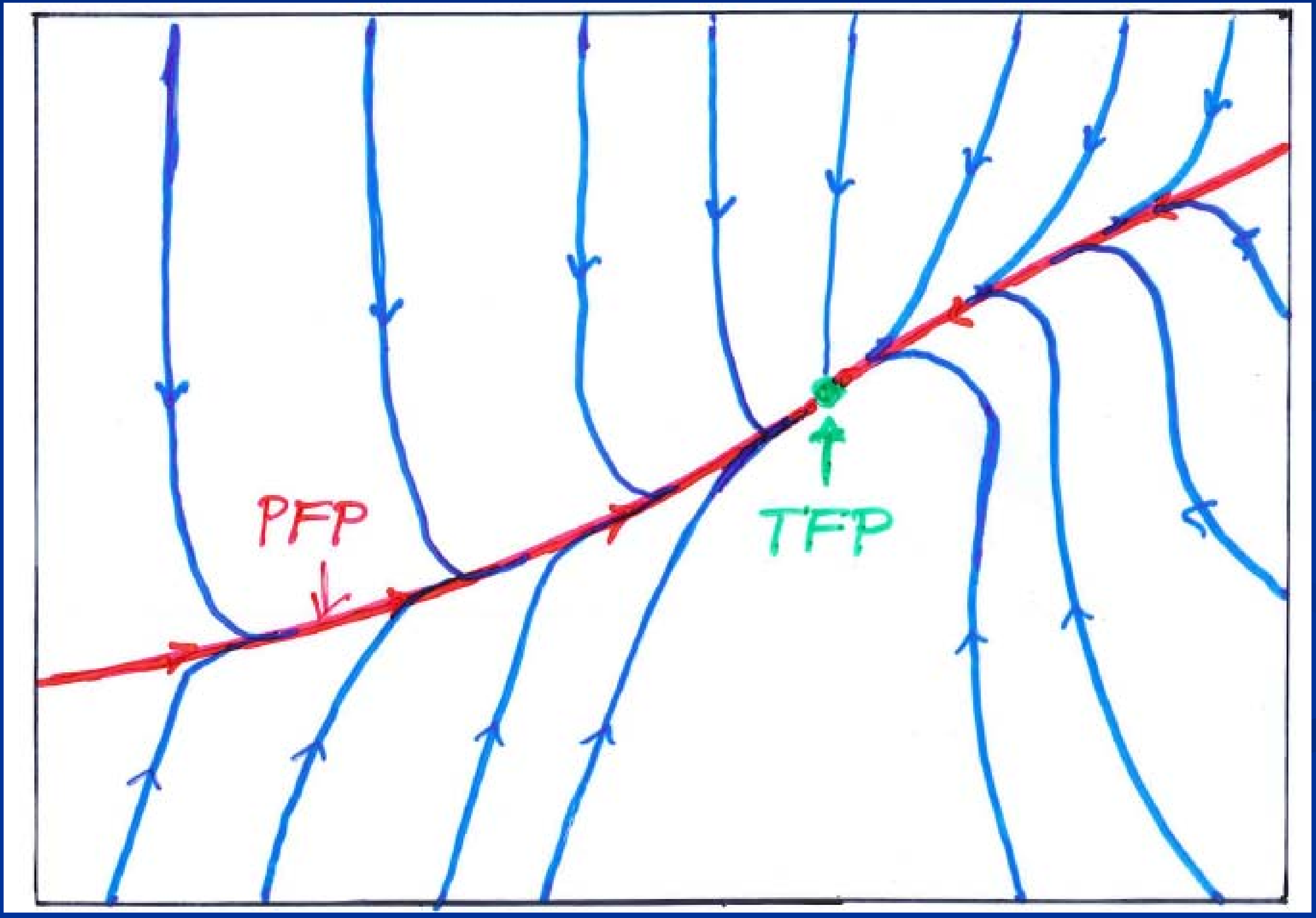
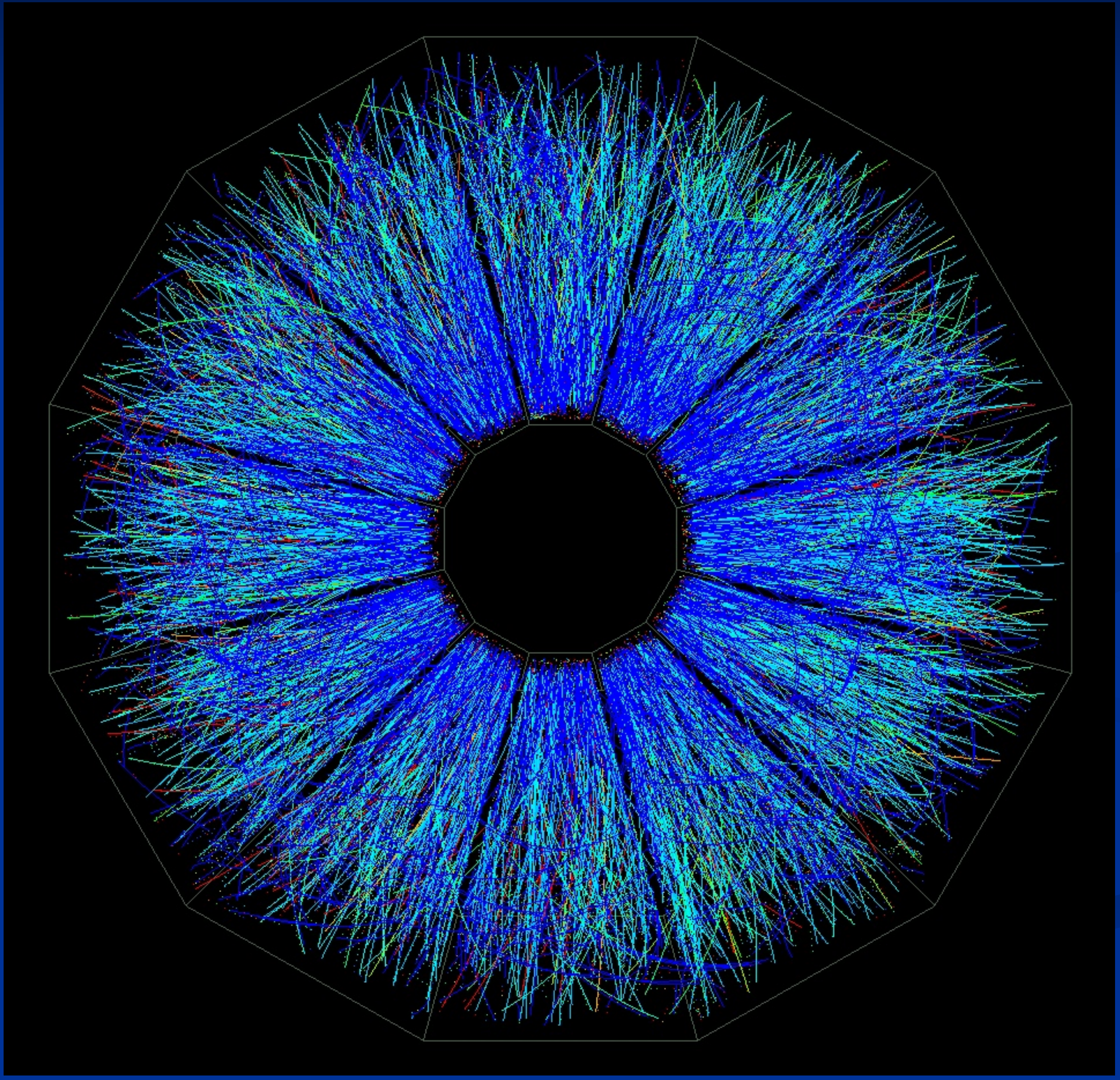
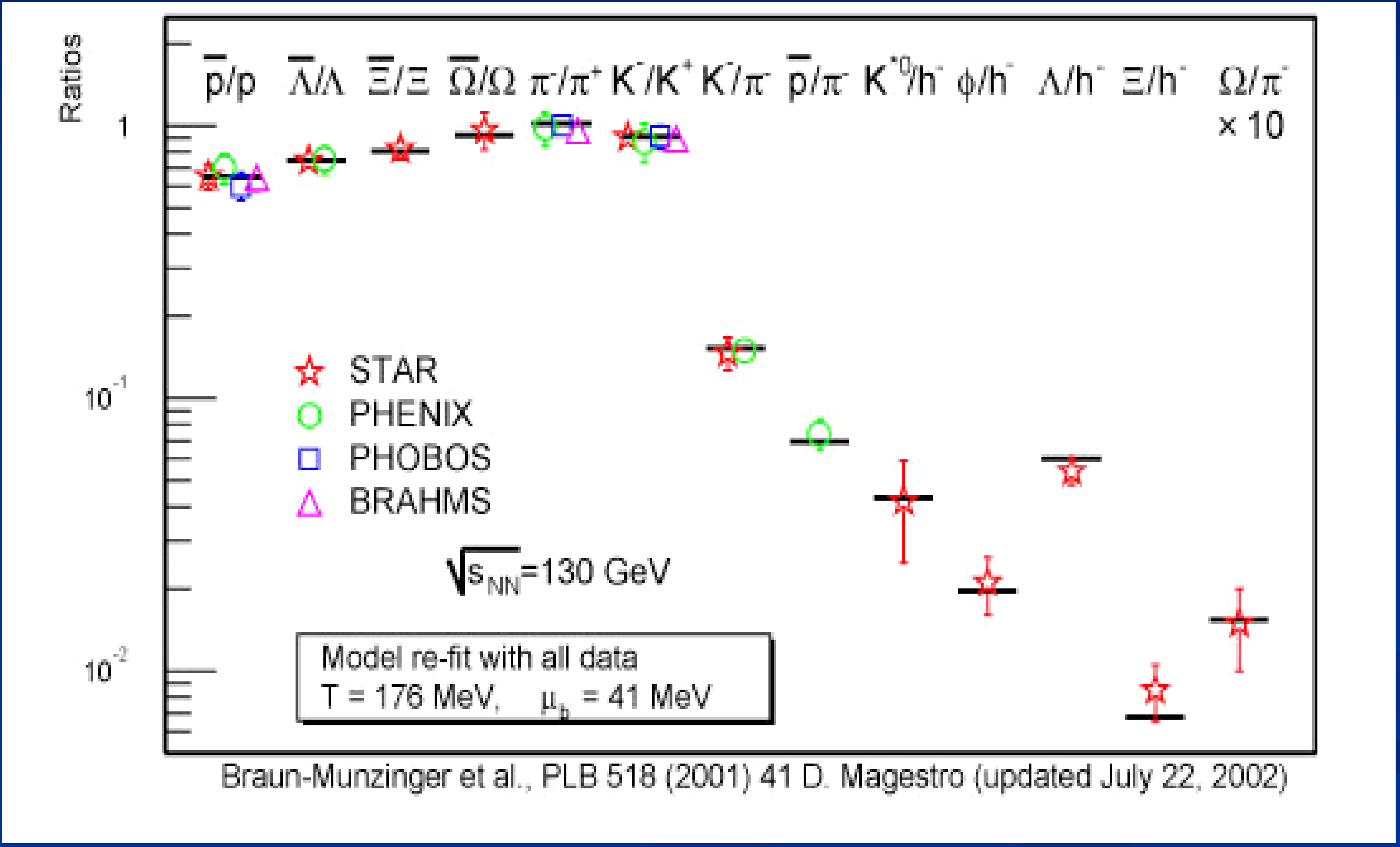
Prethermalization



Heavy ion collision

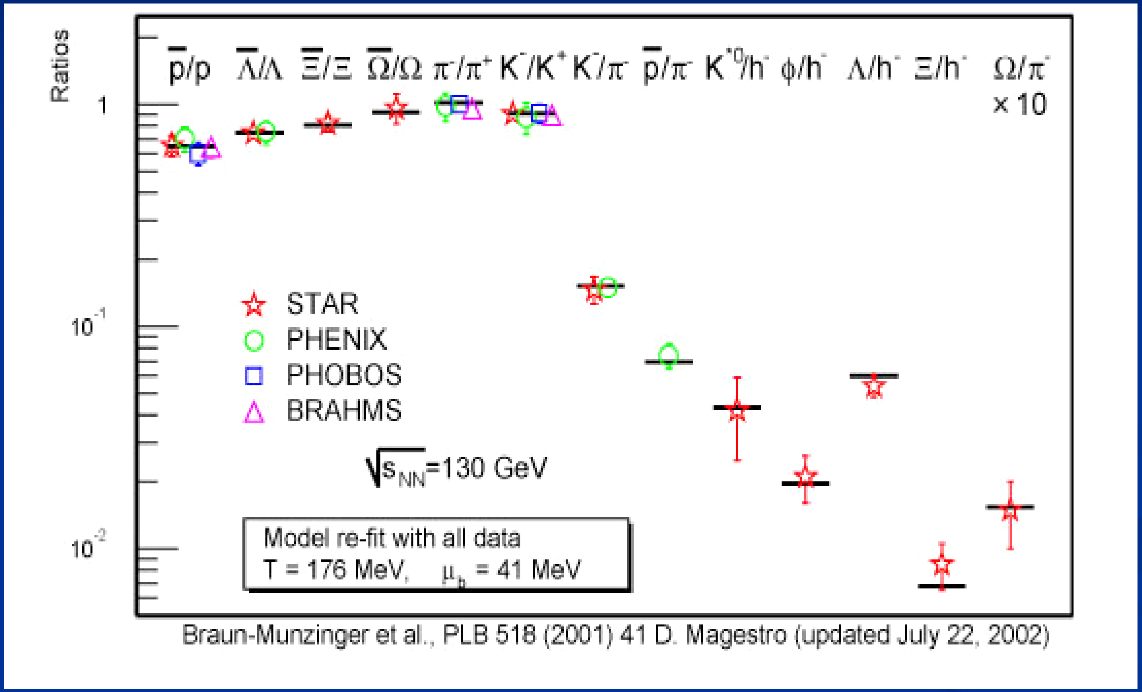


Hadron abundancies



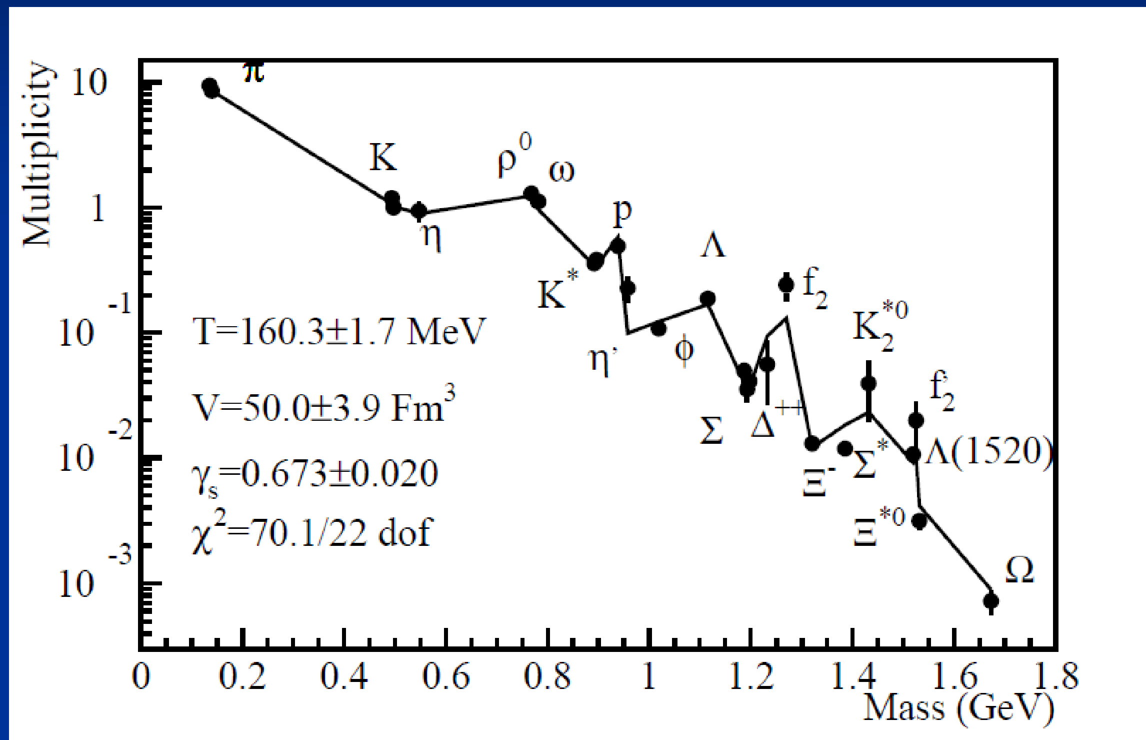
RHIC

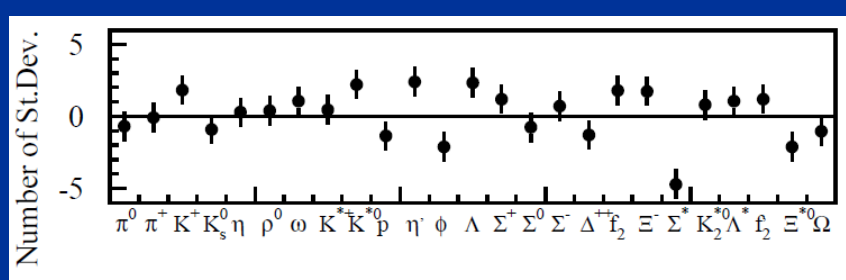
Hadron abundancies



follow thermal eqilibrium distribution for suitable temperature T and baryon chemical potential μ

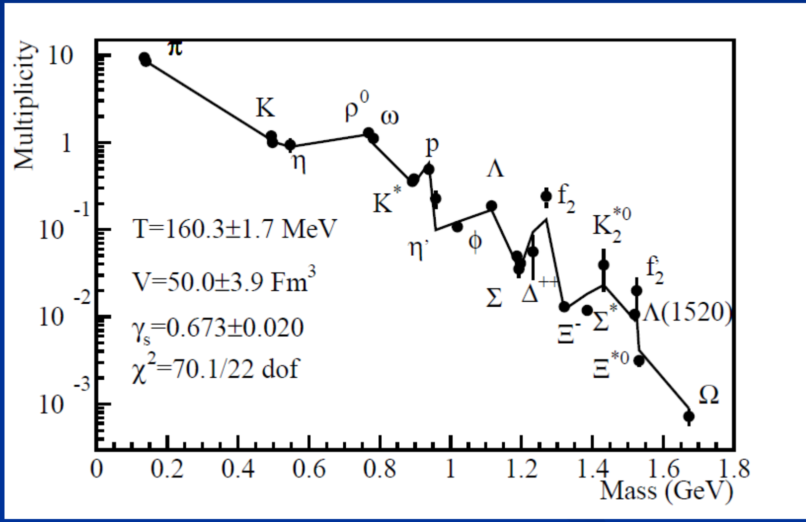
Hadron abundancies in e^+ - e^- collisions





Becattini

Hadron abundancies in $e^+ - e^-$ collisions



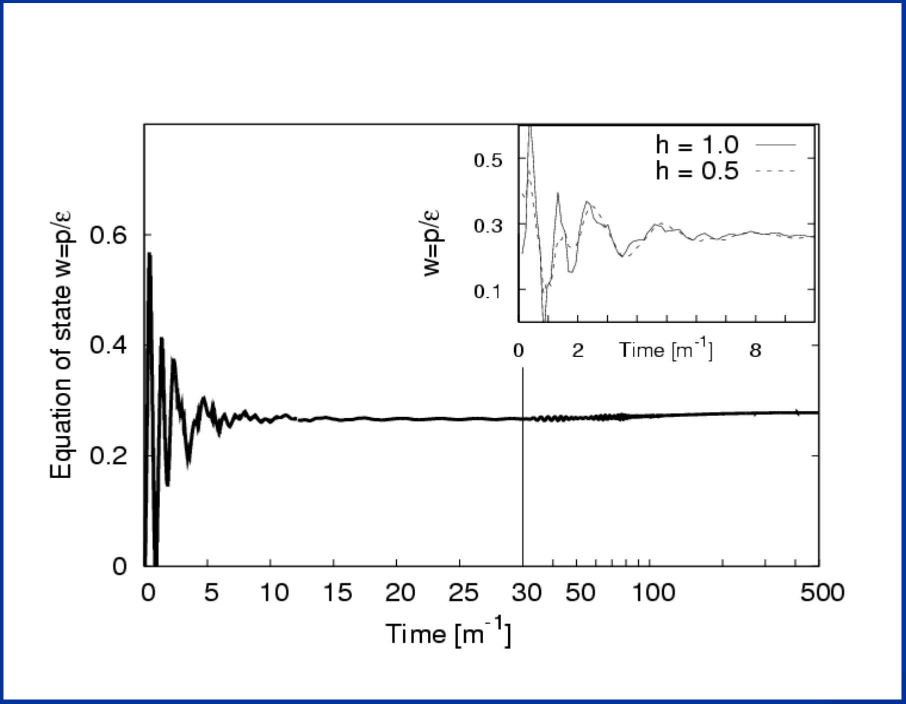
not thermal equilibrium !

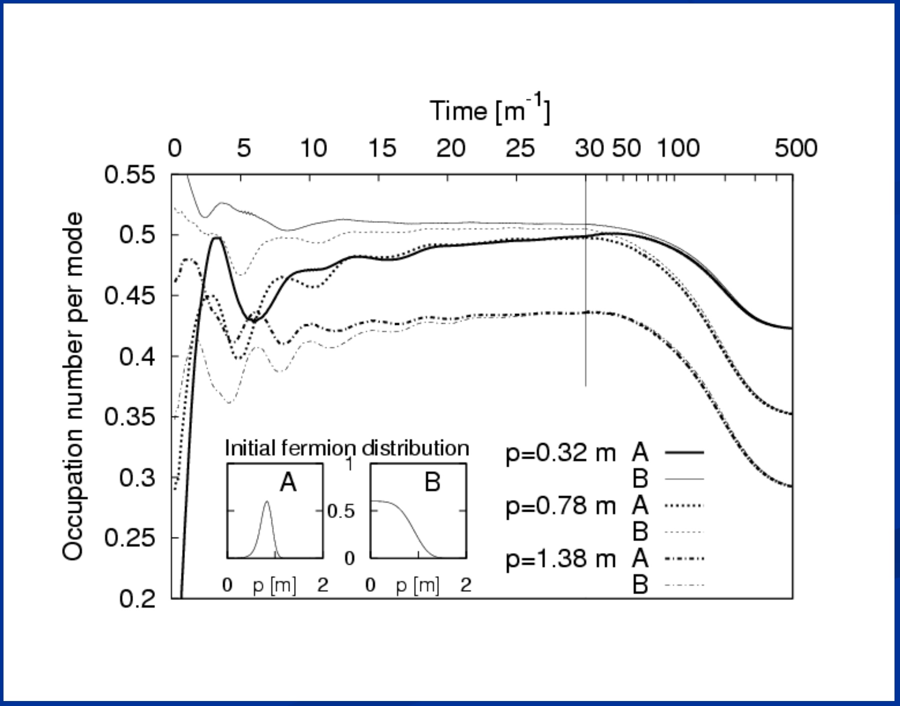
no substantial scattering of produced hadrons

no Boltzmann equations apply

Prethermalization

J.Berges, Sz.Borsanyi, CW





Thermal equilibrium

only one parameter characterizes distribution functions and correlations : temperature T

(several parameters if other quantities besides energy E are conserved , e.g. chemical potential μ or particle density n =N/V for conserved particle number N)

essence of thermalization

loss of memory of details of initial state

for fixed volume V : only energy E matters

prethermalization

only partial loss of
memory of initial state

prethermalization

- can happen on time scales much shorter than for thermalization
- nevertheless: several important features look already similar to thermal equilibrium state
- can produce states different from thermal equilibrium that persist for very long (sometimes infinite) time scales

quantities to investigate

- set of correlation functions
- or effective action as generating functional for correlation functions

- **not:** density matrix or probability distribution

reasons

- rather different density matrices / probability distributions can produce essentially identical correlation functions

  (they only differ by unobservable higher order correlations e.g. 754367-point functions, or unobservable phase correlations)

- only correlation functions are observed in practice

  (distributions : one-point function or expectation value)

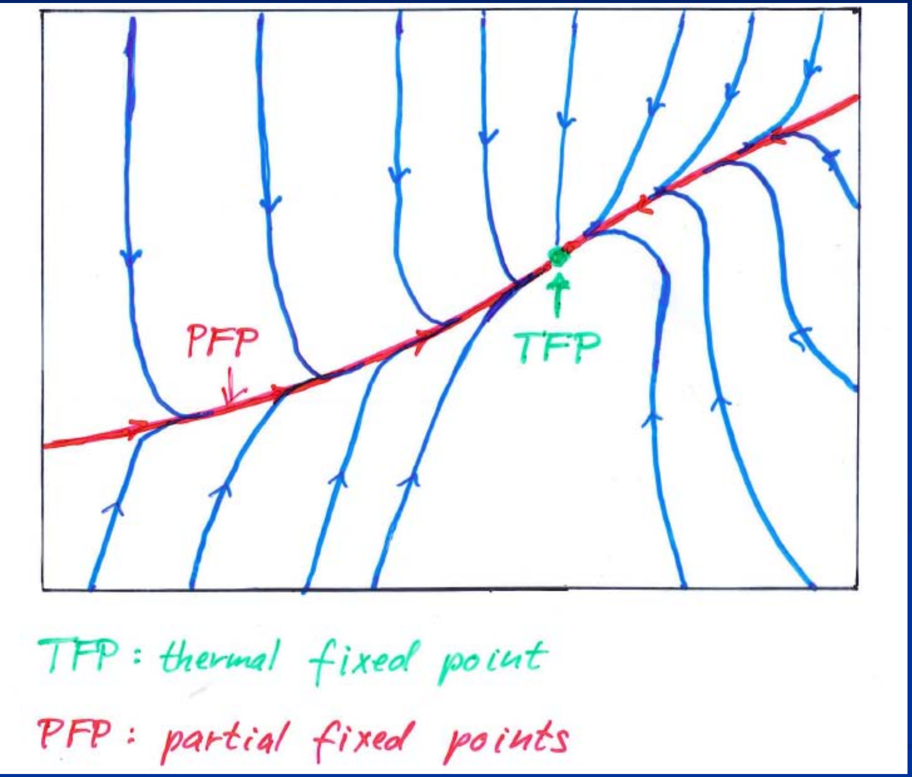
Boltzmann's conjecture

- start with arbitrary initial probability distribution
- wait long enough
- probability distribution comes arbitrarily close to thermal equilibrium distribution

probably not true !

but : observable correlations come arbitrarily close to thermal equilibrium values (not all systems)

time flow of correlation functions



prethermalized state =
partial fixed point in space of
correlation functions

time evolution of correlation functions

▷ microscopic equation of motion

$$\dot{\chi}_m = F_m[\chi] = \sum_{k=0}^{\infty} f^{(k)}_{m n_1 \dots n_k} \, \chi_{n_1} \cdots \chi_{n_k}$$

▷ ensemble with probability distribution at some initial time t_0

$$p[\chi^0] = \exp(-S_0[\chi^0])$$

time evolution of correlation functions

$$< \chi_{n_1}(t) ... \chi_{n_i}(t) >= Z^{-1} \int D\chi^0 \, \chi_{n_1}(t;\chi^0)...\chi_{n_i}(t;\chi^0) \exp -S_0[\chi^0]$$

$$Z = \int D\chi^0 \exp -S_0[\chi^0]$$

$$\frac{d}{dt}\chi_m = \dot{\chi}_m = F_m[\chi] = \sum_{k=0}^{\infty} f^{(k)}_{mn_1...n_k} \chi_{n_1} \cdots \chi_{n_k}$$

$$\frac{d}{dt} < \chi_m(t) >= \sum_{k=0}^{\infty} f^{(k)}_{mn_1...n_k} < \chi_{n_1}(t)...\chi_{n_k}(t) >$$

hierarchical system of flow equations for correlation functions

BBGKY- hierarchy

Yvon, Born, Green, Kirkwood, Bogoliubov

$$\frac{d}{dt}\chi_m = \dot{\chi}_m = F_m[\chi] = \sum_{k=0}^{\infty} f^{(k)}_{mn_1\ldots n_k}\chi_{n_1}\cdots\chi_{n_k}$$

$$\frac{d}{dt} < \chi_m(t) >= \sum_{k=0}^{\infty} f^{(k)}_{mn_1\ldots n_k} < \chi_{n_1}(t)\ldots\chi_{n_k}(t) >$$

for interacting theories : system not closed

non-equilibrium effective action

$$Z[j,t] = \int D\chi^0 \exp\{-S_0[\chi^0] + j_m^* \chi_m(t;\chi^0)\}$$

$$\Gamma[\varphi,t] = -\ln Z[j,t] + j_m^* \varphi_m,$$
$$\varphi_m = \frac{\partial \ln Z[j]}{\partial j_m^*}$$

variation of effective action yields
field equations in presence of fluctuations, and
time dependent equal-time correlation functions

exact evolution equation

$$\partial_t \Gamma[\varphi] = -\frac{\partial \Gamma[\varphi]}{\partial \varphi_m} \hat{F}_m[\varphi]$$

$$\hat{F}_m[\varphi] = f_m^{(0)} + f_{mn}^{(1)} \varphi_n + f_{mn_1,-n_2}^{(2)} \{\varphi_{n_1} \varphi_{n_2}^* + (\Gamma^{(2)})^{-1}_{n_1 n_2}\}$$
$$+ f_{mn_1 n_2,-n_3}^{(3)} \{\varphi_{n_1} \varphi_{n_2} \varphi_{n_3}^* + \varphi_{n_1} (\Gamma^{(2)})^{-1}_{n_2 n_3} + \varphi_{n_2} (\Gamma^{(2)})^{-1}_{n_1 n_3} + \varphi_{n_3^*} (\Gamma^{(2)})^{-1}_{n_1,-n_2}$$
$$-(\Gamma^{(2)})^{-1}_{n_1 p_1} (\Gamma^{(2)})^{-1}_{n_2 p_2} (\Gamma^{(2)})^{-1}_{p_3 n_3} \frac{\partial^3 \Gamma}{\partial \varphi_{p_1}^* \partial \varphi_{p_2}^* \partial \varphi_{p_3}}\}$$

$$+ \sum_{k=4}^{\infty} f_{mn_1 \dots n_{k-1},-n_k}^{(k)} (\varphi_{n_1} + (\Gamma^{(2)})^{-1}_{n_1 p_1} \frac{\partial}{\partial \varphi_{p_1}^*})$$
$$\dots (\varphi_{n_{k-2}} + (\Gamma^{(2)})^{-1}_{n_{k-2} p_{k-2}} \frac{\partial}{\partial \varphi_{p_{k-2}}^*})(\varphi_{n_{k-1}} \varphi_{n_k}^* + (\Gamma^{(2)})^{-1}_{n_{k-1} n_k})$$

$$(\Gamma^{(2)})_{mn} = \frac{\partial^2 \Gamma[\varphi]}{\partial \varphi_m^* \partial \varphi_n}$$

cosmology : evolution of density fluctuations

$$\rho(t) = \rho_0 \, t^{-2} \left(1 + \sum_k \underline{\tilde{\delta}_k}(t) \, e^{i(\vec{k}\vec{r})/a(t)} \right)$$

mean density — density contrast

$$\vec{v}(t) = H\vec{r} + i \frac{a(t)}{t} \sum_k \left(\frac{\vec{k}}{k^2} \underline{\eta_k}(t) + \ldots \right) e^{i(\vec{k}\vec{r})/a(t)}$$

Hubble flow — peculiar velocities (... : transversal)

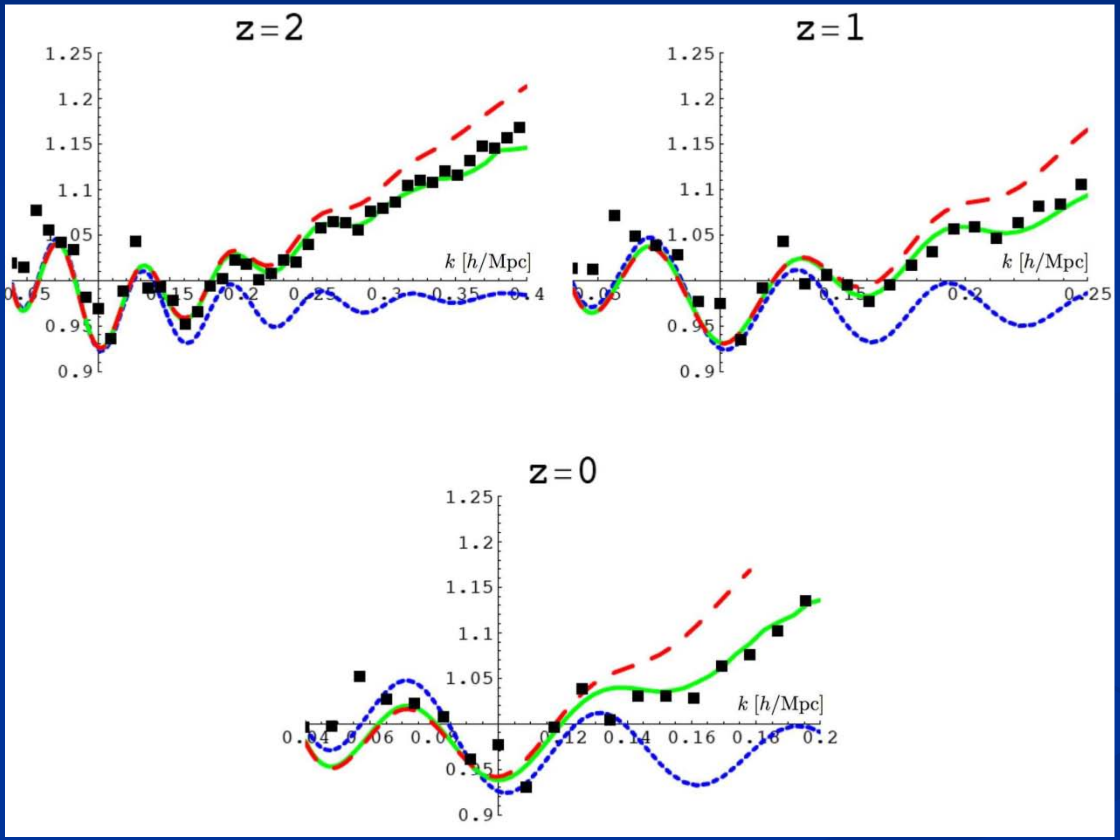
$$\tau = \ln(t/t_0)$$

$$\frac{\partial}{\partial \tau} \tilde{\delta}_k = \eta_k + \sum_{k'} \frac{\vec{k}\vec{k}'}{k'^2} \eta_{k'} \tilde{\delta}_{k-k'}$$

$$\frac{\partial}{\partial \tau} \eta_k = -\frac{1}{3} \eta_k + \frac{2}{3} \tilde{\delta}_k + \frac{1}{2} \sum_{k'} \frac{((\vec{k}-\vec{k}')\vec{k}')\, k^2}{(\vec{k}-\vec{k}')^2 \, k'^2} \eta_{k'} \eta_{k-k'}$$

J.Jaeckel,
O.Philipsen,
...

baryonic accoustic peaks



Pietroni, Matarrese

classical scalar field theory (d=1)

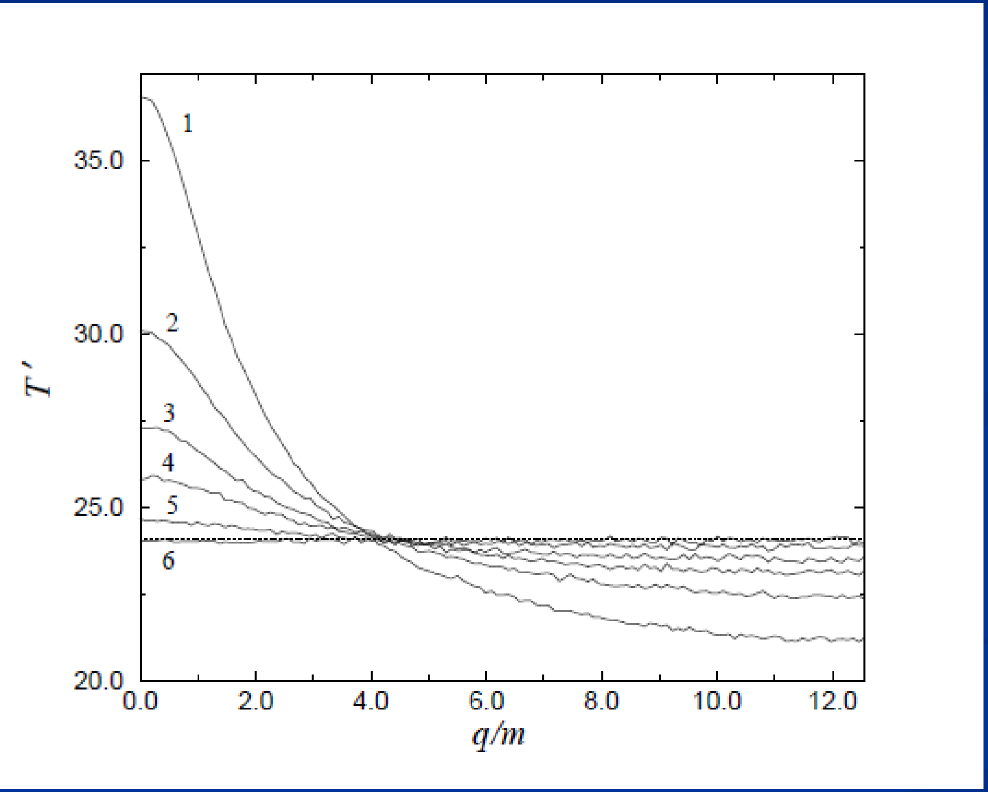
G.Aarts,G.F.Bonini,...

$$\partial_t^2 \phi(x,t) = \left[\partial_x^2 - m^2\right] \phi(x,t) - \lambda \phi^3(x,t)/2$$

$$\pi(x,t) = \partial_t \phi(x,t)$$

can be solved numerically by discretization on space-lattice

thermalization of correlation functions



mode temperature

G.Aarts, G.F.Bonini, CW, 2000

classical scalar field theory (d=1)

G.Aarts,G.F.Bonini,...

$$\partial_t^2\phi(x,t) = \left[\partial_x^2 - m^2\right]\phi(x,t) - \lambda\phi^3(x,t)/2$$

$$\pi(x,t) = \partial_t\phi(x,t)$$

$$\partial_t\Gamma[\phi,\pi;t] = -\mathcal{L}_{\rm cl}\Gamma[\phi,\pi;t],$$

$$\begin{aligned}\mathcal{L}_{\rm cl} \;=\; & \int dx \left[\pi(x)\frac{\delta}{\delta\phi(x)} + \phi(x)\left(\partial_x^2 - m^2 - \frac{1}{2}\lambda\left[\phi^2(x) + 3\bar{G}_{\phi\phi}(x,x)\right]\right)\frac{\delta}{\delta\pi(x)}\right. \\ & - \int dx_1dx_2dx_3\,\bar{G}_{\phi\psi_1}(x,x_1)\bar{G}_{\phi\psi_2}(x,x_2)\bar{G}_{\phi\psi_3}(x,x_3) \\ & \left.\times\frac{\delta^3\Gamma}{\delta\psi_1(x_1)\delta\psi_2(x_2)\delta\psi_3(x_3)}\frac{\delta}{\delta\pi(x)}\right]. \qquad (17)\end{aligned}$$

$$\begin{aligned} G_{\phi\phi}(x-y,t) &= \langle\phi(x,t)\phi(y,t)\rangle, \\ G_{\pi\pi}(x-y,t) &= \langle\pi(x,t)\pi(y,t)\rangle, \\ G_{\pi\phi}(x-y,t) &= \frac{1}{2}\langle\pi(x,t)\phi(y,t) + \phi(x,t)\pi(y,t)\rangle,\end{aligned}$$

$$\bar{G}^{-1}_{\psi\psi'}(x,y) = \frac{\delta^2\Gamma}{\delta\psi(x)\psi'(y)}.$$

truncation

$$\Gamma[\phi,\pi;t] = \int_q \left[\frac{1}{2}A(q)\phi^*(q)\phi(q) + \frac{1}{2}B(q)\pi^*(q)\pi(q) + C(q)\pi^*(q)\phi(q)\right]$$
$$+\frac{1}{8}\int_{q_1,q_2,q_3,q_4} 2\pi\delta(q_1+q_2+q_3+q_4)\Big[u(q_1,q_2,q_3)\phi(q_1)\phi(q_2)\phi(q_3)\phi(q_4)$$
$$+v(q_1,q_2,q_3)\pi(q_1)\phi(q_2)\phi(q_3)\phi(q_4) + w(q_1,q_2,q_3)\pi(q_1)\pi(q_2)\phi(q_3)\phi(q_4)$$
$$+y(q_1,q_2,q_3)\pi(q_1)\pi(q_2)\pi(q_3)\phi(q_4) + z(q_1,q_2,q_3)\pi(q_1)\pi(q_2)\pi(q_3)\pi(q_4)\Big]$$

momentum space

partial fixed points

further truncation :
momentum independent u,v,w,y,z
(N-component scalar field theory , QFT)

$$A_*(q) = \omega_q^2 \, B_*(q)$$

$$C_*(q) = 0$$

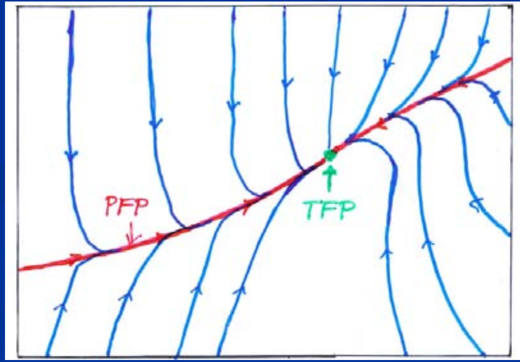
$$v_* = 0 \quad , \quad y_* = 0$$

$$u_* = \frac{\lambda B_*(0) + \frac{1}{2}\omega_0^2 w_*}{1 + \lambda B_*(0) S_0}$$

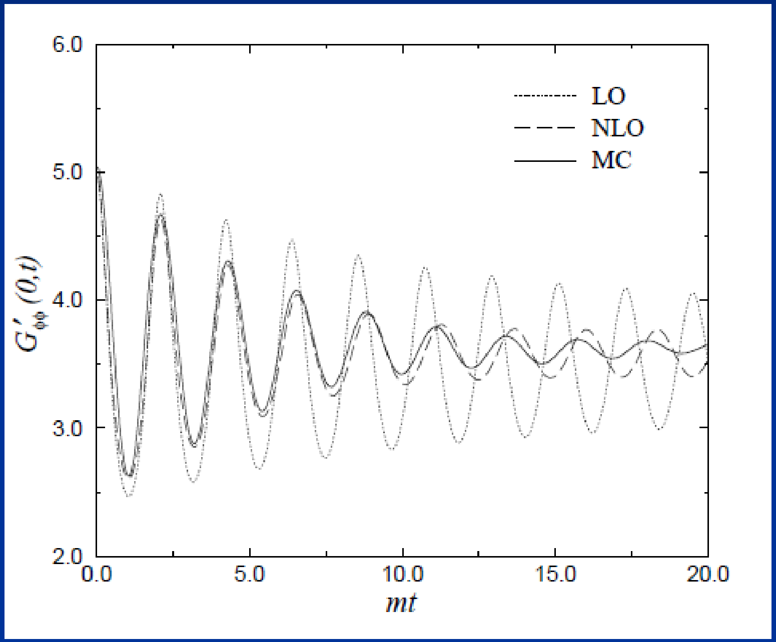
$$w_* = \frac{2\omega_0^2 z_* + \frac{N-1}{N+8}\lambda B_*(0) S_0 \, s_* - \frac{\lambda}{2}\hbar^2 B_*(0)^3}{1 + \frac{N+2}{N+8}\lambda B_*(0) S_0}$$

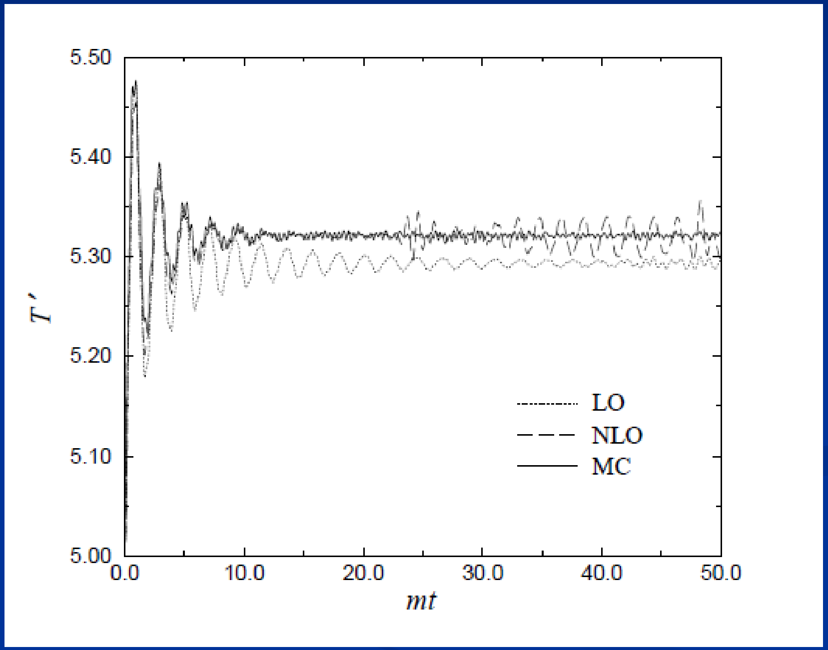
$$\omega_q^2 = q^2 + m^2 + \frac{N+2}{2}\frac{\lambda}{\beta}\Big(\int \frac{d^dq'}{(2\pi)^d}G(q') - \frac{u}{\beta}\int \frac{d^dq'}{(2\pi)^d}\frac{d^dq''}{(2\pi)^d}G(q')G(q'')G(q+q'-q'')\Big)$$

$$S_0 = \frac{N+8}{2}\int \frac{d^Dp}{(2\pi)^D}\, G^2(p)$$



comparison of approximations





quantum field theory

$$Z[j,h,t] =$$

$$\mathrm{Tr}\left\{ \exp\left(j_m Q_m(t) + h_m P_m(t)\right) \rho \right\}$$

Heisenberg - picture :

$\rho \equiv \rho_0$ independent of t

$$\partial_t \Gamma[\varphi,\pi] = -\int d^D x \left\{ \pi_a(x) \frac{\delta\Gamma}{\delta\varphi_a(x)} + F_a(x) \frac{\delta\Gamma}{\delta\pi_a(x)} + \frac{\lambda}{8} \hbar^2 \varphi_a(x) \frac{\delta\Gamma}{\delta\pi_a(x)} \frac{\delta\Gamma}{\delta\pi_b(x)} \frac{\delta\Gamma}{\delta\pi_b(x)} \right\}$$

$$F_a(x) = (\vec{\nabla}^2 - m^2)\varphi_a(x)$$

$$-\frac{\lambda}{2}\Big[\varphi_b(x)\varphi_b(x)\varphi_a(x) + \varphi_a(x) G^{\varphi\varphi}_{bb}(x,x) + 2\varphi_b(x) G^{\varphi\varphi}_{ba}(x,x)$$

$$- \int d^D x_1\, d^D x_2\, d^D x_3\, G^{\varphi\psi}_{ai}(x,x_1)\, G^{\varphi\psi}_{bj}(x,x_2)\, G^{\varphi\psi}_{bk}(x,x_3) \frac{\delta^3\Gamma}{\delta\psi_i(x_1)\delta\psi_j(x_2)\delta\psi_k(x_3)} \Big]$$

$$= \langle F_a[\chi(x)] \rangle \quad ; \quad \psi = \begin{pmatrix} \varphi \\ \pi \end{pmatrix}$$

extensions

(1) non-equal time correlation functions

$$G_n(t_1 \dots t_n) = \langle \varphi(t_1)\varphi(t_2)\dots\varphi(t_n)\rangle_c$$

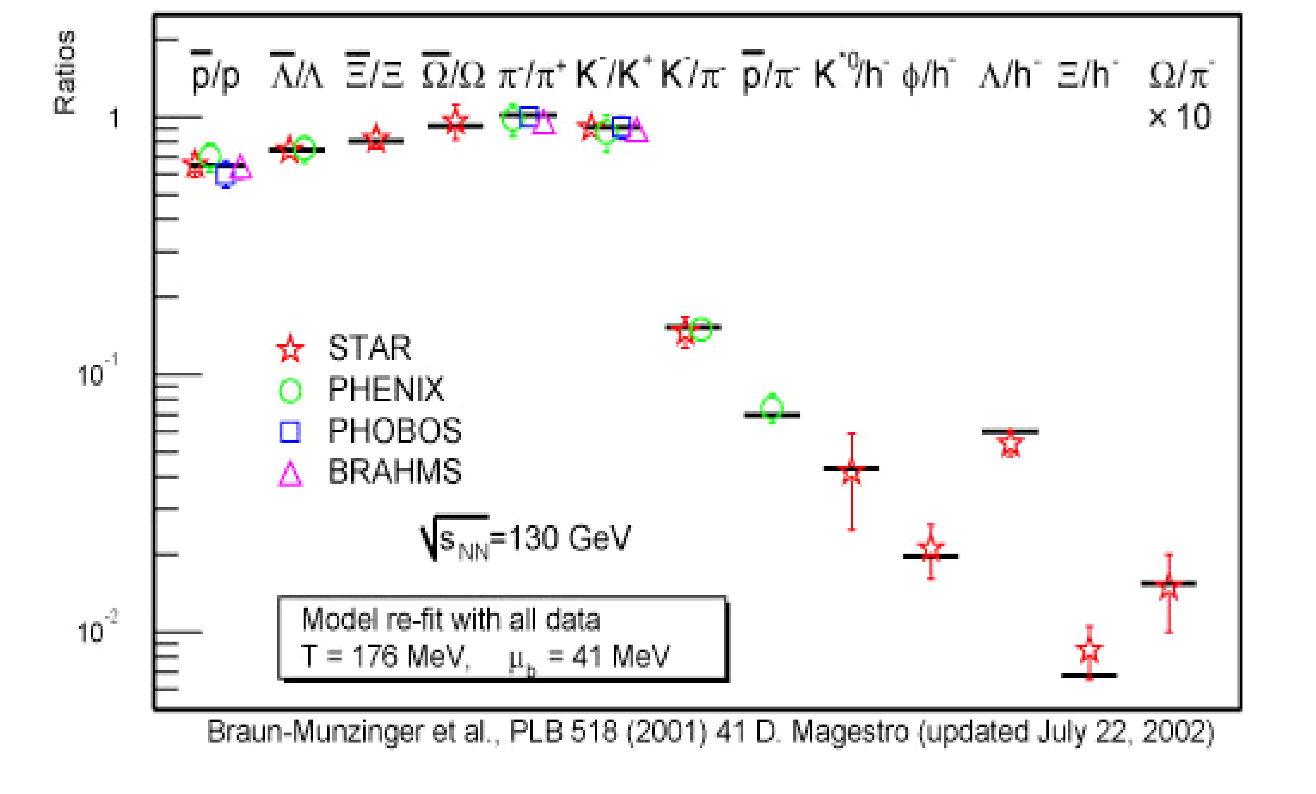
$$t_k = t + \tau_k$$

$\Rightarrow$ $G(t;\tau_1 \dots \tau_k)$

permits contact to Schwinger-Keldish formalism

(2) 2PI instead 1PI ($\Rightarrow$ J.Berges,…)

Hadron abundancies in heavy ion collisions



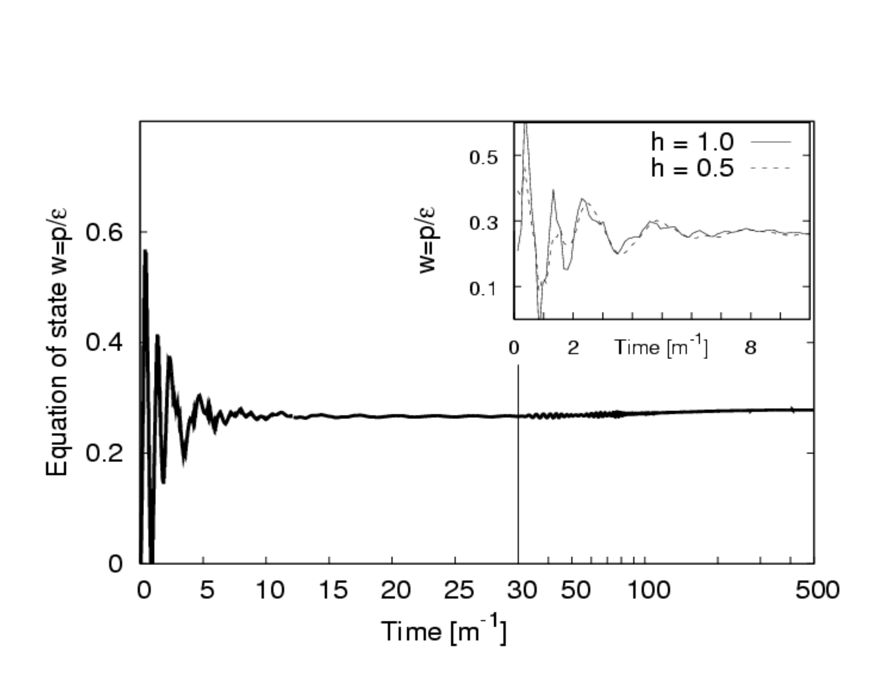
RHIC

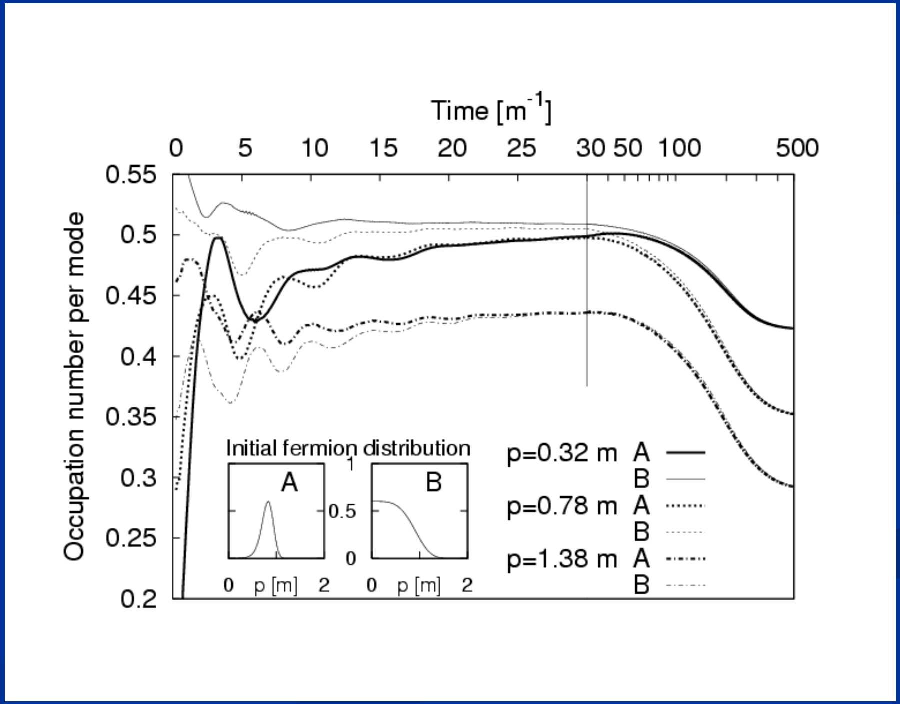
Is temperature defined ?

Does comparison with equilibrium critical temperature make sense ?

Prethermalization

J.Berges,Sz.Borsanyi,CW





Vastly different time scales

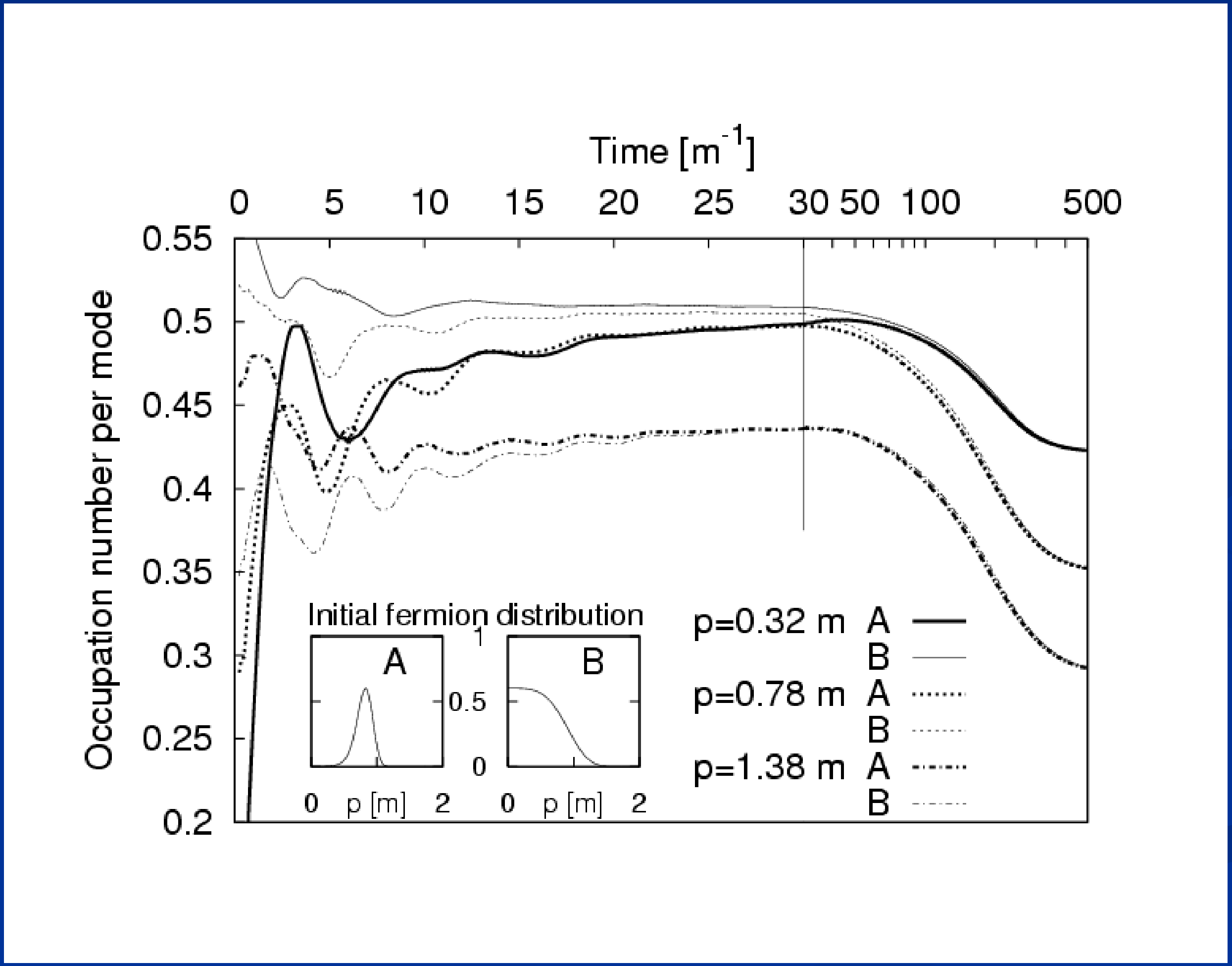
for “thermalization” of different quantities

here : scalar with mass m coupled to fermions

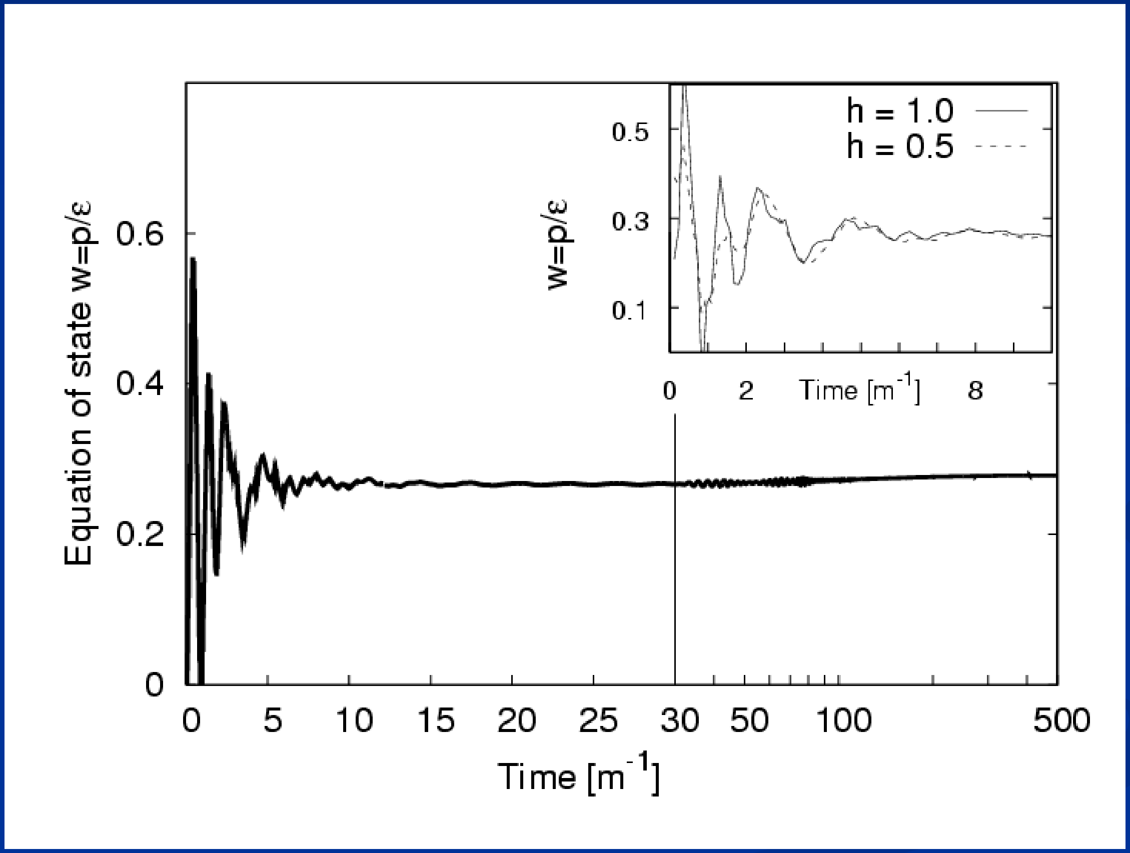
(linear quark-meson-model)

method : two particle irreducible non- equilibrium effective action (J.Berges et al)

Thermal equilibration : occupation numbers



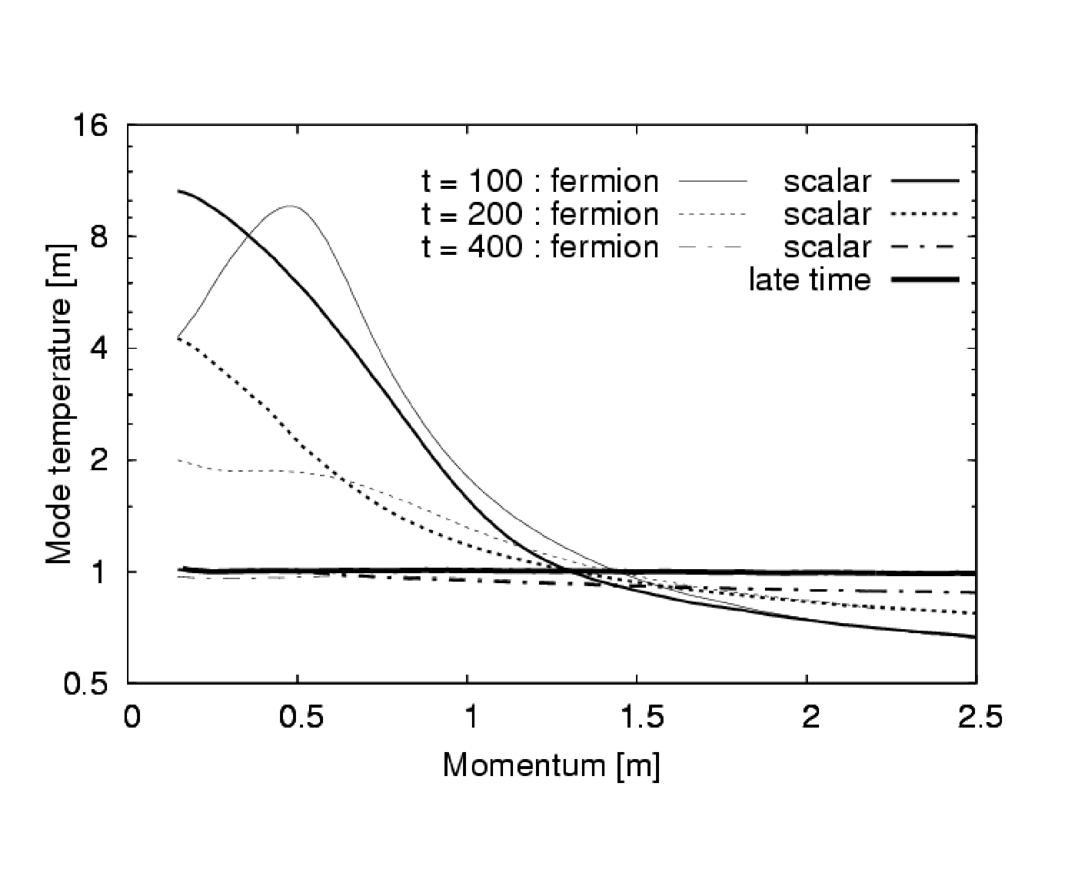
Prethermalization
equation of state p/ε



similar for kinetic temperature

different “temperatures”

Mode temperature



$$n_p(t) \stackrel{!}{=} \frac{1}{\exp\left[\omega_p(t)/T_p(t)\right] \pm 1}$$

$\omega_p^{(f,s)}(t)$ determined by peak of spectral function

n_p :occupation number for momentum p

late time:
Bose-Einstein or Fermi-Dirac distribution

Global kinetic temperature $T_{\rm kin}$

Practical definition:

- association of temperature with average kinetic energy per d.o.f.

$$T_{\rm kin}(t) = E_{\rm kin}(t)/c_{\rm eq}$$

- $c_{\rm eq} = E_{\rm kin,eq}/T_{\rm eq}$ is given solely in terms of equilibrium quantities
  (E.g. relativistic plasma: $E_{\rm kin}/N = \epsilon/n = \alpha T$)

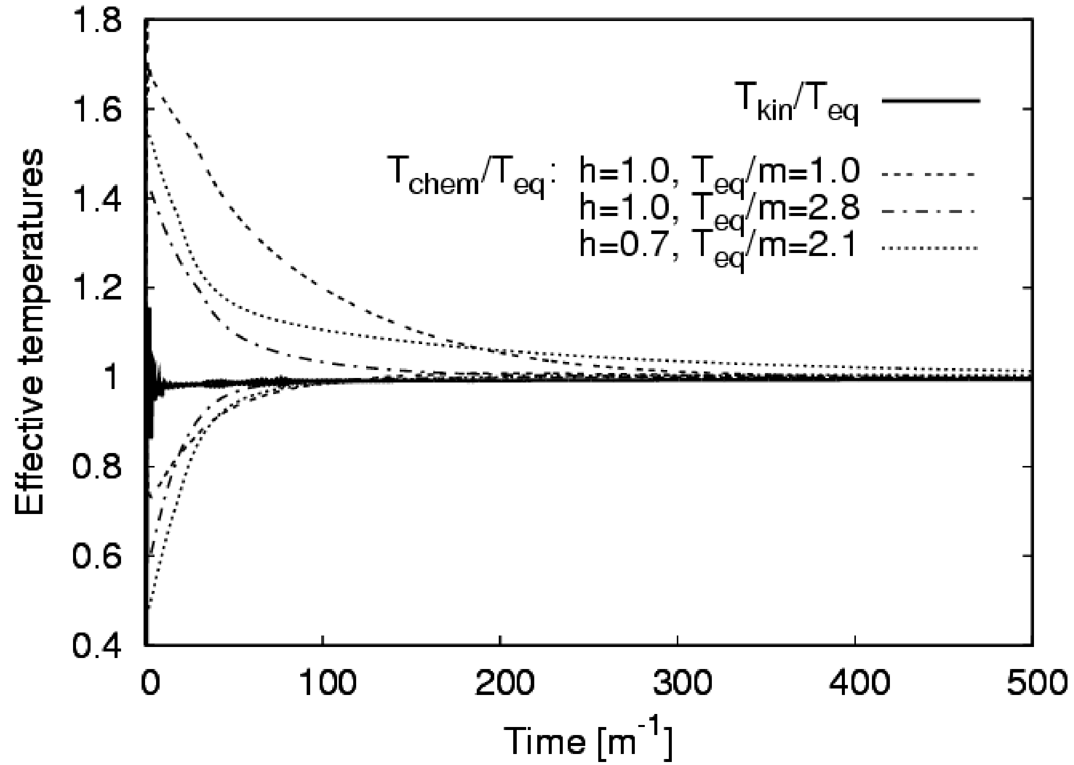
$$\boxed{\text{Kinetic equilibration:} \quad T_{\rm kin}(t) \; = \; T_{\rm eq}}$$

Consider also *chemical temperatures* $T_{\rm ch}^{(f,s)}$ from integrated number density of each species, $n^{(f,s)}(t) = g^{(f,s)} \int {\rm d}^3p/(2\pi)^3 \, n_p^{(f,s)}(t)$:

$$n(t) \stackrel{!}{=} \frac{g}{2\pi^2} \int_0^\infty {\rm d}p p^2 \left[\exp\left(\omega_p(t)/T_{\rm ch}(t)\right) \pm 1\right]^{-1}$$

$$\boxed{\text{Chemical equilibration:} \quad T_{\rm ch}^{(f)}(t) \; = \; T_{\rm ch}^{(s)}(t)}$$

Kinetic equilibration before chemical equilibration

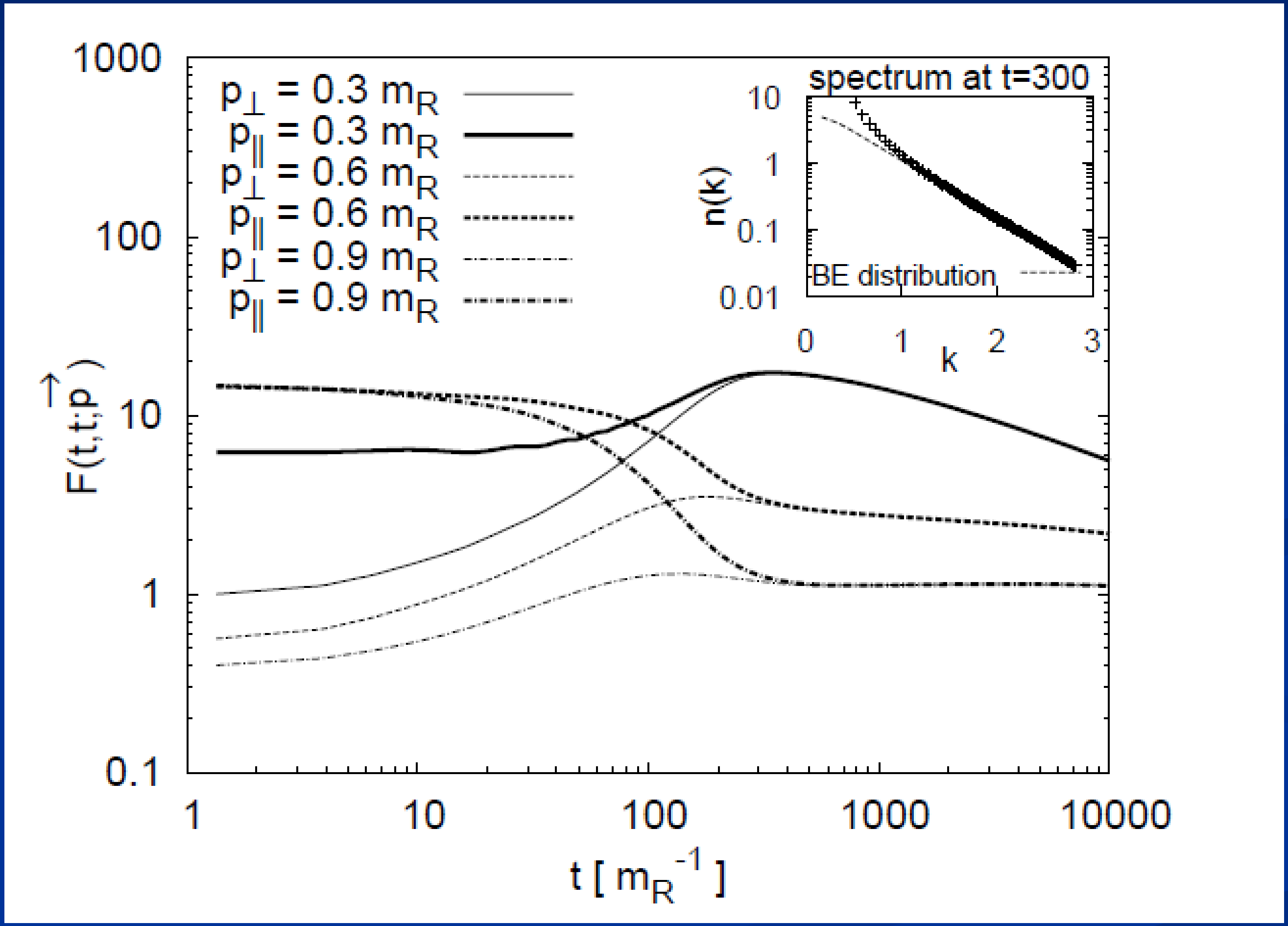


Once a temperature becomes stationary it takes the value of the equilibrium temperature.

Once chemical equilibration has been reached the chemical temperature equals the kinetic temperature and can be associated with the overall equilibrium temperature.

Comparison of chemical freeze out temperature with critical temperature of phase transition makes sense

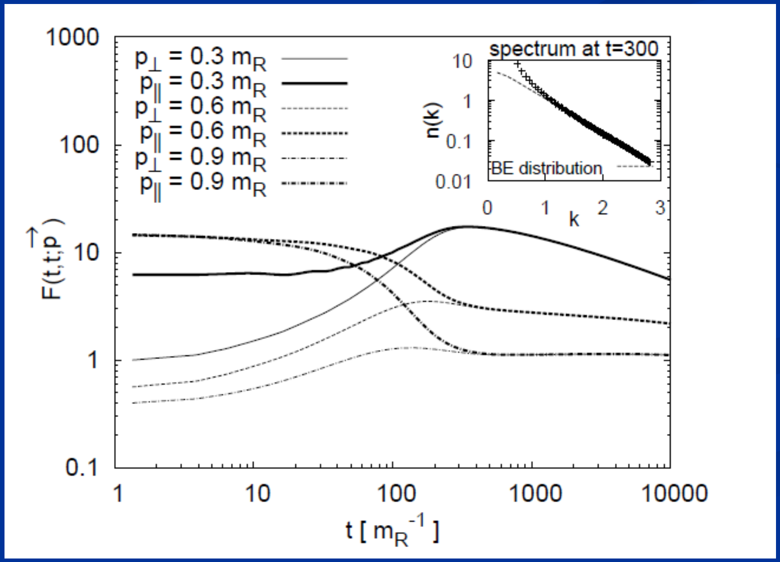
isotropization



two-point functions for momenta in different directions

isotropization

- occurs before thermalization
- different time scale
- gradient expansion, Boltzmann equations become valid only after isotropization



some questions

Can pre-thermalized state be qualitatively different from thermal equilibrium state ?

yes

e.g. e^+ - e^- collisions :

particle abundancies close to thermal ,

momentum disributions not

Is there always a common temperature T in pre-thermalized state ?

no

e.g. two components with weak coupling

Does one always reach thermal equilibrium for time going to infinity ?

no

simple obstructions : initial energy distribution

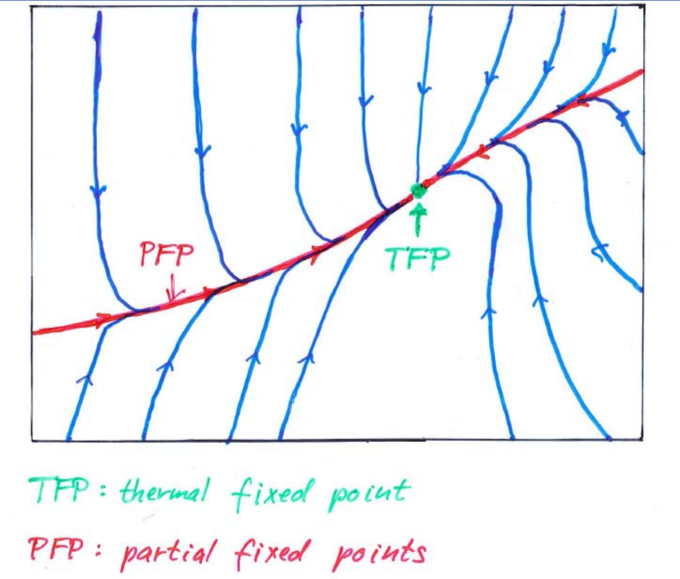
exact non-thermal fixed points are possible

instabilities from long-range forces (gravity)

in practice : metastable states

role and limitation of linear response theory ?

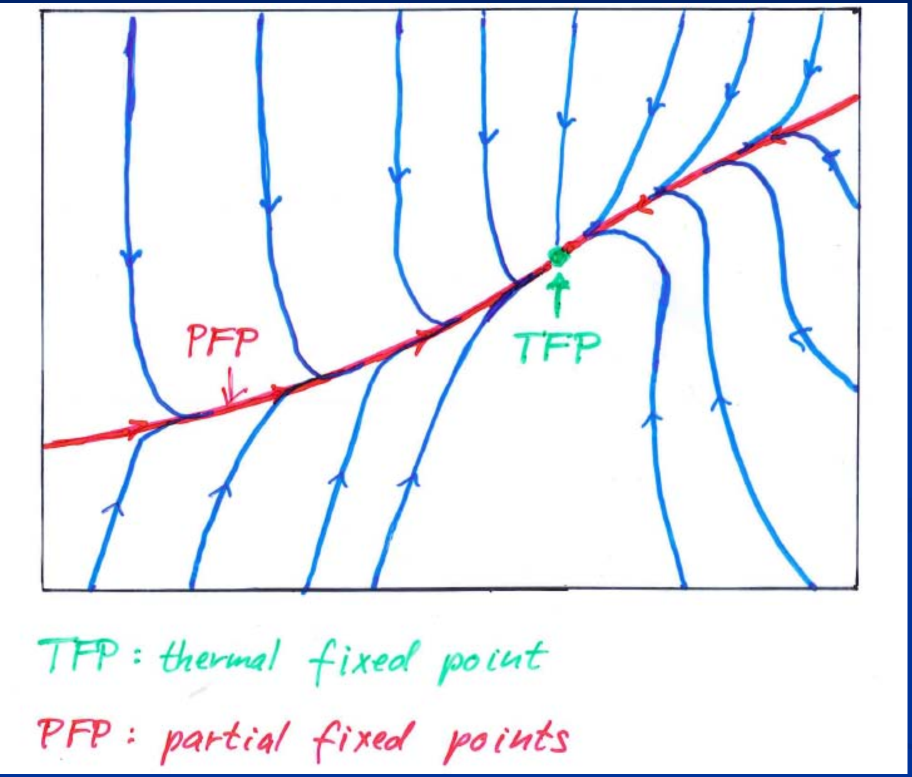
- fails for approach to thermal equilibrium
- time scales of linear response are often characteristic scales for prethermalization



conclusions

- Approach to thermal equilibrium is a complex process involving very different time scales.
- This holds already for simple models as scalar field theory.
- Observation sees often only early stages , not equilibrium state : prethermalization.
- Prethermalization can be characterized by partial fixed points in flow of correlation functons.

time flow of correlation functions



prethermalized state =
partial fixed point in space of
correlation functions

end

Numerical simulation

G. Aarts, G.F. Bonini

D = 1, periodic lattice, ϕ^4-theory

* solve discretized microscopic equations numerically
* take ensemble averages over initial conditions
* Compare correlation functions for large t with thermal values

Maxwell - velocity distribution

(classical systems)

$$\langle \pi(x)\, \pi(y) \rangle_T = \frac{T}{a}\, \delta_{xy}$$

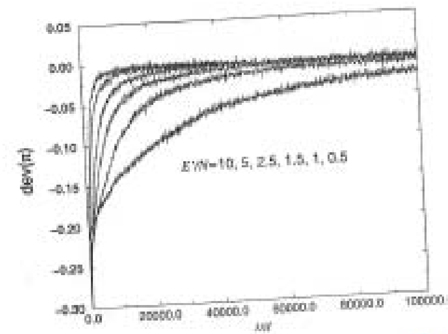
$$G_s^{(n)}(x_s, t) \equiv \frac{1}{N_s} \sum_x^{(N_s)} \pi^n(x, t)$$

$$\mathrm{dev}(\pi_s) = \frac{\langle G_s^{(4)}(x_s, t) \rangle}{3 \langle G_s^{(2)}(x_s, t) \rangle^2} - 1$$

* $\mathrm{dev}(\pi)$ measures the deviation from Gaussian momentum distribution

* thermal equilibrium : $\mathrm{dev}(\pi) = 0$

dev (π)



Aarts, Bonini, W

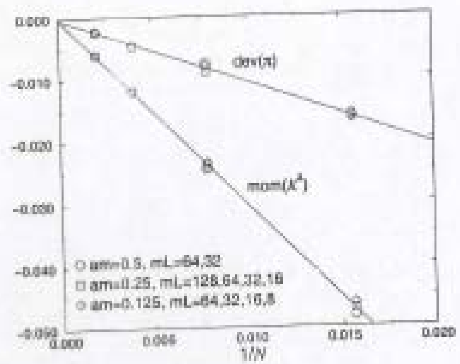
dev (π) for different initial energies

equilibration time depends strongly on $E' = \frac{3\lambda E}{m^3}$

slow relaxation for small E'

thermalization for time averaged correlation functions for fixed initial energy

dev (π)



asymptotic value of dev(π) as function of number of degrees of freedom N

„mesoscopic dynamics" for finite volume and finite cutoff

$\langle dev(\pi) \rangle$ only depends on N, not L and Λ separately

dev(π)

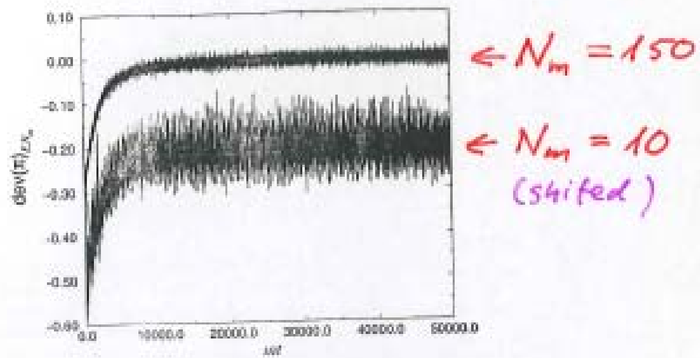


Figure 4: Time dependence of $\mathrm{dev}(\pi)_{E,N_m}$ in nonequilibrium ensembles with $N_m = 10$ members (lower curve, shifted down by 0.2 for clarity) and $N_m = 150$ members (upper curve). In a larger ensemble the size of fluctuations is reduced. Parameters are $mL = 32$, $N = 256$, $E'/N = 4$.

long equilibration time !

effective thermalization for

fixed energy initial ensemble !

fluctuations decrease for

ensembles with many

microstates !

Generic ensembles with

nonzero initial spread

in energy

consider Gaussian initial

energy distribution

$$\rho(E) = \left(\frac{\kappa}{2\pi \bar{E}^2}\right)^{1/2} \exp\left(-\frac{\kappa}{2}\frac{(E-\bar{E})^2}{\bar{E}^2}\right)$$

assume for asymptotic expectation values

$$\langle \sigma \rangle_\kappa = \int dE \, \rho(E) \, \langle \sigma \rangle_E$$

$$\overset{!}{=} \int dE \, \rho(E) \, \langle \sigma \rangle_{T(E)}$$

$$\frac{\Delta E^2}{E^2} = \varkappa \neq O\left(\frac{1}{N}\right)$$

$\Delta E^2, E$: conserved correlation functions

consequences for asymptotic behavior ?

Obstruction to thermalization !