Field transformations in functional flow equations

powerful tool for
effective action and flowing action

Wide applications

Composite fields

- mesons in QCD
- antiferromagnetic spin waves in Hubbard model
- relation between composite and fundamental fields can depend on renormalization scale k

Dressed fields

- incorporating some type of wave function renormalization

Computational simplifications by transforming away unpleasant terms in flow equations ("essential RG")

Better insights for symmetries

Linear field transformations

- Standard example : renormalized fields

- Momentum- dependent field transformations transform different Fourier modes separately
- Useful to simplify propagator if momentum dependence is sufficiently smooth
- Transforming away the Fermi surface for electrons in solids seems not to be a good idea …

- Transformation of fields in position space can get complicated

"Transforming away" the Fermi surface

inverse propagator

$$\varepsilon(q) - \mu + i(2n+1)\pi T$$

regular linear FT

$$P_F(q) \to \tilde{P}_F(q) = f(q)P_F(q)$$

$$\Downarrow$$

$$\tilde{P}_F = q^2/(2m) + i(2n+1)\pi T$$

physics moves to vertices

“Irrelevant couplings”

- Couplings that can be transformed away are not unphysical
- If one calls couplings that can be transformed away “irrelevant”: irrelevant is not unphysical
- Field transformations move physical information between different sectors of flowing action
- In general: physical observables involve combination of choice of fields and functionals of these fields (correlation functions)

Non-linear field transformations

Non- linear field transformations

- (Too?) powerful tool for effective action and flowing action
- Many possibilities : for example, one can transform away the potential in a scalar field theory
- Information is shuffled to other sectors
- While field transformations are exact, truncations can become very misleading
- Keep track of physics !

Microscopic and macroscopic field transformations

- Microscopic field transformations transform fluctuating field variables in functional integral
- Possibly combined with new sources and new cutoff for „composite fields“ in flowing action
  (Jan Pawlowski, essential RG)
- Macroscopic field transformations transform field variables in effective action or flowing action

Jacobian

- Microscopic field transformations involve a Jacobian
- Comes back indirectly through initial conditions even if integration variables in functional integral are kept fixed, and only sources and cutoff are changed (Jan Pawlowski, essential RG)
- Macroscopic field transformations have no Jacobian

Example: non-linear kinetial

- classical action $$S[\chi'] = \frac{1}{2}\int_x K(\chi')\partial^\mu \chi' \partial_\mu \chi'$$

$$p[\chi'] = Z^{-1} \exp\big(-S[\chi']\big)$$ $$Z = \int \mathcal{D}\chi' \exp\big(-S[\chi']\big)$$

- functional measure $$\int \mathcal{D}\chi' = \prod_x \int_{-\infty}^{\infty} \mathrm{d}\chi'(x)$$

- model with interactions: non-linear field classical equations $$K\partial^\mu \partial_\mu \chi' = -\frac{1}{2}\frac{\partial K}{\partial \chi'}\partial^\mu \chi' \partial_\mu \chi'$$

Microscopic field transformation

- non-linear field transformation

$$\varphi'(x) = \varphi'\big(\chi'(x)\big) \qquad \frac{\partial \varphi'}{\partial \chi'} = K^{1/2}(\chi')$$

$$S[\chi'] = \frac{1}{2}\int_x K(\chi')\partial^\mu \chi' \partial_\mu \chi' \quad \Longrightarrow \quad S = \frac{1}{2}\int_x \partial^\mu \varphi' \partial_\mu \varphi'$$

- Jacobian

$$\begin{aligned} Z &= \int \mathcal{D}\chi'\, e^{-S[\chi']} = \int \mathcal{D}\varphi'\, J[\varphi'] e^{-S\left[\chi'(\varphi')\right]} \\ &= \int \mathcal{D}\varphi'\, e^{-\left(S[\varphi'] - \ln J[\varphi']\right)} = \int \mathcal{D}\varphi'\, e^{-\bar{S}[\varphi']} \end{aligned}$$

$$J[\varphi'] = \prod_x \left| \frac{\partial \chi'(x)}{\partial \varphi'(x)} \right| = \exp\left(-\varepsilon^{-d} \int_x \ln \left| \frac{\partial \varphi'}{\partial \chi'} \right| \right)$$

FT induced potential

Jacobian results in scalar potential for classical action

$$\bar{S}[\varphi'] = \int_x \left\{ \frac{1}{2} \partial^\mu \varphi' \partial_\mu \varphi' + V(\varphi') \right\}$$

$$V(\varphi') = \frac{\varepsilon^{-d}}{2} \ln K[\varphi']$$

$$\begin{aligned} Z &= \int \mathcal{D}\chi' \, e^{-S[\chi']} = \int \mathcal{D}\varphi' \, J[\varphi'] e^{-S[\chi'(\varphi')]} \\ &= \int \mathcal{D}\varphi' \, e^{-\left(S[\varphi'] - \ln J[\varphi']\right)} = \int \mathcal{D}\varphi' \, e^{-\bar{S}[\varphi']} \end{aligned}$$

equivalent formulations

$$Z = \int \mathcal{D}\chi' \, \exp\left(-S[\chi']\right) = \int \mathcal{D}\varphi' \, \exp\left(-\bar{S}[\varphi']\right)$$

suitable kinetial can lead to spontaneous symmetry breaking

$$K = \exp\left(2\varepsilon^d V\right)$$

$$K = \exp\left\{ \frac{1}{\lambda} \tanh^2 \left[\frac{\lambda \varepsilon^{d/2}}{2} (\varphi'^2 - \varphi_0^2) \right] \right\}$$

$$V = \frac{\varepsilon^{-d}}{2\lambda} \tanh^2 \left\{ \frac{\lambda \varepsilon^{d/2}}{2} (\varphi'^2 - \varphi_0^2) \right\}$$

k- dependent field transformations

- Correction terms in flow equations
- Macroscopic FT : corrections due to simple change of the variables which are kept fixed in flow equations
- Microscopic FT : corrections due to k – dependent Jacobian
- Linear field transformations : Macroscopic and microscopic FT are equivalent
- In practice: often only small difference between microscopic and macroscopic non-linear field transformations H. Gies, …

Two- field formalism

- Keep fundamental and composite fields

  (quarks and mesons)
- often most powerful
- not covered in this talk
- Gies, Pawlowski, Floerchinger, …

Macroscopic and microscopic field transformations

Functional integral for flowing action

- Implicit functional integral with cutoff

$$\Gamma_k[\chi] = \Gamma'_k[\chi] - \Delta_k S[\chi] \ ,$$

$$\Gamma'_k[\chi] = -\ln \int \mathcal{D}\chi' \exp\Big\{ - S[\chi'] - \Delta_k S[\chi'] + \int_x \frac{\partial \Gamma'_k}{\partial \chi(x)} \big(\chi'(x) - \chi(x)\big) \Big\} \ ,$$

$$\Delta_k S[\chi'] = \frac{1}{2} \int_{x,y} \chi'(x) \mathcal{R}_k(x,y) \chi'(y)$$

- Exact flow equation

$$\partial_k \Gamma_k = \frac{1}{2} \mathrm{tr} \left\{ \partial_k \mathcal{R}_k \big(\Gamma_k^{(2)} + \mathcal{R}_k\big)^{-1} \right\}$$

Macroscopic field transformation

- Variable change in differential equation

$$\varphi = \varphi_k[\chi]$$

$$\partial_t \Gamma_k|_\varphi = \partial_t \Gamma_k|_\chi - \int_x \frac{\partial \Gamma_k}{\partial \varphi(x)} \partial_t \varphi(x)|_\chi$$

CW hep-ph/9604227
Gies,…hep-th/0107221

$$\partial_t \varphi(x)|_\chi = k \frac{\partial \varphi_k[\chi]}{\partial k}(x) = \tilde{\gamma}_k[\varphi]$$

- Can be used to derive scaling form of flow equation or “eliminate” certain couplings

Field relativity

Effective actions related by macroscopic field transformations are strictly equivalent

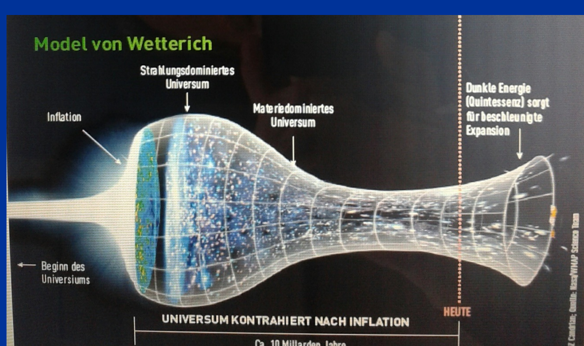
Different geometries for same physics

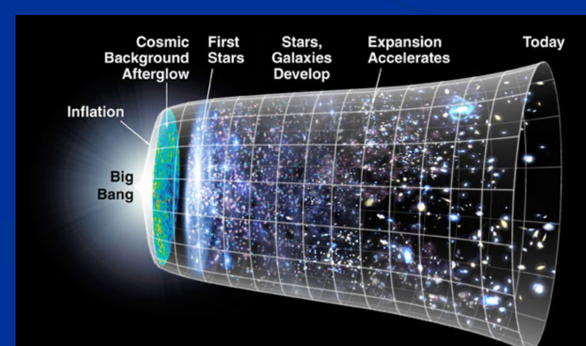
Field relativity

Weyl scaling :

$$g'_{\mu\nu} = \frac{\chi^2}{M^2} g_{\mu\nu}$$

changes geometry,
not a coordinate transformation





Microscopic field transformation

correction term in flow equation is not identical to macroscopic field transformation

$$\partial_t \Gamma_{\varphi,k} = \frac{1}{2}\int_{x,y}\left[\partial_t \mathcal{R}_k(x,y) + 2\int_z \frac{\partial\gamma(z)}{\partial\varphi(x)}\mathcal{R}_k(z,y)\right] \times \left(\Gamma^{(2)}_{\varphi,k} + \mathcal{R}_k\right)^{-1}(y,x) - \int_x \frac{\partial\Gamma_{\varphi,k}}{\partial\varphi(x)}\gamma(x) \,.$$

Pawlowski,
Baldazzi, Ben Ali Zinati, Falls, …

Flowing action for composite fields

- functional integral for flowing action for composite fields

$$\Gamma'_{\varphi,k}[\varphi] = -\ln \int \mathcal{D}\chi' \exp\Big\{ - S[\chi'] - \frac{1}{2}\int_{x,y} \varphi'_k(x)\mathcal{R}_k(x,y)\varphi'_k(y) + \int_x \frac{\partial \Gamma'_{\varphi,k}}{\partial \varphi(x)}\big(\varphi'_k(x) - \varphi(x)\big)\Big\} \quad (114)$$

$$\Gamma_{\varphi,k}[\varphi] = \Gamma'_{\varphi,k}[\varphi] - \frac{1}{2}\int_{x,y} \varphi(x)\mathcal{R}_k(x,y)\varphi(y)$$

$$\partial_k \Gamma_{\varphi,k}[\varphi] = \frac{1}{2}\mathrm{tr}\Big\{\partial_k \mathcal{R}_k \big(\Gamma^{(2)}_{\varphi,k} + \mathcal{R}_k\big)^{-1}\Big\} - \int_x \frac{\partial \Gamma_{\varphi,k}}{\partial \varphi(x)}\langle \partial_k \varphi'_k(x)\rangle + \int_{x,y} \mathcal{R}_k(x,y)\Big\langle\big(\varphi'_k(y) - \varphi(y)\big)\partial_k \varphi'_k(x)\Big\rangle \quad (11$$

RG - kernel

- expectation values for k > 0

$$\mathcal{A}[\varphi] = \langle \mathcal{A}[\varphi'] \rangle = \frac{\int \mathcal{D}\chi' \mathcal{A}[\varphi'] \exp\{-F[\chi']\}}{\int \mathcal{D}\chi' \exp\{-F[\chi']\}}$$

$$F[\chi'] = S[\chi'] + \Delta_k S[\chi'] - \int_y \frac{\partial \Gamma'_{\varphi,k}}{\partial \varphi(y)} (\varphi'(y) - \varphi(y))$$

$$\frac{\partial}{\partial \varphi(x)} \mathcal{A}[\varphi] = \int_y \frac{\partial^2 \Gamma}{\partial \varphi(x) \partial \varphi(y)} \langle (\varphi'(y) - \varphi(y)) \mathcal{A}[\varphi'] \rangle$$

- RG - kernel $\gamma(x) = \langle \partial_t \varphi'_k(x) \rangle$

- flow equation

$$\partial_t \Gamma_{\varphi,k} = \frac{1}{2} \int_{x,y} \left[\partial_t \mathcal{R}_k(x,y) + 2 \int_z \frac{\partial \gamma(z)}{\partial \varphi(x)} \mathcal{R}_k(z,y) \right] \times \left(\Gamma^{(2)}_{\varphi,k} + \mathcal{R}_k \right)^{-1}(y,x) - \int_x \frac{\partial \Gamma_{\varphi,k}}{\partial \varphi(x)} \gamma(x) \, .$$

Macroscopic and microscopic field transformations do not yield identical flow equations

- different correction terms
- macroscopic FT

$$\partial_t \Gamma_k|_\varphi = \partial_t \Gamma_k|_\chi - \int_x \frac{\partial \Gamma_k}{\partial \varphi(x)} \partial_t \varphi(x)|_\chi$$

- microscopic FT

$$\partial_t \Gamma_{\varphi,k} = \frac{1}{2} \int_{x,y} \left[\partial_t \mathcal{R}_k(x,y) + 2 \int_z \frac{\partial \gamma(z)}{\partial \varphi(x)} \mathcal{R}_k(z,y) \right] \times \left(\Gamma^{(2)}_{\varphi,k} + \mathcal{R}_k \right)^{-1}(y,x) - \int_x \frac{\partial \Gamma_{\varphi,k}}{\partial \varphi(x)} \gamma(x) \ .$$

Different flowing actions

- macroscopic FT

$$\Gamma_k[\chi] = \Gamma'_k[\chi] - \Delta_k S[\chi] \ ,$$
$$\Gamma'_k[\chi] = -\ln \int \mathcal{D}\chi' \ \exp\Big\{ - S[\chi'] - \Delta_k S[\chi'] + \int_x \frac{\partial \Gamma'_k}{\partial \chi(x)} \big(\chi'(x) - \chi(x)\big) \Big\} \ ,$$

$$\Delta_k S[\chi'] = \frac{1}{2}\int_{x,y} \chi'(x) \mathcal{R}_k(x,y) \chi'(y)$$

- microscopic FT

$$\Gamma'_{\varphi,k}[\varphi] = -\ln \int \mathcal{D}\chi' \exp\Big\{ - S[\chi'] - \frac{1}{2}\int_{x,y} \varphi'_k(x) \mathcal{R}_k(x,y) \varphi'_k(y) + \int_x \frac{\partial \Gamma'_{\varphi,k}}{\partial \varphi(x)} \big(\varphi'_k(x) - \varphi(x)\big) \Big\} \qquad (114)$$

$$\Gamma_{\varphi,k}[\varphi] = \Gamma'_{\varphi,k}[\varphi] - \frac{1}{2}\int_{x,y} \varphi(x) \mathcal{R}_k(x,y) \varphi(y)$$

- different sources, different cutoff

Variable transformation in functional integral

- One could transform variables in functional integral

$$\Gamma'_{\varphi,k}[\varphi] = -\ln \int \mathcal{D}\chi' \exp\Big\{ -S[\chi'] \qquad (114)$$
$$-\frac{1}{2}\int_{x,y} \varphi'_k(x)\mathcal{R}_k(x,y)\varphi'_k(y) + \int_x \frac{\partial\Gamma'_{\varphi,k}}{\partial\varphi(x)}(\varphi'_k(x) - \varphi(x))\Big\}$$

- k- dependent classical action for composite fields yield corrections to flow equation

Initial conditions

- For microscopic FT the Jacobian strikes back !
- Relation between classical action and flowing action at large k involves Jacobian
- For linear measure:

$$\Gamma_{\chi,k}[\chi] = -\ln \int \mathcal{D}\hat{\chi} \exp\left\{ - S[\chi + \hat{\chi}] + \int_x \frac{\partial \Gamma_k}{\partial \chi(x)} \hat{\chi}(x) \right.$$
$$\left. -\frac{1}{2} \int_{x,y} \hat{\chi}(x) \mathcal{R}_k(x,y) \hat{\chi}(y) \right\} . \qquad (75)$$

- General measure:

$$\Gamma'_{\varphi,k}[\varphi] = -\ln \int \mathcal{D}\chi' \exp\left\{ - S[\chi'] \right. \qquad (114)$$
$$\left. -\frac{1}{2} \int_{x,y} \varphi'_k(x) \mathcal{R}_k(x,y) \varphi'_k(y) + \int_x \frac{\partial \Gamma'_{\varphi,k}}{\partial \varphi(x)} (\varphi'_k(x) - \varphi(x)) \right\}$$

- Proof: Change of integration variable

Advantages and disadvantages

microscopic FT:

- advantages: cutoff can be adapted to composite fields,
  richer possibilities of "eliminating couplings"
- disadvantages: no direct access to fundamental correlation functions,
  Jacobian in initial conditions

macroscopic FT:

- advantage: contact to fundamental correlation functions remains straightforward if field transformation is reconstructed

Flowing action and correlation functions

Relation RG- kernel to correlations of fundamental fields

- for non-linear microscopic FT the relation between the RG- kernel $\gamma(x) = \langle \partial_t \varphi'_k(x) \rangle$ and correlation functions of fundamental fields gets complex
- example: $\varphi'_k(x) = \chi'(x) + g(k)\chi'(x)^2$ $\chi'(\varphi'_k) = \frac{1}{2g}(\sqrt{1+4g\varphi'_k} - 1)$

$$\partial_k \varphi'_k(x) = \partial_k g(k) \chi'(x)^2 = \partial_k \ln g(k) \big(\varphi'_k - \chi'(\varphi'_k)\big)$$

- RG – kernel involves correlation functions of composite fields of arbitrary order

"Transforming away" the potential…

- For a suitably chosen RG kernel contributions to the scalar potential can be "transformed away"
- Assume fermion fluctuations induce $\Delta \partial_t U = A_1 k^2 \varphi^2 + A_2 \varphi^4$

For truncation $\Gamma_k = (Z/2) \int_x \partial^\mu \varphi \partial_\mu \varphi$ the fermion contribution is cancelled if one chooses RG – kernel $\gamma = -\frac{A_1 \varphi^3}{3Ck^2} - \frac{A_2 \varphi^5}{5Ck^4}$, with

$$\partial_t \left(\frac{\partial U}{\partial \varphi} \right) = \frac{\partial^2 \gamma}{\partial \varphi^2} \int_q \frac{\mathcal{R}_k(q^2)}{Zq^2 + \mathcal{R}_k(q^2)} = Ck^4 \frac{\partial^2 \gamma}{\partial \varphi^2}$$

- Physical information moved from potential to Yukawa couplings
- Spontaneous symmetry breaking now through complicated Yukawa couplings

“Irrelevant couplings”

- Couplings that can be transformed away are not unphysical
- If one calls couplings that can be transformed away “irrelevant”: irrelevant is not unphysical
- Field transformations move physical information between different sectors of flowing action
- In general: physical observables involve combination of choice of fields and functionals of these fields (correlation functions)

Flowing action and correlations

- Meaningful truncations need understanding of relation between flowing action and correlation functions at least qualitatively
- Defining flowing action by arbitrary flow equation and boundary conditions is not sufficient
- Danger of truncating away relevant physical information

microscopic FT with k – dependent macroscopic field

integrate $k\partial_k \varphi(k;\chi) = \gamma_k[\varphi(k;\chi)]$

with initial value $\varphi(\Lambda;\chi) = \chi$

- initial value may not need Jacobian
- yields relation for expectation value, not microscopic field transformation
- direct contact to correlations of fundamental or composite fields is not available

Conclusions

- Field transformations are powerful tool for functional flow equations
- Control of physical properties and relation to correlation functions is necessary
- “Irrelevant couplings” are not unphysical
- Universality can help, but is not guaranteed
- Macroscopic and microscopic non-linear field transformations lead to different flowing actions

end