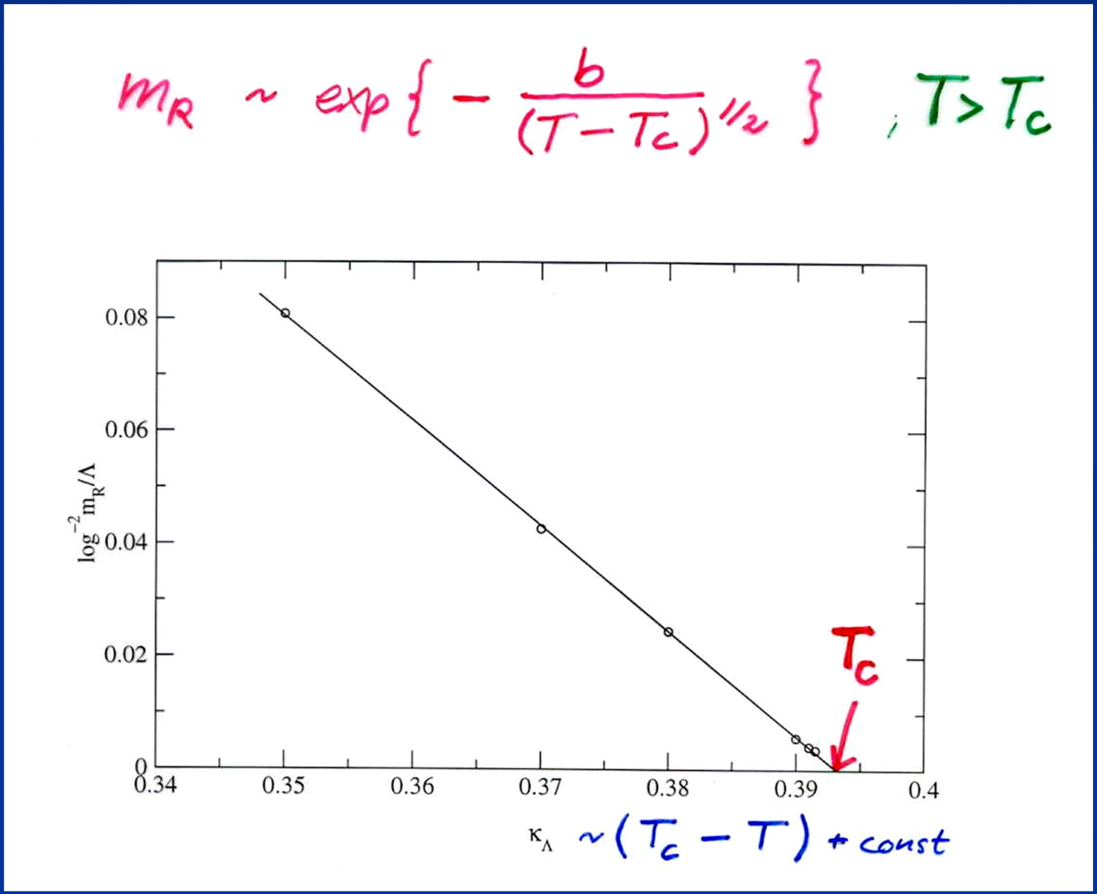
Solution of partial differential equation :

yields highly nontrivial non-perturbative results despite the one loop structure !

Example:

Kosterlitz-Thouless phase transition

Essential scaling : d=2, N=2

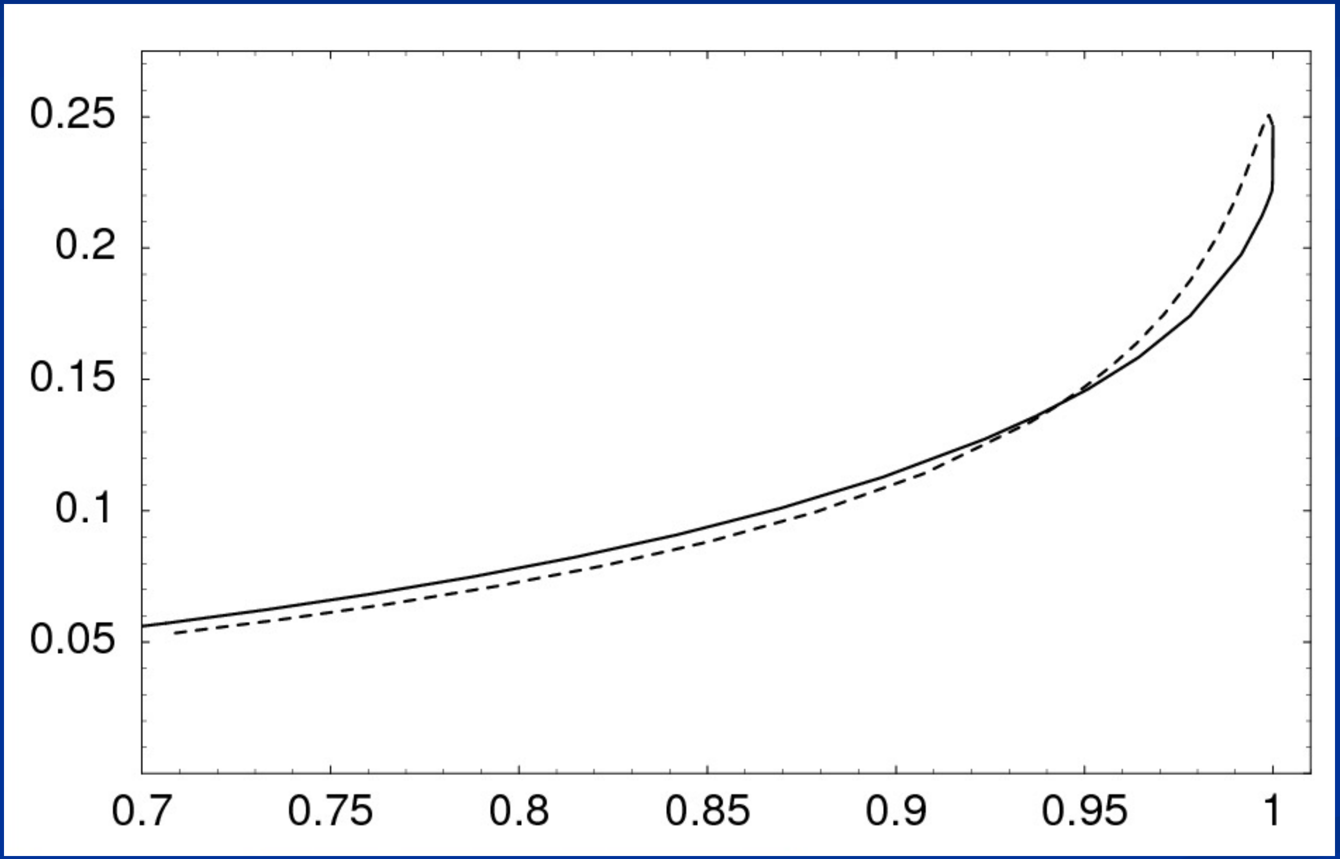


- Flow equation contains correctly the non-perturbative information !
- (essential scaling usually described by vortices)

Kosterlitz-Thouless phase transition (d=2, N=2)

Correct description of phase with
Goldstone boson
(infinite correlation length)
for $T<T_c$

Temperature dependent anomalous dimension η



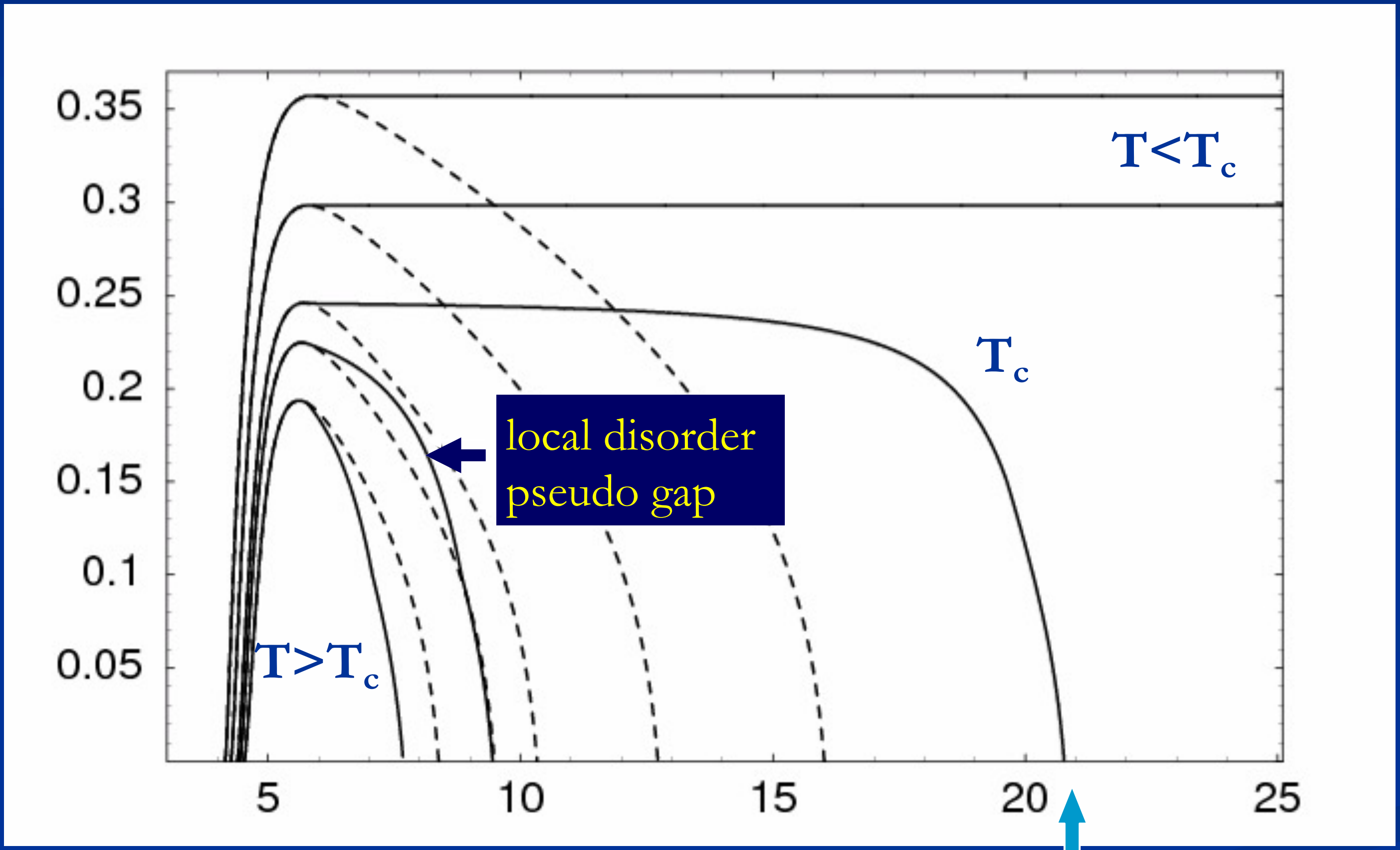
η

T/T_c

Running renormalized d-wave superconducting order parameter $\varkappa$ in doped Hubbard (-type) model

$\varkappa$

location of minimum of u

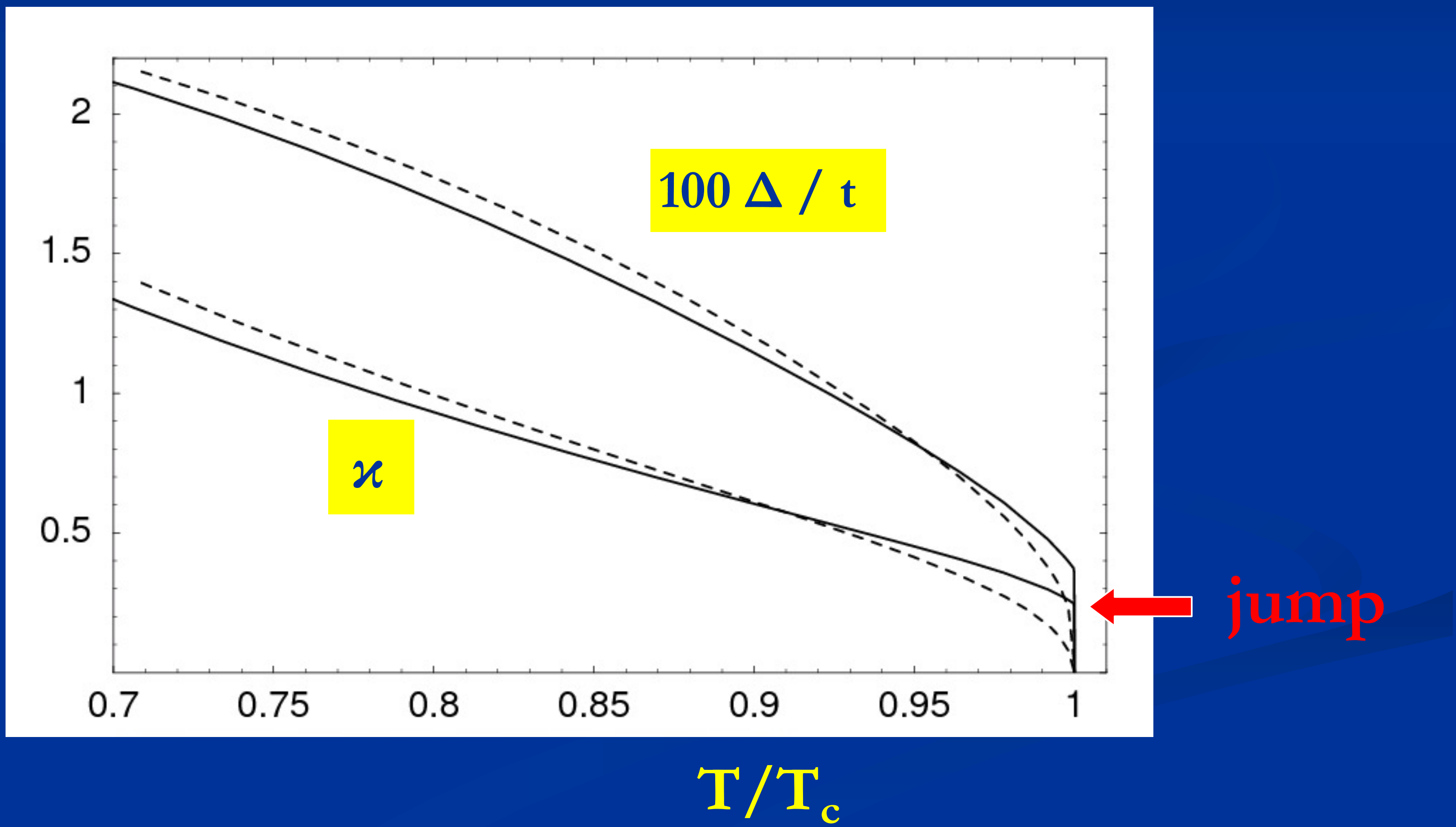


C.Krahl,…

- ln (k/Λ)

macroscopic scale 1 cm

Renormalized order parameter $\varkappa$ and gap in electron propagator Δ in doped Hubbard model



Anti-ferromagnetic and superconducting order in the Hubbard model

A functional renormalization group study

T. Baier, E. Bick,
C. Krahl, J. Mueller, S. Friederich, …

Action for Hubbard model

$$
\begin{aligned}
S \;=\; & \sum_{Q} \hat{\psi}^{\dagger}(Q)[i\omega_Q + \xi_Q]\hat{\psi}(Q) \\
& + \frac{U}{2} \sum_{K_1,K_2,K_3,K_4} [\hat{\psi}^{\dagger}(K_1)\hat{\psi}(K_2)]\,[\hat{\psi}^{\dagger}(K_3)\hat{\psi}(K_4)] \\
& \qquad \times \delta\,(K_1 - K_2 + K_3 - K_4)\ ,
\end{aligned}
$$

$$
\hat{\psi}(Q) = \left(\hat{\psi}_{\uparrow}(Q), \hat{\psi}_{\downarrow}(Q)\right)^{T}
$$

$$
\xi(\mathbf{q}) = -\mu - 2t(\cos q_x + \cos q_y) - 4t' \cos q_x \cos q_y
$$

$$
\sum_{Q} = T \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{d^2 q}{(2\pi)^2}\,,
$$

$$
\delta(Q - Q') = T^{-1}\delta_{n,n'}(2\pi)^2\delta^{(2)}(\mathbf{q} - \mathbf{q}')
$$

Truncation for flowing action

$$\Gamma_k[\chi] = \Gamma_{F,k} + \Gamma_{Fm,k} + \Gamma_{F\rho,k} + \Gamma_{Fs,k} + \Gamma_{Fd,k} + \Gamma_{a,k} + \Gamma_{\rho,k} + \Gamma_{s,k} + \Gamma_{d,k} + \sum_X U_{B,k}(\mathbf{a}, \rho, s, d)$$

$$\Gamma_F = \Gamma_{F\mathrm{kin}} + \Gamma_F^U$$

$$\Gamma_{F\mathrm{kin}} = \sum_Q \psi^\dagger(Q) P_F(Q) \psi(Q)$$

$$P_F(Q) = Z_F(\omega_Q)\left(i\omega_Q + \xi(\mathbf{q})\right)$$

$$\Gamma_F^U = \frac{1}{2} \sum_{K_1,K_2,K_3,K_4} U\,\delta\,(K_1 - K_2 + K_3 - K_4) \times [\psi^\dagger(K_1)\psi(K_2)]\,[\psi^\dagger(K_3)\psi(K_4)]\,.$$

Additional bosonic fields

- anti-ferromagnetic
- charge density wave
- s-wave superconducting
- d-wave superconducting

initial values for flow : bosons are decoupled auxiliary fields (microscopic action)

Effective potential for bosons

$$
\begin{aligned}
\sum_X U_B(\mathbf{a},\rho,s,d) = & \sum_Q \frac{1}{2}\left(\bar{m}_a^2\,\mathbf{a}^T(-Q)\mathbf{a}(Q) + \bar{m}_\rho^2\,\rho(-Q)\rho(Q)\right) \\
& + \bar{m}_s^2\, s^*(Q)s(Q) + \bar{m}_d^2\, d^*(Q)d(Q) \\
& + \frac{1}{2}\sum_{Q_1,Q_2,Q_3,Q_4} \delta\,(Q_1+Q_2+Q_3+Q_4) \\
& \times\left(\bar{\lambda}_a\,\alpha(Q_1,Q_2)\alpha(Q_3,Q_4)\right. \qquad (23) \\
& + \bar{\lambda}_d\,\delta(Q_1,Q_2)\delta(Q_3,Q_4) \\
& \left. + 2\bar{\lambda}_{ad}\,\alpha(Q_1,Q_2)\delta(Q_3,Q_4)\right),
\end{aligned}
$$

$$
\begin{aligned}
\sum_X U_B(\mathbf{a},d) = & \frac{1}{2}\sum_{Q_1,Q_2,Q_3,Q_4} \delta\,(Q_1+Q_2+Q_3+Q_4) \\
& \left(\bar{\lambda}_a\{\alpha(Q_1,Q_2)-\alpha_0\delta(Q_1)\delta(Q_2)\}\right. \\
& \qquad \times\{\alpha(Q_3,Q_4)-\alpha_0\delta(Q_3)\delta(Q_4)\} \\
& + \bar{\lambda}_d\{\delta(Q_1,Q_2)-\delta_0\delta(Q_1)\delta(Q_2)\} \qquad (24) \\
& \qquad \times\{\delta(Q_3,Q_4)-\delta_0\delta(Q_3)\delta(Q_4)\} \\
& + 2\bar{\lambda}_{ad}\{\alpha(Q_1,Q_2)-\alpha_0\delta(Q_1)\delta(Q_2)\} \\
& \qquad \left.\times\{\delta(Q_3,Q_4)-\delta_0\delta(Q_3)\delta(Q_4)\}\right).
\end{aligned}
$$

SYM

microscopic :
only “mass terms”

Yukawa coupling between fermions and bosons

$$
\begin{aligned}
\Gamma_{Fa} &= -\sum_{K,Q,Q'} \bar{h}_a(K)\, \mathbf{a}(K) \cdot [\psi^\dagger(Q)\boldsymbol{\sigma}\psi(Q')] \\
&\qquad \delta(K - Q + Q' + \Pi)\,, \\
\Gamma_{F\rho} &= -\sum_{K,Q,Q'} \bar{h}_\rho(K)\, \rho(K)\, [\psi^\dagger(Q)\psi(Q')]\, \delta(K - Q + Q')\,, \\
\Gamma_{Fs} &= -\sum_{K,Q,Q'} \bar{h}_s(K) \left(s^*(K)\, [\psi^T(Q)\epsilon\psi(Q')] \right. \qquad (12) \\
&\qquad \left. - s(K)\, [\psi^\dagger(Q)\epsilon\psi^*(Q')] \right) \delta(K - Q - Q')\,, \\
\Gamma_{Fd} &= -\sum_{K,Q,Q'} \bar{h}_d(K) f_d\, ((Q - Q')/2) \left(d^*(K)\, [\psi^T(Q)\epsilon\psi(Q')] \right. \\
&\qquad \left. - d(K)\, [\psi^\dagger(Q)\epsilon\psi^*(Q')] \right) \delta(K - Q - Q')\,,
\end{aligned}
$$

$$
f_d(Q) = f_d(\mathbf{q}) = \frac{1}{2}\left(\cos(q_x) - \cos(q_y)\right)
$$

Microscopic Yukawa couplings vanish !

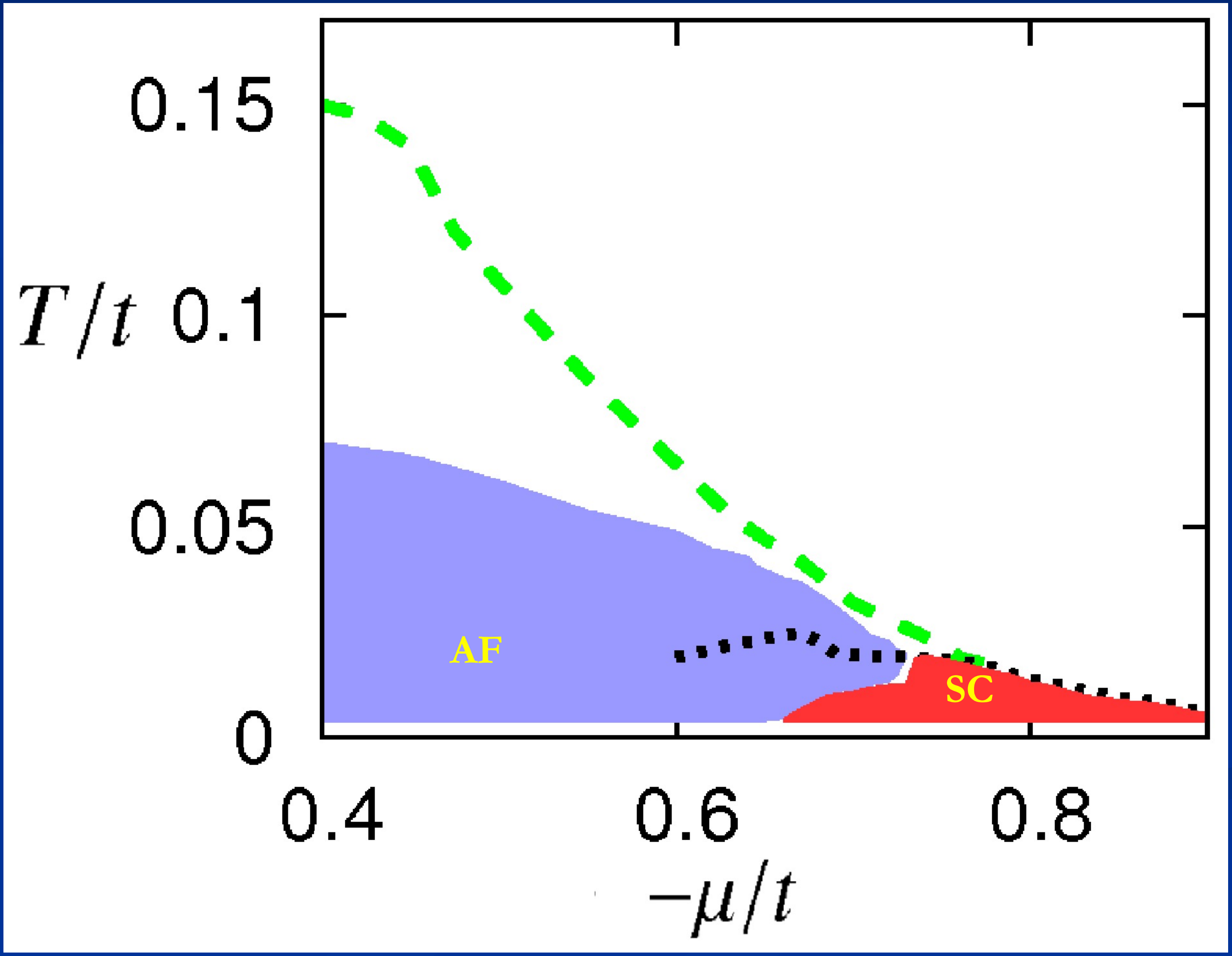
Kinetic terms for bosonic fields

$$
\begin{aligned}
\Gamma_a &= \frac{1}{2}\sum_Q \mathbf{a}^T(-Q)P_a(Q)\mathbf{a}(Q)\,, \\
\Gamma_\rho &= \frac{1}{2}\sum_Q \rho(-Q)P_\rho(Q)\rho(Q)\,, \\
\Gamma_s &= \sum_Q s^*(Q)P_s(Q)s(Q)\,, \\
\Gamma_d &= \sum_Q d^*(Q)P_d(Q)d(Q)\,.
\end{aligned}
$$

anti-ferromagnetic
boson

$$\mathbf{a}(Q) = \mathbf{m}(Q + \Pi)$$

Phase diagram



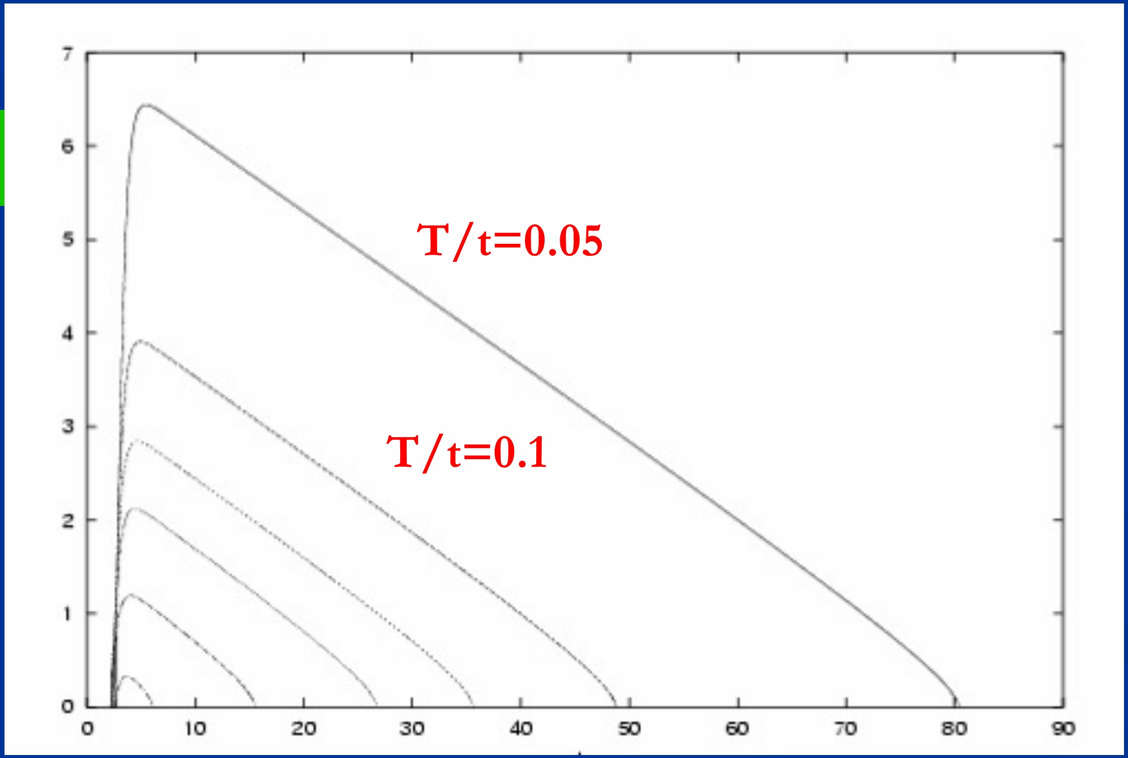
Anti-ferromagnetism in Hubbard model

- SO(3) – symmetric scalar model coupled to fermions
- For low enough k : fermion degrees of freedom decouple effectively
- crucial question : running of $\varkappa$ (location of minimum of effective potential , renormalized , dimensionless)

Critical temperature

For $T<T_c$: $\varkappa$ remains positive for $k/t > 10^{-9}$
size of probe > 1 cm

$\varkappa$



$$\kappa_a = \frac{Z_a t^2}{T}\alpha_0$$

local disorder
pseudo gap

SSB

$-\ln(k/t)$

Below the pseudocritical temperature

the reign of the goldstone bosons

critical behavior

for interval $T_c < T < T_{pc}$
evolution as for classical Heisenberg model

$$k\partial_k\kappa = \frac{1}{4\pi} + \frac{1}{16\pi^2\kappa} + 0(\kappa^{-2})$$

Mermin-Wagner theorem ?

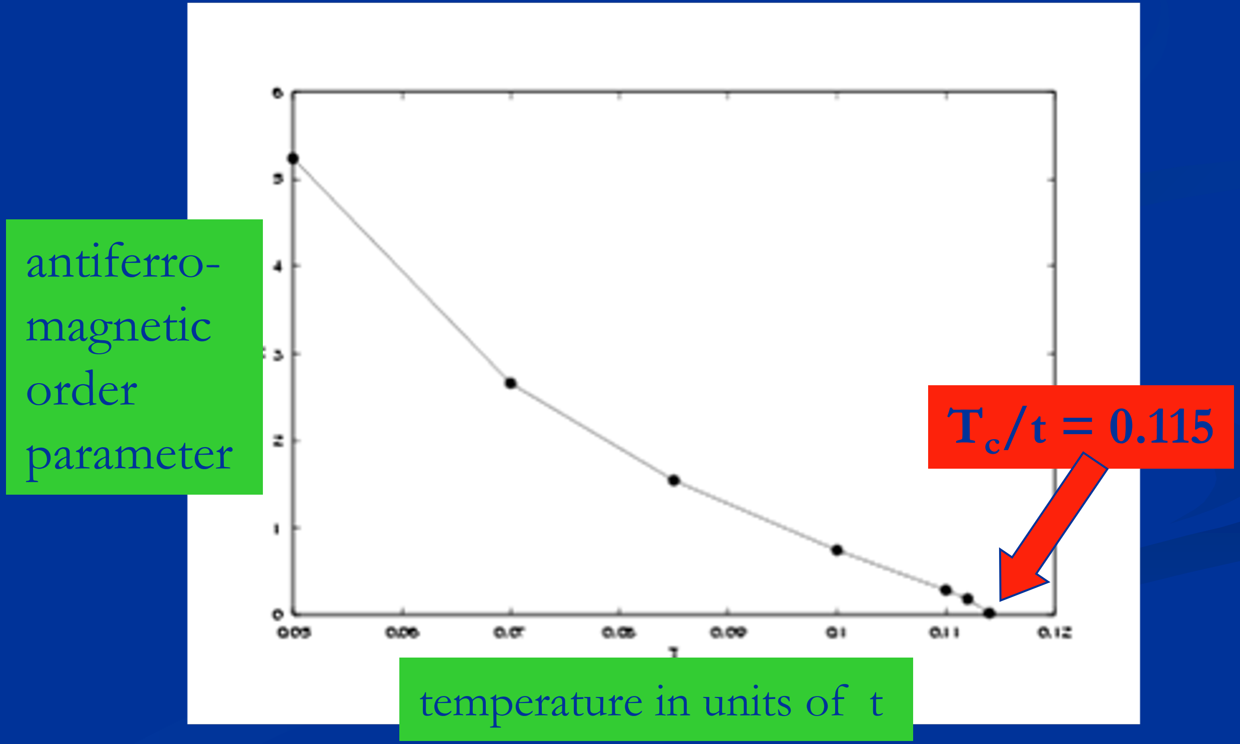
No spontaneous symmetry breaking of continuous symmetry in d=2 !

not valid in practice !

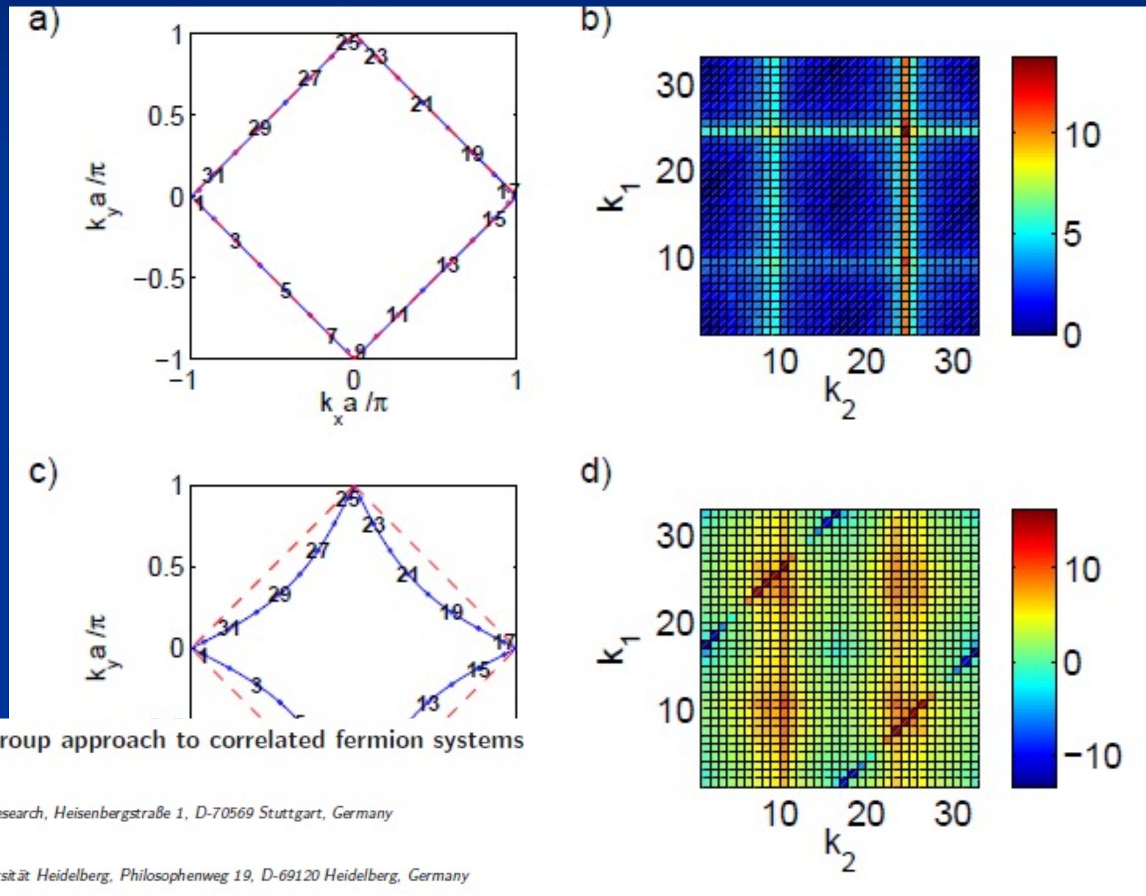
Below the critical temperature :

Infinite-volume-correlation-length becomes larger than sample size

finite sample ≈ finite k   :   order remains effectively



Flow of four point function
Hubbard model



Functional renormalization group approach to correlated fermion systems

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Many applications

- Ultracold atoms (quantum statistics)
- Disorder
- Turbulence (non-equilibrium physics)
- Density functional
- Active matter (biophysics)
- Economics

Conclusions

- Functional renormalisation has worked out in many areas of physics,

  even biology and economics…
- try it out !

end