Universality in ultra-cold fermionic atom gases

Universality in ultra-cold fermionic atom gases

with

S. Diehl , H.Gies , J.Pawlowski

BEC – BCS crossover

Bound molecules of two atoms
on microscopic scale:

Bose-Einstein condensate (BEC) for low T

Fermions with attractive interactions
(molecules play no role) :

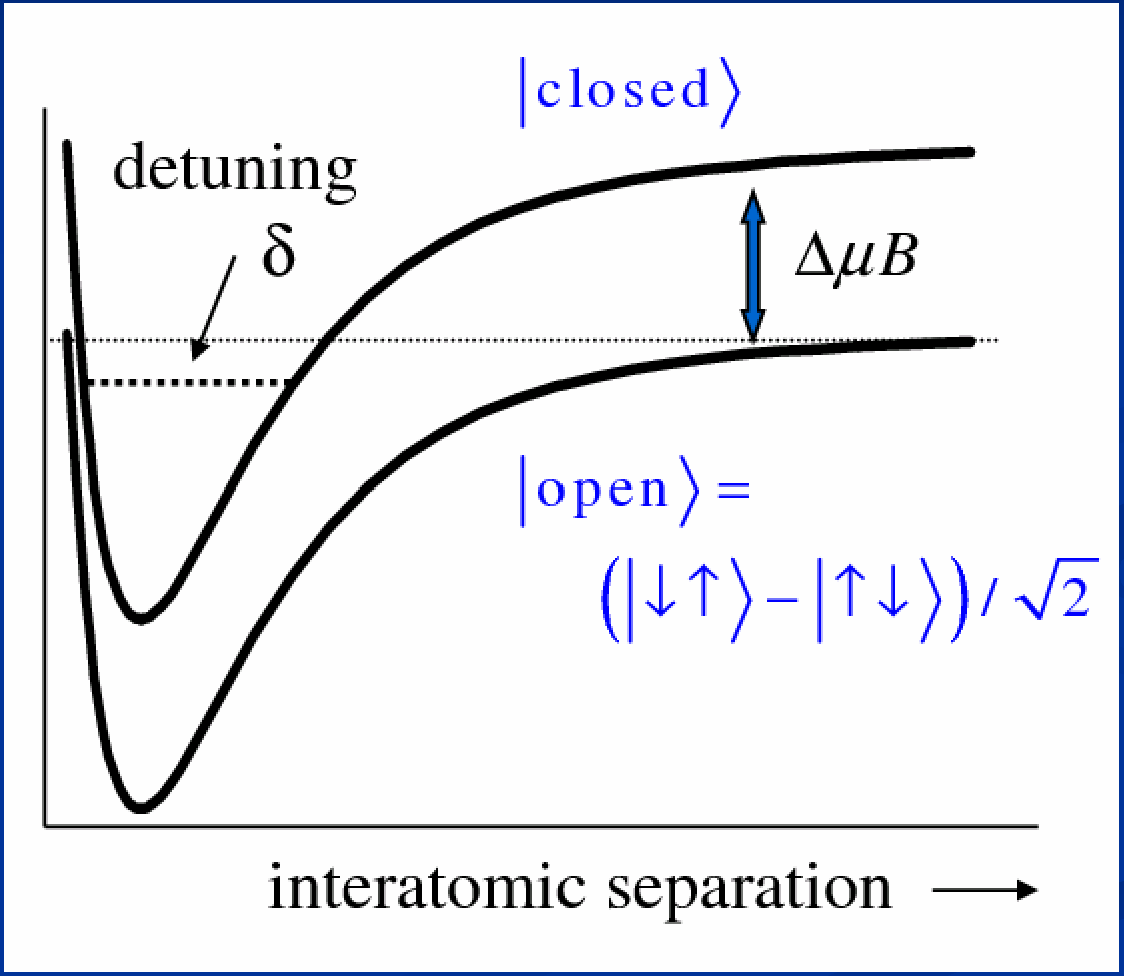
BCS – superfluidity at low T
by condensation of Cooper pairs

Crossover by Feshbach resonance
as a transition in terms of external magnetic field

microphysics

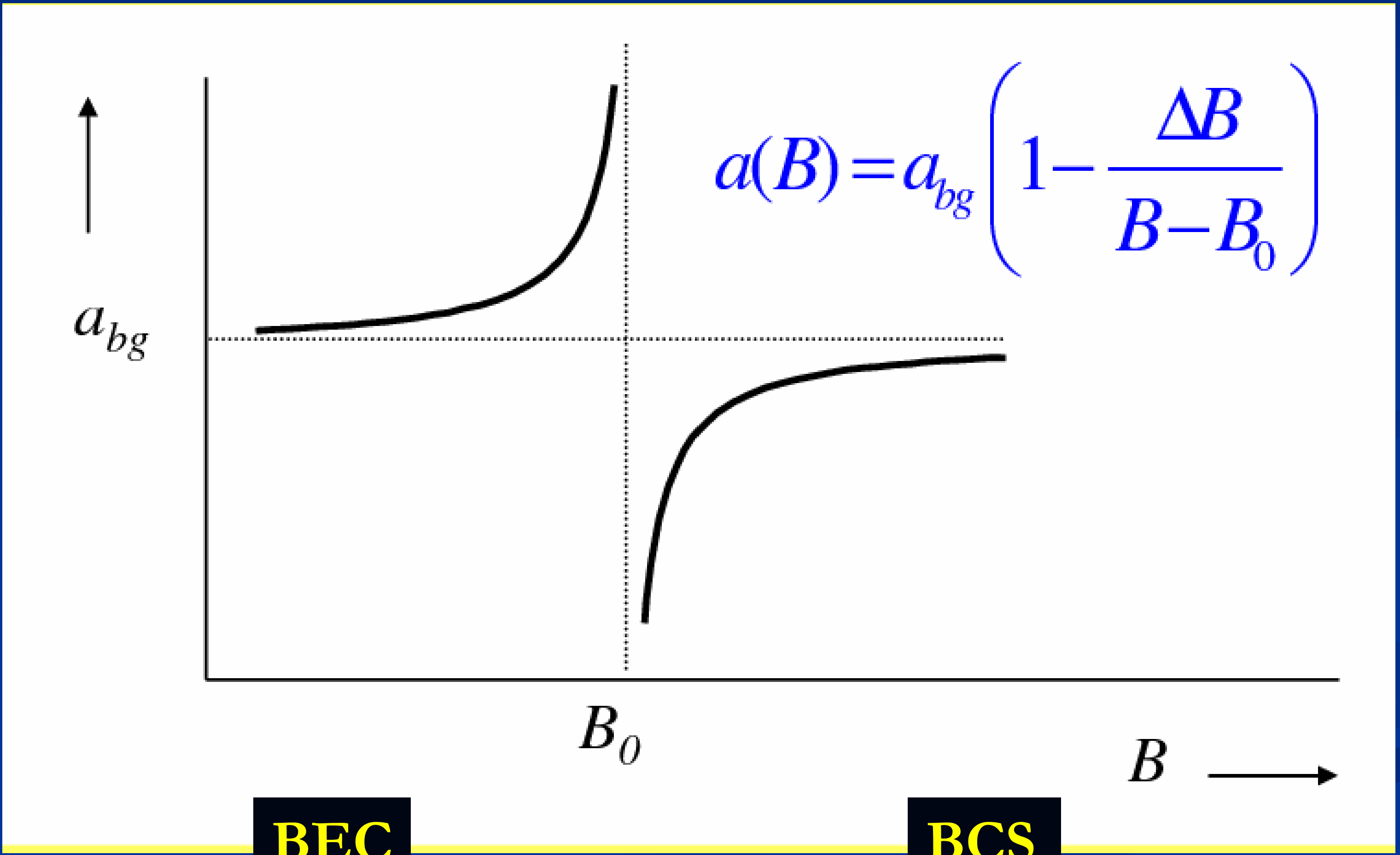
- determined by interactions between two atoms
- length scale : atomic scale

Feshbach resonance



H.Stoof

scattering length



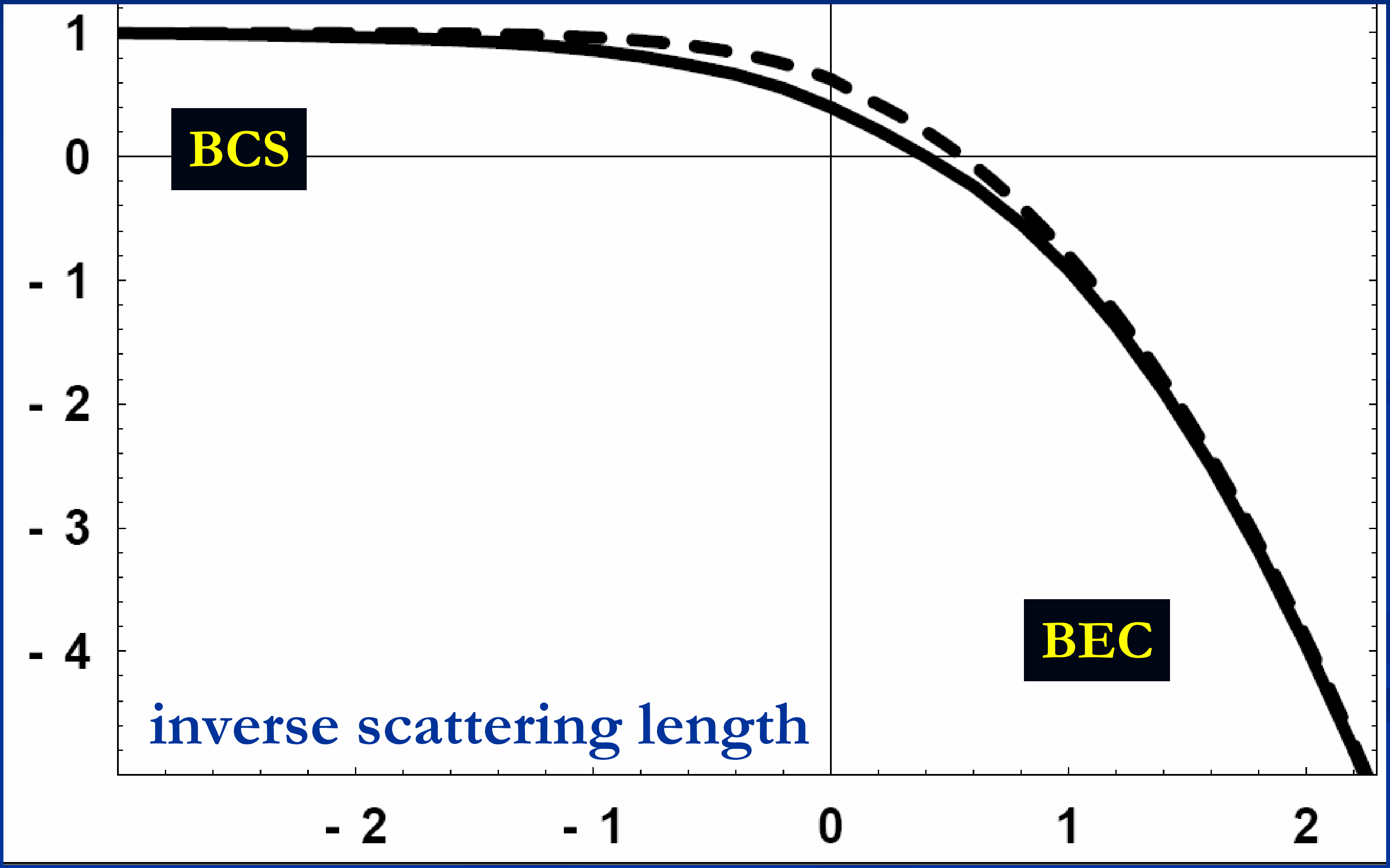
BEC

BCS

many body physics

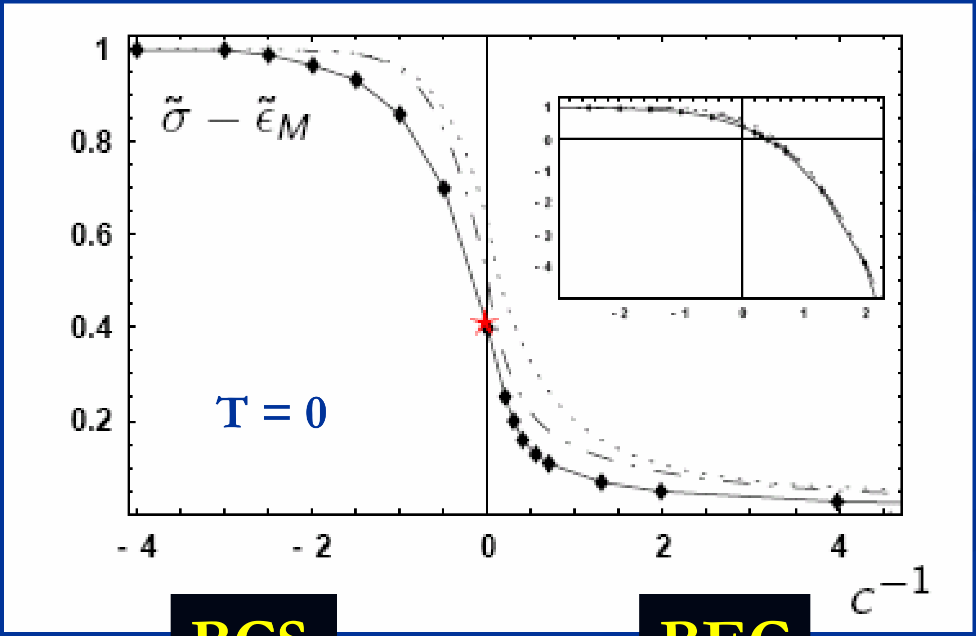
- dilute gas of ultra-cold atoms
- length scale : distance between atoms

chemical potential



BEC – BCS crossover

- qualitative and partially quantitative theoretical understanding
- mean field theory (MFT) and first attempts beyond



BCS

BEC

concentration : **c = a k_F**

reduced chemical potential : **σ˜ = μ/ε_F**

Fermi momemtum : k_F

Fermi energy : ε_F

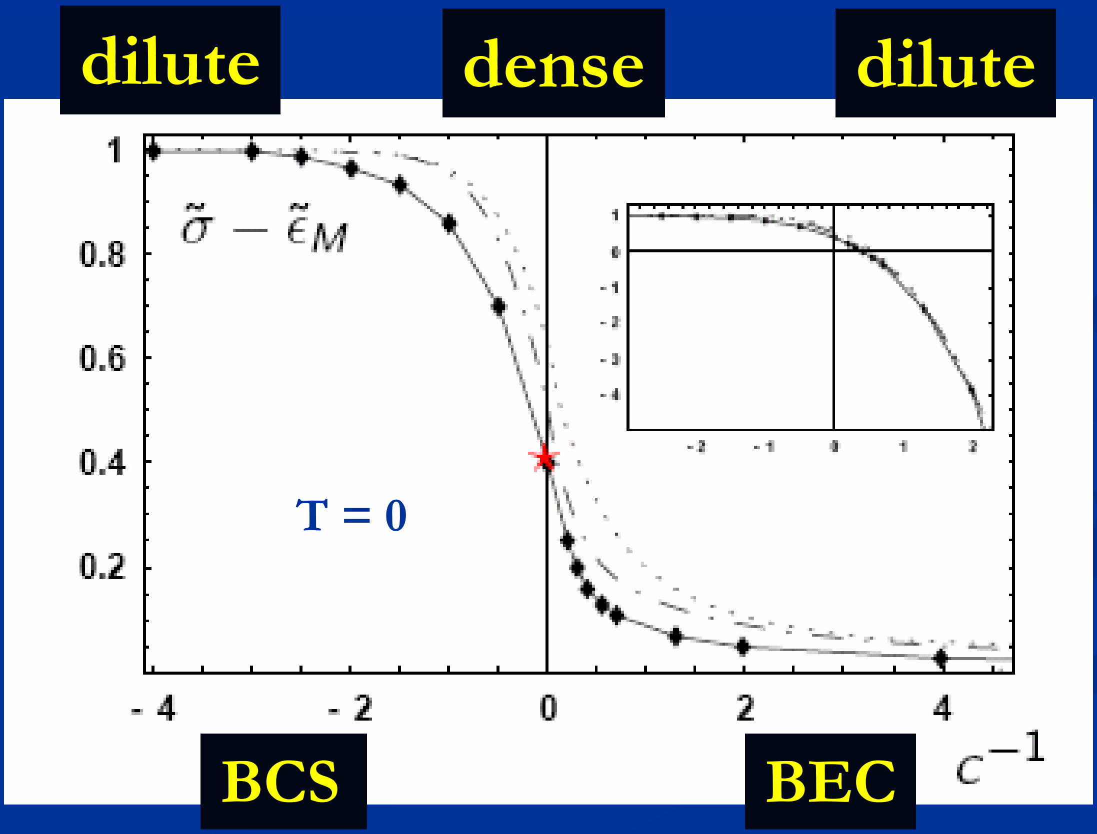
binding energy :

$$\tilde{\epsilon}_M = -\theta(c^{-1})c^{-2}$$

concentration

- $c = a\,k_F$, $a(B)$: scattering length
- needs computation of density $n = k_F{}^3/(3\pi^2)$

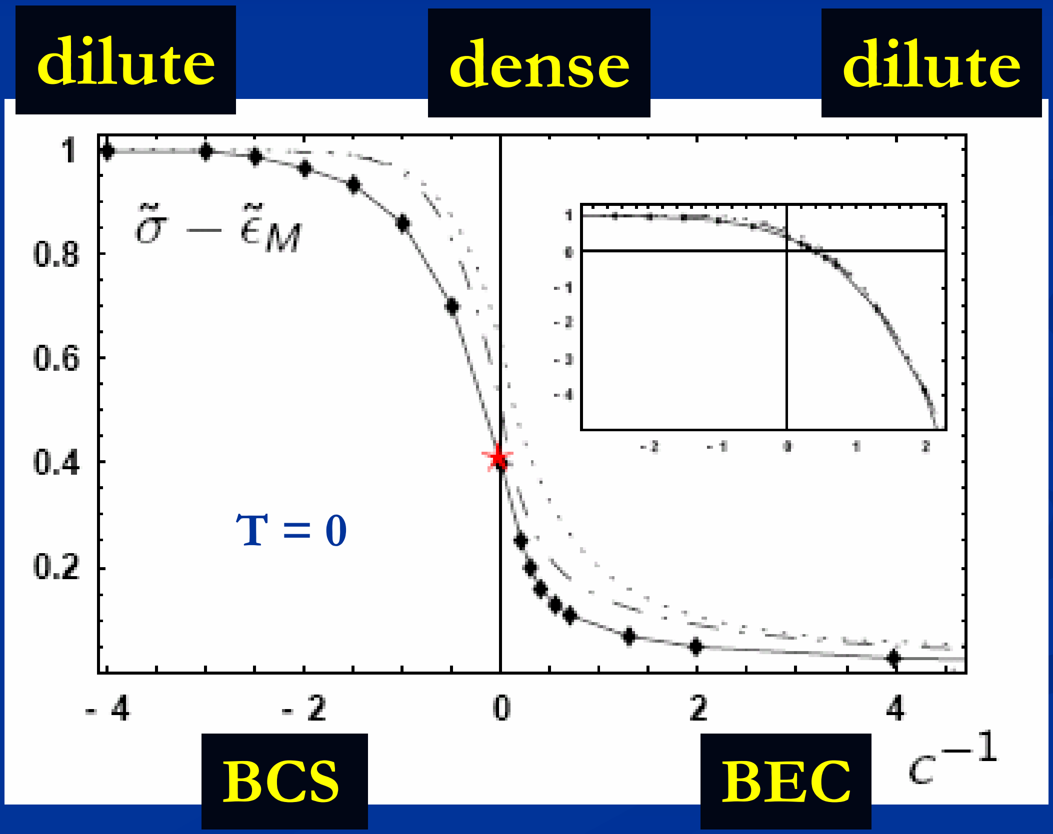
non-interacting Fermi gas



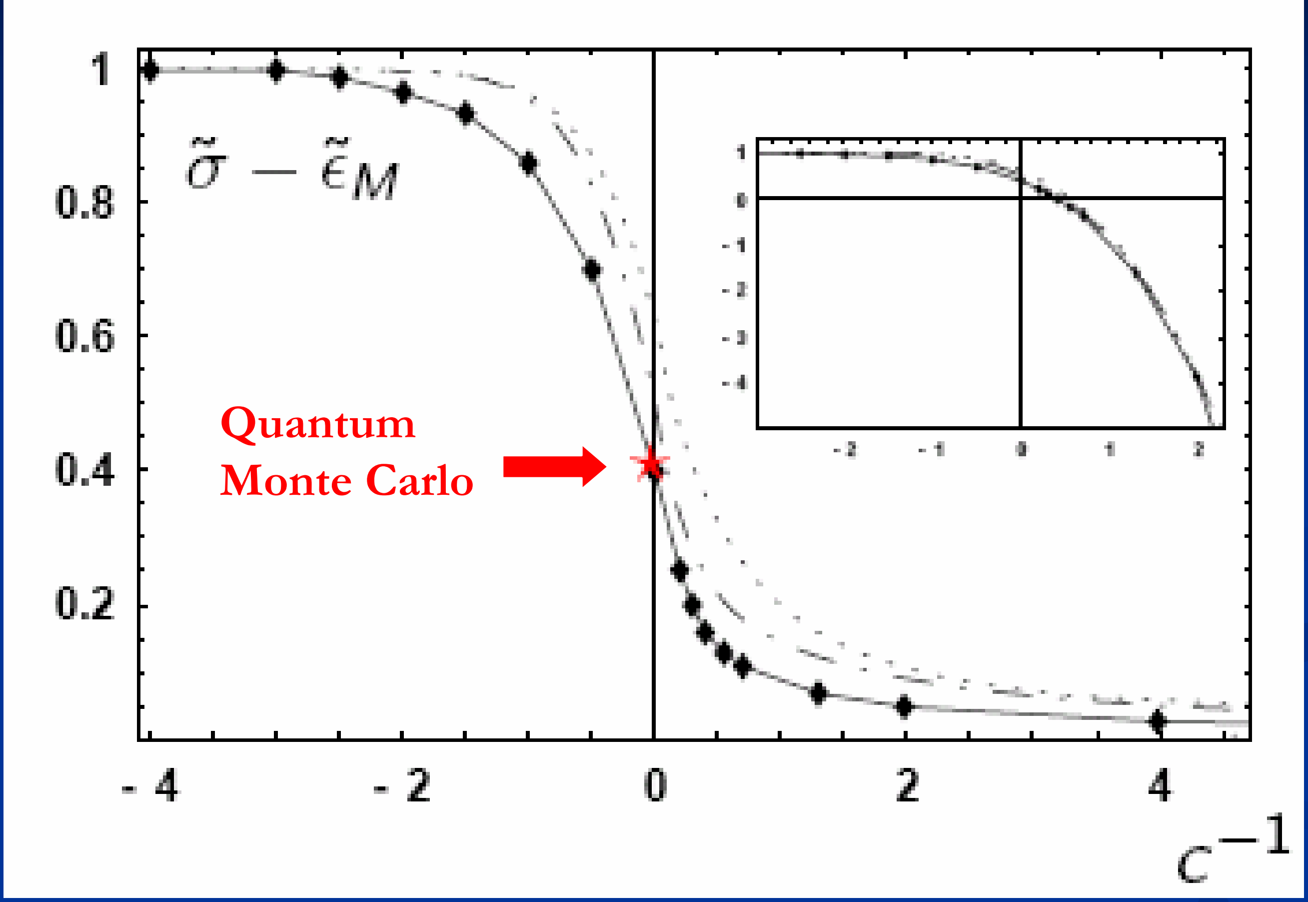
non-interacting Bose gas

universality

same curve for Li and K atoms ?



different methods



- Compare RGE (diamonds), SDE (dashed-dotted) and MFT (dashed) approximation schemes.

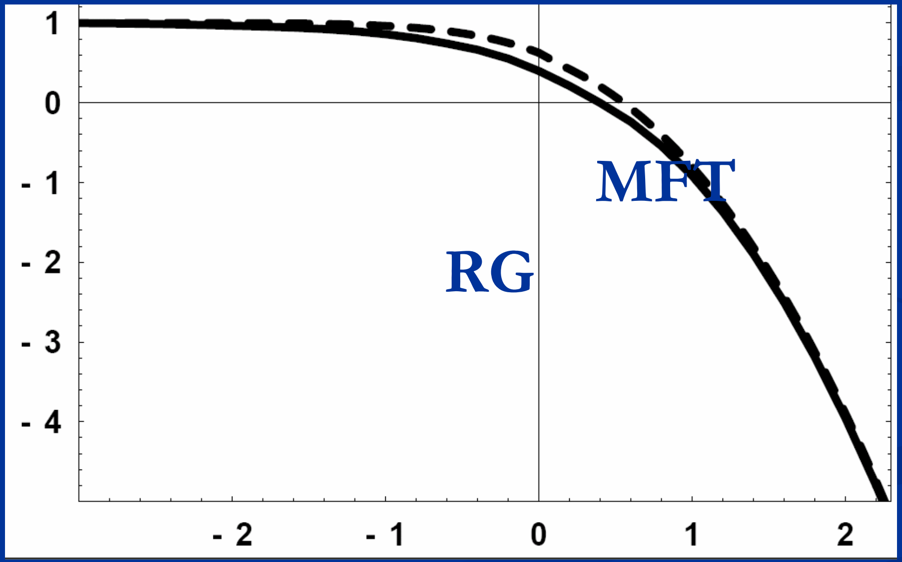
- Compare to QMC calculations at $c^{-1} = 0$

	QMC	RGE	SDE	MFT
$\tilde{\sigma}$	0.44(2)*,0.42(2)†	0.40	0.50	0.63

(* Carlson *et al.*, PRL 91, 050401 (2003),
† Giorgini et al., PRL 93, 200404 (2004)).

who cares about details ?

a theorists game …?



a theorists dream :

reliable method for strongly interacting fermions

“ solving fermionic quantum field theory “

experimental precision tests are crucial !

precision many body theory
- quantum field theory -

so far :

- particle physics : perturbative calculations

  magnetic moment of electron :

  g/2 = 1.001 159 652 180 85 (76) (Gabrielse et al.)

- statistical physics : universal critical exponents for second order phase transitions : $\nu = 0.6308\ (10)$

  renormalization group

- lattice simulations for bosonic systems in particle and statistical physics (e.g. QCD)

QFT with fermions

needed:

universal theoretical tools for complex fermionic systems

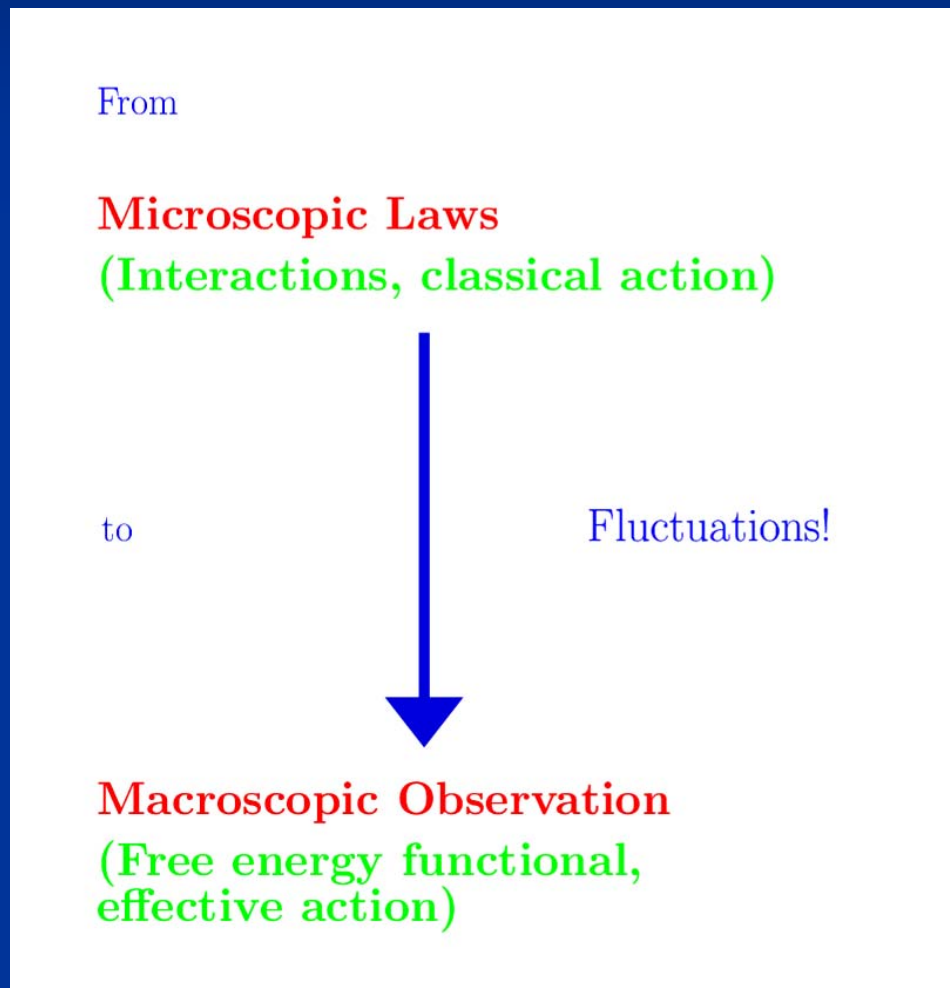
wide applications :

electrons in solids ,

nuclear matter in neutron stars , ….

problems

(1) bridge from microphysics to macrophysics



(2) different effective degrees of freedom

microphysics : single atoms

(+ molecules on BEC – side)

macrophysics : bosonic collective degrees of freedom

compare QCD : from quarks and gluons to mesons and hadrons

(3) no small coupling

ultra-cold atoms :

- microphysics known
- coupling can be tuned

- for tests of theoretical methods these are important advantages as compared to solid state physics !

challenge for ultra-cold atoms :

Non-relativistic fermion systems with precision similar to particle physics !

(QCD with quarks)

functional renormalization group

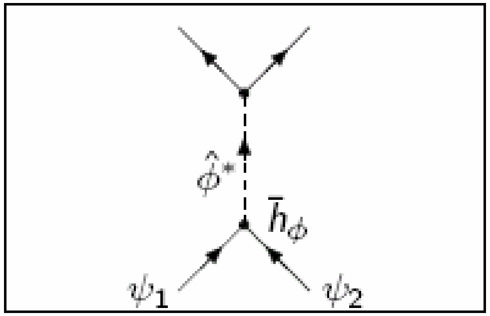
- conceived to cope with the above problems
- should be tested by ultra-cold atoms

QFT for non-relativistic fermions

- functional integral, action

$$S = \int_x \{\psi^\dagger(\partial_\tau - \frac{\Delta}{2M} - \sigma)\psi$$
$$+\varphi^*(\partial_\tau - \frac{\Delta}{4M} + \bar{\nu}_\Lambda - 2\sigma)\varphi$$
$$-\bar{h}_\varphi(\varphi^*\psi_1\psi_2 - \varphi\psi_1^*\psi_2^*)\}$$

Molecule exchange



perturbation theory:
Feynman rules

τ : euclidean time on torus with circumference 1/T
σ : effective chemical potential

variables

- ψ : Grassmann variables
- φ : bosonic field with atom number two

What is φ ?
microscopic molecule,
macroscopic Cooper pair ?

All !

parameters

- detuning ν(B)

$$\bar{\nu}_\Lambda = \bar{\nu}_{\Lambda,0} + \bar{\mu}_B(B - B_0)$$

$$\frac{\partial \bar{\nu}_\Lambda}{\partial B} = \bar{\mu}_B$$

$$\begin{aligned} S = & \int_x \{\psi^\dagger(\partial_\tau - \frac{\Delta}{2M} - \sigma)\psi \\ & + \varphi^*(\partial_\tau - \frac{\Delta}{4M} + \bar{\nu}_\Lambda - 2\sigma)\varphi \\ & - \bar{h}_\varphi(\varphi^*\psi_1\psi_2 - \varphi\psi_1^*\psi_2^*)\} \end{aligned}$$

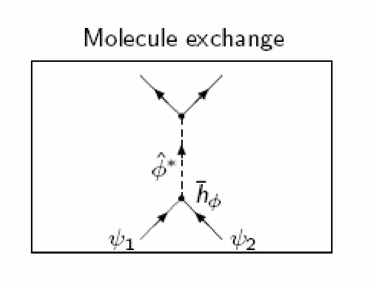
- Yukawa or Feshbach coupling h_φ

fermionic action

equivalent fermionic action , in general not local

$$S_F = \int_x \psi^\dagger\left(\partial_\tau - \frac{\Delta}{2M} - \sigma\right)\psi + S_{\text{int}}$$

$$S_{\text{int}} = -\frac{1}{2}\int_{Q_1,Q_2,Q_3} \left(\psi^\dagger(-Q_1)\psi(Q_2)\right)\left(\psi^\dagger(Q_4)\psi(-Q_3)\right) \frac{\bar{h}_\varphi^2}{\bar{\nu}_\Lambda - 2\sigma + (\vec{q}_1 - \vec{q}_4)^2/4M + 2\pi i T(n_1 - n_4)}$$



scattering length a

$$\bar{\lambda} = -\frac{\bar{h}_\varphi^2}{\bar{\nu}_\Lambda}$$

a= M λ/4π

- broad resonance : pointlike limit
- large Feshbach coupling

$$\bar{h}_\varphi^2 \to \infty, \ \bar{\nu}_\Lambda \to \infty, \ \bar{\lambda} \text{ fixed}$$

$$S_{\text{int}} = -\frac{1}{2} \int_{Q_1,Q_2,Q_3} (\psi^\dagger(-Q_1)\psi(Q_2))(\psi^\dagger(Q_4)\psi(-Q_3)) \frac{\bar{h}_\varphi^2}{\bar{\nu}_\Lambda - 2\sigma + (\vec{q}_1 - \vec{q}_4)^2/4M + 2\pi i T(n_1 - n_4)}$$

parameters

- Yukawa or Feshbach coupling h_φ
- scattering length a

Set of microscopic parameters:

$$\{\nu(B), \quad h_{\phi,0}\} \leftrightarrow \{a(B), \quad h_{\phi,0}\}.$$

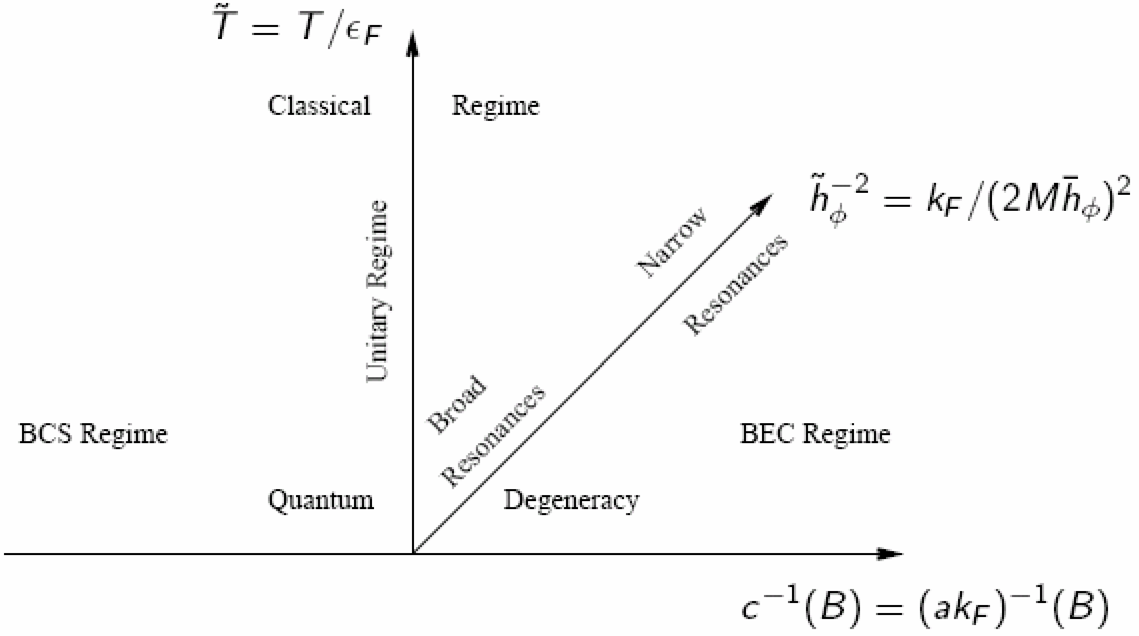
- **broad resonance : h_φ drops out**

concentration c

$$c = a k_F = -\frac{M k_F \bar{h}_\varphi^2}{4\pi \bar{\mu}_B (B - B_0)}$$

$$n = \frac{k_F^3}{3\pi^2}$$

- Dimensionless axes: measure in units of Fermi momentum, $k_F = (3\pi^2 n)^{1/3}$ and Fermi energy, $\epsilon_F = k_F^2/(2M)$.
- Crossover induced by magnetic field (B) dependence of scattering length: Feshbach resonance.
- Narrow resonances: Nonlocal interactions, exact solution possible (S. Diehl, C. Wetterich, Phys. Rev. A **73** 033615 (2006)).
- Focus on the broad resonance limit $\tilde{h}_\phi \to \infty$: pointlike interactions.



universality

- Are these parameters enough for a quantitatively precise description ?
- Have Li and K the same crossover when described with these parameters ?
- Long distance physics looses memory of detailed microscopic properties of atoms and molecules !

universality for $c^{-1} = 0$: Ho,…(valid for broad resonance)
here: whole crossover range

analogy with particle physics

microscopic theory not known -

nevertheless "macroscopic theory" characterized by a finite number of

"renormalizable couplings"

m_e , α ; g_w , g_s , M_w , …

here : **c** , h_φ (only **c** for broad resonance)

analogy with universal critical exponents

only one relevant parameter :

$$T - T_c$$

universality

- issue is not that particular Hamiltonian with two couplings ν , h_φ gives good approximation to microphysics

- large class of different microphysical Hamiltonians lead to a macroscopic behavior described only by ν , h_φ

- **difference in length scales matters !**

units and dimensions

- $c = 1$; $\hbar = 1$; $k_B = 1$
- momentum ~ length^{-1} ~ mass ~ eV
- energies : 2ME ~ (momentum)^{2}

  (M : atom mass)
- typical momentum unit : Fermi momentum
- typical energy and temperature unit : Fermi energy
- time ~ (momentum) $^{-2}$
- **canonical dimensions different from relativistic QFT !**

rescaled action

$$S = \int_{\hat{x}} \{ \hat{\psi}^\dagger (\hat{\partial}_\tau - \hat{\Delta} - \hat{\sigma}) \hat{\psi} + \hat{\varphi}^* (\hat{\partial}_\tau - \frac{1}{2}\hat{\Delta} + \hat{\nu} - 2\hat{\sigma}) \hat{\varphi} - \hat{h}_\varphi (\hat{\varphi}^* \hat{\psi}_1 \hat{\psi}_2 - \hat{\varphi} \hat{\psi}_1^* \hat{\psi}_2^*) \}$$

$$\hat{\psi} = \hat{k}^{-3/2} \psi \ , \ \hat{\varphi} = \hat{k}^{-3/2} \varphi ,$$
$$\hat{x} = \hat{k} x \ , \ \hat{\tau} = \frac{\hat{k}^2}{2M} \tau ,$$
$$\hat{\sigma} = \frac{2M\sigma}{\hat{k}^2} \ , \ \hat{h}_\varphi = \frac{2M\bar{h}_\varphi}{\sqrt{\hat{k}}}$$

- M drops out
- all quantities in units of k_F , ε_F if $\hat{k} = k_F$

what is to be computed ?

Inclusion of fluctuation effects
via **functional integral**
leads to **effective action.**

This contains all relevant information for arbitrary T and n !

effective action

- integrate out all quantum and thermal fluctuations
- quantum effective action
- generates full propagators and vertices
- richer structure than classical action

$$\Gamma = \int_x \{\psi^\dagger(\partial_\tau - A_\psi\Delta - \sigma)\psi + \varphi^*(\partial_\tau - A_\varphi\Delta)\varphi + u(\varphi) - h_\varphi(\varphi^*\psi_1\psi_2 - \varphi\psi_1^*\psi_2^*) + \dots\}$$

effective potential

minimum determines order parameter

$$u = m_\varphi^2 \rho + \frac{\lambda_\varphi}{2}\rho^2 \quad , \quad SYM$$
$$u = \frac{\lambda_\varphi}{2}(\rho - \rho_0)^2 \quad , \quad SSB$$

$$\rho = \varphi^* \varphi$$

condensate fraction

$$\Omega_c = 2\,\rho_0 / n$$

$$\Gamma = \int_x \{\psi^\dagger(\partial_\tau - A_\psi \Delta - \sigma)\psi + \varphi^*(\partial_\tau - A_\varphi \Delta)\varphi + u(\varphi) - h_\varphi(\varphi^* \psi_1 \psi_2 - \varphi \psi_1^* \psi_2^*) + \ldots\}$$

renormalized fields and couplings

$$\psi = Z_\psi^{1/2}\hat{\psi}\ ,\ \varphi = Z_\varphi^{1/2}\hat{\varphi}$$

$$h_\varphi = Z_\varphi^{-1/2} Z_\psi^{-1} \hat{h}_\varphi$$

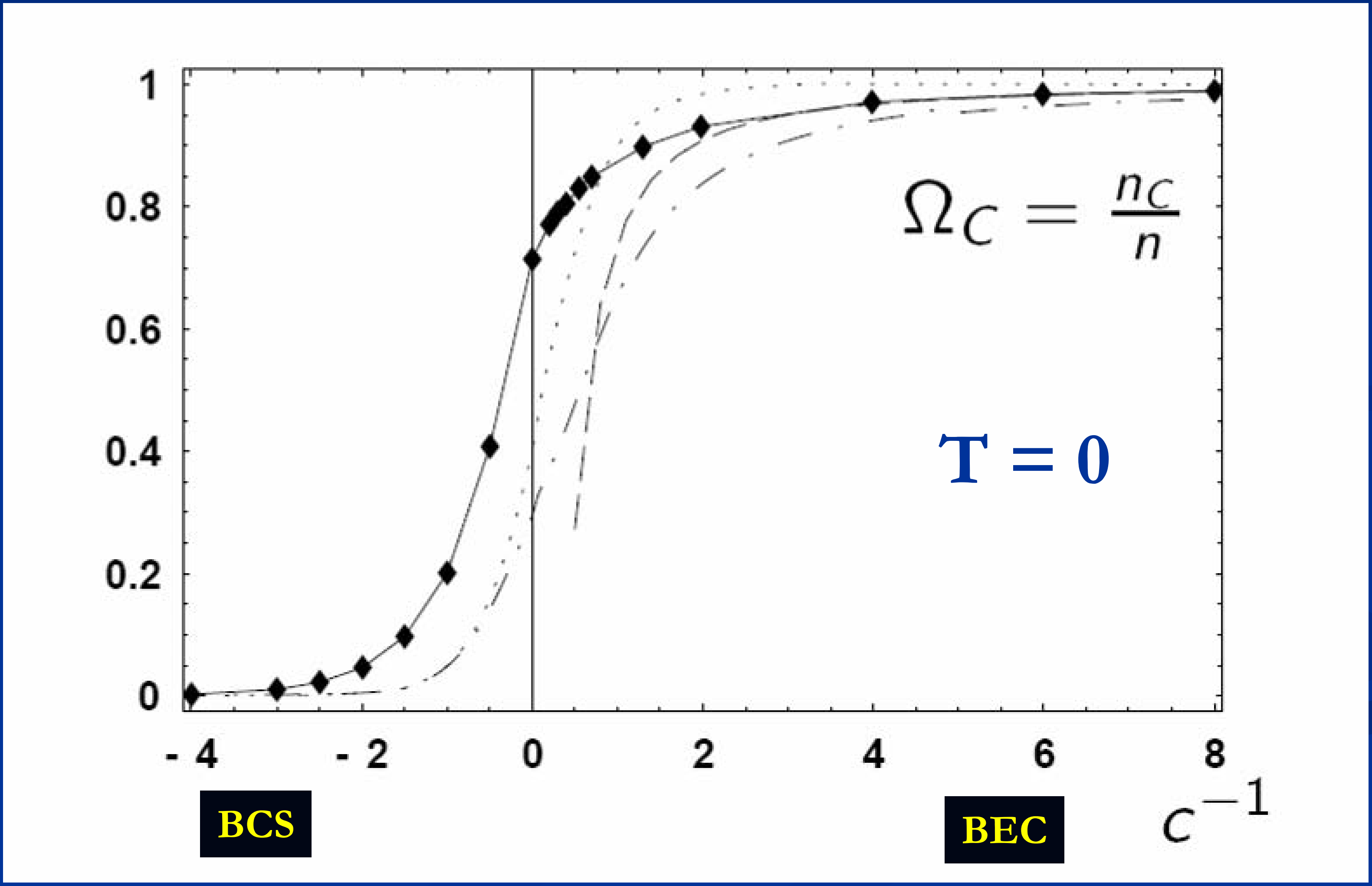
$$\Gamma = \int_x \{\psi^\dagger(\partial_\tau - A_\psi \Delta - \sigma)\psi$$
$$+\varphi^*(\partial_\tau - A_\varphi \Delta)\varphi + u(\varphi)$$
$$-h_\varphi(\varphi^* \psi_1 \psi_2 - \varphi \psi_1^* \psi_2^*) + \dots\}$$

results

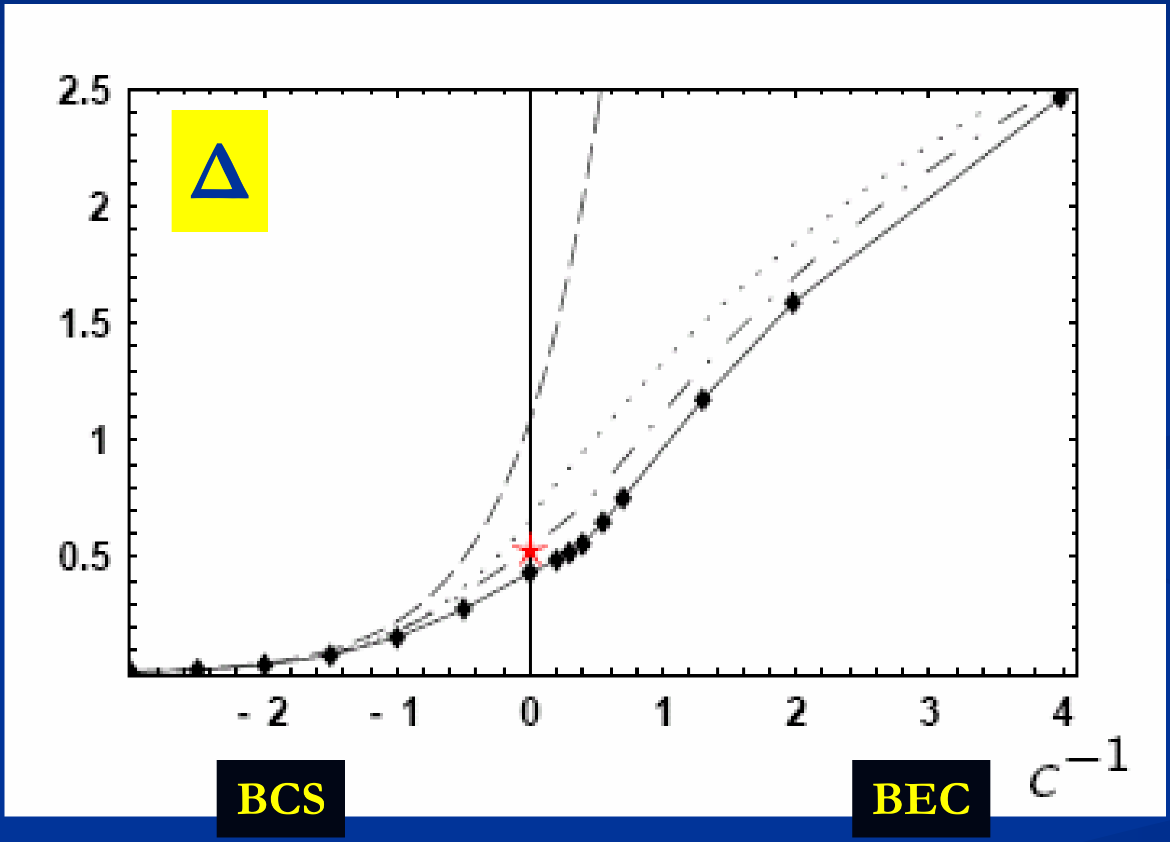
from

functional renormalization group

condensate fraction



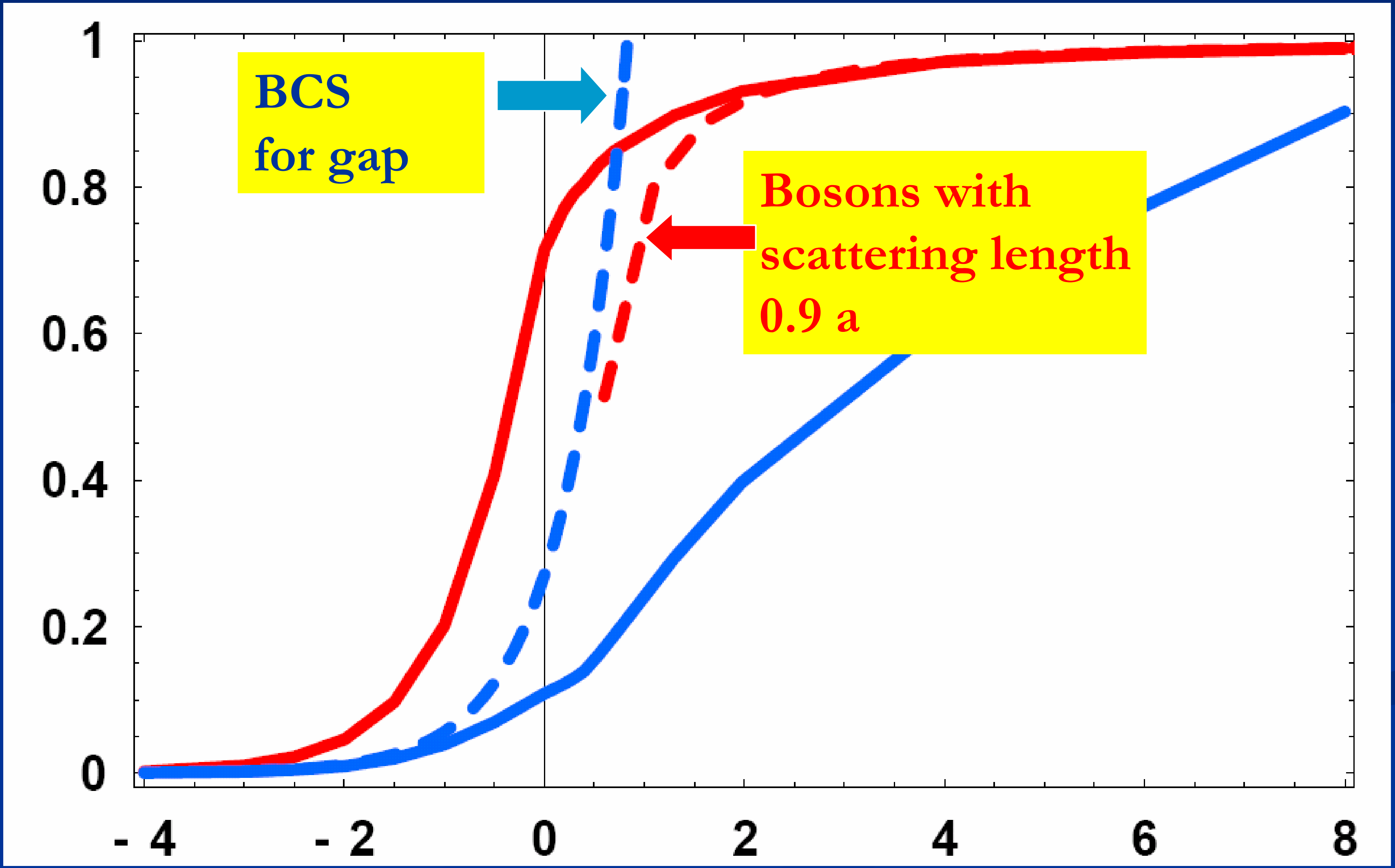
gap parameter



BCS regime: recover BCS gap result $\Delta/\Delta^{BCS}(c^{-1}) \approx 0.9$ for $c^{-1} < -2$.

MFT (dashed): No boson interactions.
SDE (dashed-dotted): Overestimates interactions, $a_M = 2$.

limits


BCS
for gap
Bosons with
scattering length
0.9 a
1
0.8
0.6
0.4
0.2
0
- 4
- 2
0
2
4
6
8

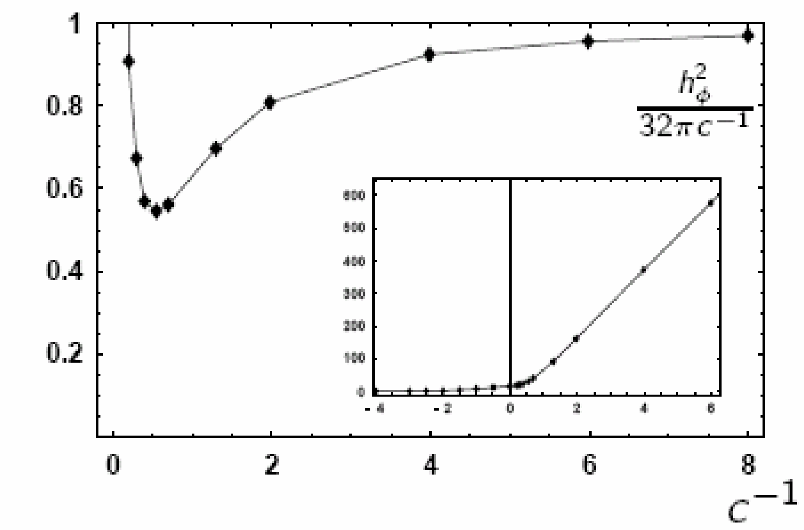
Yukawa coupling

Extract both BCS-type gap parameter $\tilde{\Delta}$ and BEC-type condensate fraction Ω_C.

- Both quantities intimately connected by renormalized Yukawa coupling

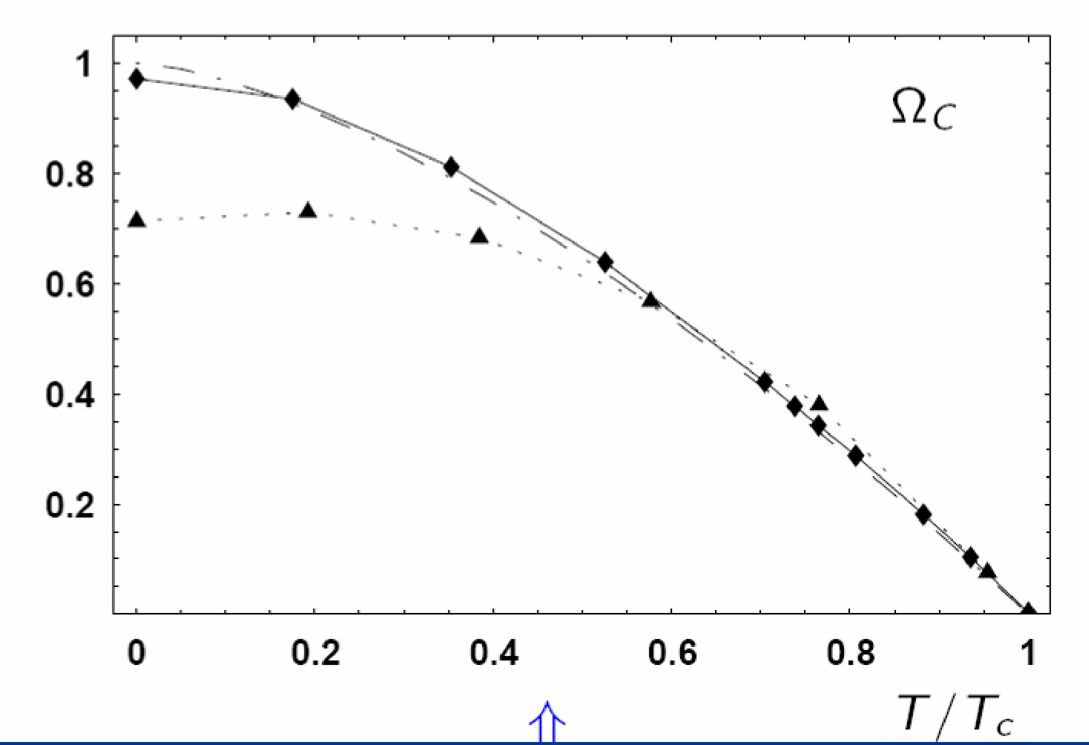
$$h_\phi^2 = \frac{\tilde{h}_{\phi,0}^2}{Z_\phi}, \quad \Omega_C = 6\pi^2 \left(\frac{\tilde{\Delta}}{h_\phi}\right)^2.$$

- Broad resonance universality confirmed: $\tilde{h}_{\phi,0} \to \infty$ drops out as physical scale.



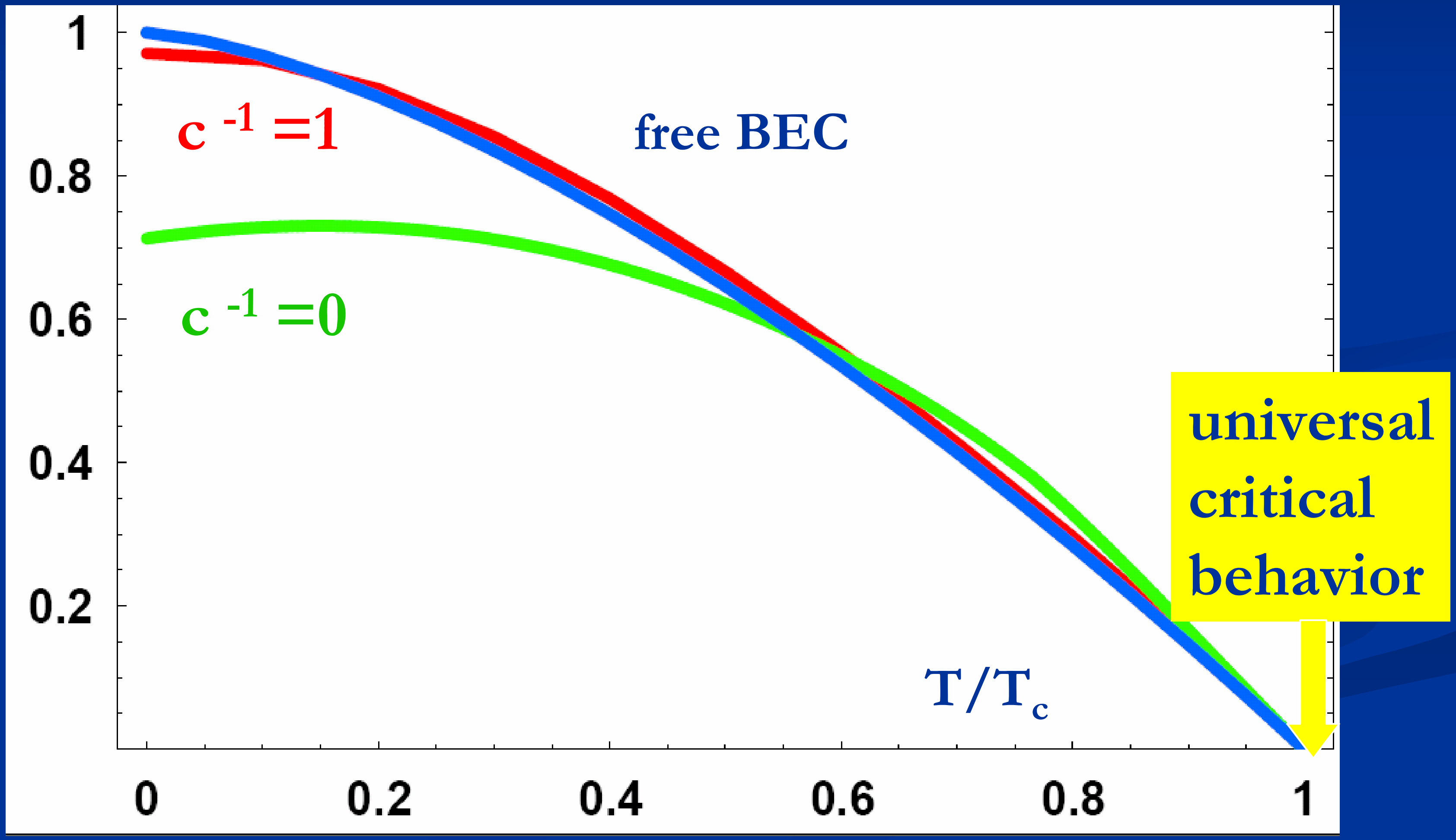
- Reduced Yukawa coupling settles to 2-body fixed point $32\pi c^{-1}$.

temperature dependence of condensate

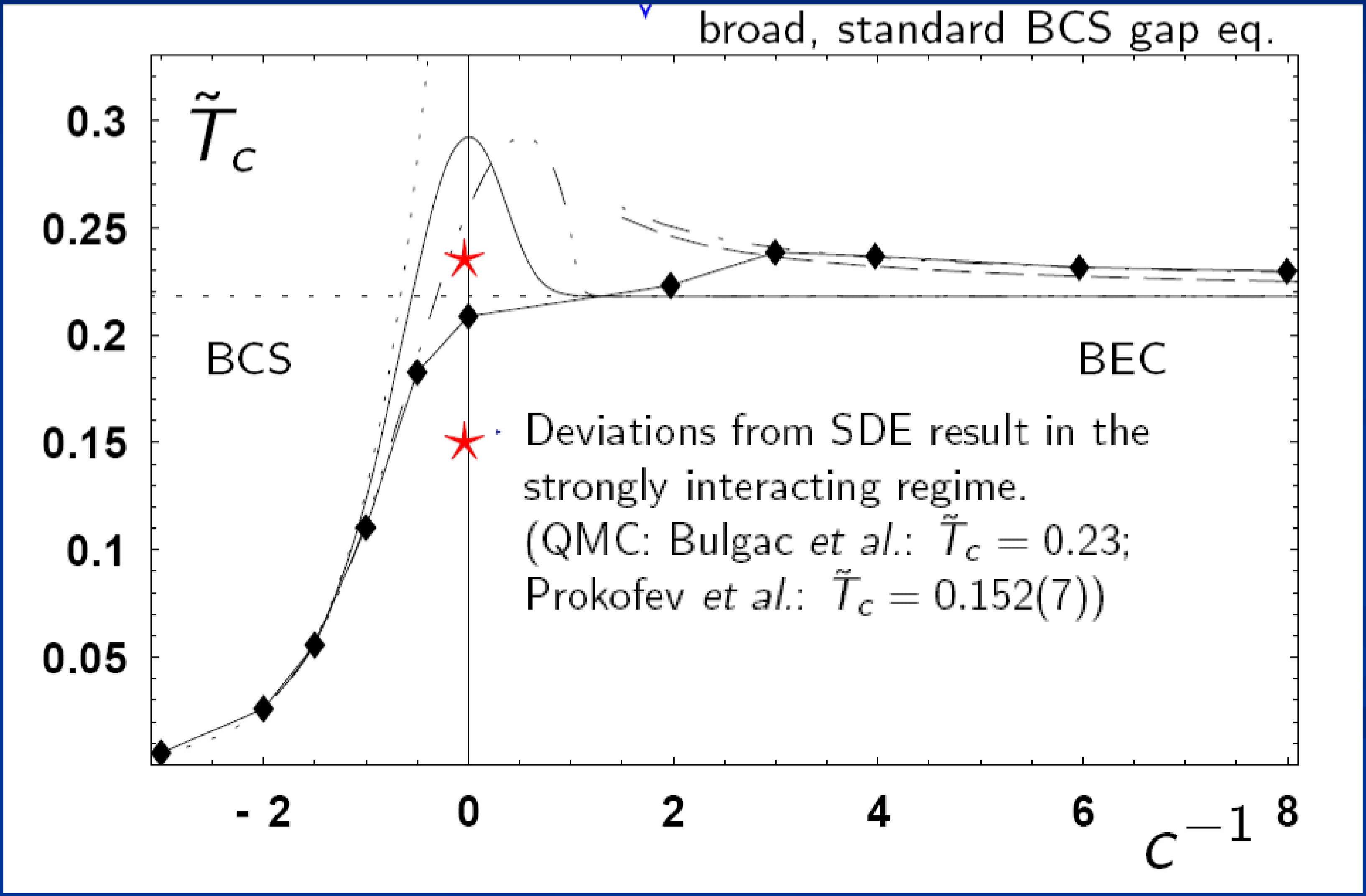


- Compare free BE condensate fraction to result for $c^{-1} = 0$ (resonance, triangles) and $c^{-1} = 1$ (BEC regime, diamonds).
- Low temperature: Condensate fraction strongly depends on c^{-1}.
- Close to criticality:
  - Second order phase transition.
  - Similar approach to T_c: dominance of boson fluctuations, system attracted to universal critical point.

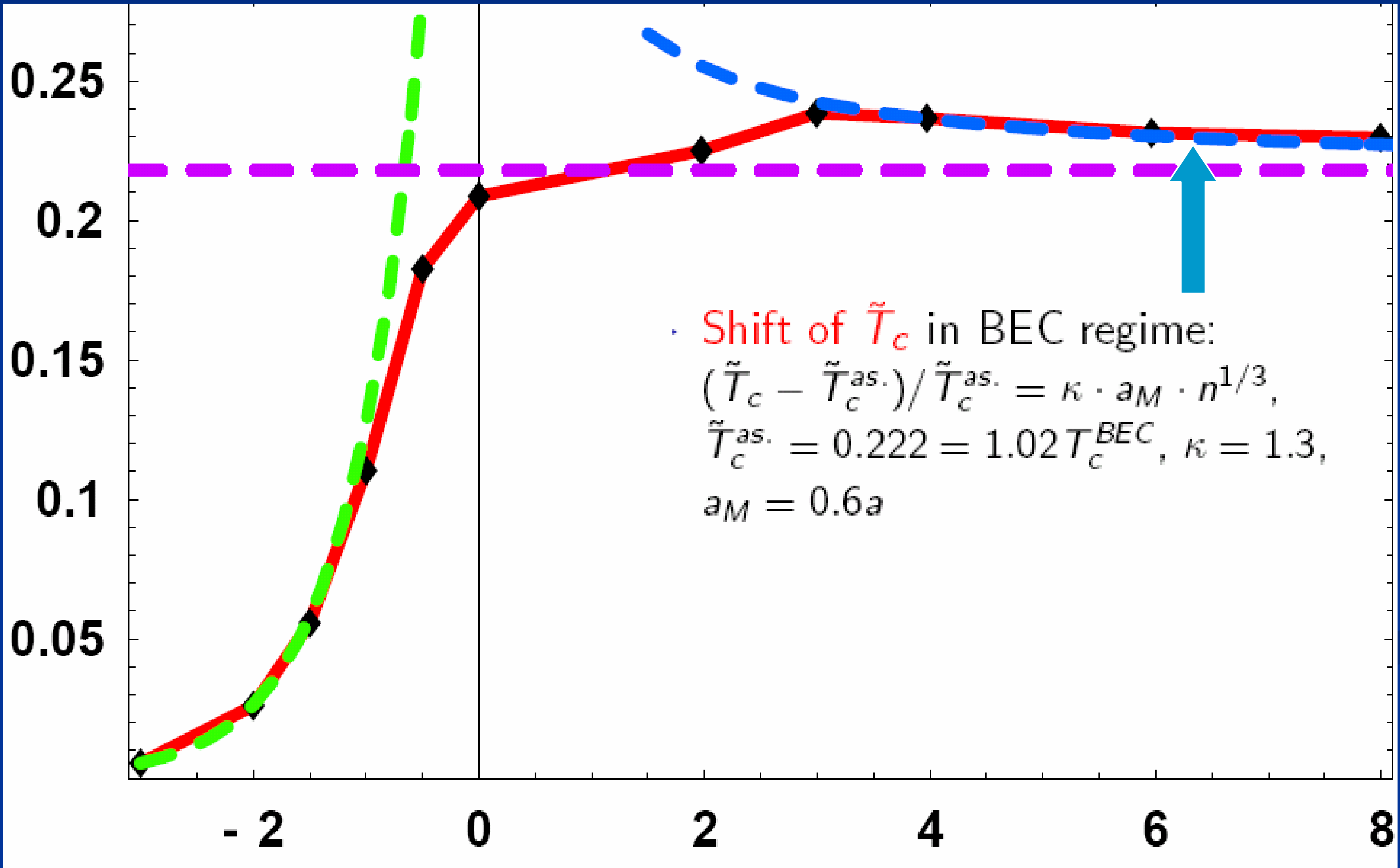
condensate fraction : second order phase transition



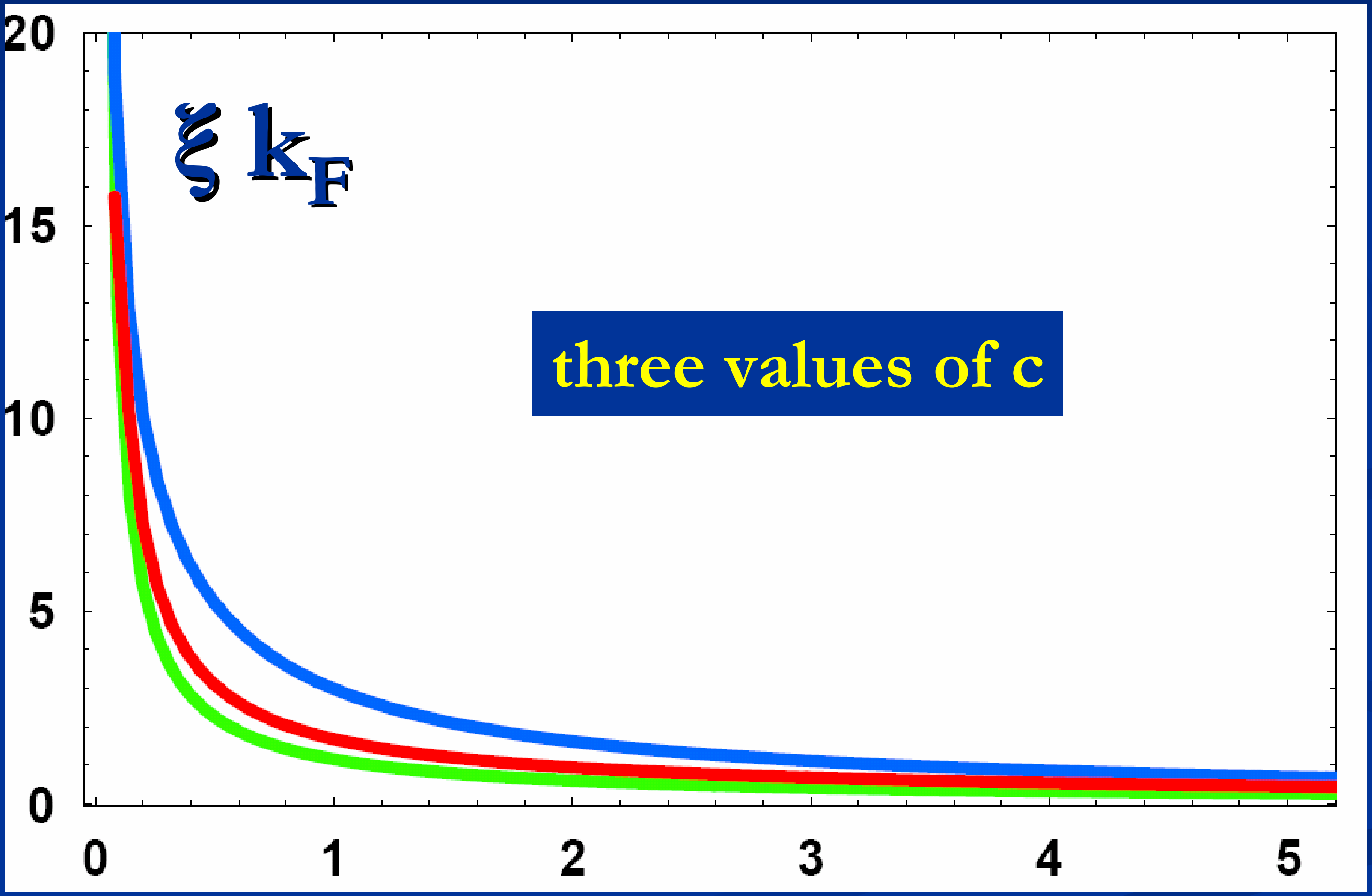
crossover phase diagram



shift of BEC critical temperature



correlation length



$(T - T_c)/T_c$

universality

universality for broad resonances

for large Yukawa couplings h_φ :

- only one relevant parameter c
- all other couplings are strongly attracted to partial fixed points
- macroscopic quantities can be predicted

  in terms of c and T/ε_F

  (in suitable range for c^{-1} ; density sets scale)

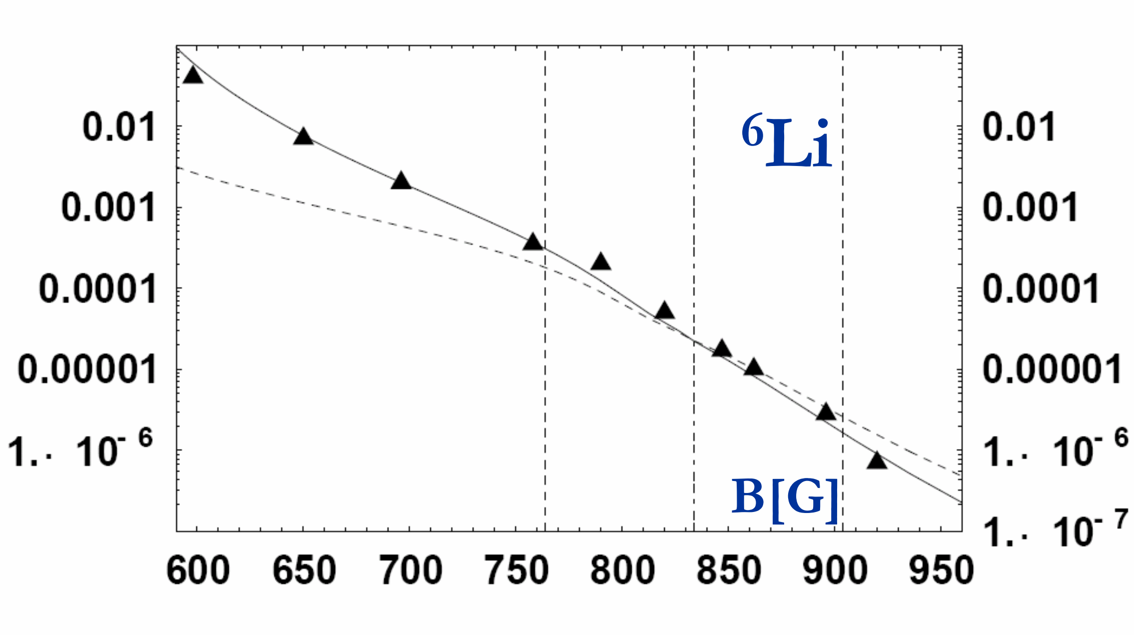
universality for narrow resonances

- Yukawa coupling becomes additional parameter (marginal coupling)
- also background scattering important

bare molecule fraction

(fraction of microscopic closed channel molecules)

- not all quantities are universal
- bare molecule fraction involves wave function renormalization that depends on value of Yukawa coupling



Experimental points by Partridge et al.

method

effective action

$$\Gamma[\psi,\phi] \;=\; \int_0^{1/T} d\tau \int d^3x \Big\{ \psi^\dagger(\partial_\tau - A_\psi \triangle - \sigma)\psi +$$

$$\phi^*(\partial_\tau - A_\phi \triangle)\phi + U(\phi^*\phi) - \frac{h_\phi}{2}\Big(\phi^*\psi^T\epsilon\psi - \phi\psi^\dagger\epsilon\psi^*\Big) + \ldots \Big\}.$$

- includes all quantum and thermal fluctuations
- formulated here in terms of renormalized fields
- involves renormalized couplings

effective potential

- value of φ at potential minimum :
  order parameter , determines condensate fraction
- second derivative of U with respect to φ yields correlation length
- derivative with respect to σ yields density

Quartic truncation for bosonic potential (displayed in symmetric phase):

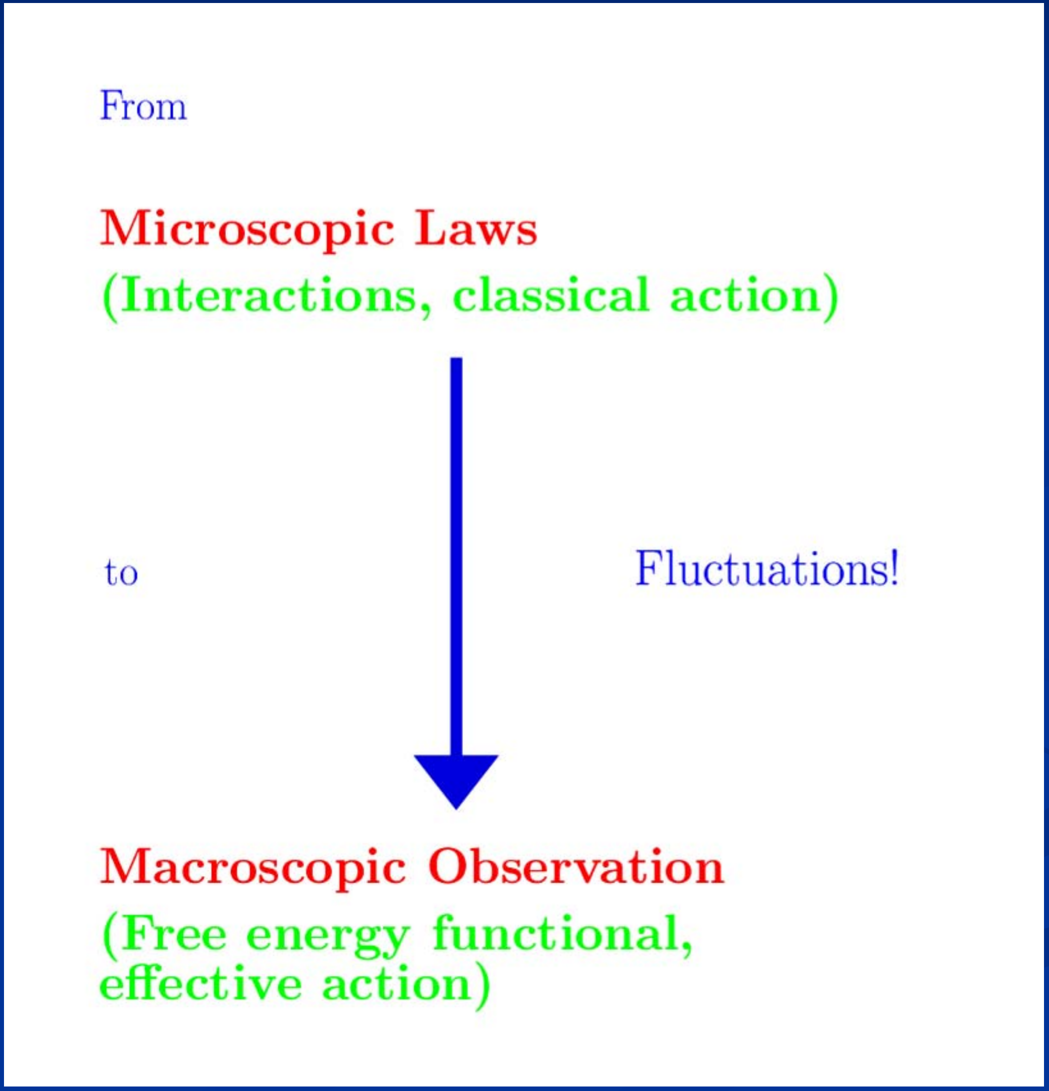
$$U(\phi^*\phi) = (\nu(B) + \Delta m_\phi^2)\phi^*\phi + \frac{\lambda_\phi}{2}(\phi^*\phi)^2 + \ldots$$

functional renormalization group

- make effective action depend on scale k :
  include only fluctuations with momenta larger than k
  (or with distance from Fermi-surface larger than k)

- k large : no fluctuations , classical action
- k $\rightarrow$ 0 : quantum effective action
- effective average action (same for effective potential)
- running couplings

microscope with variable resolution

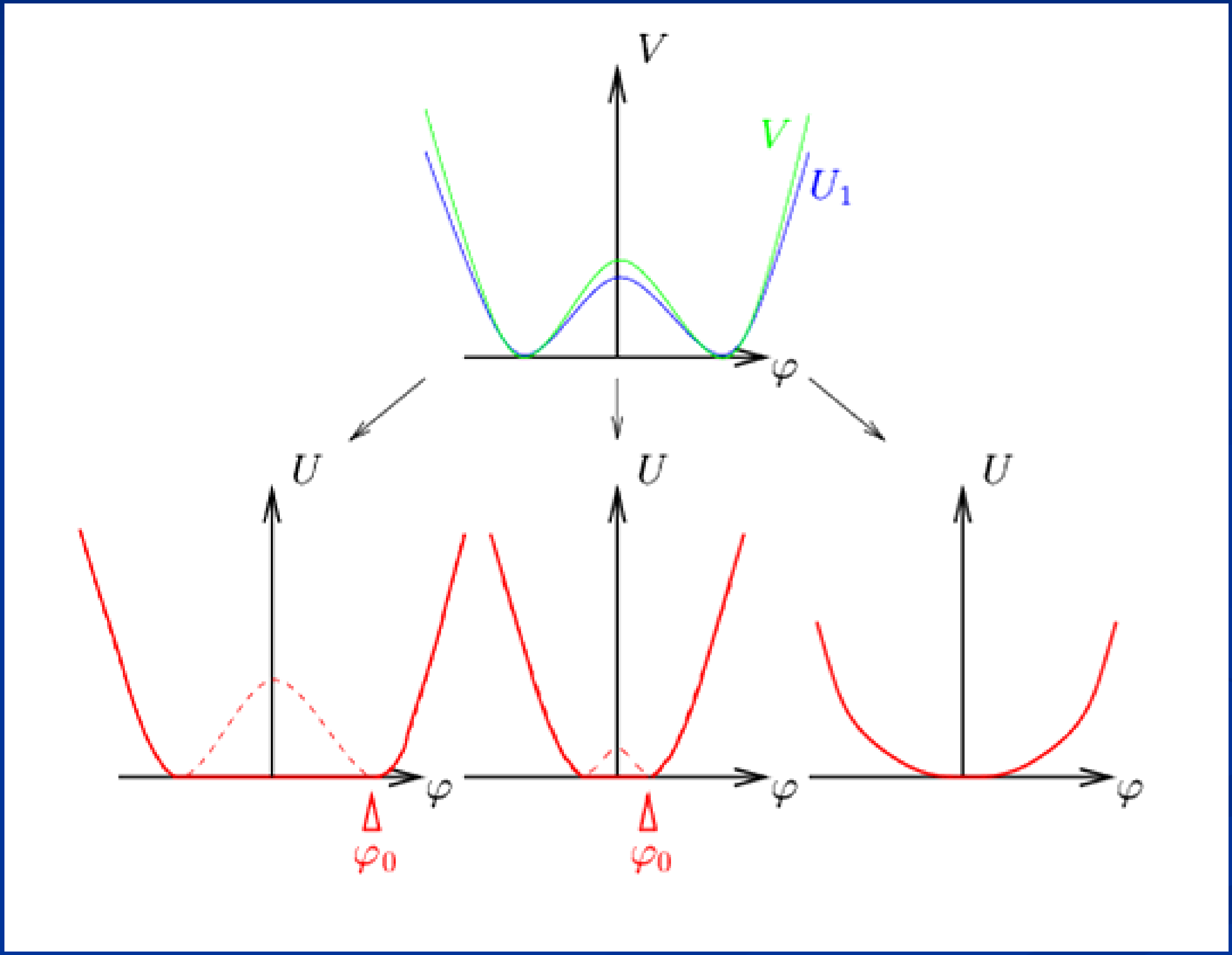


running couplings : crucial for universality

for large Yukawa couplings h_φ :

- only one relevant parameter c
- all other couplings are strongly attracted to **partial fixed points**
- macroscopic quantities can be predicted in terms of c and T/ε_F (in suitable range for c^{-1})

running potential



micro

macro

here for scalar theory

physics at different length scales

- microscopic theories : where the laws are formulated
- effective theories : where observations are made
- effective theory may involve different degrees of freedom as compared to microscopic theory
- example: microscopic theory only for fermionic atoms , macroscopic theory involves bosonic collective degrees of freedom (φ)

Functional Renormalization Group

describes flow of effective action from small to large length scales

perturbative renormalization : case where only couplings change , and couplings are small

conclusions

the challenge of precision :

- substantial theoretical progress needed
- "phenomenology" has to identify quantities that are accessible to precision both for experiment and theory
- dedicated experimental effort needed

challenges for experiment

- study the simplest system
- identify quantities that can be measured with precision of a few percent and have clear theoretical interpretation
- precise thermometer that does not destroy probe
- same for density

functional renormalization group

- block spins

Kadanoff, Wilson

- exact renormalization group equations

Wegner, Houghton
Wagner, Houghton
Weinberg
Polchinski
Hasenfratz^2

- Lattice finite size scaling

Lüscher,...

- coarse grained free energy/average action

effective average action

here only for bosons , addition of fermions straightforward

Flow equation for average potential

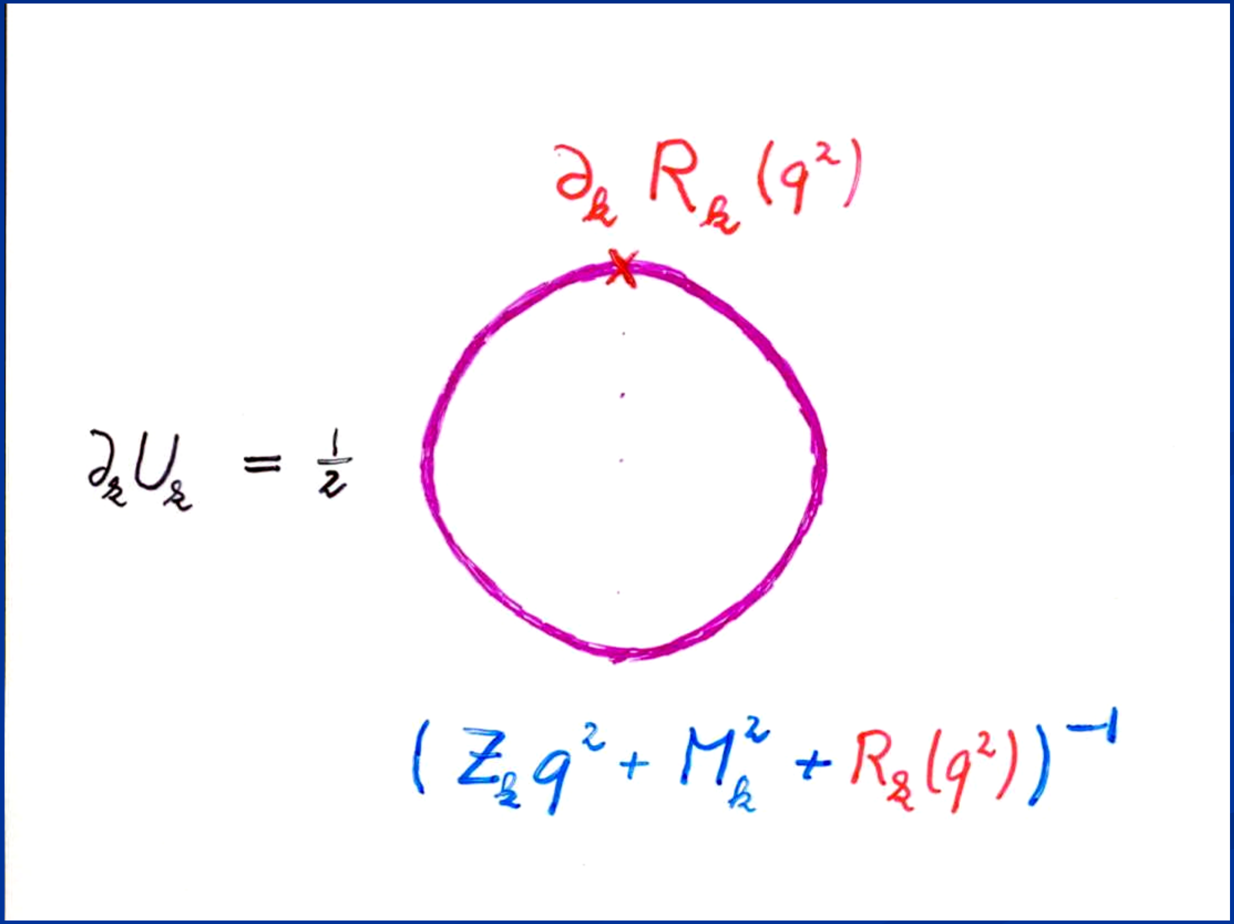
$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

+ contribution from fermion fluctuations

Simple one loop structure – nevertheless (almost) exact



Infrared cutoff

$R_k \quad : \quad$ IR-cutoff

$$\text{e.g} \qquad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

$$\text{or} \qquad R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2) \qquad \text{(Litim)}$$

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Flow equation for U_k

$$\boxed{\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}}$$

'91

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

R_k : IR-cutoff

e.g $R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$ (Litim)

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Partial differential equation for function U(k,φ) depending on two variables

$Z_k = c\, k^{-\eta}$

Regularisation

For suitable R_k :

$$R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$
$$R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2)$$

- Momentum integral is ultraviolet and infrared finite
- Numerical integration possible
- Flow equation defines a regularization scheme (ERGE –regularization)

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Integration by momentum shells

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Momentum integral
is dominated by
$q^2 \sim k^2$.

Flow only sensitive to
physics at scale k

Wave function renormalization and anomalous dimension

Z_k: wave function renormalization

$$k\partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

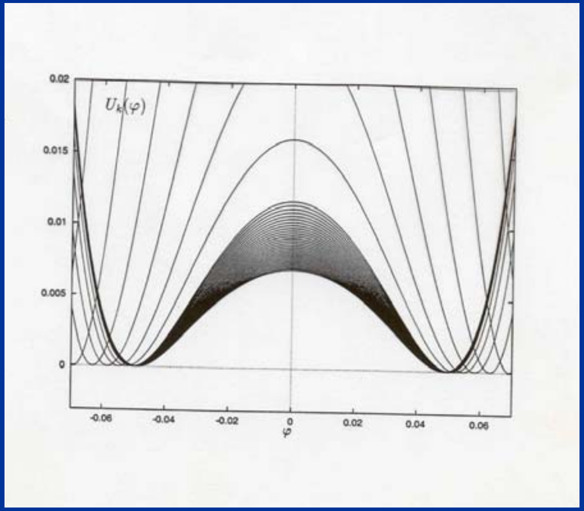
$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

for $Z_k(\varphi, q^2)$: flow equation is **exact !**

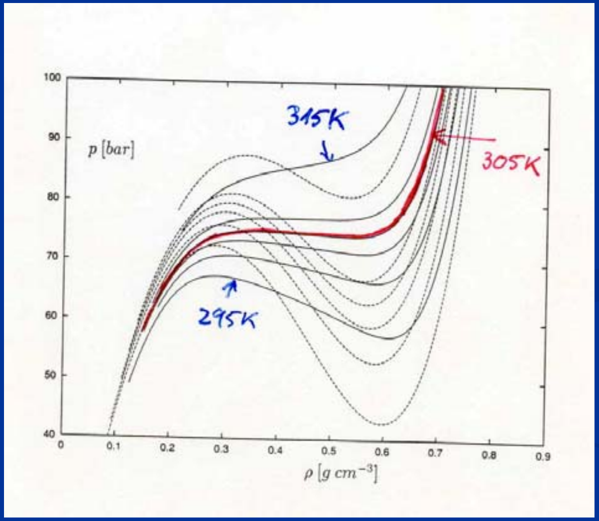
Flow of effective potential

Ising model



S.Seide …

CO_2



Experiment :

T_* =304.15 K
p_* =73.8.bar
ρ_* = 0.442 g cm-2

Critical exponents

$d = 3$

Critical exponents ν and η

N	ν		η	
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
10	0.881	0.886	0.028	0.025
100	0.990	0.980	0.0030	0.003
		↑		↑

"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

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Solution of partial differential equation :

yields highly nontrivial non-perturbative results despite the one loop structure !

Example:
Kosterlitz-Thouless phase transition

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left\{ \left(\Gamma_k^{(2)}[\varphi] + R_k \right)^{-1} \partial_k R_k \right\}$$

'92

$\left(\Gamma_k^{(2)}\right)_{ab}(q,q') = \frac{\delta^2 \Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$

$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$

(fermions : STr)

end

Effective average action

and

exact renormalization group equation

Generating functional

generating functional for connected Green's functions in presence of quadratic infrared cutoff

$$W_k[j] = \ln \int \mathcal{D}\chi \, \exp\left(-S[\chi] - \Delta_k S[\chi] + \int d^d x \, j_a \chi_a \right)$$

$$\Delta_k S = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q^2) \chi_a(-q) \chi_a(q)$$

$$\text{e.g.} \quad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

$$\lim_{k \to 0} R_k = 0$$

$$R_{k \to \infty} \to \infty$$

Effective average action

$$\Gamma_k[\varphi] = -W_k[j] + \int d^dx\, j_a \varphi_a - \Delta_k S[\varphi]$$

$\Gamma_0[\varphi]$: quantum effective action
generates 1PI vertices
free energy: $F = \Gamma T + \mu n V$

Γ_k includes all fluctuations (quantum, thermal) with $q^2 > k^2$

Γ_Λ specifies microphysics

$$\varphi_a = \langle \chi_a \rangle = \frac{\delta W_k}{\delta j_a}$$

Loop expansion : perturbation theory with infrared cutoff in propagator

Quantum effective action

for $k \to 0$
all fluctuations (quantum + thermal)
are included

knowledge of $\Gamma_{k \to 0} \hat{=}$ solution of model

Truncations

Functional differential equation –

cannot be solved exactly

Approximative solution by truncation of

most general form of effective action

Exact flow equation for effective potential

- Evaluate exact flow equation for homogeneous field φ .
- R.h.s. involves exact propagator in homogeneous background field φ.

two body limit (vacuum)

- Motivation – Physical parameters measured in low energy scattering experiments $\rightarrow$ include vacuum fluctuations.
- Project on physical vacuum by

$$\Gamma_k(vak) = \lim_{k_F \rightarrow 0} \Gamma_k \big|_{\tilde{T} > \tilde{T}_c}$$

$\Longrightarrow$ massive simplification of full diagrammatic structure.

- Picture: Smooth crossover terminates in second order vacuum phase transition

  (i) Atom phase ($a^{-1} < 0$): $\sigma_A = 0,\ \bar{m}_\phi^2 > 0,$

  (ii) Molecule phase ($a^{-1} > 0$) $\sigma_A < 0,\ \bar{m}_\phi^2 = 0,$

  (iii) Resonance ($a^{-1} = 0$) $\sigma_A = 0,\ \bar{m}_\phi^2 = 0.$

  with "order parameter" $\sigma_A = \epsilon_M/2$: half the binding energy ϵ_M of a molecule.

- Nontrivial vacuum physics: scaling of molecular scattering length a_M with fermion scattering length a (a_M/a)

Fermion fluct.s	Fermion and boson fluct.s	Four-body Schrödinger eq.*
2	0.81	0.6

(* Shlyapnikov *et al.*, PRL **93**,090404 (2004))