Simplified flow equation

$$k\partial_k \Gamma_k[\varphi] = k^{2n} \mathrm{tr}\Big\{ \Big(1 - \frac{1}{2} E[\varphi]\Big) \Big(\Gamma_R^{(2)}[\varphi] + k^2\Big)^{-n} \Big\}$$

Simplified flow equation

- Flow equation for effective average action is sometimes difficult to handle in practice…
- Simplified flow equation

$$k\partial_k \Gamma_k[\varphi] = k^{2n} \mathrm{tr}\Big\{ \Big(1 - \frac{1}{2} E[\varphi]\Big) \Big(\Gamma_R^{(2)}[\varphi] + k^2\Big)^{-n} \Big\}$$

$$\Gamma_R^{(2)}[\varphi] = g^{-1/2} N^{-1}[\varphi] \Gamma_k^{(2)}[\varphi] \big(N^T\big)^{-1}[\varphi]$$

$$E[\varphi] = -N^{-1} \partial_t \big(N N^T\big) \big(N^T\big)^{-1}[\varphi]$$

Simplified flow equation

- Particular field-dependent cutoff function is used

$$k\partial_k\Gamma_k[\varphi] = k^{2n}\mathrm{tr}\left\{\left(1-\frac{1}{2}E[\varphi]\right)\left(\Gamma_R^{(2)}[\varphi]+k^2\right)^{-n}\right\}$$

- Field-dependent root of wave function renormalization N

$$\Gamma_R^{(2)}[\varphi] = g^{-1/2}N^{-1}[\varphi]\Gamma_k^{(2)}[\varphi]\left(N^T\right)^{-1}[\varphi]$$

- Field-dependent anomalous dimension E

$$E[\varphi] = -N^{-1}\partial_t\left(NN^T\right)\left(N^T\right)^{-1}[\varphi]$$

- n needs to be large enough for UV- finiteness and decoupling of massive modes, d=4 : n=3

Properties of simplified flow equation

Local gauge symmetry

- second functional derivative of gauge invariant effective action involves covariant derivatives and transforms as tensor
- sufficient that N is chosen to transform as a tensor
- add universal measure term from regularized Faddeev-Popov determinant

$$k\partial_k \Gamma_k[\varphi] = k^{2n}\mathrm{tr}\Big\{\Big(1 - \frac{1}{2}E[\varphi]\Big)\Big(\Gamma_R^{(2)}[\varphi] + k^2\Big)^{-n}\Big\}$$

$$\Gamma_R^{(2)}[\varphi] = g^{-1/2}N^{-1}[\varphi]\Gamma_k^{(2)}[\varphi]\big(N^T\big)^{-1}[\varphi]$$

$$E[\varphi] = -N^{-1}\partial_t\big(NN^T\big)\big(N^T\big)^{-1}[\varphi]$$

Arbitrary metric and signature

- g= det $g_{\mu\nu}$
- all factors of i for Minkowski signature from g<0

$$k\partial_k\Gamma_k[\varphi] = k^{2n}\mathrm{tr}\Big\{\Big(1-\frac{1}{2}E[\varphi]\Big)\Big(\Gamma_R^{(2)}[\varphi]+k^2\Big)^{-n}\Big\}$$

$$\Gamma_R^{(2)}[\varphi] = g^{-1/2}N^{-1}[\varphi]\Gamma_k^{(2)}[\varphi]\Big(N^T\Big)^{-1}[\varphi]$$

Analytic continuation

- for analytic E: all poles are given by zeros of

$$\Gamma_R^{(2)}[\varphi] + k^2$$

$$k\partial_k \Gamma_k[\varphi] = k^{2n} \mathrm{tr}\Big\{ \big(1 - \frac{1}{2} E[\varphi]\big) \big(\Gamma_R^{(2)}[\varphi] + k^2\big)^{-n} \Big\}$$

$$\Gamma_R^{(2)}[\varphi] = g^{-1/2} N^{-1}[\varphi] \Gamma_k^{(2)}[\varphi] \big(N^T\big)^{-1}[\varphi] \qquad E[\varphi] = -N^{-1} \partial_t \big(N N^T\big) \big(N^T\big)^{-1}[\varphi]$$

Automatic cutoff of dangerous modes with large effects in perturbation theory

- IR –cutoff for massless modes
- cutoff of momenta near Fermi surface

$$k\partial_k \Gamma_k[\varphi] = k^{2n}\mathrm{tr}\Big\{\Big(1-\frac{1}{2}E[\varphi]\Big)\Big(\Gamma_R^{(2)}[\varphi]+k^2\Big)^{-n}\Big\}$$

$$\Gamma_R^{(2)}[\varphi] = g^{-1/2}N^{-1}[\varphi]\Gamma_k^{(2)}[\varphi]\big(N^T\big)^{-1}[\varphi]$$

Status of simplified flow equation

Field dependent cutoff function

- field dependence of cutoff is essential ingredient
- needed for gauge invariance

  see previous work on gauge invariant flow equation
- needed for derivation of simple form
- often used in practice as truncation by background field method

Simplified flow equation is lowest order of systematic expansion

- Exact flow equation for field dependent cutoff

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{tr} \left\{ \partial_t R_k \left(\Gamma_k^{(2)} + R_k + Q_k \right)^{-1} \right\}$$

$$Q_{k,\alpha\beta} = -\frac{1}{2} \left[\frac{\partial}{\partial \varphi_\alpha} \mathrm{tr} \left\{ \frac{\partial R_k}{\partial \varphi_\beta} G \right\} + \mathrm{tr} \left\{ \frac{\partial R_k}{\partial \varphi_\alpha} \tilde{H}_\beta \right\} \right]$$

$$G_{\alpha\beta} = \langle (\chi_\alpha - \varphi_\alpha)(\chi_\beta - \varphi_\beta) \rangle$$

$$G = \left(\Gamma_k^{(2)} + R_k + Q_k \right)^{-1}$$

$$\tilde{H}_{\beta,\gamma\delta} = H_{\gamma\delta\varepsilon} G_{\varepsilon\beta}^{-1}$$

$$H_{\gamma\delta\beta} = \langle (\chi_\gamma - \varphi_\gamma)(\chi_\delta - \varphi_\delta)(\chi_\beta - \varphi_\beta) \rangle$$

- Q can be expanded in loops
- SFE : Q=0

Particular form of simplified flow equation

- Start with $\partial_t \Gamma_k = \frac{1}{2}\mathrm{tr}\left\{\partial_t R_k \left(\Gamma_k^{(2)} + R_k\right)^{-1}\right\}$
- Field dependent cutoff $R_k = g^{1/2} k^2 N r(B) N^T$ $B = k^{-2}\Gamma_R^{(2)}$
- Block diagonal constant anomalous dimension

$$\gamma_i = (1 - \eta_i/2)^{-1}$$

$$\partial_t \Gamma_k^{(i)} = \mathrm{tr}^{(i)}\left\{ \left(1 - \frac{\eta_i}{2}\right)\left(r_i - \gamma_i B_i \frac{\partial r_i}{\partial B_i}\right)(B_i + r_i)^{-1}\right\}$$

Particular choice of cutoff function

- Choose r(x) obeying $(r - \gamma_i x \partial_x r)(x+r)^{-1} = (x+1)^{-n}$

$$\partial_t \Gamma_k^{(i)} = \mathrm{tr}^{(i)} \left\{ \left(1 - \frac{\eta_i}{2}\right) \left(r_i - \gamma_i B_i \frac{\partial r_i}{\partial B_i} \right) (B_i + r_i)^{-1} \right\}$$

- For n=3, γ=1: $$\frac{\mathrm{d}r}{\mathrm{d}x} = (1+x)^{-3} \left[r \left(3 + 3x + x^2\right) - 1 \right]$$

- Solution $$r = 1 + \frac{1}{z} \left[\exp\left(-z - \frac{z^2}{2} \right) - 1 \right] , \quad z = \frac{1}{1+x}$$

  yields wanted form

Computation of correction term Q

- Exact flow equation for field dependent cutoff

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{tr}\left\{\partial_t R_k \left(\Gamma_k^{(2)} + R_k + Q_k\right)^{-1}\right\}$$

$$Q_{k,\alpha\beta} = -\frac{1}{2}\left[\frac{\partial}{\partial\varphi_\alpha}\mathrm{tr}\left\{\frac{\partial R_k}{\partial\varphi_\beta}G\right\} + \mathrm{tr}\left\{\frac{\partial R_k}{\partial\varphi_\alpha}\tilde{H}_\beta\right\}\right]$$

$$G_{\alpha\beta} = \langle(\chi_\alpha - \varphi_\alpha)(\chi_\beta - \varphi_\beta)\rangle \qquad G = \left(\Gamma_k^{(2)} + R_k + Q_k\right)^{-1}$$

$$\tilde{H}_{\beta,\gamma\delta} = H_{\gamma\delta\varepsilon}G^{-1}_{\varepsilon\beta} \qquad H_{\gamma\delta\beta} = \langle(\chi_\gamma - \varphi_\gamma)(\chi_\delta - \varphi_\delta)(\chi_\beta - \varphi_\beta)\rangle$$

- Q can be expanded in loops
- SFE : Q=0

Expansion of correction term

- Lowest order is a one-loop expression

$$Q_k = Q_k^{(1)} + Q_k^{(2)} \; ,$$
$$Q^{(1)}_{k,\alpha\beta} = -\frac{1}{2}\mathrm{tr}\Big\{\partial_\alpha\partial_\beta R_k \big(\Gamma_k^{(2)} + R_k\big)^{-1}\Big\} \; ,$$
$$Q^{(2)}_{k,\alpha\beta} = -\frac{1}{2}\mathrm{tr}\Big\{\partial_\alpha R_k \partial_\beta \big(\Gamma_k^{(2)} + R_k\big)^{-1} + \alpha \leftrightarrow \beta\Big\}$$

- Higher orders involve higher loops
- Loop expansion of correction term, not flow equation
- Dominant error from truncations

Conclusions

- Simplified flow equation yields qualitatively correct physics in many situations, in particular for gauge theories or for Minkowski space
- SFE is easy to handle (heat kernel methods)
- SFE is often quantitatively reliable. Corrections can be computed in loop expansion and turn often out to be small. Suppressed by small couplings, loop factors…

New fixed point for scalar QFT in d=4

Truncation

$$\Gamma_k = \int_x \sqrt{g}\left\{\frac{1}{2}K(\varphi)\partial^\mu\varphi\partial_\mu\varphi + U(\varphi)\right\}$$

$$\Gamma_R^{(2)} = -\partial^\mu\partial_\mu - C + K^{-1}\partial_\varphi^2 U$$

$$C = \frac{1}{4}(\partial_\varphi \ln K)^2\partial^\mu\varphi\partial_\mu\varphi + \frac{1}{2}\partial_\varphi \ln K\partial^\mu\partial_\mu\varphi$$

Flow of potential

$$\partial_t U = \frac{(1-\eta/2)k^6}{32\pi^2(k^2+\partial_\varphi^2 U K^{-1})}$$

Flow of kinetial

$$\begin{aligned}\partial_t K = \frac{(1-\eta/2)k^6}{16\pi^2}\Bigg[&\frac{(\partial_\varphi \ln K)^2 - 2\partial_\varphi^2 \ln K}{4(k^2+\partial_\varphi^2 U K^{-1})^2}\\ &+\frac{(\partial_\varphi^3 U K^{-1} - \partial_\varphi^2 U\partial_\varphi K K^{-2})\partial_\varphi \ln K}{(k^2+\partial_\varphi^2 U K^{-1})^3}\\ &-\frac{(\partial_\varphi^3 U K^{-1} - \partial_\varphi^2 U\partial_\varphi K K^{-2})^2}{2(k^2+\partial_\varphi^2 U K^{-1})^4}\Bigg].\end{aligned}$$

Fixed point

$$U = u_0 k^4\ ,\quad K = \frac{\kappa_0}{\varphi^2}\ ,\quad u_0 = \frac{1}{128\pi^2}$$

Quantum scale symmetry

Fixed point is invariant under multiplicative rescaling of φ

$$\Gamma_k = \int_x \sqrt{g} \left\{ \frac{1}{2} K(\varphi) \partial^\mu \varphi \partial_\mu \varphi + U(\varphi) \right\}$$

$$U = u_0 k^4 \ , \quad K = \frac{\kappa_0}{\varphi^2} \ , \quad u_0 = \frac{1}{128\pi^2}$$

SFE preserves this symmetry

corrections preserve this symmetry

If crossover trajectory to trivial fixed point for large φ exists:

UV completion of non-trivial scalar QFT in d=4 end

Expansion of correction term

Use

$$\partial_\alpha G_{\gamma\delta} = \left(\tilde{H}_\alpha\right)_{\gamma\delta} + N_{\alpha\gamma\delta} \ ,$$
$$N_{\alpha\gamma\delta} = \frac{1}{2}\mathrm{tr}\{\partial_\alpha R_k \tilde{H}_\eta\} H_{\eta\gamma\delta} - \frac{1}{2}\mathrm{tr}\{\partial_\alpha R_k \tilde{V}_{\gamma\delta}\}$$
$$+ \frac{1}{2}\mathrm{tr}\{\partial_\alpha R_k G\} G_{\gamma\delta} \ ,$$
$$\left(\tilde{V}_{\gamma\delta}\right)_{\varepsilon\eta} = V_{\varepsilon\eta\gamma\delta} \ ,$$

$$V_{\varepsilon\eta\gamma\delta} = \langle(\chi_\varepsilon - \varphi_\varepsilon)(\chi_\eta - \varphi_\eta)(\chi_\gamma - \varphi_\gamma)(\chi_\delta - \varphi_\delta)\rangle$$

yields

$$\left(Q_k\right)_{\alpha\beta} = -\frac{1}{2}\mathrm{tr}\{\partial_\alpha\partial_\beta R_k G + \partial_\alpha R_k \partial_\beta G + \partial_\beta R_k \partial_\alpha G\} + D_{k,\alpha\beta} \ , \qquad (67)$$

$$D_{k,\alpha\beta} = \frac{1}{2}\partial_\alpha R_{k,\gamma\delta} N_{\beta\gamma\delta}$$

Set D=0

$$Q_k = Q_k^{(1)} + Q_k^{(2)} \ ,$$
$$Q_{k,\alpha\beta}^{(1)} = -\frac{1}{2}\mathrm{tr}\left\{\partial_\alpha\partial_\beta R_k \left(\Gamma_k^{(2)} + R_k\right)^{-1}\right\} \ ,$$
$$Q_{k,\alpha\beta}^{(2)} = -\frac{1}{2}\mathrm{tr}\left\{\partial_\alpha R_k \partial_\beta \left(\Gamma_k^{(2)} + R_k\right)^{-1} + \alpha \leftrightarrow \beta\right\}$$