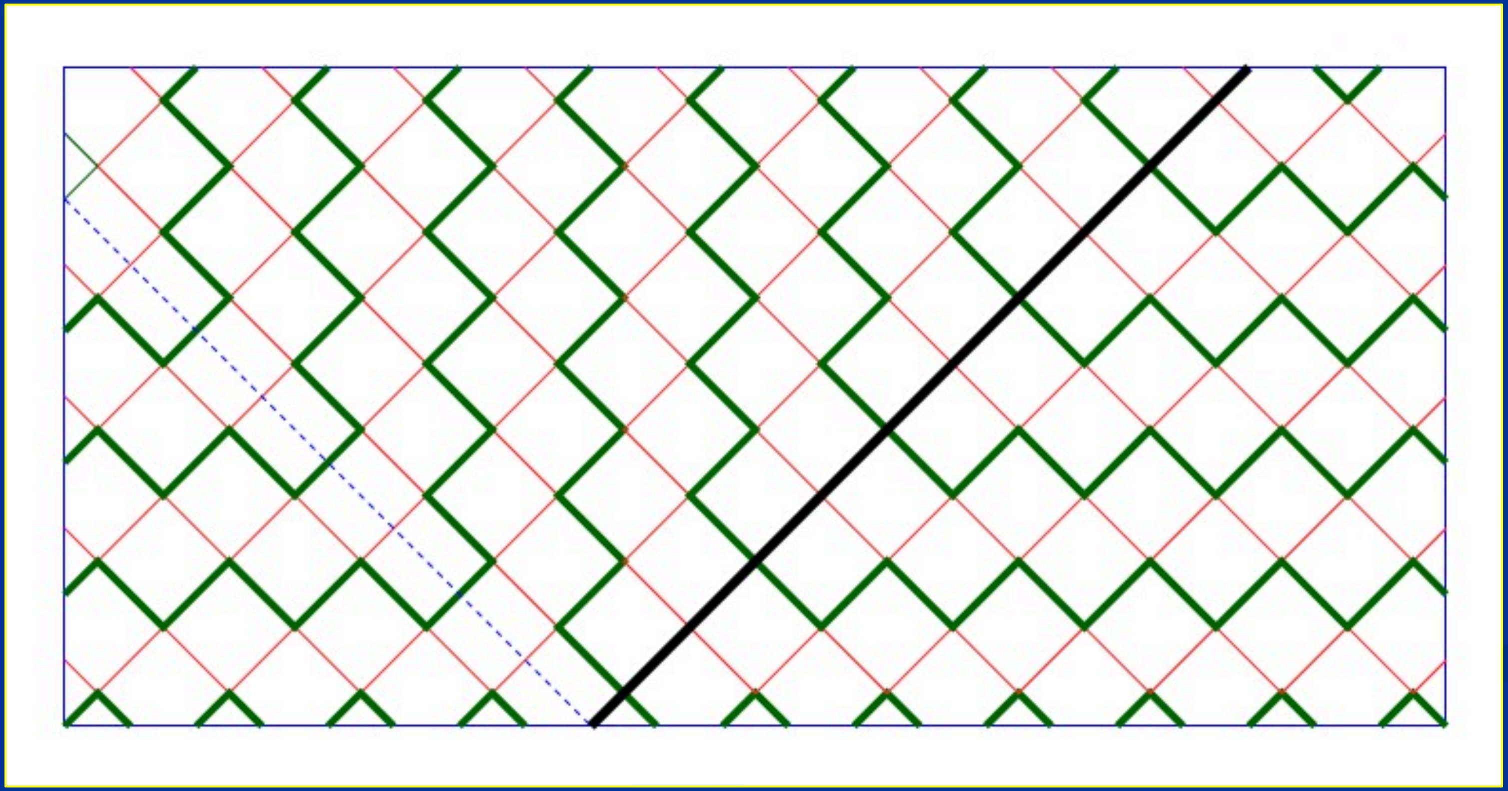
Quantum fermions from classical bits



Fermions

- quantum objects
- wave function totally antisymmetric
  (Pauli principle)
- anticommutator for annihilation and creation operators
- anticommuting Grassmann variables
- functional integral or partition function for many body systems or quantum field theories is Grassmann functional integral

Some interacting fermionic quantum field theories or many body systems are equivalent to probabilistic cellular automata

Cellular automaton

- Deterministic manipulation of bits
- Updating rule of bit configurations in sequential steps
- usually: repetition

(Classical computer is a type of cellular automaton)

Cellular automaton

At each step :

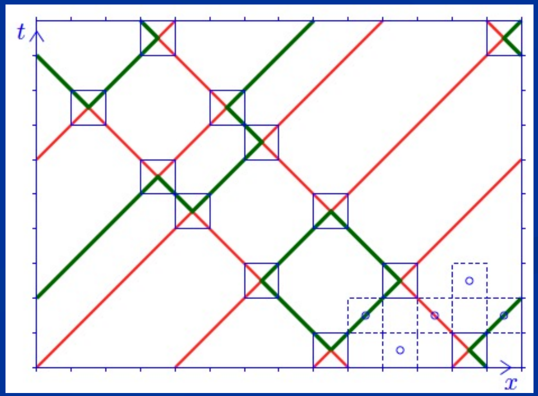
- each bit configuration changes to a unique new bit configuration according to an updating rule
- for a fixed initial configuration : classical deterministic computing

Updating rule for Thirring automaton

- one – dimensional chain, x : discrete lattice sites
- at each x : red and green right movers and left movers

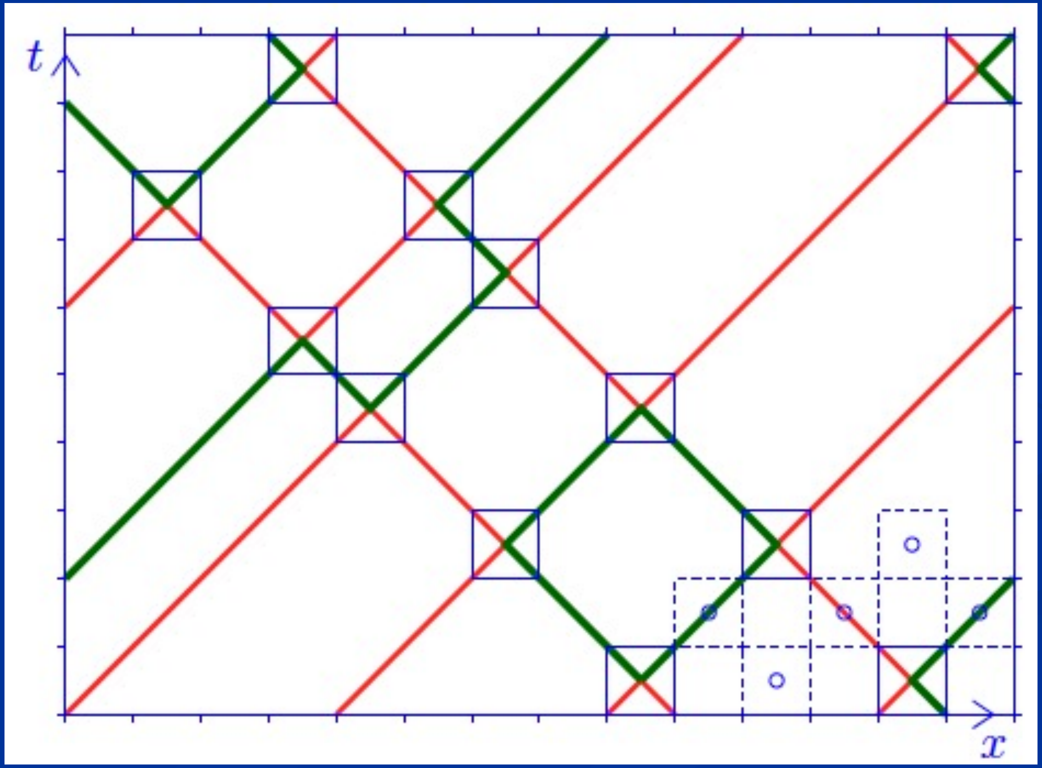
  (4 different species at each site)
- at each time step: configuration for right(left) movers moves one position to the right(left)
- if two single particles meet:

  colors are exchanged

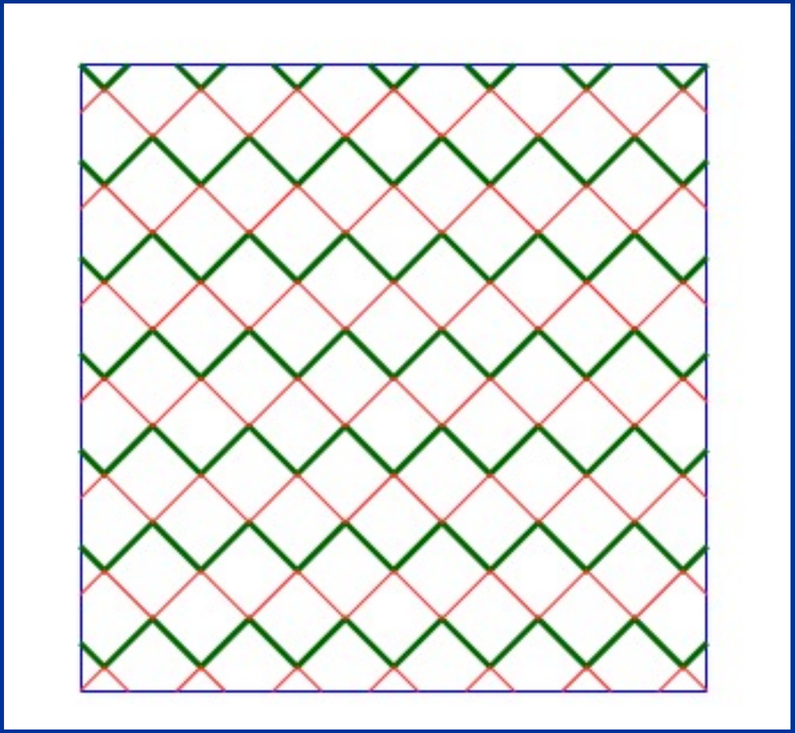


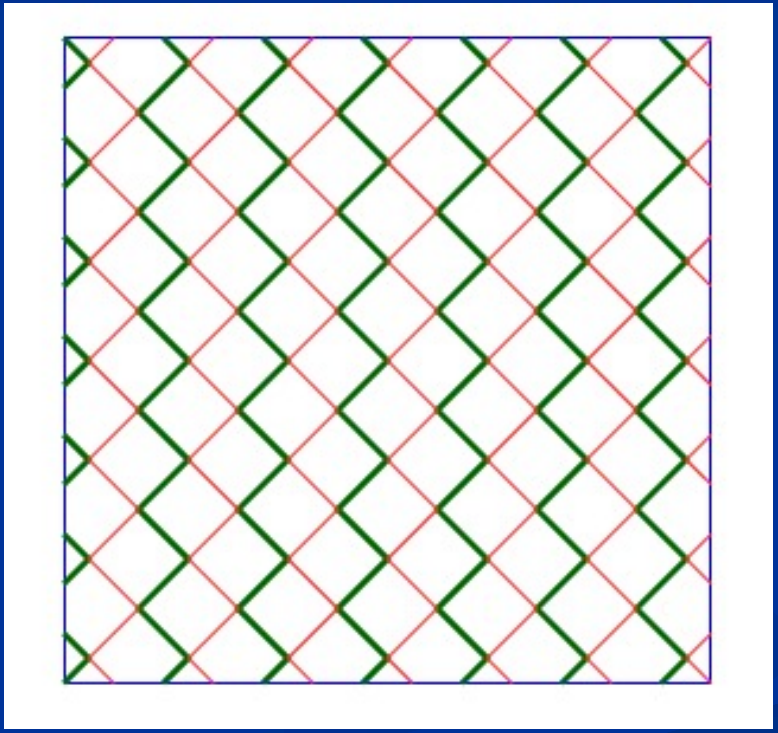
Updating rule

- at each time step configuration for right(left) movers moves one position to the right(left), periodicity in x
- if precisely two single particles meet at a site : colors are exchanged

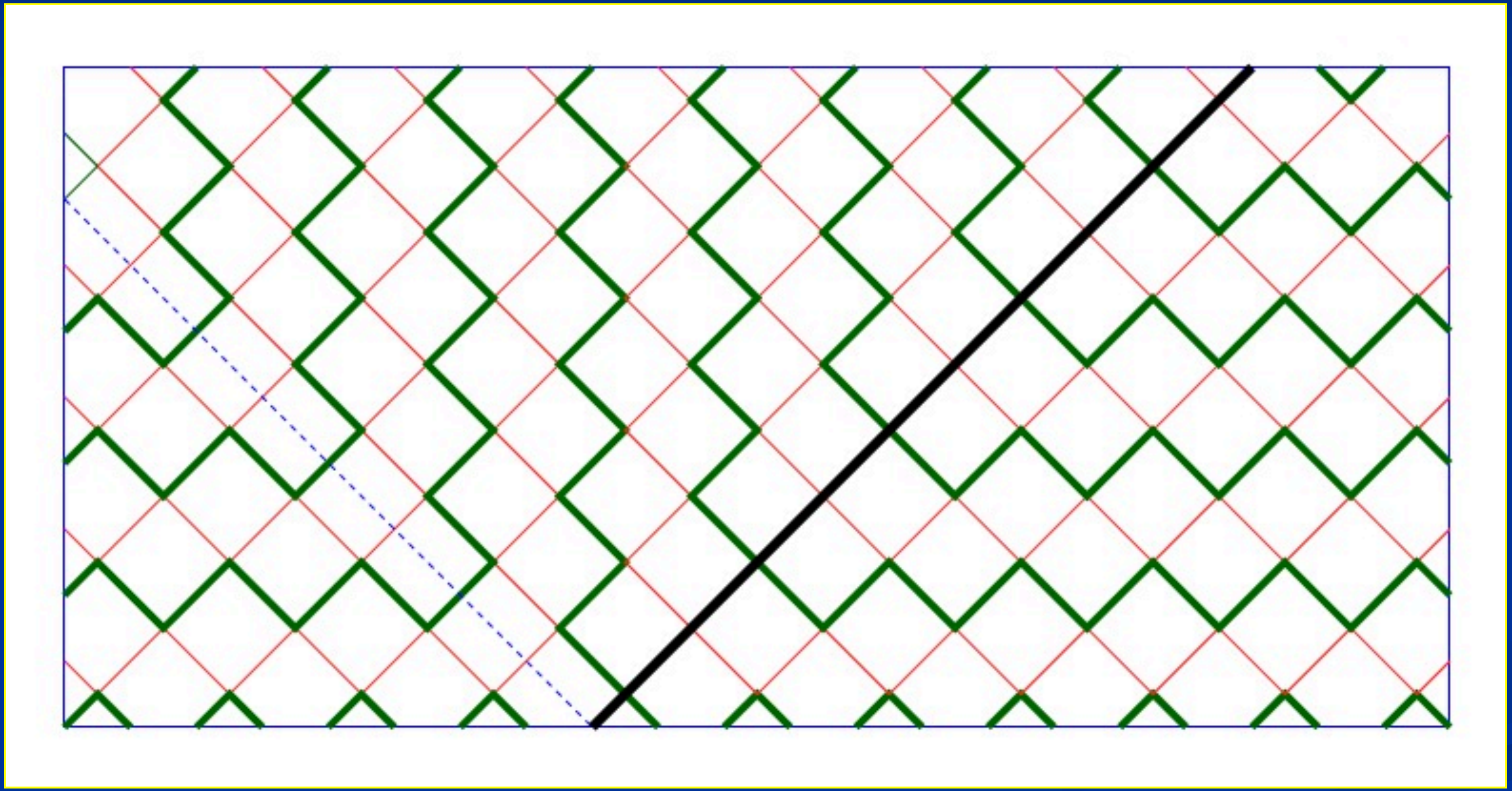


Half filled ground states



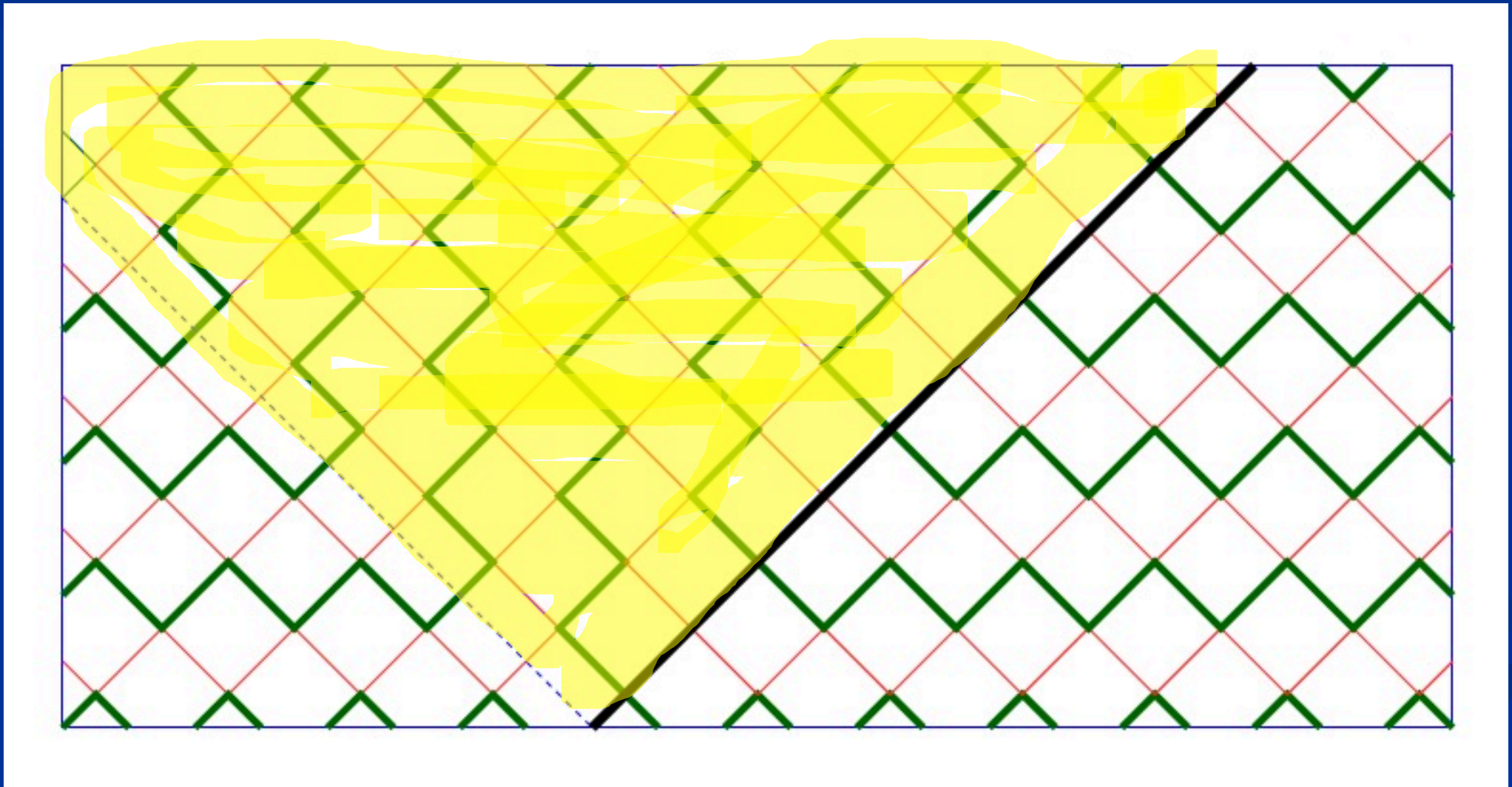


Soliton



black line : two right movers with different colors

Soliton separates different vacua



Probabilistic cellular automaton

Probability distribution for initial configurations

(or other probabilistic boundary condition)

Equivalence with fermionic QFT

The probabilistic Thirring automaton is equivalent with a discretized fermionic quantum field theory in 1+1 dimensions, namely a type of Thirring model

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

two colors: a,b = 1,2 = red, green

Important conceptual consequences

- Probabilistic cellular automata are **classical statistical systems**
- Fermionic quantum field theories are **quantum systems**
- Quantum mechanics emerges from classical statistics
- Quantum formalism can be used for classical statistics

Fermions are Ising spins or bits

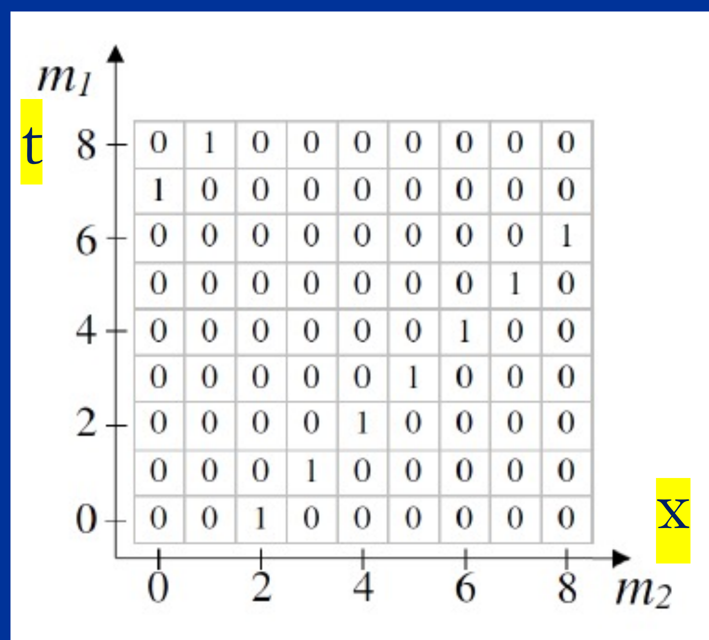
- Fermionic occupation numbers $n = 0 , 1$
- Classical bits
- Ising spins $s = 2n - 1$
- Bit configurations = many body states of fermions

Free fermion in d=1+1

m_1										
t 8	0	1	0	0	0	0	0	0	0	
	1	0	0	0	0	0	0	0	0	
6	0	0	0	0	0	0	0	0	1	
	0	0	0	0	0	0	0	1	0	
4	0	0	0	0	0	0	1	0	0	
	0	0	0	0	0	1	0	0	0	
2	0	0	0	0	1	0	0	0	0	
	0	0	0	1	0	0	0	0	0	
0	0	0	1	0	0	0	0	0	0	x
	0		2		4		6		8	m_2

Single particle wave function for free fermion in d=1+1

q^2 (t_{in} , x) determines probability to find fermion at initial time t_{in} at position x



Probabilistic cellular automaton

Probability distribution for initial configurations

(or other probabilistic boundary condition)

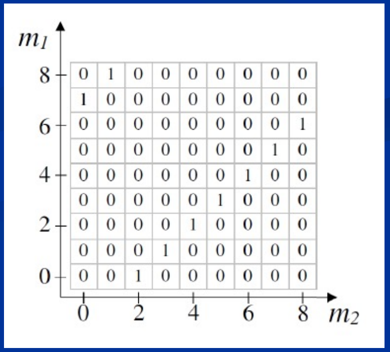
Probabilistic cellular automaton

Probabilistic initial condition: Specify
at initial time t_{in} for each bit configuration $\overline{\rho}$
a probability $p_{\overline{\rho}}(t_{\text{in}})$

Evolution: every given configuration $\overline{\rho}$ at t_{in}
propagates at t to a configuration $\tau(t, \overline{\rho})$

$$p_{\tau}(t) = p_{\overline{\rho}(\tau)}(t_{\text{in}})$$

Updating rule: specifies $\tau\big(t + \varepsilon, \rho(t)\big)$



Wave function for probabilistic cellular automaton

Probability distribution: at every time t a bit configuration τ occurs with probability $p_\tau(t)$

Real wave function q(t): probability amplitude

$$p_\tau(t) = (q_\tau(t))^2 \qquad q_\tau(t) q_\tau(t) = 1$$

N – component unit vector

Deterministic and probabilistic cellular automaton

- Deterministic CA : sharp wave function $q_\rho(t_{\text{in}}) = \delta_{\rho,\overline{\rho}}$
- Probabilistic CA : arbitrary wave function

Particle wave duality

Particle aspect:

- Bits: yes/no decisions
- Possible measurement values 1 or 0

Discrete spectrum of observables

Wave aspect : continuous wave function

more generally: continuity of probabilistic information

Step evolution operator

- Evolution for basic time step is encoded in the step evolution operator

$$q(t+\varepsilon) = \widehat{S}(t)q(t)$$

$$q_\tau(t+\varepsilon) = \widehat{S}_{\tau\rho}(t)q_\rho(t)$$

$$\widehat{S}_{\tau\rho}(t) = \delta_{\tau,\overline{\tau}(\rho)} = \delta_{\overline{\rho}(\tau),\rho}$$

- Contains the updating rule for CA

$$q_\tau(t+\varepsilon) = q_{\overline{\rho}(\tau)}(t)\,, \quad p_\tau(t+\varepsilon) = p_{\overline{\rho}(\tau)}(t)$$

step evolution operator

- step evolution operator is orthogonal matrix

- rotates the wave function
- in presence of complex structure : unitary evolution

- unique jump operator : in each row and column exactly one element one , all other elements zero

example : switch between two bits

$$S : (1,0) \leftrightarrow (0,1) ,$$
$$(0,0) , (1,1) \text{ invariant.}$$

$$\widehat{S}_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\widehat{S}_S^2 = 1$$

Step evolution operator for Thirring automaton

- Sequence of kinetic (free) and interaction part

$$\widehat{S} = \widehat{S}_{\text{int}}\,\widehat{S}_{\text{free}}$$

- Local interaction

$$\widehat{S}_{\text{int}} = \widehat{S}_i(x_{\text{in}}) \otimes \widehat{S}_i(x_{\text{in}} + \varepsilon) \otimes \widehat{S}_i(x_{\text{in}} + 2\varepsilon) \otimes \dots$$

$$\widehat{S}_S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(1) at each time step configuration for right(left) movers moves one position to the right(left),

(2) if precisely two single particles meet at a site : colors are exchanged

Annihilation and creation operators

Step evolution operator can be written in terms of fermionic annihilation and creation operators

$$\{a^\dagger_\gamma(x), a_\delta(y)\} = \delta_{\gamma\delta}\delta_{xy}$$

$$\{a_\gamma(x), a_\delta(y)\} = \{a^\dagger_\gamma(x), a^\dagger_\delta(y)\} = 0$$

$$\widehat{S}_i(x) = \exp\left\{\frac{i\pi}{2}\left[a^\dagger_{\mathrm{R1}}(x)a_{\mathrm{R2}}(x) - a^\dagger_{\mathrm{R2}}(x)a_{\mathrm{R1}}(x)\right]\left[a^\dagger_{\mathrm{L1}}(x)a_{\mathrm{L2}}(x) - a^\dagger_{\mathrm{L2}}(x)a_{\mathrm{L1}}(x)\right]\right\}$$

$$\widehat{S}_{\mathrm{free}} = \widehat{S}^{(\mathrm{R})}_1 \otimes \widehat{S}^{(\mathrm{R})}_2 \otimes \widehat{S}^{(\mathrm{L})}_1 \otimes \widehat{S}^{(\mathrm{L})}_2$$

$$\widehat{S}^{(\mathrm{R,L})}_a = N\left[\exp\left\{\sum_x a^\dagger(x \pm \varepsilon)\left[a(x) - a(x \pm \varepsilon)\right]\right\}\right]$$

Hamiltonian

- Define H by $\widehat{S} = \exp\left(-i\varepsilon H \right)$

- Interpolating continuous time evolution

  $q(t_2) = U(t_2, t_1) q(t_1)$ $U(t_1, t_2) = \exp\left(-i(t_1 - t_2) H \right)$

- Agrees with discrete evolution for $t_{\text{in}} + m\varepsilon$

- Schrödinger equation $i\partial_t q = Hq$

Continuum limit

- Hamiltonian simplifies in the continuum limit

$$H = H_{\text{free}} + H_{\text{int}} + \Delta H$$

$$\Delta H = \mathcal{O}(\varepsilon[H_{\text{int}}, H_{\text{free}}])$$

$$H_{\text{free}} = \frac{i}{\varepsilon}\int \mathrm{d}x \sum_a \left\{ a^\dagger_{La}(x)\partial_x a_{La}(x) - a^\dagger_{Ra}(x)\partial_x a_{Ra}(x) \right\}$$

$$H_{\text{int}} = -\frac{\pi}{2\varepsilon^2}\int \mathrm{d}x \left[a^\dagger_{\mathrm{R}1}a_{\mathrm{R}2} - a^\dagger_{\mathrm{R}2}a_{\mathrm{R}1}\right]\left[a^\dagger_{\mathrm{L}1}a_{\mathrm{L}2} - a^\dagger_{\mathrm{L}2}a_{\mathrm{L}1}\right]$$

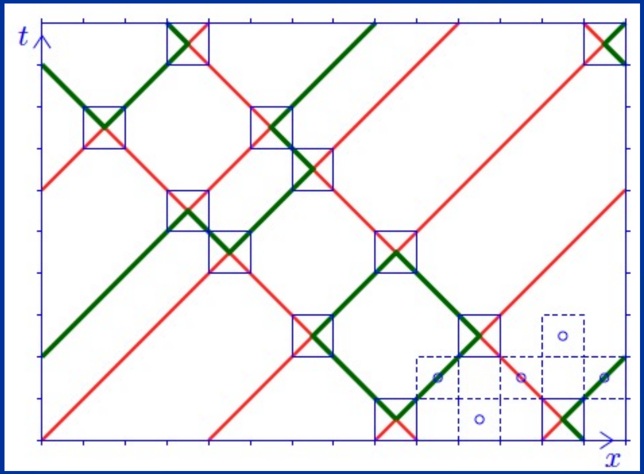
- Standard form of Hamiltonian for fermions

Quantum rules

The wave function of the cellular automaton is identical to the wave function of a quantum multi-fermion system for all discrete times

The quantum rule for expectation values follows from the classical statistical rule for expectation values

Probabilistic cellular automaton is equivalent to quantum field theory for fermions with interactions



General bit- fermion map

- Fermion model in terms of Grassmann functional integral is equivalent to generalized Ising model
- Isomorphism for evolution and observables
- Map is general. But not always positive weight distribution for Ising type model, and unitary evolution.

Discrete fermion model in 1+1 dimensions

Grassmann functional integral

$$Z = \int \mathcal{D}\psi \exp(-S[\psi]) = \int \mathcal{D}\psi w[\psi] \,, \quad S = \sum_t \mathcal{L}(t)$$

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x) \\ - \Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x)\Big]\big(1+\overline{D}(x)\big)\Big\}$$

$$\overline{D} = -(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}})(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}) - (\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}})(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}})$$

Continuum limit

$$S = \int_{t,x} \{\overline{\psi}_{R\alpha}(t,x)(\partial_t + \partial_x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)(\partial_t - \partial_x)\psi_{L\alpha}(t,x) + 2\overline{D}(t,x)\}$$

$$\overline{D} = -\left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}\right) - \left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)$$

$$(\partial_t + \partial_x)\psi(t,x) = \frac{1}{\varepsilon}\left[\psi(t,x) - \psi(t-\varepsilon, x-\varepsilon)\right]$$

$$(\partial_t - \partial_x)\psi(t,x) = \frac{1}{\varepsilon}\left[\psi(t,x) - \psi(t-\varepsilon, x+\varepsilon)\right]$$

$$\int dt \int dx = \int_{t,x} = 2\varepsilon^2 \sum_{t,x}$$

$$\psi(t,x) = \sqrt{2\varepsilon}\,\psi_N(t,x)$$

Lorentz symmetry

Dirac spinor

$$\psi_a = \begin{pmatrix} \psi_{\mathrm{R}a} \\ \psi_{\mathrm{L}a} \end{pmatrix} , \quad \overline{\psi}_a = (\overline{\psi}_{\mathrm{L}a}, -\overline{\psi}_{\mathrm{R}a})$$

Action

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

$$\gamma^0 = -i\tau_2 , \quad \gamma_1 = \tau_1 , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Infinitesimal Lorentz transformation

$$\delta\psi = -\eta \Sigma^{01} \psi , \quad \delta\overline{\psi} = \eta \overline{\psi} \Sigma^{01}$$

$$\Sigma^{01} = \frac{1}{4}[\gamma^0, \gamma^1] = \frac{1}{2}\tau_3$$

Quantum field theory for fermions with interactions

- Thirring model with two colors
- Particular value of coupling

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

Complex structure

- So far real wave function
- In the presence of a complex structure this can be mapped to a complex wave function

- complex conjugation : exchange of 0 and 1, or exchange of particles and holes
- half filled ground state : antiparticles = holes

Quantum mechanics

from classical statistics

Quantum mechanics from classical statistics

For particular quantum model:

Isomorphism between classical statistics
(probabilistic cellular automaton,
generalized Ising model)

and quantum mechanics
(many body quantum system for fermions)

Equivalence

- Expectation values of all observables are the same in both models
- Two equivalent descriptions of the same physical reality

Can quantum physics be described by classical probabilities ?

“ No go “ theorems

Bell , Clauser , Horne , Shimony , Holt

implicit assumption : use of classical correlation function for all correlations between measurements

Kochen , Specker

assumption : unique map from quantum operators to classical observables

Quantum formalism for classical statistics

- Formalism for information transport from one hypersurface to the next:
- Classical wave functions and density matrix
- Transfer matrix formalism : Heisenberg picture
- Wave functions : Schrödinger picture
- Non commuting operators for observables
- Quantum rules from classical statistical rules

structural elements of quantum mechanics

wave function as probability amplitude

quantum rule for expectation values of observables

follows from

classical statistical rule for expectation values

unitary time evolution

ν

h

Simple conversion factor for units

i

presence of complex structure

$$[A,B]=C$$

non – commuting operators

examples : particle numbers, Hamilton operator, momentum

necessary to represent observables in incomplete statistics

Conclusion

- Particular quantum field theory for interacting fermions is equivalent to the classical statistical model of a particular probabilistic cellular automaton.
- Example for quantum mechanics from classical statistics
- Useful for simulating fermionic models and understanding of statistical properties of cellular automata?

end

Determinism vs. Probabilism

" Does god throw dices ? "

… an old dispute

Gott würfelt

Gott würfelt nicht

"Es scheint hart, dem Herrgott in die Karten zu gucken. Aber dass er würfelt und sich telepatischer Mittel bedient (wie es ihm von der gegenwärtigen Quantentheorie zugemutet wird), kann ich keinen Augenblick glauben.."

Einstein: Brief an Cornelius Lanczos am 21. März 1942

not todays topic

Gott würfelt　　　　Gott würfelt nicht

humans can only deal with probabilities

determinism vs. probabilism

my personal view :

- determinism not needed
- one can start with probabilistic theory and probabilistic evolution
- nevertheless : deterministic evolution remains a possible logical option
  - in practice fictitious

Probabilistic subsystems

- quantum mechanics of single atoms arises from

  quantum subsystems
- subsystems are genuinely probabilistic
- part of information is lost by focus on subsystem
- partially ”integrating out” degrees of freedom

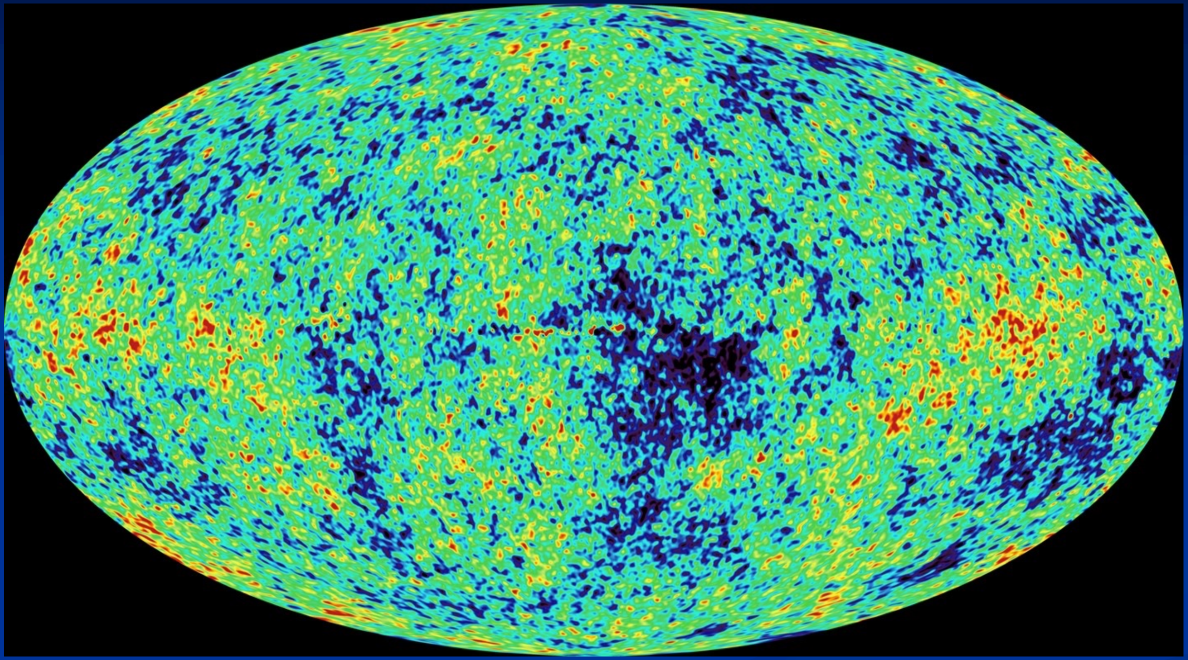
Reduction of wave function

- Reduction of wave function is a convenient technical method to describe conditional probabilities
- This must not be a physical process during the measurement

conditional probability

sequences of events(measurements)
are described by
conditional probabilities

both in classical statistics
and in quantum statistics

$w(t_1)$  :

not very suitable
for statement , if here and now
a pointer falls down

Schrödinger’s cat

conditional probability :
if nucleus decays
then cat dead with $w_c = 1$
(reduction of wave function)

Reduction of wave function

- Reduction of wave function is a convenient technical method to describe conditional probabilities
- This must not be a physical process during the measurement

correlation and operator product

- Classical statistical systems admit many product structures of observables
- Many different definitions of correlation functions possible , not only classical correlation !
- Type of measurement determines correct selection of correlation function !
- Ideal quantum measurement should be compatible with information in quantum subsystem

General bit fermion map

- Isomorphism between generalized Ising model and Grassmann functional integral
- Based on identical step evolution operator for both models, with associated map of observables
- In our case: proof that discrete Thirring model with two colors has the same step evolution operator as the cellular automaton

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x) \\ - \Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x) \Big] \left(1 + \overline{D}(x)\right) \Big\}$$

Probabilistic formalism for cellular automata

Generalized Ising model: $$w[s] = Z^{-1} \exp\big(-S[s]\big) b(s_{in}, s_f)$$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

limit : beta to infinity , sigma to zero :
only one possibility for change , unique jump

probabilistic aspects only in boundary term :

$$b(s_{in}, s_f) = \bar{f}_f(s_f) f_{in}(s_{in})$$

Functional integral for cellular automata (free fermions)

Generalized Ising model: $$w[s] = Z^{-1} \exp\big(- S[s]\big) b(s_{in}, s_f)$$

$$S = -\frac{\beta}{2} \sum_{x,t} s(t,x) \Big[s(t+1, x+1) + \sigma s(t+1, x-1) \Big]$$

limit : beta to infinity , sigma to zero :
only one possibility for change , unique jump

Functional renormalization for cellular automata