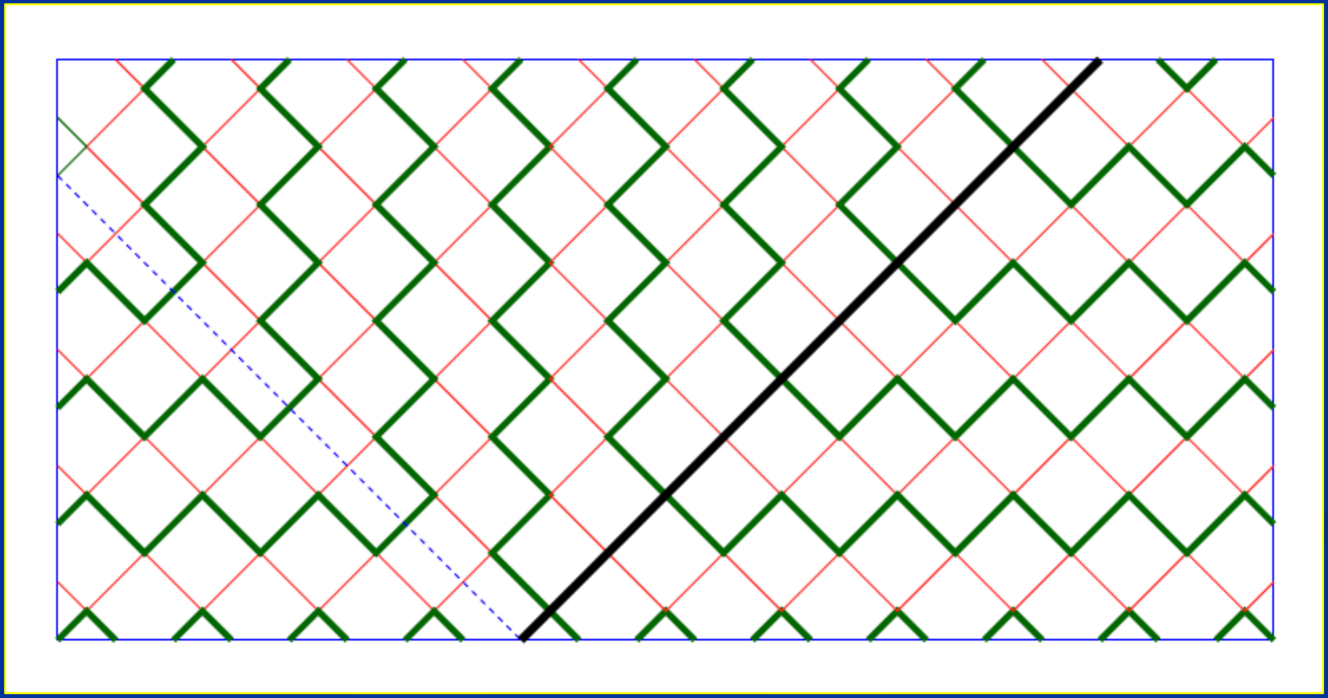
Probabilisitic cellular automata for fermionic quantum field theories



Some interacting fermionic quantum field theories or many body systems
are equivalent to
probabilistic cellular automata

Quantum field theory and quantum mechanics

- Is the Thirring model a model for quantum mechanics ?

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

- Yes, for a given vacuum consider the one-particle state.

Quantum mechanics from classical statistics

- Probabilistic Cellular automata are classical statistical systems
- Quantum mechanics emerges from a classical statistical system.
- All no go theorems (Bell etc.) are circumvented

Fermions

- quantum objects
- wave function totally antisymmetric

  (Pauli principle)
- anticommutator for annihilation and creation operators
- anticommuting Grassmann variables
- functional integral or partition function for many body systems or quantum field theories is Grassmann functional integral

Cellular automaton

- Deterministic manipulation of bits
- Updating rule of bit configurations in sequential steps
- usually: repetition

(Classical computer is a type of cellular automaton without repetition)

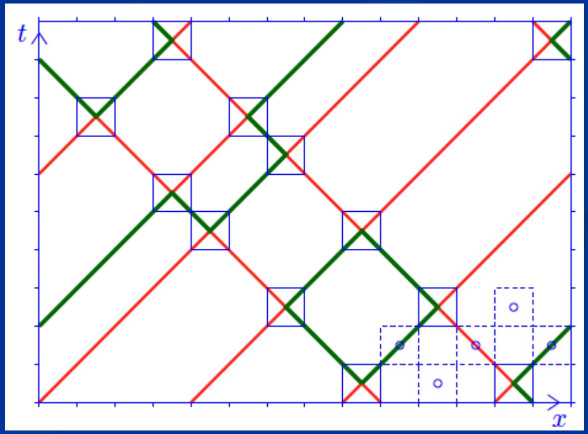
Cellular automaton

At each step :

- each bit configuration changes to a unique new bit configuration according to an updating rule
- for a fixed initial configuration : classical deterministic computing

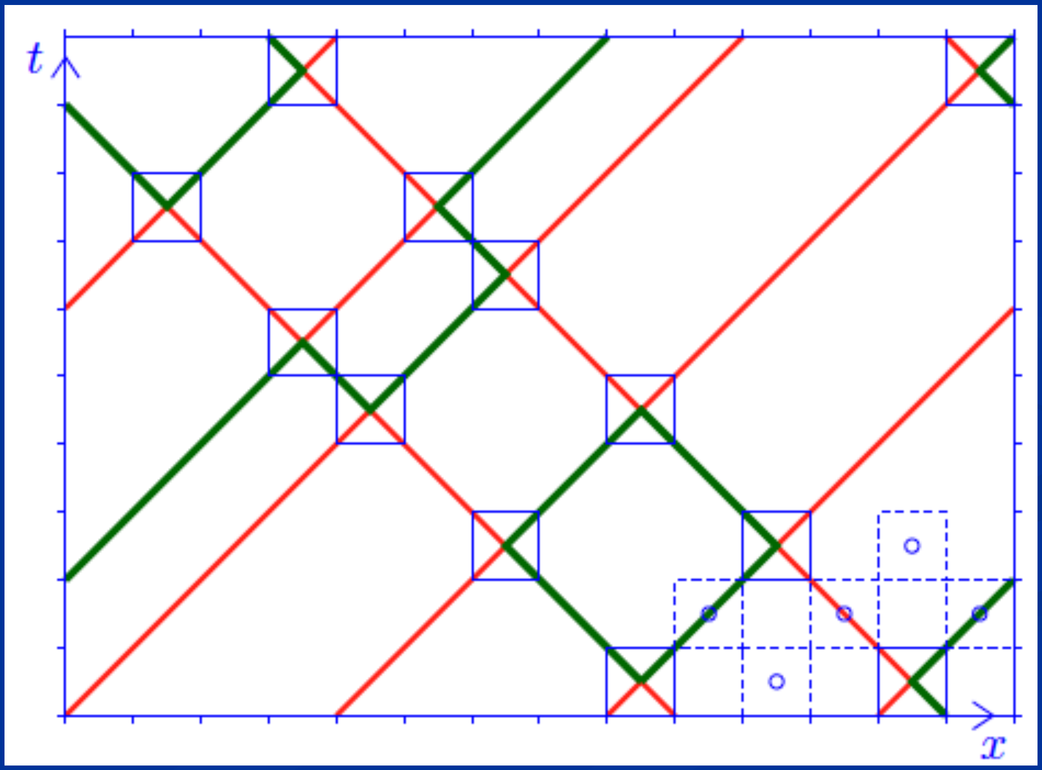
Updating rule for Thirring automaton

- one – dimensional chain, x : discrete lattice sites
- at each x : red and green right movers and left movers (4 different species at each site)
- at each time step: configuration for right(left) movers moves one position to the right(left)
- if two single particles meet:
  colors are exchanged

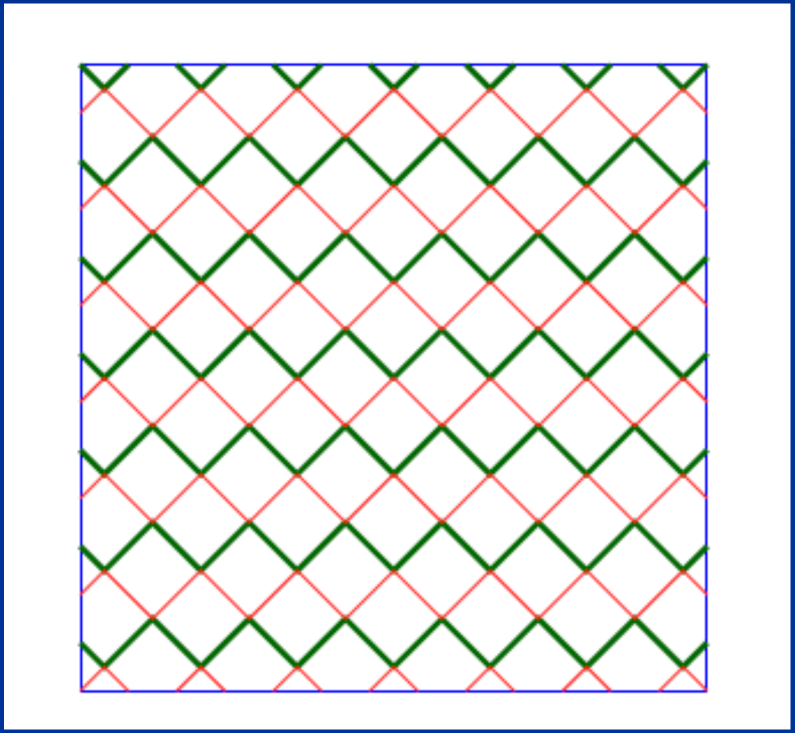


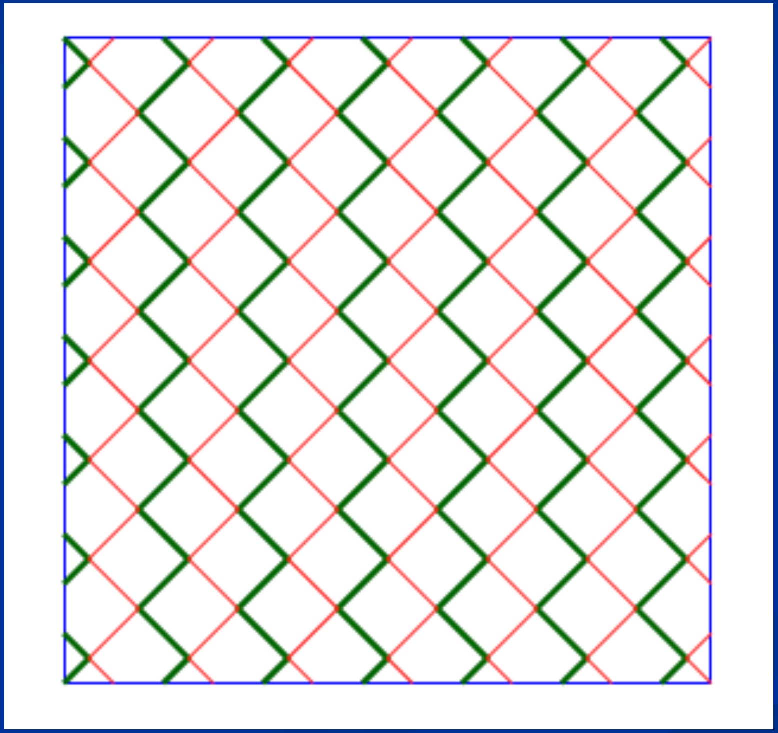
Updating rule

- at each time step configuration for right(left) movers moves one position to the right(left), periodicity in x
- if precisely two single particles meet at a site : colors are exchanged

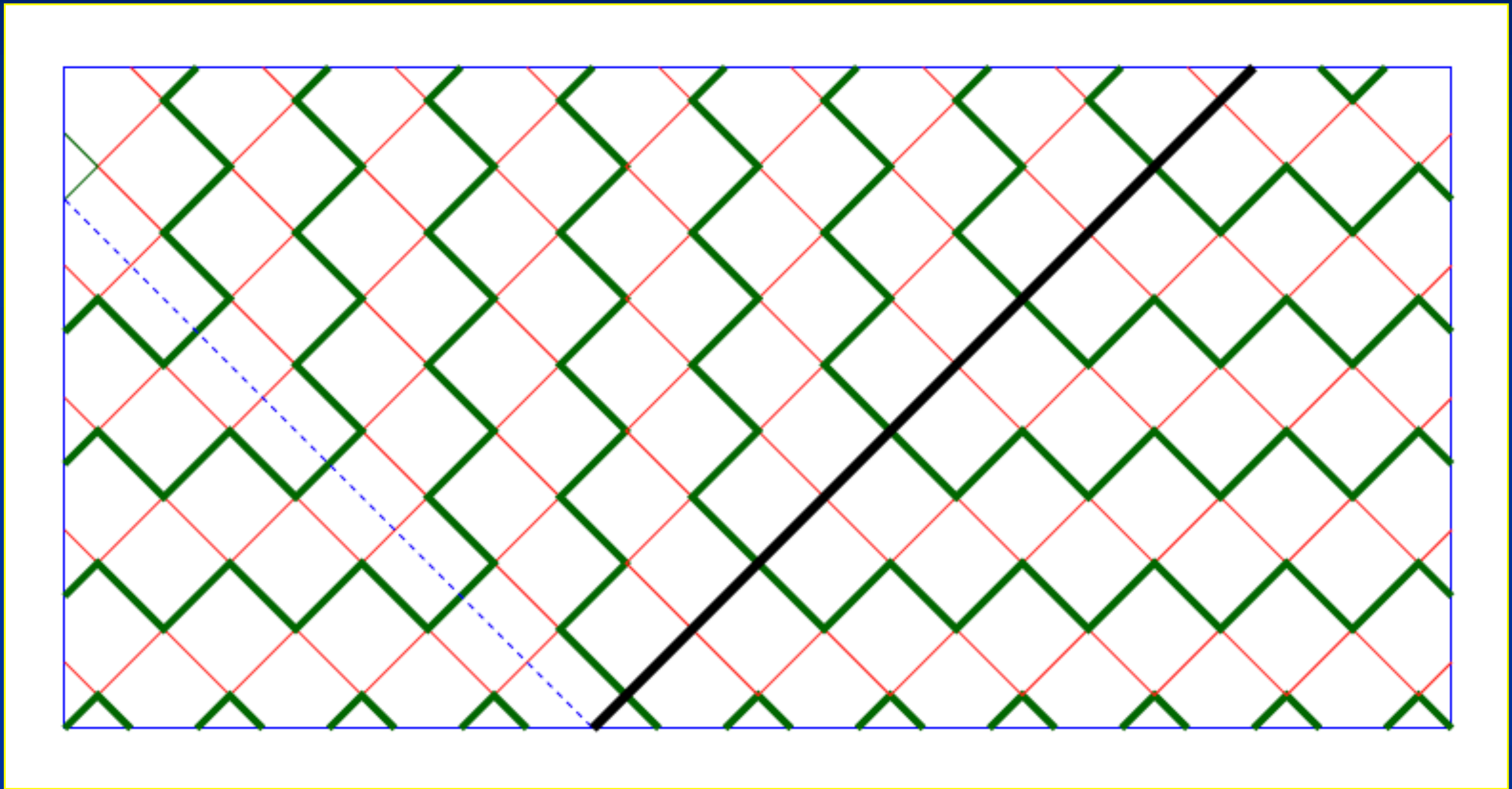


Half filled ground states



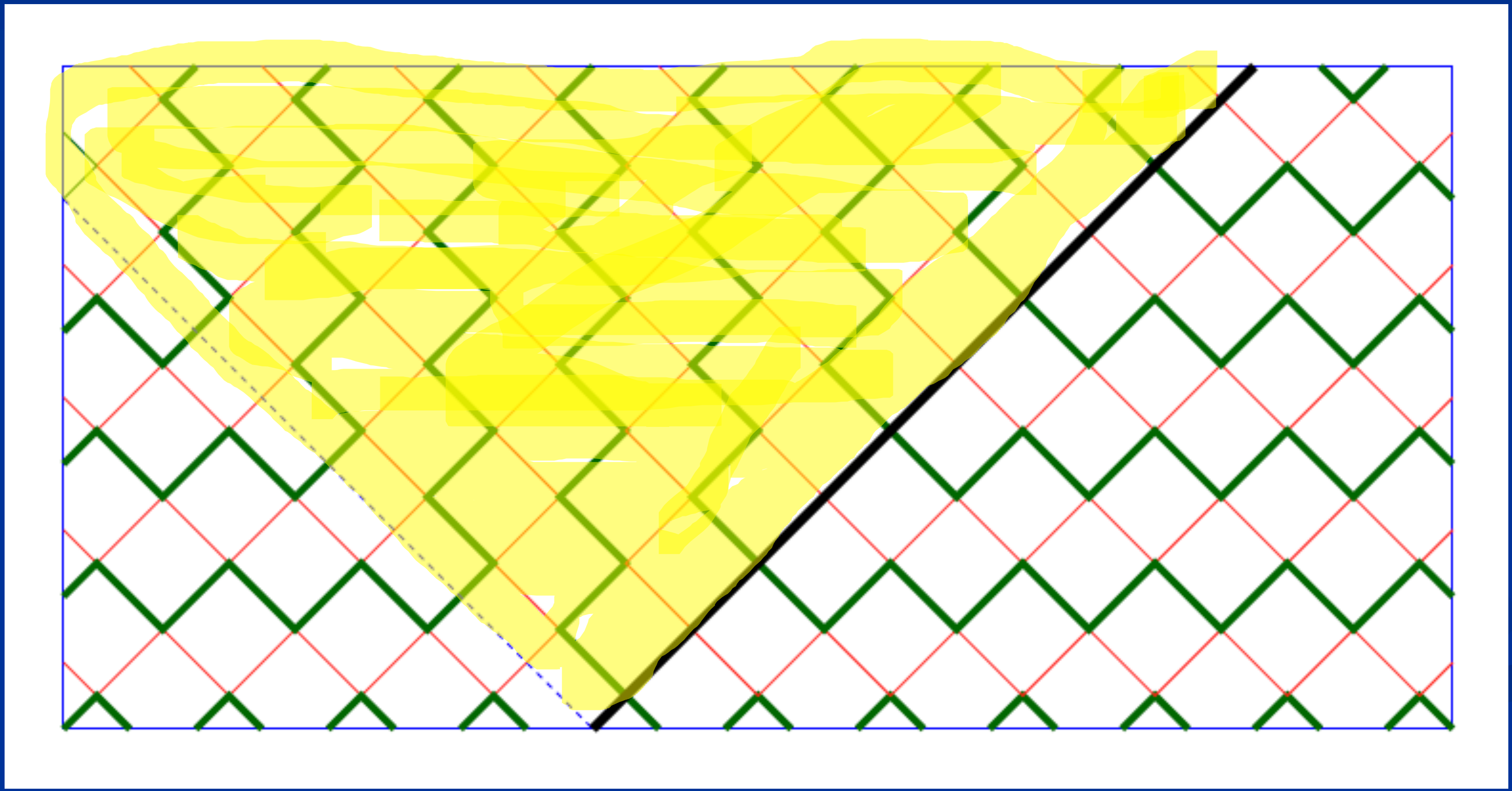


Soliton



black line : no right movers,
or two right movers with different colors

Soliton separates different vacua



Probabilistic cellular automaton

Probability distribution for initial configurations

(or other probabilistic boundary condition)

Equivalence with fermionic QFT

The probabilistic Thirring automaton is equivalent with a fermionic quantum field theory in 1+1 dimensions, namely a type of Thirring model

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

two colors: a,b = 1,2 = red, green

Fermions are Ising spins or bits

- Fermionic occupation numbers $n = 0 , 1$
- Classical bits
- Ising spins $s = 2n - 1$
- Bit configurations = many body states of fermions

Fermionic wave function

- Occupation number basis for multi-fermion systems:

- To each bit configuration one associates an element of the wave function
- Occupation numbers for different space points and species

Probabilistic cellular automaton

Probabilistic initial condition: Specify at initial time t_{in} for each bit configuration $\overline{\rho}$ a probability $p_{\overline{\rho}}(t_{\text{in}})$

Evolution: every given configuration $\overline{\rho}$ at t_{in} propagates at t to a configuration $\tau(t,\overline{\rho})$

$$\Rightarrow \quad p_{\tau}(t) = p_{\overline{\rho}(\tau)}(t_{\text{in}})$$

Updating rule: specifies $\tau\big(t+\varepsilon, \rho(t)\big)$

Wave function for probabilistic cellular automaton

Probability distribution: at every time t a bit configuration τ occurs with probability $p_\tau(t)$, which equals the probability for the initial bit configuration from which it originates.

Real wave function q(t): probability amplitude

$$p_\tau(t) = (q_\tau(t))^2$$

$$q_\tau(t) q_\tau(t) = 1$$

N – component unit vector

Deterministic and probabilistic cellular automaton

- Deterministic CA : sharp wave function

$$q_\rho(t_{\text{in}}) = \delta_{\rho,\overline{\rho}}$$

- Probabilistic CA : arbitrary wave function

Particle wave duality

Particle aspect:

- Bits: yes/no decisions
- Possible measurement values 1 or 0

Discrete spectrum of observables

Wave aspect : continuous wave function

more generally: continuity of probabilistic information

Step evolution operator

- Evolution for basic time step is encoded in the step evolution operator

$$q(t+\varepsilon) = \widehat{S}(t)q(t)$$

$$q_\tau(t+\varepsilon) = \widehat{S}_{\tau\rho}(t)q_\rho(t)$$

- Contains the updating rule for CA

$$\widehat{S}_{\tau\rho}(t) = \delta_{\tau,\overline{\tau}(\rho)} = \delta_{\overline{\rho}(\tau),\rho}$$

$$q_\tau(t+\varepsilon) = q_{\overline{\rho}(\tau)}(t)\,, \quad p_\tau(t+\varepsilon) = p_{\overline{\rho}(\tau)}(t)$$

Unique jump matrix

- Step evolution operator for cellular automata is unique jump matrix
- In every row and coloumn: precisely one element +1 or -1, all other elements zero

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

QFT- CA equivalence

A fermionic quantum field theory is equivalent to a probabilistic cellular automaton if the evolution operator for discrete time steps is a

unique jump matrix

(in a real formulation of the evolution equation)

Continuum limit

- Discretized fermionic QFT (e.g. on lattice) makes model well defined
- Regularized functional integral
- Step evolution operator
- Needs continuum limit

Step evolution operator

- Sequence of kinetic (free) and interaction part

$$\widehat{S} = \widehat{S}_{\text{int}}\, \widehat{S}_{\text{free}}$$

- Local interaction

$$\widehat{S}_{\text{int}} = \widehat{S}_i(x_{\text{in}}) \otimes \widehat{S}_i(x_{\text{in}} + \varepsilon) \otimes \widehat{S}_i(x_{\text{in}} + 2\varepsilon) \otimes \dots$$

(1) at each time step configuration for right(left) movers moves one position to the right(left),

(2) if precisely two single particles meet at a site : colors are exchanged

Annihilation and creation operators

Step evolution operator for Thirring automaton can be written in terms of fermionic annihilation and creation operators

$$\{a^\dagger_\gamma(x), a_\delta(y)\} = \delta_{\gamma\delta}\delta_{xy} \qquad \{a_\gamma(x), a_\delta(y)\} = \{a^\dagger_\gamma(x), a^\dagger_\delta(y)\} = 0$$

$$\widehat{S}_i(x) = \exp\left\{\frac{i\pi}{2}\left[a^\dagger_{\mathrm{R}1}(x)a_{\mathrm{R}2}(x) - a^\dagger_{\mathrm{R}2}(x)a_{\mathrm{R}1}(x)\right]\left[a^\dagger_{\mathrm{L}1}(x)a_{\mathrm{L}2}(x) - a^\dagger_{\mathrm{L}2}(x)a_{\mathrm{L}1}(x)\right]\right\}$$

$$\widehat{S}_{\mathrm{free}} = \widehat{S}_1^{(\mathrm{R})} \otimes \widehat{S}_2^{(\mathrm{R})} \otimes \widehat{S}_1^{(\mathrm{L})} \otimes \widehat{S}_2^{(\mathrm{L})}$$

$$\widehat{S}_a^{(\mathrm{R,L})} = N\left[\exp\left\{\sum_x a^\dagger(x \pm \varepsilon)\left[a(x) - a(x \pm \varepsilon)\right]\right\}\right]$$

Hamiltonian

- Define H by $\widehat{S} = \exp\left(-i\varepsilon H\right)$

- Interpolating continuous time evolution

$$q(t_2) = U(t_2, t_1) q(t_1) \qquad U(t_1, t_2) = \exp\left(-i(t_1 - t_2)H\right)$$

- Agrees with discrete evolution for $t_{\text{in}} + m\varepsilon$

- Schrödinger equation $i\partial_t q = Hq$

Naïve continuum limit

- Hamiltonian simplifies in the continuum limit

$$H = H_{\text{free}} + H_{\text{int}} + \Delta H \qquad \Delta H = \mathcal{O}(\varepsilon[H_{\text{int}}, H_{\text{free}}])$$

- Standard form of Hamiltonian for fermions

$$H_{\text{free}} = \frac{i}{\varepsilon}\int \mathrm{d}x \sum_a \left\{ a^\dagger_{La}(x)\partial_x a_{La}(x) - a^\dagger_{Ra}(x)\partial_x a_{Ra}(x) \right\}$$

$$H_{\text{int}} = -\frac{\pi}{2\varepsilon^2}\int \mathrm{d}x \left[a^\dagger_{\text{R}1} a_{\text{R}2} - a^\dagger_{\text{R}2} a_{\text{R}1}\right]\left[a^\dagger_{\text{L}1} a_{\text{L}2} - a^\dagger_{\text{L}2} a_{\text{L}1}\right]$$

General bit fermion map

- Isomorphism between generalized Ising model and Grassmann functional integral
- Based on identical step evolution operator for both models, with associated map of observables
- In our case: proof that discrete Thirring model with two colors has the same step evolution operator as the cellular automaton

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x) \\ - \Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x)\Big](1+\overline{D}(x)) \Big\}$$

Quantum field theory and quantum mechanics

- Is the Thirring model a model for quantum mechanics ?

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

- Yes, for a given vacuum consider the one-particle state.

Discretization remains a quantum model, if evolution is unitary

Grassmann functional integral

$$Z = \int \mathcal{D}\psi \exp(-S[\psi]) = \int \mathcal{D}\psi w[\psi] \,, \quad S = \sum_t \mathcal{L}(t)$$

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x)$$
$$- \Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x) \Big] \big(1 + \overline{D}(x)\big) \Big\}$$

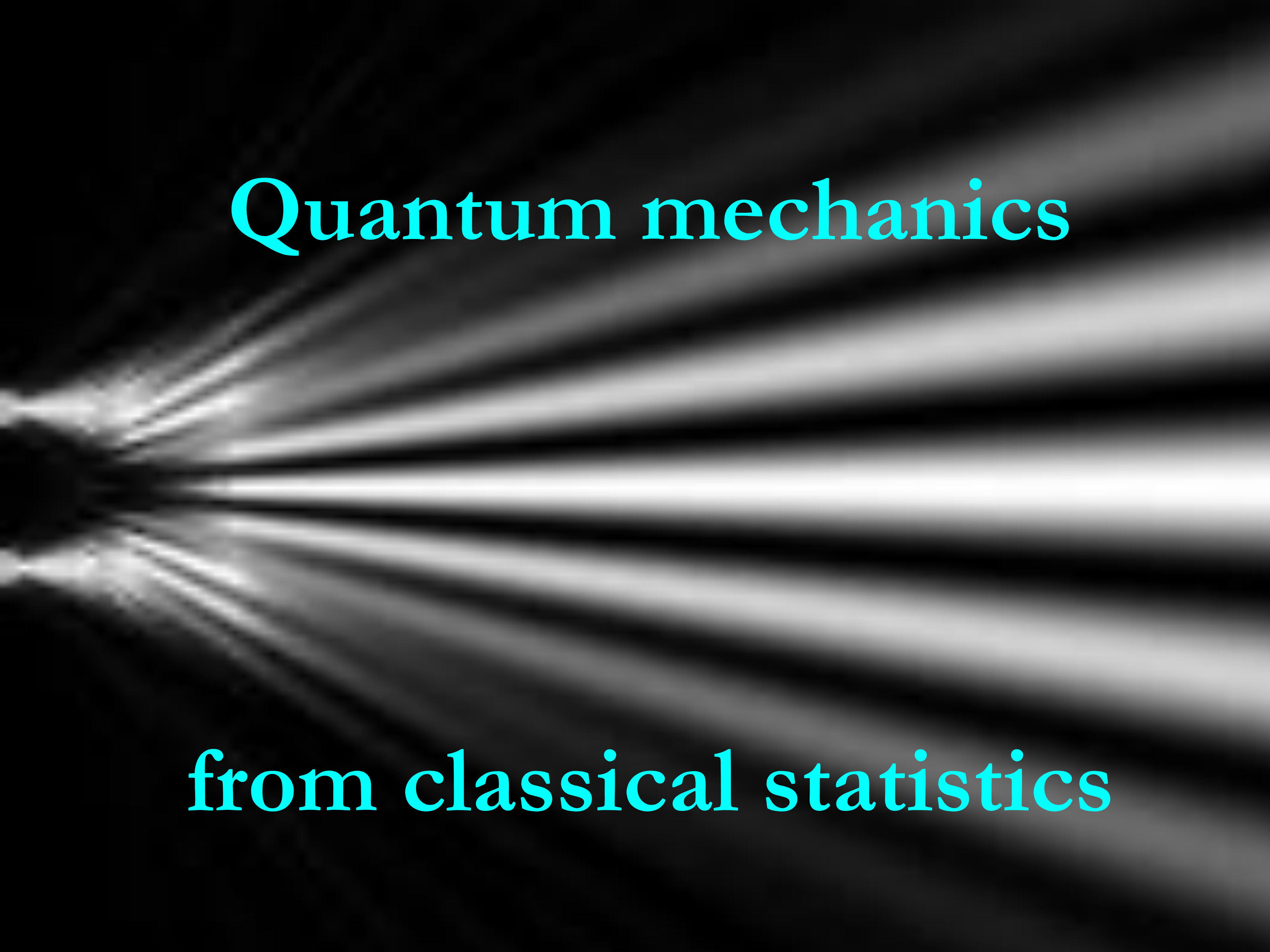
$$\overline{D} = -\left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}\right) - \left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)$$

Quantum formalism for classical statistics

- Formalism for information transport from one hypersurface to the next:
- Classical wave functions and density matrix
- Transfer matrix formalism : Heisenberg picture
- Wave functions : Schrödinger picture
- Non commuting operators for observables
- Quantum rules from classical statistical rules

Momentum observable

- Measures periodicity of wave function
- Statistical observable, similar to temperature
- No fixed value in microstate
- Classical correlation function with occupation numbers does not exist
- Needs probabilistic information


Quantum mechanics
from classical statistics

Quantum mechanics from classical statistics

For particular quantum model:

Isomorphism between classical statistics
(probabilistic cellular automaton,
generalized Ising model)

and quantum mechanics
(many body quantum system for fermions)

Equivalence

- Expectation values of all observables are the same in both models
- Two equivalent descriptions of the same physical reality

Important conceptual consequences

- Probabilistic cellular automata are **classical statistical systems**
- Fermionic quantum field theories are **quantum systems**
- Quantum mechanics emerges from classical statistics
- Quantum formalism can be used for classical statistics

Can quantum physics be described by classical probabilities ?

“ No go “ theorems

Bell , Clauser , Horne , Shimony , Holt

implicit assumption : use of classical correlation function for all correlations between measurements

Kochen , Specker

assumption : unique map from quantum operators to classical observables

Conclusion

- Particular quantum field theory for interacting fermions is equivalent to the classical statistical model of a particular probabilistic cellular automaton.
- Large family of models – not all models!
- Examples for quantum mechanics from classical statistics
- Useful for simulating fermionic models and understanding of statistical properties of cellular automata?

end

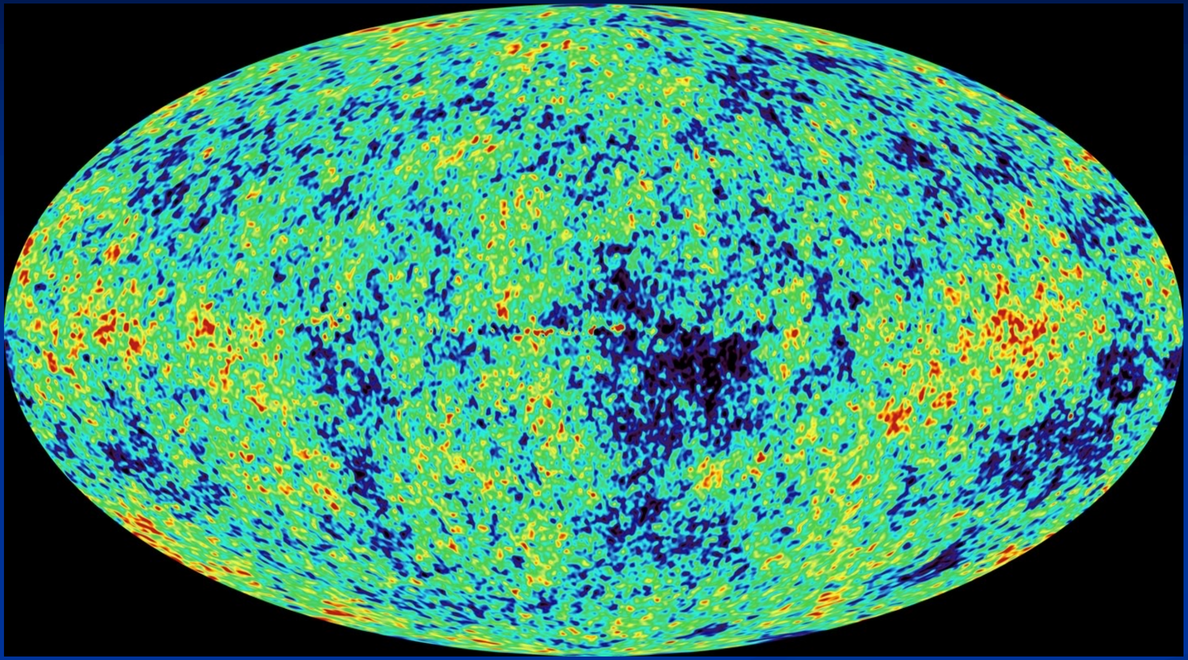
Reduction of wave function

- Reduction of wave function is a convenient technical method to describe conditional probabilities for sequences of measurements
- This must not be a physical process during the measurement

conditional probability

sequences of events(measurements)
are described by
conditional probabilities

both in classical statistics
and in quantum statistics

$w(t_1)$  :

not very suitable
for statement , if here and now
a pointer falls down

Schrödinger’s cat

conditional probability :
if nucleus decays
then cat dead with $w_c = 1$
(reduction of wave function)

Probabilistic cellular automaton

Probability distribution for initial configurations

(or other probabilistic boundary condition)

Probabilistic formalism for cellular automata

Generalized Ising model:

$$w[s] = Z^{-1} \exp\big(-S[s]\big) b(s_{in}, s_f)$$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

limit : β to infinity , σ to zero :
only one possibility for change , unique jump

probabilistic aspects only in boundary term :

$$b(s_{in}, s_f) = \bar{f}_f(s_f) f_{in}(s_{in})$$

Functional integral for cellular automata

Generalized Ising model: $w[s] = Z^{-1} \exp\big(- S[s]\big) b(s_{in}, s_f)$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

limit : beta to infinity , sigma to zero :
only one possibility for change , unique jump

Functional renormalization for cellular automata

Probabilistic computing with static memory materials ?

- Let general equilibrium classical statistics transport information from one layer to the next

- Simulation, with D. Sexty

Static memory materials

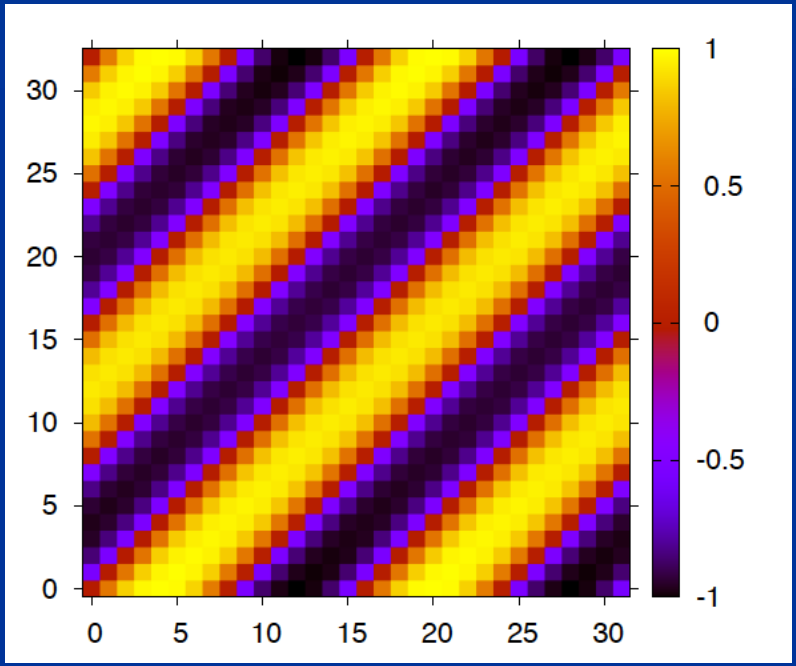
Generalized Ising model: $w[s] = Z^{-1} \exp\big(-S[s]\big) b(s_{in}, s_f)$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

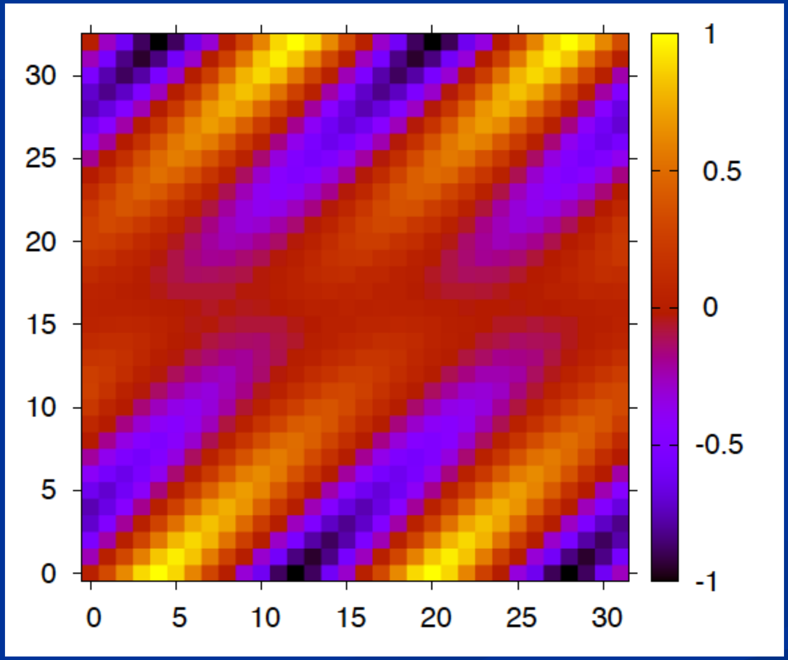
Boundary term : $b(s_{in}, s_f) = \bar{f}_f(s_f) f_{in}(s_{in})$

Classical interference

Depending on boundary conditions :



Positive interference



Negative interference

Static memory material for
two dimensional Ising spins
on Euclidean square lattice
can describe propagation of Weyl fermion
in two- dimensional Minkowski space

Discrete fermion model in 1+1 dimensions

Grassmann functional integral

$$Z = \int \mathcal{D}\psi \exp(-S[\psi]) = \int \mathcal{D}\psi w[\psi]\,, \quad S = \sum_t \mathcal{L}(t)$$

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x)$$
$$- \Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x) \Big] \left(1 + \overline{D}(x)\right) \Big\}$$

$$\overline{D} = -\left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}\right) - \left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)$$

Continuum limit

$$S = \int_{t,x} \{\overline{\psi}_{R\alpha}(t,x)(\partial_t + \partial_x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)(\partial_t - \partial_x)\psi_{L\alpha}(t,x) + 2\overline{D}(t,x)\}$$

$$\overline{D} = -(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}})(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}) - (\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}})(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}})$$

$$(\partial_t + \partial_x)\psi(t,x) = \frac{1}{\varepsilon}[\psi(t,x) - \psi(t-\varepsilon, x-\varepsilon)]$$
$$(\partial_t - \partial_x)\psi(t,x) = \frac{1}{\varepsilon}[\psi(t,x) - \psi(t-\varepsilon, x+\varepsilon)]$$

$$\int dt \int dx = \int_{t,x} = 2\varepsilon^2 \sum_{t,x}$$

$$\psi(t,x) = \sqrt{2\varepsilon}\,\psi_N(t,x)$$

Lorentz symmetry

Dirac spinor

$$\psi_a = \begin{pmatrix} \psi_{Ra} \\ \psi_{La} \end{pmatrix} , \quad \overline{\psi}_a = (\overline{\psi}_{La}, -\overline{\psi}_{Ra})$$

Action

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

$$\gamma^0 = -i\tau_2 , \quad \gamma_1 = \tau_1 , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Infinitesimal Lorentz transformation

$$\delta\psi = -\eta \Sigma^{01} \psi , \quad \delta\overline{\psi} = \eta \overline{\psi} \Sigma^{01}$$

$$\Sigma^{01} = \frac{1}{4}[\gamma^0, \gamma^1] = \frac{1}{2}\tau_3$$

structural elements of quantum mechanics

unitary time evolution

ν

h

Simple conversion factor for units

i

presence of complex structure

$$[A,B]=C$$

non – commuting operators are necessary to represent observables in incomplete statistics

correlation and operator product

- Classical statistical systems admit many product structures of observables
- Many different definitions of correlation functions possible , not only classical correlation !
- Type of measurement determines correct selection of correlation function !
- Ideal quantum measurement should be compatible with information in quantum subsystem

Deterministic evolution – probabilistic interpretation

- quantum mechanics arises from
  quantum subsystems
- subsystems are genuinely probabilistic
- part of information is lost by focus on subsystem
- partially ”integrating out” degrees of freedom

Determinism vs. Probabilism

“ Does god throw dices ? ”

… an old dispute

Gott würfelt

Gott würfelt nicht

"Es scheint hart, dem Herrgott in die Karten zu gucken. Aber dass er würfelt und sich telepatischer Mittel bedient (wie es ihm von der gegenwärtigen Quantentheorie zugemutet wird), kann ich keinen Augenblick glauben.."

Einstein: Brief an Cornelius Lanczos am 21. März 1942

not todays topic

Gott würfelt　　　Gott würfelt nicht

humans can only deal with probabilities

determinism vs. probabilism

my personal view :

- determinism not needed, nor useful
- start with probabilities as basic concept for the description of the world (not related to lack of knowledge for deterministic state !)

- nevertheless : deterministic evolution is a possible option