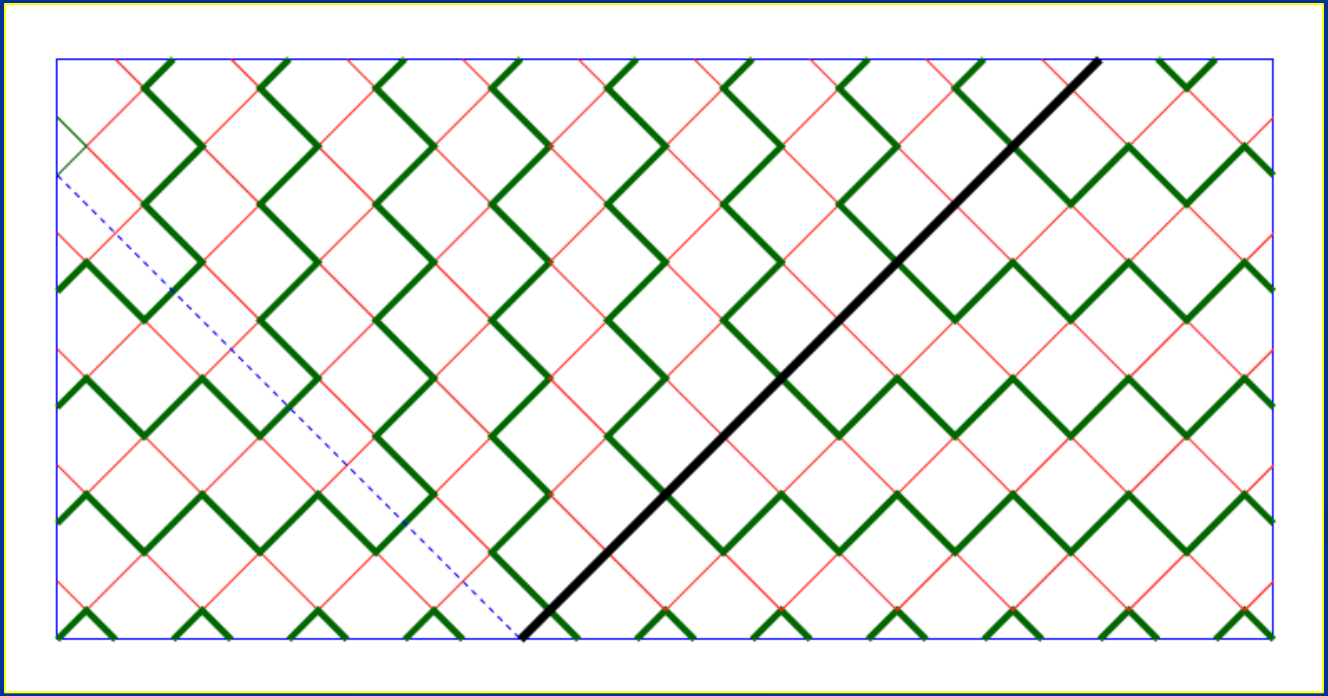
Probabilistic cellular automata for fermionic quantum field theories



Some interacting fermionic quantum field theories or many body systems
are equivalent to
probabilistic cellular automata

Quantum field theory and quantum mechanics

- Is the Thirring model a model for quantum mechanics ?

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

- Yes, for a given vacuum consider the one-particle state.

Quantum mechanics from classical statistics

- Probabilistic Cellular automata are classical statistical systems
- Quantum mechanics emerges from a classical statistical system.
- All no go theorems (Bell etc.) are circumvented

Fermions

- quantum objects
- wave function totally antisymmetric
  (Pauli principle)
- anticommutator for annihilation and creation operators
- anticommuting Grassmann variables
- functional integral or partition function for many body systems or quantum field theories is Grassmann functional integral

Cellular automaton

- Deterministic manipulation of bits
- Updating rule of bit configurations in sequential steps
- usually: repetition

(Classical computer is a type of cellular automaton without repetition)

Cellular automaton

At each step :

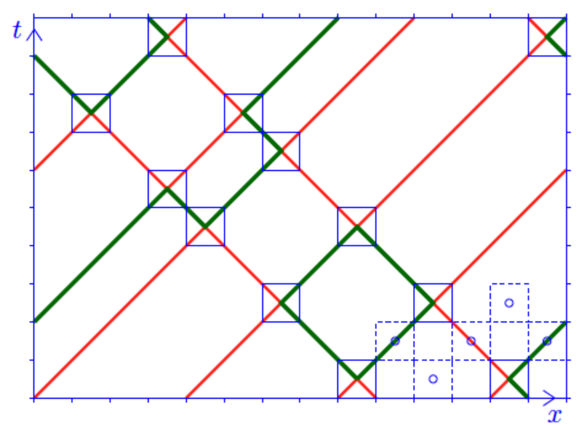
- each bit configuration changes to a unique new bit configuration according to an updating rule

- for a fixed initial configuration : classical deterministic computing

Updating rule for Thirring automaton

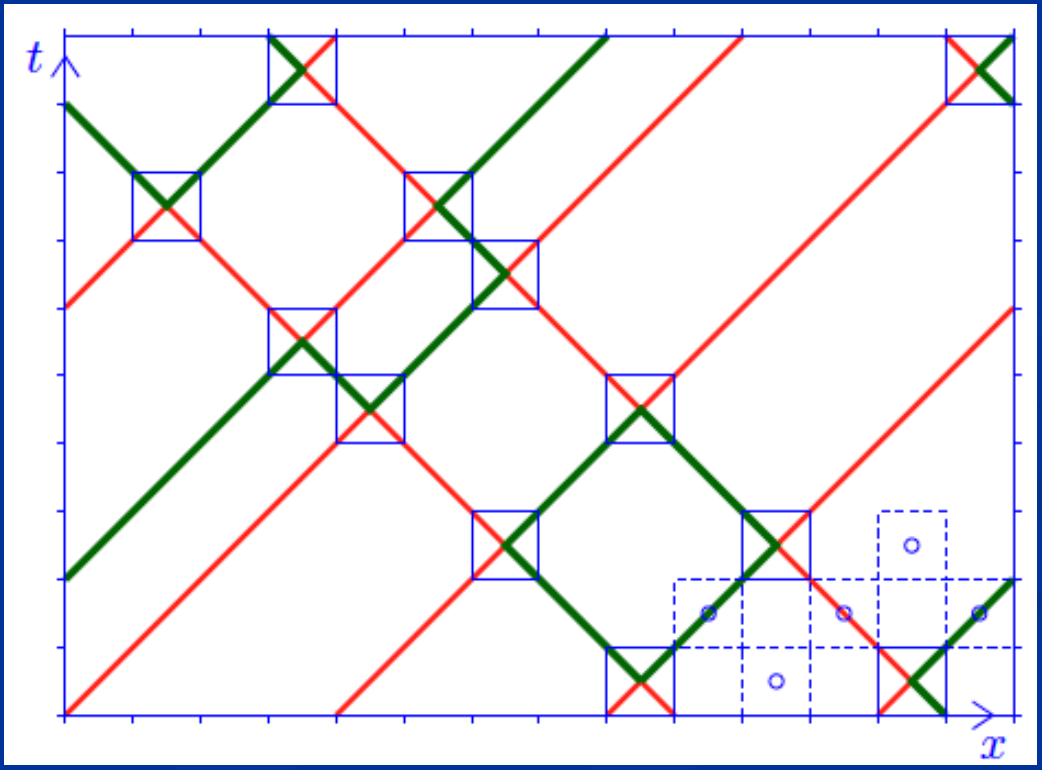
- one – dimensional chain, x : discrete lattice sites
- at each x : red and green right movers and left movers (4 different species at each site)
- at each time step: configuration for right(left) movers moves one position to the right(left)
- if two single particles meet:

  colors are exchanged

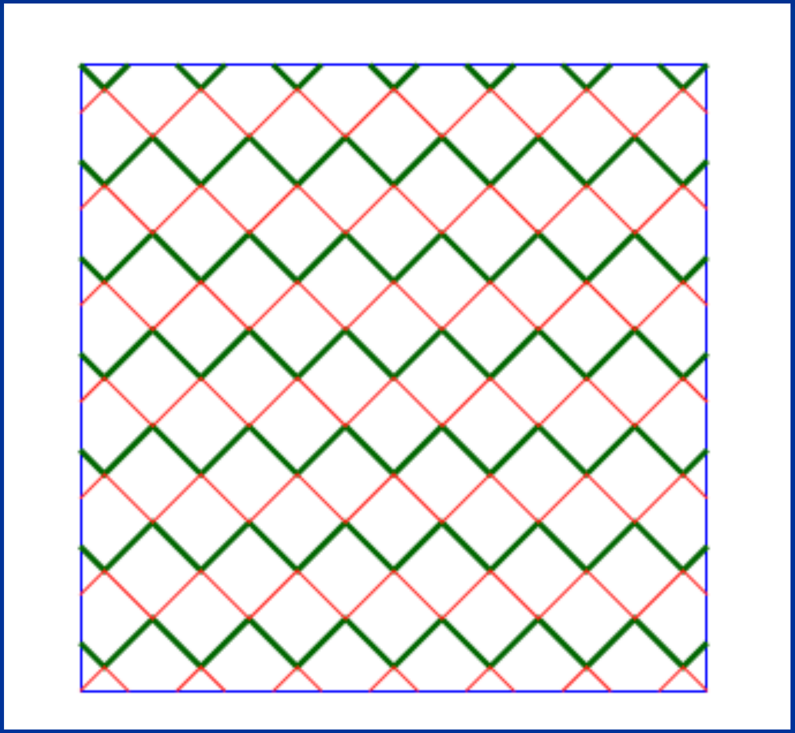


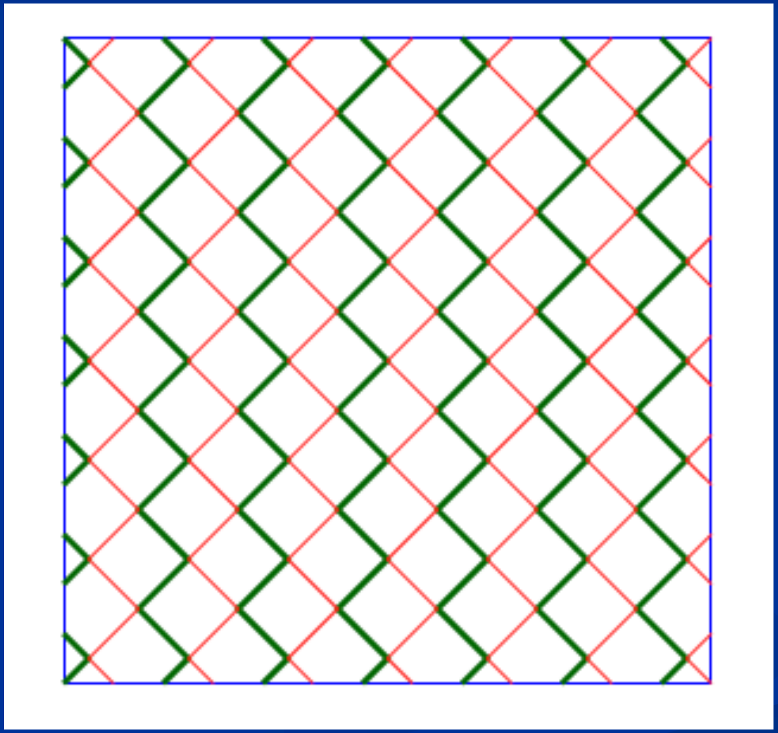
Updating rule

- at each time step configuration for right(left) movers moves one position to the right(left), periodicity in x
- if precisely two single particles meet at a site : colors are exchanged

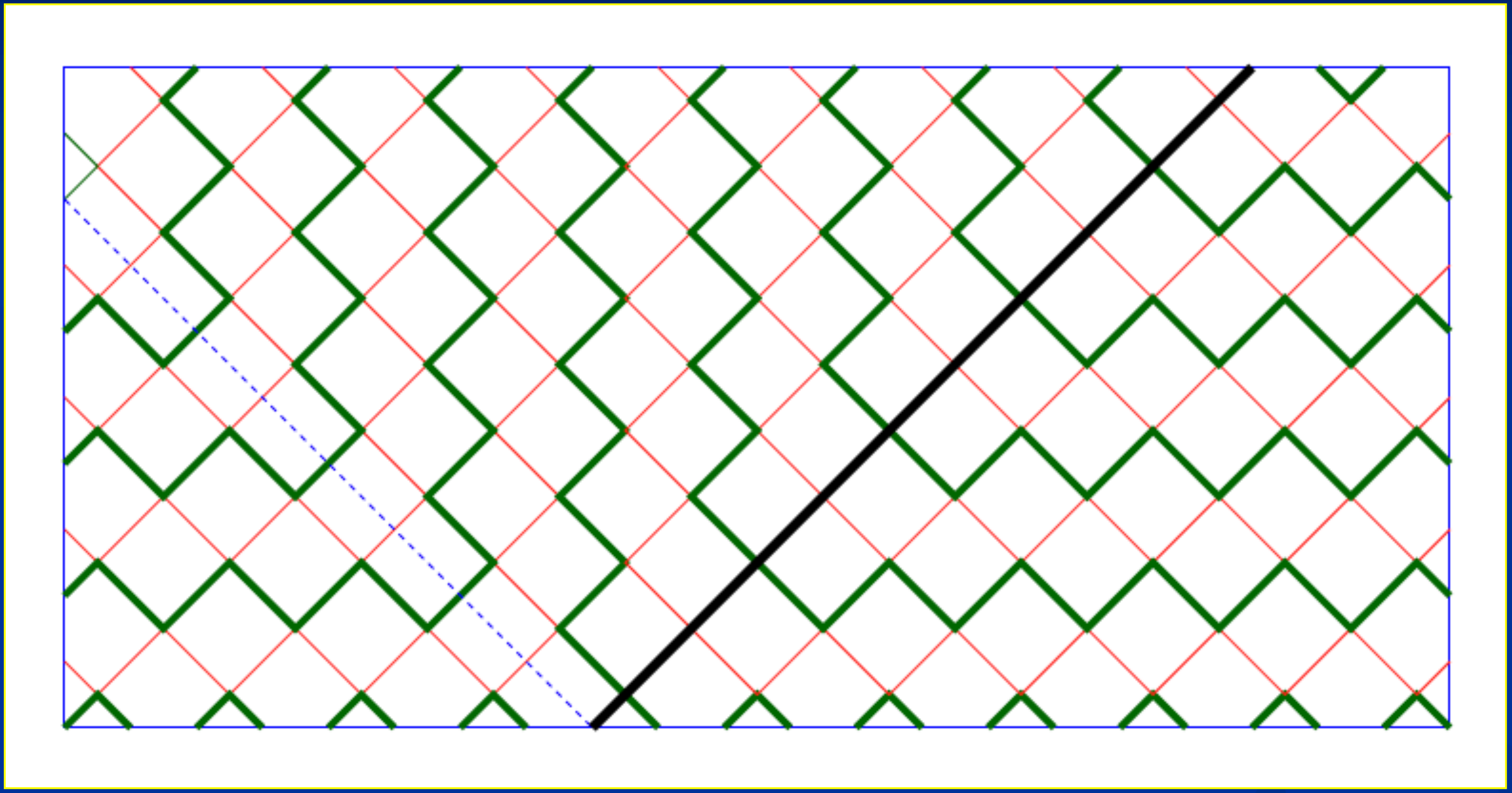


Half filled ground states



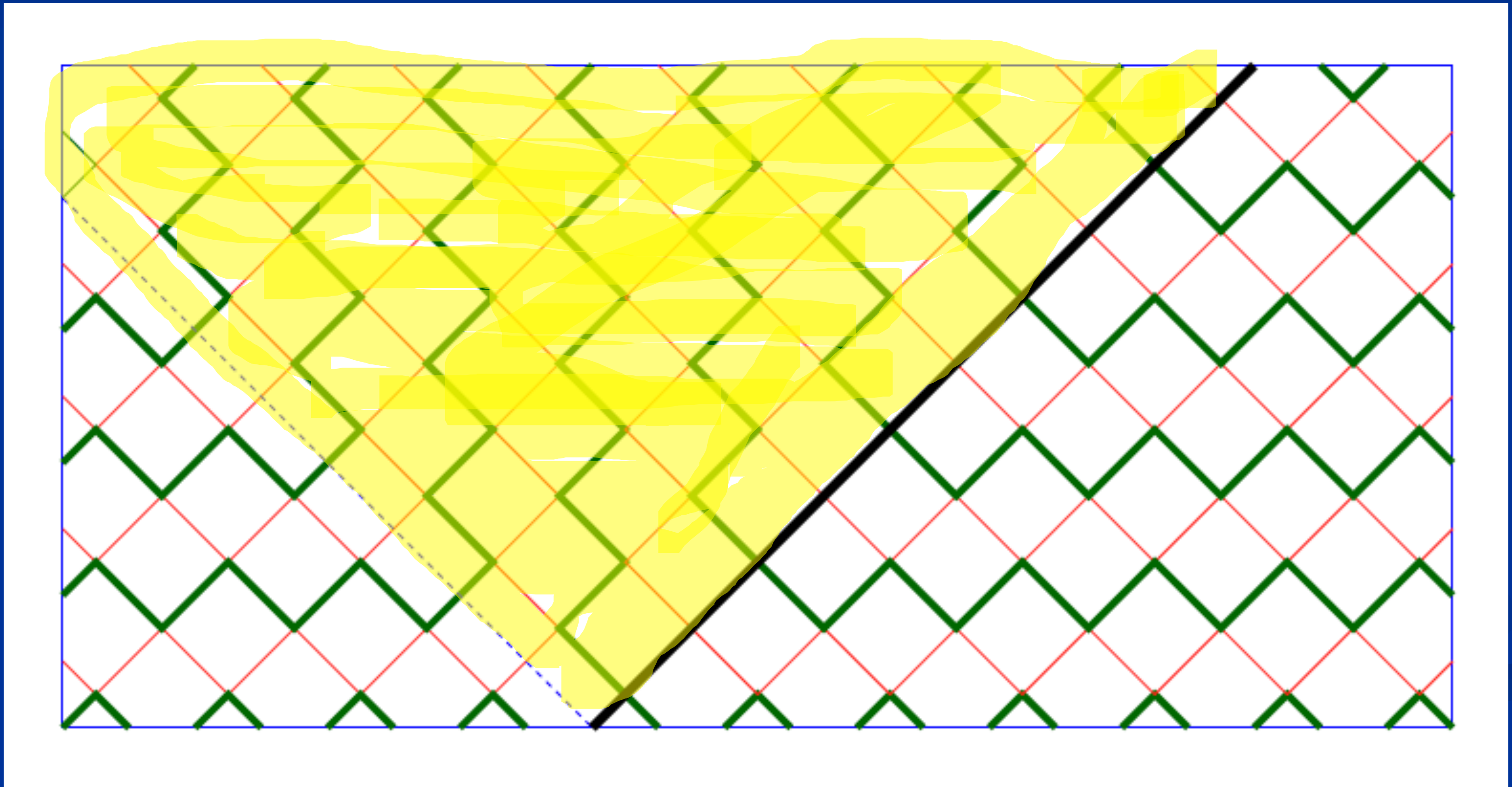


Soliton



black line : no right movers,
or two right movers with different colors

Soliton separates different vacua



Probabilistic cellular automaton

Probability distribution for initial configurations

(or other probabilistic boundary condition)

Probabilistic cellular automaton

Probabilistic initial condition: Specify at initial time t_{in} for each bit configuration $\overline{\rho}$ a probability $p_{\overline{\rho}}(t_{\text{in}})$

Evolution: every given configuration $\overline{\rho}$ at t_{in} propagates at t to a configuration $\tau(t, \overline{\rho})$

$$\Rightarrow \quad p_{\tau}(t) = p_{\overline{\rho}(\tau)}(t_{\text{in}})$$

Updating rule: specifies $\tau\big(t+\varepsilon, \rho(t)\big)$

Wave function for probabilistic cellular automaton

Probability distribution: at every time t a bit configuration τ occurs with probability $p_\tau(t)$, which equals the probability for the initial bit configuration from which it originates.

Real wave function q(t): probability amplitude

$$p_\tau(t) = (q_\tau(t))^2$$

$$q_\tau(t) q_\tau(t) = 1$$

N – component unit vector

Deterministic and probabilistic cellular automaton

- Deterministic CA : sharp wave function

$$q_\rho(t_{\text{in}}) = \delta_{\rho,\overline{\rho}}.$$

- Probabilistic CA : arbitrary wave function

Particle wave duality

Particle aspect:

- Bits: yes/no decisions
- Possible measurement values 1 or 0

Discrete spectrum of observables

Wave aspect : continuous wave function

more generally: continuity of probabilistic information

Step evolution operator

- Evolution for basic time step is encoded in the step evolution operator

$$q(t+\varepsilon) = \widehat{S}(t)q(t)$$

$$q_\tau(t+\varepsilon) = \widehat{S}_{\tau\rho}(t)q_\rho(t)$$

- Contains the updating rule for CA

$$\widehat{S}_{\tau\rho}(t) = \delta_{\tau,\overline{\tau}(\rho)} = \delta_{\overline{\rho}(\tau),\rho}$$

$$q_\tau(t+\varepsilon) = q_{\overline{\rho}(\tau)}(t)\,, \quad p_\tau(t+\varepsilon) = p_{\overline{\rho}(\tau)}(t)$$

Unique jump matrix

- Step evolution operator for cellular automata is unique jump matrix
- In every row and coloumn: precisely one element +1 or -1, all other elements zero

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Equivalence with fermionic QFT

The probabilistic Thirring automaton is equivalent with a fermionic quantum field theory in 1+1 dimensions, namely a discretization of a type of Thirring model

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

two colors: a,b = 1,2 = red, green

Fermions are Ising spins or bits

- Fermionic occupation numbers n = 0 , 1
- Classical bits
- Ising spins s = 2n -1
- Bit configurations = many body states of fermions

Fermionic wave function

- Occupation number basis for multi-fermion systems:

- To each bit configuration one associates an element of the wave function
- Occupation numbers for different space points and species

Complex structure

- Real formulation of complex wave function always possible
- Complex formulation of real wave function requires complex structure based on appropriate involution
- particle - hole conjugation

QFT- CA equivalence

A (discrete) fermionic quantum field theory is equivalent to a probabilistic cellular automaton if the evolution operator for discrete time steps is a

unique jump matrix

(in a real formulation of the evolution equation)

Continuum limit

- Discretized fermionic QFT (e.g. on lattice) makes model well defined
- Regularized functional integral
- Step evolution operator
- Needs continuum limit

FRG ?

Overall probability distribution for cellular automata

Generalized Ising model: $w[s] = Z^{-1} \exp\big(-S[s]\big) b(s_{in}, s_f)$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

limit : β to infinity , σ to zero :
only one possibility for change , unique jump

probabilistic aspects only in boundary term : $b(s_{in}, s_f) = \bar{f}_f(s_f) f_{in}(s_{in})$

Functional integral for cellular automata

Generalized Ising model: $w[s] = Z^{-1} \exp\big(-S[s]\big) b(s_{in}, s_f)$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

limit : β to infinity , σ to zero :
only one possibility for change , unique jump

Functional renormalization for cellular automata

Probabilistic computing with static memory materials ?

- Let general equilibrium classical statistics transport information from one layer to the next

- Simulation, with D. Sexty

Static memory materials

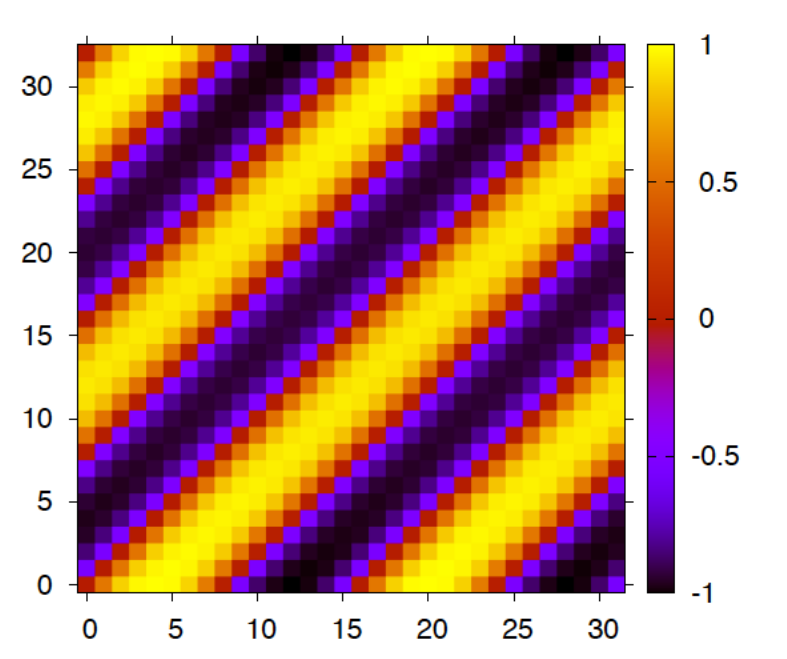
Generalized Ising model: $w[s] = Z^{-1} \exp\left(-S[s]\right) b(s_{in}, s_f)$

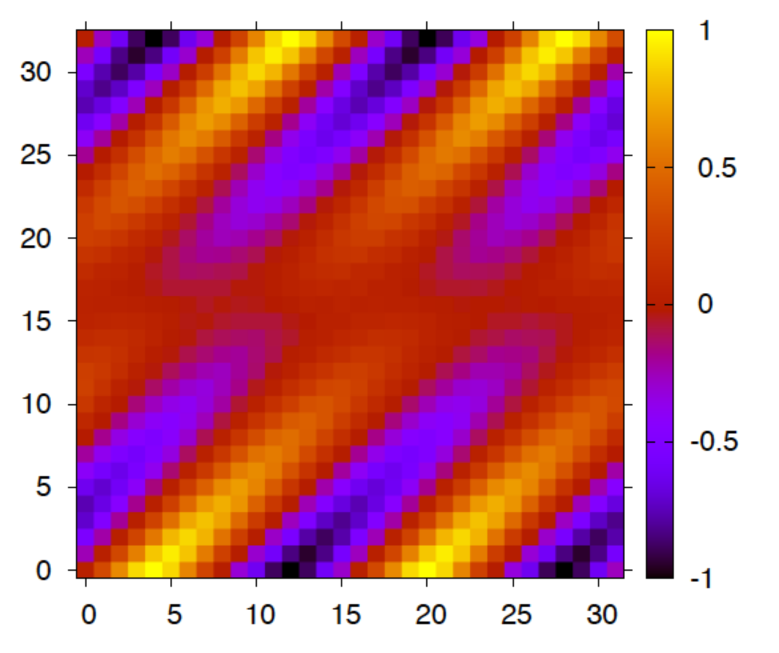
$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \left[s(t+1, x+1) + \sigma s(t+1, x-1) \right]$$

Boundary term : $b(s_{in}, s_f) = \bar{f}_f(s_f) f_{in}(s_{in})$

Classical interference

Depending on boundary conditions :





Positive interference

Negative interference

Static memory material for
two dimensional Ising spins
on Euclidean square lattice
can describe propagation of Weyl fermion
in two- dimensional Minkowski space

Continuum limit

- Similar generalized Ising model for Thirring automaton
- Continuum limit involves

  coarse graining
- Functional renormalization such that pure quantum state remains pure quantum state on coarse grained level

Step evolution operator

- Sequence of kinetic (free) and interaction part

$$\widehat{S} = \widehat{S}_{\text{int}} \, \widehat{S}_{\text{free}}$$

- Local interaction

$$\widehat{S}_{\text{int}} = \widehat{S}_i(x_{\text{in}}) \otimes \widehat{S}_i(x_{\text{in}} + \varepsilon) \otimes \widehat{S}_i(x_{\text{in}} + 2\varepsilon) \otimes \dots$$

(1) at each time step configuration for right(left) movers
moves one position to the right(left),
(2) if precisely two single particles meet at a site : colors are exchanged

Annihilation and creation operators

Step evolution operator for Thirring automaton can be written in terms of fermionic annihilation and creation operators

$$\{a_\gamma^\dagger(x), a_\delta(y)\} = \delta_{\gamma\delta}\delta_{xy} \qquad \{a_\gamma(x), a_\delta(y)\} = \{a_\gamma^\dagger(x), a_\delta^\dagger(y)\} = 0$$

$$\widehat{S}_i(x) = \exp\left\{\frac{i\pi}{2}\left[a_{\mathrm{R}1}^\dagger(x)a_{\mathrm{R}2}(x) - a_{\mathrm{R}2}^\dagger(x)a_{\mathrm{R}1}(x)\right]\left[a_{\mathrm{L}1}^\dagger(x)a_{\mathrm{L}2}(x) - a_{\mathrm{L}2}^\dagger(x)a_{\mathrm{L}1}(x)\right]\right\}$$

$$\widehat{S}_{\mathrm{free}} = \widehat{S}_1^{(\mathrm{R})} \otimes \widehat{S}_2^{(\mathrm{R})} \otimes \widehat{S}_1^{(\mathrm{L})} \otimes \widehat{S}_2^{(\mathrm{L})}$$

$$\widehat{S}_a^{(\mathrm{R,L})} = N\left[\exp\left\{\sum_x a^\dagger(x \pm \varepsilon)\left[a(x) - a(x \pm \varepsilon)\right]\right\}\right]$$

Hamiltonian

- Define H by $\widehat{S} = \exp\left(-i\varepsilon H\right)$

- Interpolating continuous time evolution

$q(t_2) = U(t_2, t_1) q(t_1)$ $U(t_1, t_2) = \exp\left(-i(t_1 - t_2)H\right)$

- Agrees with discrete evolution for $t_{\text{in}} + m\varepsilon$

- Schrödinger equation $i\partial_t q = Hq$

Naïve continuum limit

- Hamiltonian simplifies in the continuum limit

$$H = H_{\text{free}} + H_{\text{int}} + \Delta H$$

$$\Delta H = \mathcal{O}(\varepsilon[H_{\text{int}}, H_{\text{free}}])$$

- Standard form of Hamiltonian for fermions

$$H_{\text{free}} = \frac{i}{\varepsilon}\int \mathrm{d}x \sum_a \left\{ a^\dagger_{La}(x)\partial_x a_{La}(x) - a^\dagger_{Ra}(x)\partial_x a_{Ra}(x) \right\}$$

$$H_{\text{int}} = -\frac{\pi}{2\varepsilon^2}\int \mathrm{d}x \left[a^\dagger_{\text{R}1} a_{\text{R}2} - a^\dagger_{\text{R}2} a_{\text{R}1}\right]\left[a^\dagger_{\text{L}1} a_{\text{L}2} - a^\dagger_{\text{L}2} a_{\text{L}1}\right]$$

General bit fermion map

- Isomorphism between generalized Ising model and Grassmann functional integral
- Based on identical step evolution operator for both models, with associated map of observables
- In our case: proof that discrete Thirring model with two colors has the same step evolution operator as the cellular automaton

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x) - \Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x)\Big]\left(1+\overline{D}(x)\right)\Big\}$$

Discrete fermionic quantum QFT evolution is unitary

Grassmann functional integral

$$Z = \int \mathcal{D}\psi \exp(-S[\psi]) = \int \mathcal{D}\psi w[\psi]\,, \quad S = \sum_t \mathcal{L}(t)$$

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x)$$
$$- \left[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x)\right]\left(1 + \overline{D}(x)\right)\Big\}$$

$$\overline{D} = -\left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}\right) - \left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)$$

Functional renormalization of probabilistic cellular automata

- Functional integral for discretized fermionic QFT: Grassmann variables
- Functional integral for generalized Ising model

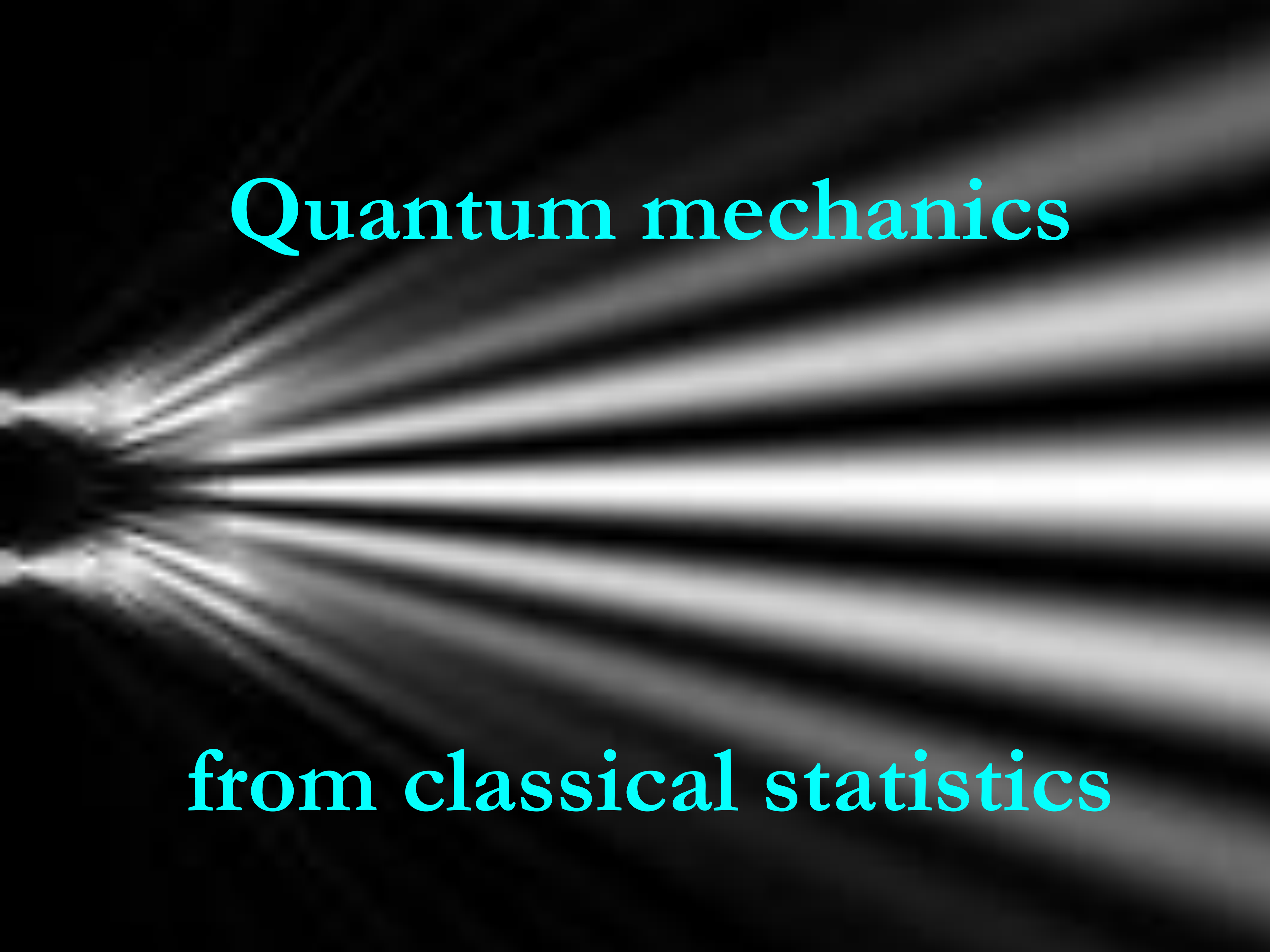
- Extract effective evolution equation for one-particle excitations from two point function
- Two point function from effective action
- No need for investigation with explicit boundary terms
- Boundary terms as source terms

Quantum formalism for classical statistics

- Formalism for information transport from one hypersurface to the next:
- Classical wave functions and density matrix
- Transfer matrix formalism : Heisenberg picture
- Wave functions : Schrödinger picture
- Non commuting operators for observables
- Quantum rules from classical statistical rules

Momentum observable

- Measures periodicity of wave function
- Statistical observable, similar to temperature
- No fixed value in microstate
- Classical correlation function with occupation numbers does not exist
- Needs probabilistic information


Quantum mechanics
from classical statistics

Quantum mechanics from classical statistics

For particular quantum model:

Isomorphism between classical statistics
(probabilistic cellular automaton,
generalized Ising model)

and quantum mechanics
(many body quantum system for fermions)

Equivalence

- Expectation values of all observables are the same in both models
- Two equivalent descriptions of the same physical reality

Important conceptual consequences

- Probabilistic cellular automata are **classical statistical systems**
- Fermionic quantum field theories are **quantum systems**
- Quantum mechanics emerges from classical statistics
- Quantum formalism can be used for classical statistics

Can quantum physics be described by classical probabilities ?

“ No go “ theorems

Bell , Clauser , Horne , Shimony , Holt

implicit assumption : use of classical correlation function for all correlations between measurements

Kochen , Specker

assumption : unique map from quantum operators to classical observables

Conclusion

- Particular quantum field theory for interacting fermions is equivalent to the classical statistical model of a particular probabilistic cellular automaton.
- Large family of models – not all models!
- Examples for quantum mechanics from classical statistics
- Useful for simulating fermionic models and understanding of statistical properties of cellular automata?

Question

- What is the true continuum limit ?
- Can one formulate FRG such that unitary evolution is preserved?

end

Reduction of wave function

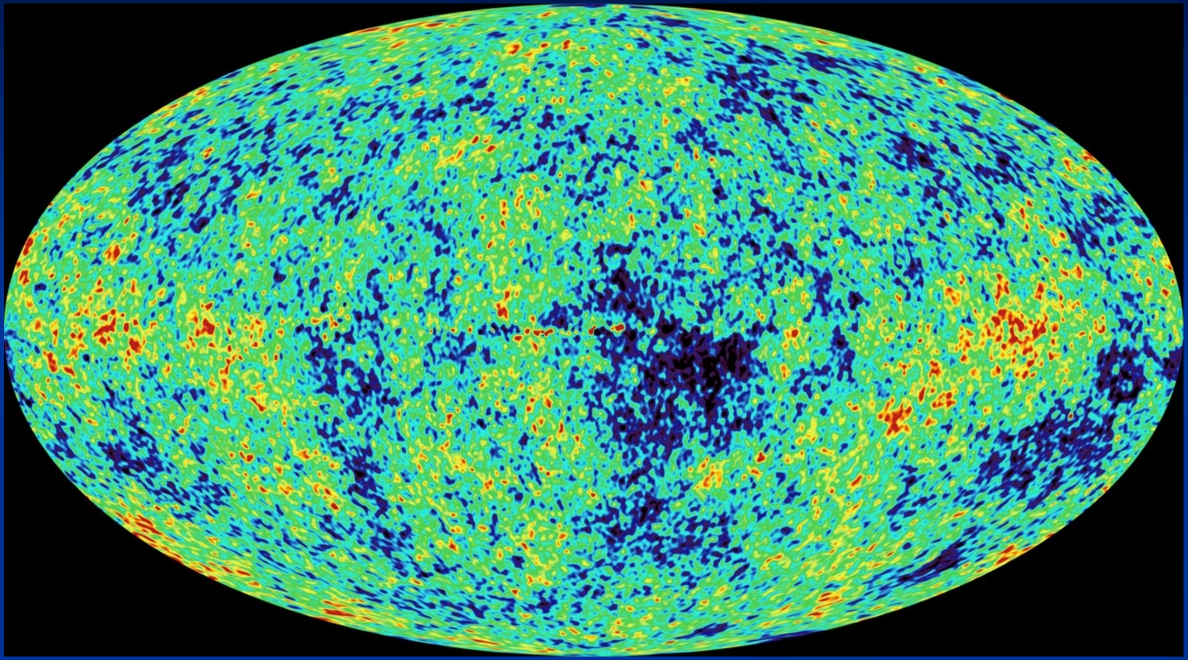
- Reduction of wave function is a convenient technical method to describe conditional probabilities for sequences of measurements
- This must not be a physical process during the measurement

conditional probability

sequences of events(measurements)
are described by
conditional probabilities

both in classical statistics
and in quantum statistics

$w(t_1)$



:

not very suitable
for statement , if here and now
a pointer falls down

Schrödinger's cat

conditional probability :
if nucleus decays
then cat dead with $w_c = 1$
(reduction of wave function)

Continuum limit

$$S = \int_{t,x} \{\overline{\psi}_{R\alpha}(t,x)(\partial_t + \partial_x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)(\partial_t - \partial_x)\psi_{L\alpha}(t,x) + 2\overline{D}(t,x)\}$$

$$\overline{D} = -\left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}\right) - \left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)$$

$$(\partial_t + \partial_x)\psi(t,x) = \frac{1}{\varepsilon}\left[\psi(t,x) - \psi(t-\varepsilon, x-\varepsilon)\right]$$
$$(\partial_t - \partial_x)\psi(t,x) = \frac{1}{\varepsilon}\left[\psi(t,x) - \psi(t-\varepsilon, x+\varepsilon)\right]$$

$$\int dt \int dx = \int_{t,x} = 2\varepsilon^2 \sum_{t,x}$$

$$\psi(t,x) = \sqrt{2\varepsilon}\,\psi_N(t,x)$$

Lorentz symmetry

Dirac spinor

$$\psi_a = \begin{pmatrix} \psi_{Ra} \\ \psi_{La} \end{pmatrix} , \quad \overline{\psi}_a = (\overline{\psi}_{La}, -\overline{\psi}_{Ra})$$

Action

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

$$\gamma^0 = -i\tau_2 , \quad \gamma_1 = \tau_1 , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Infinitesimal Lorentz transformation

$$\delta\psi = -\eta \Sigma^{01} \psi , \quad \delta\overline{\psi} = \eta \overline{\psi} \Sigma^{01}$$

$$\Sigma^{01} = \frac{1}{4}[\gamma^0, \gamma^1] = \frac{1}{2}\tau_3$$