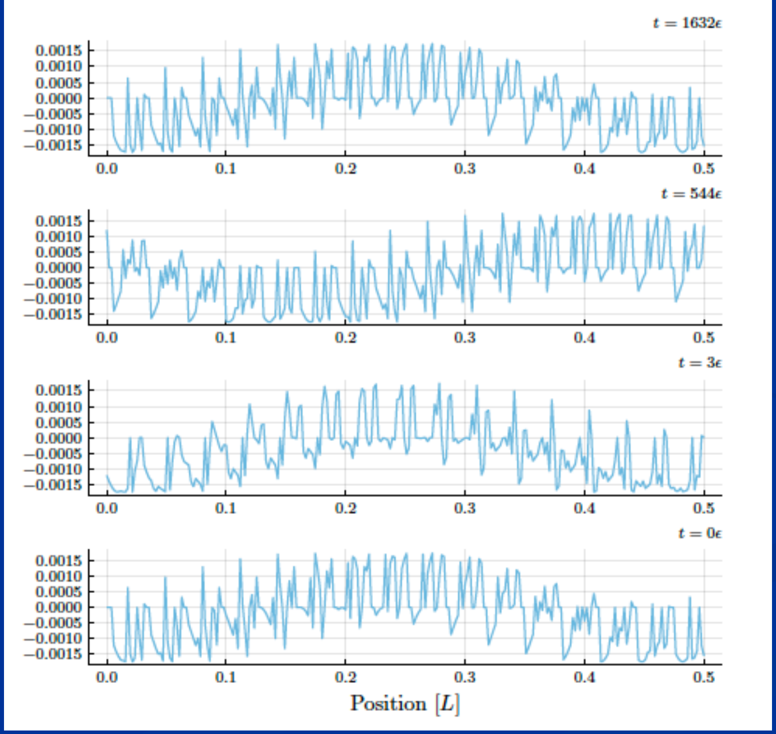
Quantum systems from probabilistic cellular automata



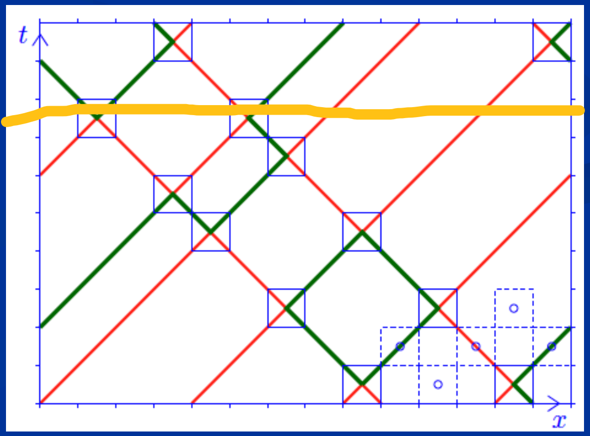
The probabilistic world

- Physicists describe the Universe by a probability distribution for events at all times and positions
- Classical statistics
- Quantum mechanics follows by focus on time-local subsystems

Overall view on quantum mechanics

- Quantum mechanics from quantum field theory
- Functional integral : variables for all times (fields, bit configurations, paths)
- Local time physics :

  Focus on hypersurfaces labeled by t



Quantum mechanics

- Projection on local time subsystem

(Feynman)

Wave function, operators,

linear evolution law, $\psi(t+\varepsilon)=U(t)\,\psi(t)$

superposition of solutions,

formalism of quantum mechanics

Overall probability distribution for events at all times and positions

- Euclidean functional integral

$$p[\chi'] = Z^{-1} \exp\left(- S[\chi'] \right)$$

$$Z = \int \mathcal{D}\chi' \, \exp\left(- S[\chi'] \right)$$

$$\int \mathcal{D}\chi' = \prod_x \int_{-\infty}^{\infty} \mathrm{d}\chi'(x)$$

Euclidean functional integral

- Projection on local time subsystem

(Feynman)

Wave function, operators,

linear evolution law, $q(t+\varepsilon) = \widehat{S}(t)q(t)$

superposition of solutions,

formalism of quantum mechanics

Euclidean functional integral

- Wave function q(t) is real
- Loss of information during evolution, e.g. approach to equilibrium state with some correlation time or length

Not always !

Probabilistic cellular automata with deterministic invertible updating are classical probabilistic systems

No loss of information during evolution

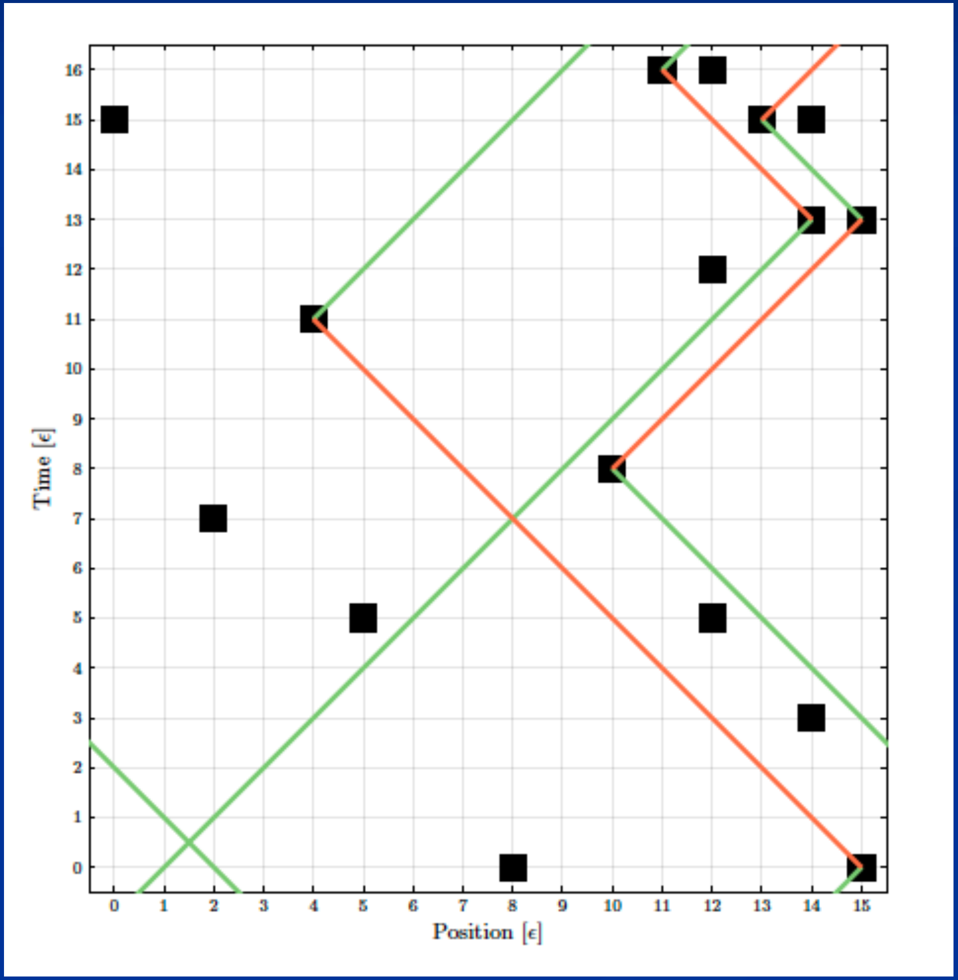
They are discrete quantum systems

Cellular automaton

- Deterministic manipulation of bit configurations
- Updating rule of bit configurations in sequential steps
- Updating of a cell depends only on some neighboring cells
- Repetition (at least after certain number of time steps)

(Classical computer is an automaton without repetition)

Updating rule for random automaton



Four types of bits

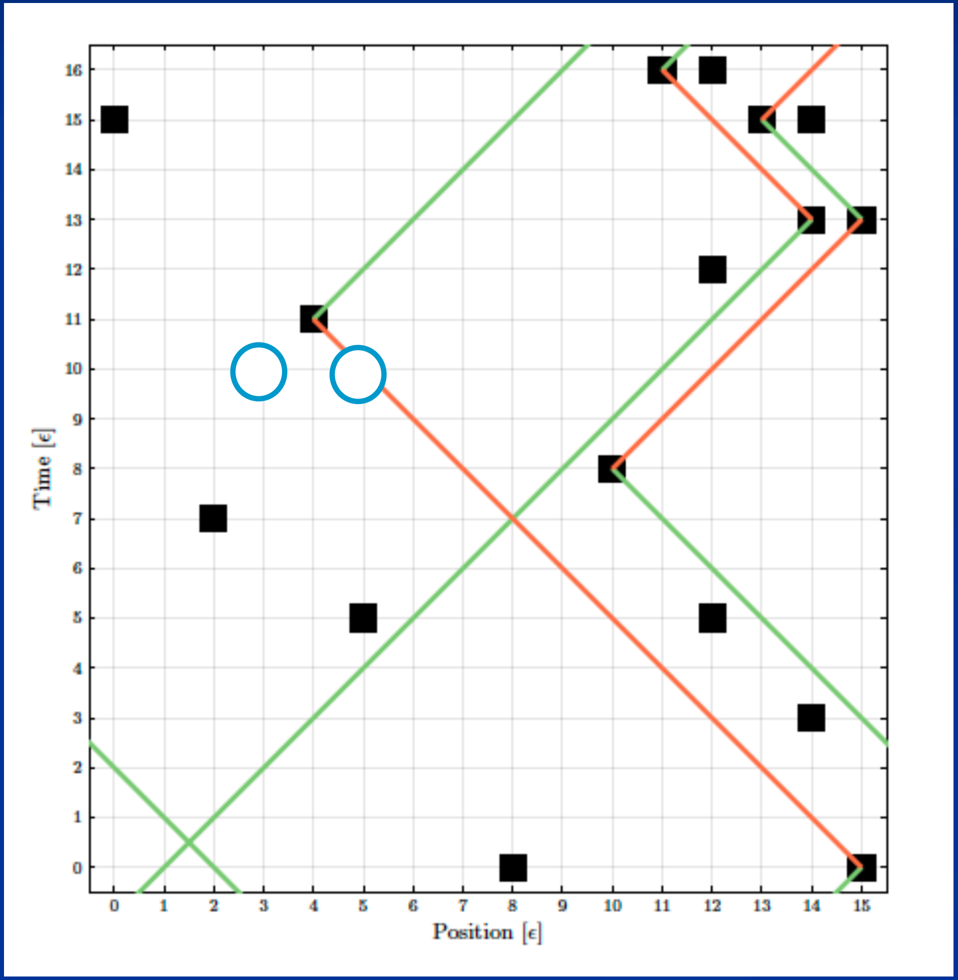
right- and left- movers

red and green

At randomly distributed scattering points: occupied bits change direction and color.

Repetition of distribution after certain number of time steps

Cellular automaton



Updating of bits
in cell (t, x)
is influenced only
by the cells
(t-ε, x-ε) and
(t-ε, x+ε).

Causal structure of
QFT with light
cones

Probabilistic cellular automaton

Probability distribution for initial configurations

deterministic updating

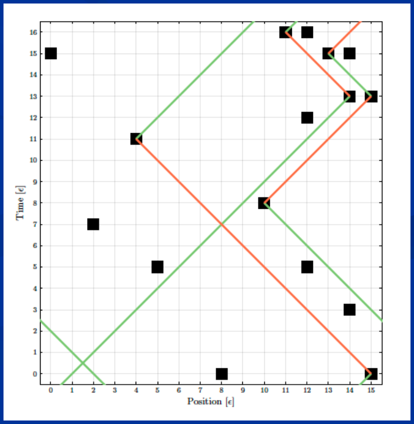
Probabilistic cellular automaton

Probabilistic initial condition: Specify at initial time t_{in} for each bit configuration $\overline{\rho}$ a probability $p_{\overline{\rho}}(t_{\text{in}})$

Evolution: every given configuration $\overline{\rho}$ at t_{in} propagates at t to a configuration $\tau(t,\overline{\rho})$

$$\Rightarrow \quad p_\tau(t) = p_{\overline{\rho}(\tau)}(t_{\text{in}})$$

Updating rule: specifies $\tau(t+\varepsilon,\rho(t))$



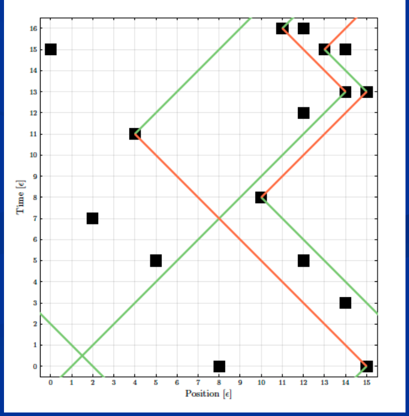
Overall probability distribution

Follow trajectory of some initial configuration

Probabilities are equal for each point on trajectory

Probabilities for arbitrary bit configurations in time and space. They differ from zero only for configurations that can be reached allowed trajectories

$$p[\chi'] = Z^{-1} \exp\left(- S[\chi'] \right)$$



Classical probabilistic system

Wave function for probabilistic cellular automaton

Local time probability distribution: at every time t a bit configuration τ occurs with probability $p_\tau(t)$, which equals the probability for the initial bit configuration from which it originates.

Real wave function q(t): probability amplitude

$$p_\tau(t) = (q_\tau(t))^2$$

$$q_\tau(t) q_\tau(t) = 1$$

q(t) is a unit vector

Deterministic and probabilistic cellular automaton

- Deterministic CA : sharp wave function

$$q_\rho(t_{\text{in}}) = \delta_{\rho,\bar{\rho}}$$

- Probabilistic CA : arbitrary wave function

Step evolution operator

- Evolution for basic time step is encoded in the step evolution operator

$$q(t+\varepsilon) = \widehat{S}(t)q(t) \qquad q_\tau(t+\varepsilon) = \widehat{S}_{\tau\rho}(t)q_\rho(t)$$

- Contains the updating rule for CA

$$\widehat{S}_{\tau\rho}(t) = \delta_{\tau,\overline{\tau}(\rho)} = \delta_{\overline{\rho}(\tau),\rho}$$

$$q_\tau(t+\varepsilon) = q_{\overline{\rho}(\tau)}(t)\,, \quad p_\tau(t+\varepsilon) = p_{\overline{\rho}(\tau)}(t)$$

Unique jump matrix

- Step evolution operator for cellular automata is unique jump matrix
- In every row and column: precisely one element +1 or -1, all other elements zero

$\widehat{S}_{\tau\rho}(t)$ is orthogonal and therefore unitary

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Hamiltonian

- Define H by $\widehat{S} = \exp\left(-i\varepsilon H\right)$

- H is Hermitian and piecewise constant

- Interpolating continuous time evolution

  $q(t_2) = U(t_2, t_1) q(t_1)$ $\quad U(t_1, t_2) = \exp\left(-i(t_1 - t_2)H\right)$

- Schrödinger equation $i\partial_t q = Hq$

- Solution agrees with discrete evolution for t= $t_{\text{in}} + m\varepsilon$

Complex structure

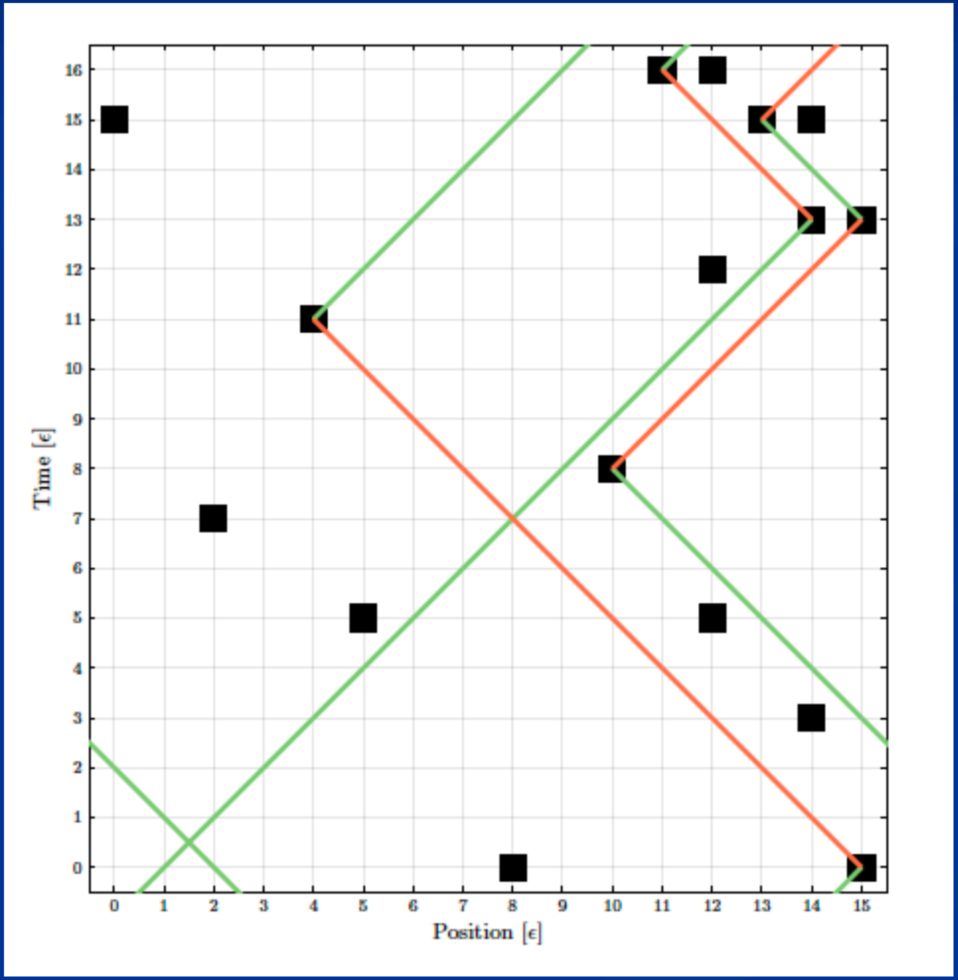
Suitable set of two discrete transformations for complex conjugation and multiplication by i

Configurations at given t with single occupied bit: (x, γ)
Wave function for a single occupied bit: $q_\gamma(t, x)$

red and green correspond to
real and imaginary parts of complex wave function

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \psi_R = q_1 + iq_2, \quad \psi_L = q_3 + iq_4.$$

Random probabilistic automaton



Four types of bits

right- and left- movers

red and green

At randomly distributed scattering points:
change of direction and color
Repetition after certain number of time steps

Free massless Dirac fermions

- Without scattering: automaton describes quantum field theory for free massless Dirac fermions in one space and one time dimension
- Arbitrary number of fermions
- Half filled vacuum with all negative energy states filled

- Single fermion state: $\psi_\alpha(t, x)$

Momentum observable for single fermion state

Fourier transform

$$\psi(x) = N_x^{-\frac{1}{2}} \sum_q \exp(iqx)\psi(q),$$
$$\psi(q) = N_x^{-\frac{1}{2}} \sum_x \exp(-iqx)\psi(x).$$

Momentum operator

$$P(q, q') = q\tilde{\delta}_{q,q'}$$

Continuum limit

$$\tilde{P}(x, x') = -i\partial_x \delta(x - x')$$

Momentum distribution

$$w(q) = \psi^\dagger(q)\psi(q), \quad \langle f(P)\rangle = \sum_q f(q)w(q)$$

Expectation value

$$\langle f(P)\rangle = \sum_{q,q'} \psi^\dagger(q)(f(P))(q,q')\psi(q)$$

Conserved momentum

Momentum is a conserved quantity $\overline{H}_f = P\tau_3.$

The expectation values $\langle f(P)\rangle = \sum_{q,q'} \psi^\dagger(q)(f(P))(q,q')\psi(q)$

do not depend on time

Momentum eigenstates : plane waves

They require probabilistic automaton with smooth wave functions

Statistical observables

- Momentum is a statistical observable
- No fixed value for given bit configuration
- Characterizes properties of probabilistic information (similar to temperature)
- Does not commute with position operator
- Bell's inequalities do not apply to pair position and momentum since no classical correlation function can be defined

Numerical solution

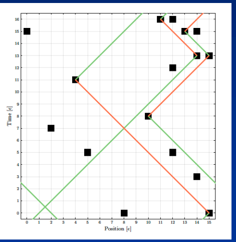
Simulate simple system by following trajectories numerically

Quantum Systems from Random Probabilistic Automata

A. Kreuzkamp[1] and C. Wetterich[1]

Brownian automaton

- Single bit occupied

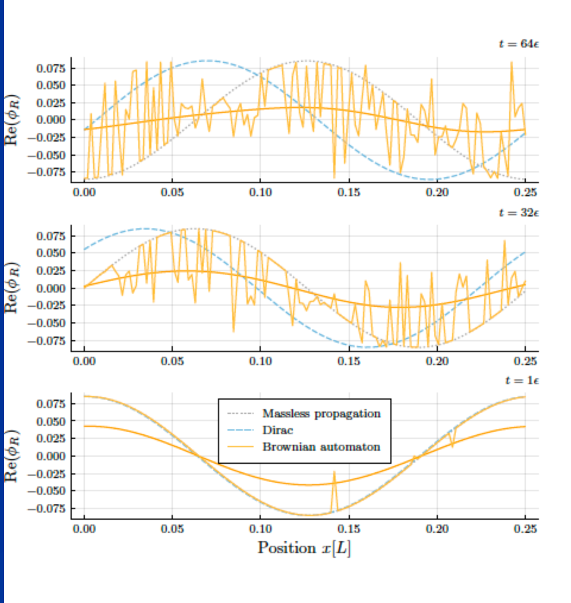


- Start with smooth probability distribution

  (harmonics corresponding to solution of Dirac equation with mass)

- evolution at different

  time steps



Conserved energy

- Mesoscopic Hamiltonian defines conserved energy observable
- H is not known explicitly
- Single time step $U(t) = U_s(t)U_f$

$$U(0) = \exp\{-i\epsilon\overline{V}(0,x)\}\exp\{-i\epsilon P\tau_3\} = \exp\{-i\epsilon\overline{H}(0)\}$$

- Generalized potential

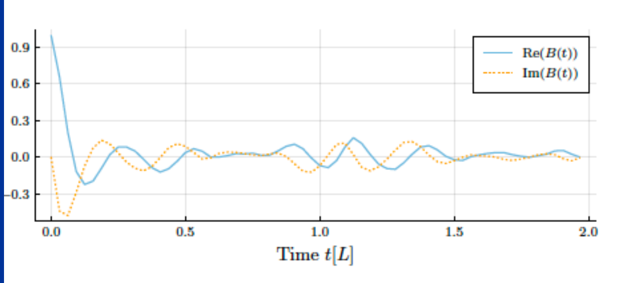
$$U_s(t;x,x') = \exp\{-i\epsilon\overline{V}(t,x)\}\delta_{x,x'},$$
$$\overline{V}(t,x) = \frac{\pi}{2\epsilon}(\tau_2 - c)\sum_j \delta_{x,\overline{x}_j(t)}.$$

- Repeat M_t times: mesoscopic Hamiltonian H

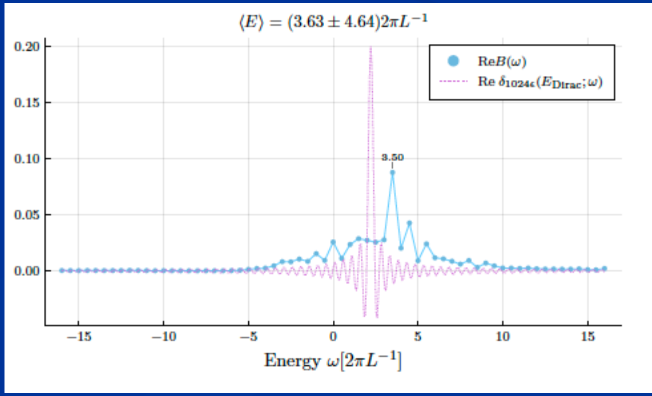
Conserved energy

Energy is conserved, but spectrum and eigenfunctions of Hamiltonian not known.
Analyze energy spectrum from transition element

$$B(t;\bar{t}) = \sum_x \psi^\dagger(\bar{t}, x)\psi(\bar{t}+t, x)$$



Superposition of energy eigenstates

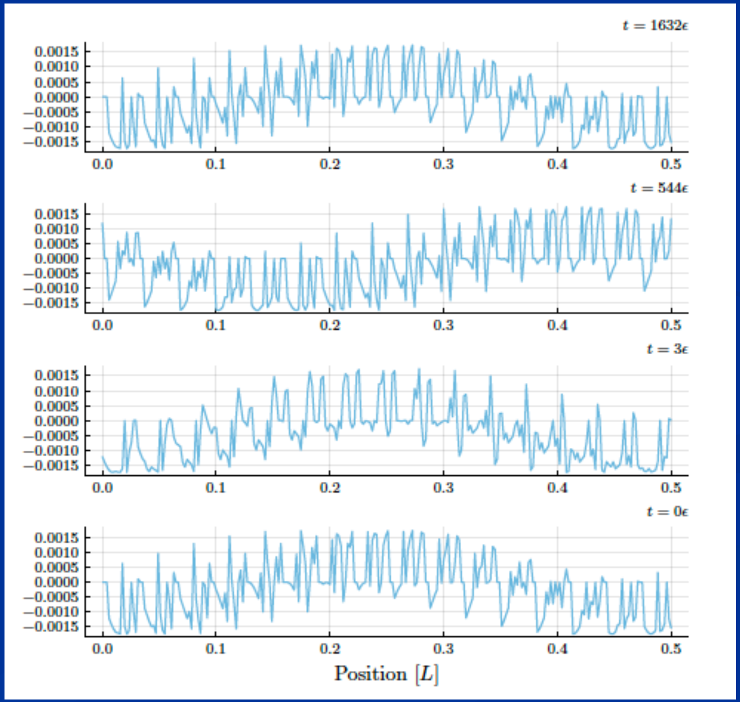


Periodic evolution of probability distribution for energy eigenstates

Automaton with periodicity in space:

Energy eigenstate found explicitly

Evolution

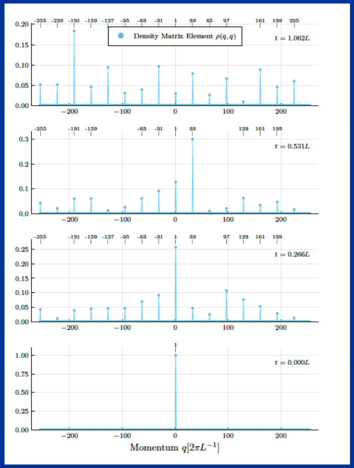


Conserved coarse grained momentum

Space translation invariance by M_x introduces conserved coarse grained momentum

Evolution of

momentum distribution



Diagonalize H on subspace with given coarse grained momentum

Naïve continuum limit

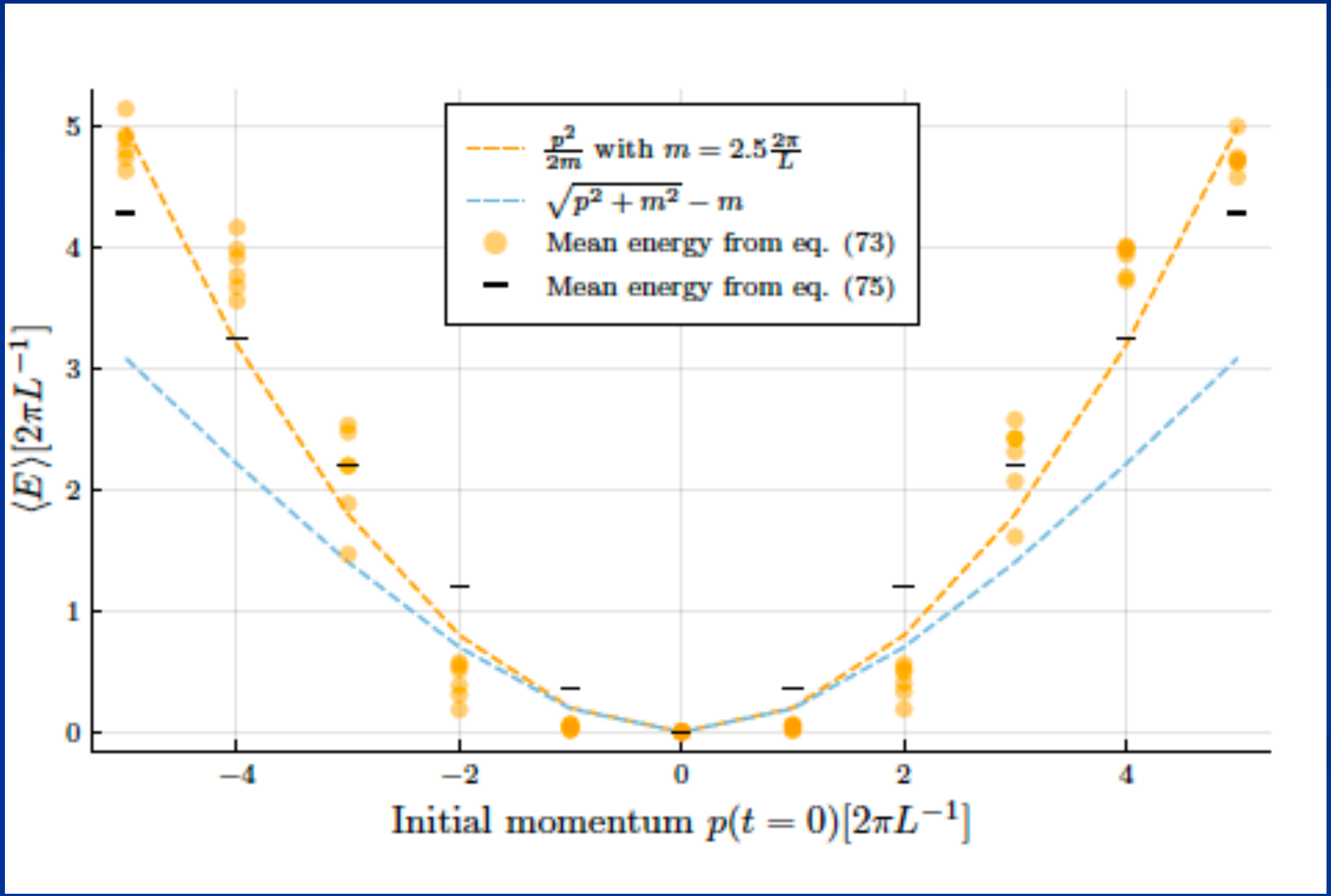
If commutator terms in expansion of U can be neglected for smooth enough wave functions

$$U(0) = \exp\{-i\epsilon \overline{V}(0,x)\}\exp\{-i\epsilon P\tau_3\} = \exp\{-i\epsilon \overline{H}(0)\}$$

$$\overline{H}(0) = P\tau_3 + \overline{V}(0,x) + O(\epsilon[P,\overline{V}])$$

Hamiltonian for free massive Dirac particle with mass proportional to mean number of scattering points

Dispersion relation for early evolution of Brownian automaton



Conclusion

- Quantum formalism very useful for investigation of random probabilistic automaton
- Continuum limit for random automaton not established
- Extended model for quantum particle in a potential ?

Quantum mechanics from classical statistics

- Probabilistic cellular automata are classical statistical systems.
- Probabilistic cellular automata are discrete quantum systems.
- Quantum mechanics emerges from a classical statistical system.
- No go theorems (Bell etc.) do not apply to all pairs of observables.

Outlook: How to find overall classical probability distribution for quantum particle in an arbitrary potential ?

Top down approach: Find automaton for interesting QFT. Construct vacuum and one-particle excitation. Find continuum limit. Realistic setting, but hard to implement.

Bottom up approach: Explicit construction of probabilistic automaton is already done for

- quantum particle in harmonic potential
- single qubit with arbitrary time-dependent Hamiltonian

Classical probability distribution not realistic,

but useful conceptually. No contradictions.

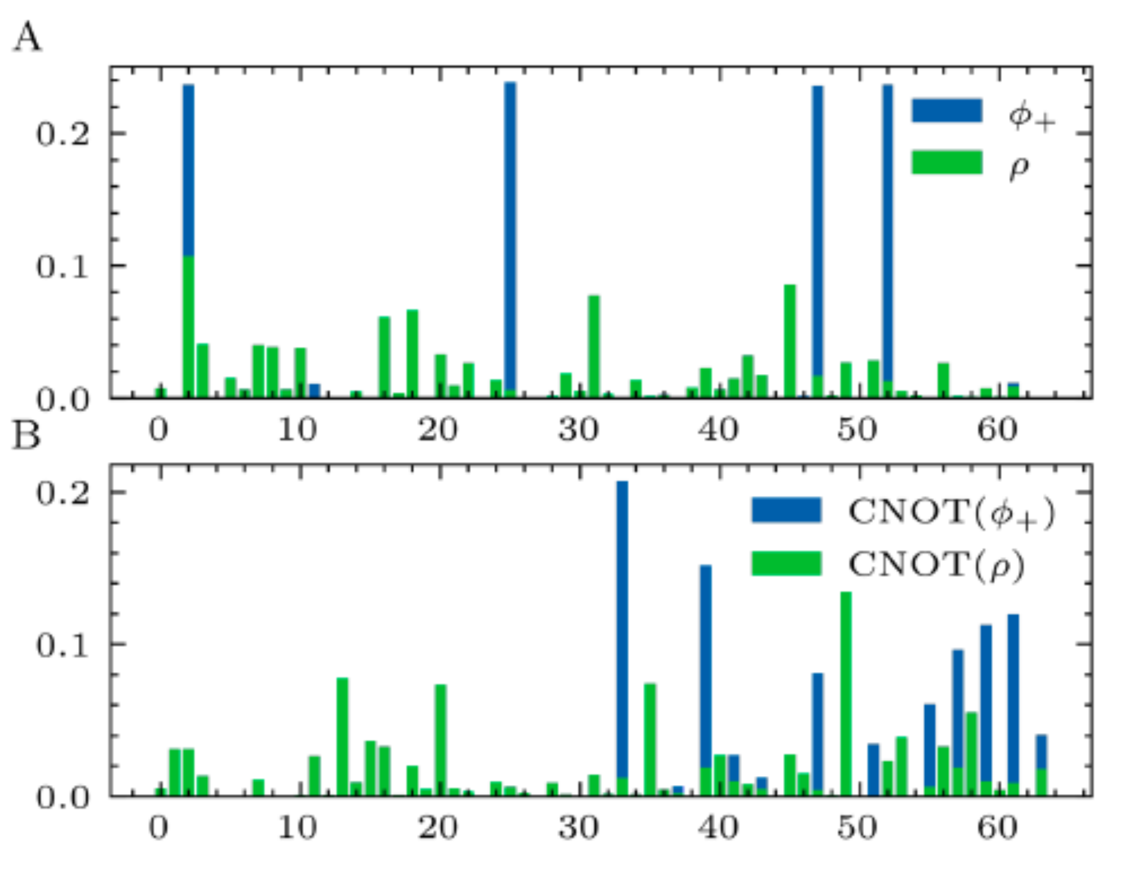
Beyond probabilistic automata

- "Unitary" change of probability distribution more general than probabilistic automata
- Step evolution operator can be orthogonal, but not a unique jump matrix

Small neuromorphic computer has learned how to find the classical probability distribution for arbitrary (entangled) two-qubit states and to perform arbitrary unitary transformations

(with C. Pehle)

Classical probability distribution for maximally entangled two-qubit quantum state



$$\psi_+ = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

6 classical bits,
64 configurations

bit quantum map from probability distribution to quantum density matrix

Automaton has been constructed for discrete fermion model with interactions in 1+1 dimensions

Grassmann functional integral

$$Z = \int \mathcal{D}\psi \exp(-S[\psi]) = \int \mathcal{D}\psi w[\psi] \,, \quad S = \sum_t \mathcal{L}(t)$$

$$\begin{aligned}\mathcal{L}(t) = -\sum_x \Big\{ & \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x) \\ & - \left[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x)\right]\left(1+\overline{D}(x)\right)\Big\}\end{aligned}$$

$$\overline{D} = -\left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}\right) - \left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)$$

Naïve continuum limit is Lorentz invariant

Dirac spinor $\psi_a = \begin{pmatrix} \psi_{Ra} \\ \psi_{La} \end{pmatrix}, \quad \overline{\psi}_a = (\overline{\psi}_{La}, -\overline{\psi}_{Ra})$ a=1,2

Action $S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$

$$\gamma^0 = -i\tau_2 , \quad \gamma_1 = \tau_1 , \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Infinitesimal Lorentz transformation

$\delta\psi = -\eta \Sigma^{01} \psi , \quad \delta\overline{\psi} = \eta \overline{\psi} \Sigma^{01}$ $\Sigma^{01} = \frac{1}{4}[\gamma^0, \gamma^1] = \frac{1}{2}\tau_3$

Is this all useful?

- Quantum formalism offers new insights for the dynamics of probabilistic cellular automata
- New forms of correlated computing
- Clarification of origin of quantum concepts – demystification
- Probabilistic realism is a philosophical concept
- Restrictions on fundamental theory?

end

Probabilistic automata
for
fermionic quantum field theories

Fermions

- quantum objects
- wave function totally antisymmetric

  (Pauli principle)
- anticommutator for annihilation and creation operators
- anticommuting Grassmann variables
- functional integral or partition function for many body systems or quantum field theories is Grassmann functional integral

Fermions are Ising spins or bits

- Fermionic occupation numbers n = 0 , 1
- Classical bits
- Ising spins s = 2n -1
- Bit configurations = many body states of fermions

Fermionic wave function

- Occupation number basis for multi-fermion systems:

- To each bit configuration one associates a component of the wave function
- Occupation numbers for different space points and species

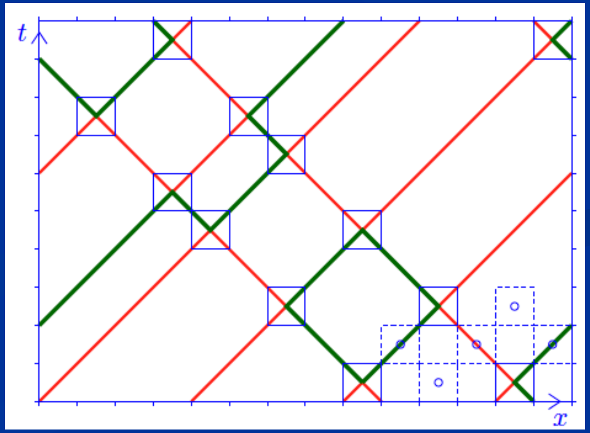
Probabilistic cellular automata for fermionic quantum field theories

- Wave function in same Hilbert space
- If step evolution operator for automaton is the same as for the fermionic quantum field theory:
  Both are equivalent

Updating rule for Thirring automaton

- one – dimensional chain, x : discrete lattice sites
- at each x : red and green right movers and left movers (4 different species at each site)
- at each time step: configuration for right(left) movers moves one position to the right(left)
- if two single particles meet:

  colors are exchanged



Particle wave duality

Particle aspect:

- Bits: yes/no decisions
- Possible measurement values 1 or 0

Discrete spectrum of observables

Wave aspect : continuous wave function

more generally: continuity of probabilistic information

QFT- CA equivalence

A fermionic quantum field theory is equivalent to a probabilistic cellular automaton if the evolution operator for discrete time steps is a

unique jump matrix

(in a real formulation of the evolution equation)

General bit fermion map

- Isomorphism between generalized Ising model and Grassmann functional integral
- Based on identical step evolution operator for both models, with associated map of observables
- In our case: proof that discrete Thirring model with two colors has the same step evolution operator as the cellular automaton

$$\begin{aligned}\mathcal{L}(t) = -\sum_x \Big\{ & \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x) \\ & - \Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x)\Big]\left(1+\overline{D}(x)\right)\Big\}\end{aligned}$$

Discrete fermion model in 1+1 dimensions

Grassmann functional integral

$$Z = \int \mathcal{D}\psi \exp(-S[\psi]) = \int \mathcal{D}\psi w[\psi] \,, \quad S = \sum_t \mathcal{L}(t)$$

$$\mathcal{L}(t) = -\sum_x \Big\{ \overline{\psi}_{R\alpha}(t+\varepsilon, x+\varepsilon)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t+\varepsilon, x-\varepsilon)\psi_{L\alpha}(t,x)$$
$$-\Big[\overline{\psi}_{R\alpha}(t,x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)\psi_{L\alpha}(t,x) + \overline{D}(x)\Big]\left(1+\overline{D}(x)\right)\Big\}$$

$$\overline{D} = -\left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}\right) - \left(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}}\right)\left(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}}\right)$$

Naïve continuum limit

$$S = \int_{t,x} \{\overline{\psi}_{R\alpha}(t,x)(\partial_t + \partial_x)\psi_{R\alpha}(t,x) + \overline{\psi}_{L\alpha}(t,x)(\partial_t - \partial_x)\psi_{L\alpha}(t,x) + 2\overline{D}(t,x)\}$$

$$\overline{D} = -(\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L2}} - \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L1}})(\psi_{\mathrm{R1}}\psi_{\mathrm{L2}} - \psi_{\mathrm{R2}}\psi_{\mathrm{L1}}) - (\overline{\psi}_{\mathrm{R1}}\overline{\psi}_{\mathrm{L1}} + \overline{\psi}_{\mathrm{R2}}\overline{\psi}_{\mathrm{L2}})(\psi_{\mathrm{R1}}\psi_{\mathrm{L1}} + \psi_{\mathrm{R2}}\psi_{\mathrm{L2}})$$

$$(\partial_t + \partial_x)\psi(t,x) = \frac{1}{\varepsilon}[\psi(t,x) - \psi(t-\varepsilon, x-\varepsilon)]$$
$$(\partial_t - \partial_x)\psi(t,x) = \frac{1}{\varepsilon}[\psi(t,x) - \psi(t-\varepsilon, x+\varepsilon)]$$

$$\int dt \int dx = \int_{t,x} = 2\varepsilon^2 \sum_{t,x}$$

$$\psi(t,x) = \sqrt{2\varepsilon}\,\psi_N(t,x)$$

Lorentz symmetry

Dirac spinor

$$\psi_a = \begin{pmatrix} \psi_{Ra} \\ \psi_{La} \end{pmatrix} , \quad \overline{\psi}_a = (\overline{\psi}_{La}, -\overline{\psi}_{Ra})$$

Action

$$S = -\int_{t,x} \left\{ \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_a \overline{\psi}_b \gamma_\mu \psi_b + \frac{1}{2} \overline{\psi}_a \gamma^\mu \psi_b \varepsilon^{ab} \overline{\psi}_c \gamma_\mu \psi_d \varepsilon^{cd} \right\}$$

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$$\Sigma^{01} = \frac{1}{4}[\gamma^0, \gamma^1] = \frac{1}{2}\tau_3$$

Can quantum physics be described by classical probabilities ?

“ No go “ theorems

Bell , Clauser , Horne , Shimony , Holt

implicit assumption : use of classical correlation function for all correlations between measurements

Kochen , Specker

assumption : unique map from quantum operators to classical observables

Conclusion

- Particular quantum field theory for interacting fermions is equivalent to the classical statistical model of a particular probabilistic cellular automaton.
- Large family of models – not all models!
- Examples for quantum mechanics from classical statistics
- Useful for simulating fermionic models and understanding of statistical properties of cellular automata?