The Weak Gravity Conjecture

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Abstract

This paper was developed in line with a seminar on QFT in curved spacetimes at the Institute for Theoretical Physics in Heidelberg. We discuss the conjecture of “gravity as the weakest force” proposed by Arkani-Hamed et al. [2]. In this context we comment on charged black holes as well as global symmetries in quantum gravity.

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1 Introduction

Up to now it seems that consistent 4D theories of quantum gravity can be constructed in terms of string theory. This can be done in many ways and leads one to expect that basically any consistent looking effective field theory can arise (“swamp-land”). But a huge number of them cannot be completed to a full theory of quantum gravity (“landscape”). Thus we are forced to find criteria, common to all consistent quantum gravity theories, to distinguish the string landscape from the swampland. In this paper we will study the criterion proposed by Arkani-Hamed et. al [2].

Consider a theory of gravity coupled to a $U(1)$ gauge theory with coupling $g$. From the perspective of an effective field theory, $g$ and Newton’s constant $G$ are assumed to be independent. Furthermore, if $g$ is small enough so that the Landau pole is above the Planck scale, we would expect a cutoff of order $M_P$. In contradiction to this Arkani-Hamed et al. claim that:

1. There exists a light, charged particle of mass $m < g M_P$
2. The effective theory breaks down at a low scale $\Lambda < g M_P$

where assumption 2 follows from 1.

It turns out that this conjecture can be rephrased as ”gravity is the weakest force” which is nicely in line with our own world. Rearranging condition 1 as $g > \frac{m}{M_P}$, this bounds the gauge coupling away from zero and thus generalizes the statement of non-existence of continuous global symmetries in a theory of quantum gravity. If this has not been satisfied we would come across the following problems:

- Charged Black Holes can exhibit very large entropies violating the Bekenstein bound
- Charged Black Holes can evaporate down to stable Planck scale remnants

The paper is structured as follows: We will start with a short reminder on charged black holes in 4 dimensions. After that we will discuss the statement of non-existence of continuous global symmetries in quantum gravity. In section 3 we will formulate and discuss conjecture 1 on the existence of light charged particles. Before we can derive the new scale, predicted by conjecture 2, in section 6, we will first review some basic facts concerning monopoles. In the end we will comment on how the conjecture can be sharpened and generalized to further gauge groups.

2 Charged Black Holes in $3 + 1$ Dimensions

The description of electrically charged black holes is given by classical Einstein-Maxwell theory

$$S = \frac{1}{16\pi G} \int \sqrt{\det g} R - \frac{1}{4g^2} \int \sqrt{\det g} F^2$$ (1)
which has the Reissner-Nordström solution
\[
    ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega_2^2
\]
with \( f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2g^2}{r^2} \)

where \( \mathcal{F} = dA \) and \( A = \frac{Q^2}{r} \). Here \( M \) denotes the ADM mass, \( Q \) the integral charge and \( \mathcal{R} \) is the Ricci scalar. The black hole horizons correspond to the roots of \( f(r) \) and are given by
\[
    r_{\pm} = GM \pm \sqrt{(GM)^2 - GQ^2g^2}.
\]

For the case of an imaginary square root we get a naked singularity which is considered to be unphysical. Thus we infer
\[
    (GM)^2 \geq GQ^2g^2 \iff M \geq QgM_P
\]
which is the extremality bound for charged black holes. Furthermore, the appropriate Hawking temperature for a charged black hole is given by
\[
    T_H = \frac{r_+ - r_-}{4\pi r_+^2}
\]
which in the case \( Q = 0 \) reduces to
\[
    T_H = \frac{1}{8\pi GM}
\]
and leads to \( T = 0 \) for extremal black holes since the two horizon radii coincide in this case.

3 Global Symmetries in Quantum Gravity [1]

In this section we will argue that quantum gravity forbids the presence of any continuous global symmetry. This idea is natural, because even classical General Relativity makes everything local. The absence of global symmetries will follow from the fact that a continuous global symmetry implies a conserved charge which cannot be detected from the outside of a black hole.

A first hint comes from the classical No-Hair theorem, which states that a black hole’s data is completely specified by it’s mass, angular momentum and possibly charge under a gauge symmetry. So in particular, a black hole doesn’t care about it’s charge under a global symmetry. While gauge charges can be detected by field measurements outside the horizon, a global symmetry charge can never be detected after falling into a black hole. Since we do not know what happens near the singularity, this is not necessarily in conflict with charge conservation but still looks fairly strange.
To convince ourselves of the non-existence of continuous global symmetries we need a stronger argument. Consider a black hole with global symmetry charge $Q$ and mass $M$. The black hole will eventually evaporate via Hawking radiation. Now we see a problem: the black hole might not be able to evaporate because of charge conservation.

Hawking radiation at low temperatures (or high masses) consists mainly of massless particles. But as the black hole evaporates, at some time the Hawking temperature will hit the mass of the lightest charged particle. Let us take it’s mass to be $m$ and normalize charges such that it has unit charge. Then after $T_H > m$ we expect Hawking radiation to contain an appreciable amount of charged particles. Neglecting numerical factors this can be reformulated as $M \leq \frac{M_p^2}{m}$.

In the best case, the black hole will just spit out $Q$ times the lightest charged particle (to which we assign unit integral charge for simplicity) and get rid of all of it’s charge. From kinematics we get $M \geq Qm$ and combining these two inequalities yields

$$Q \leq \left(\frac{M_p}{m}\right)^2$$

Obviously, such a bound is in contradiction with the possibility to create arbitrarily high charges (at fixed mass) by just dropping in further charged particles (and letting the excess mass evaporate). So we get an infinite family of black holes which cannot decay and thus will finally evaporate down to what is usually know as a “Planck scale remnant”. A Planck scale remnant is an unknown object with mass around $M_p$ and radius around $l_p$ which is protected from decay e.g. by the existence of a charge.

We will see that the argument presented above will generalize to gauge symmetries. For global symmetries, we see a further difficulty. Although black holes satisfying (8) can in principle decay, they will not do so in practice. This is because of the thermal nature of Hawking radiation. The black hole will not lose any net charge simply because of the equality in numbers of radiated particles and antiparticles.

Now we can finally see what goes wrong. An external observer of the black hole will not be able to measure it’s charge even if he tracks every particle in the Hawking radiation. The charge is forever lost in the black hole and cannot be retrieved. The external observer thus associates an infinite additional entropy to the black hole because he must count microstates with different charges all equally and there is no bound on the black hole’s charge. This then leads to two problems. First of all, it is generally believed that black hole entropy in quantum gravity is given by the Bekenstein-Hawking formula and the entropy contained in a spherical region is bounded by the corresponding entropy of a black hole (Bekenstein-Bound). This is due to exact calculations in string theory and the Holographic Principle. Secondly, Susskind [4] claims that an infinite number of stable Planck scale remnants as above lead to production of an infinite number of remnants per volume in the thermal atmosphere of a Rindler observer.
Before we go on with gauge symmetries, let us briefly think about the relation between global and gauge symmetries: As we send the gauge coupling $g$ of a gauge symmetry to zero, all covariant derivatives become ordinary ones and so the theory only retains a global symmetry in the limit $g = 0$. Gauge bosons and charged matter decouple. In particular, in the quantum theory each interaction vertex carries a factor of $g$ so for very small $g$, every scattering amplitude including only particles with small charges gets tiny.

For gauge symmetries, an external observer can classically measure the charge of a black hole by Gauss law. Quantum mechanically, a charge measurement is done by scattering experiments. But as we have seen above, scattering amplitudes go to zero as $g \to 0$ and we have to conclude that for very small gauge coupling, one can only detect charges up to an uncertainty of order $\frac{1}{g}$ since for $q = \frac{1}{g}$ we get amplitudes of order $g \cdot \frac{1}{g} = 1$.

We now have a first hint to suspect that something could go wrong also for weakly coupled gauge symmetries.

### 4 Existence of Light Charged Particles [2]

Having understood the problems of global symmetries and arbitrarily low couplings, we are now in a position to discuss the claim of Arkani-Hamed et al. of the existence of a light charged particle. Therefore, let us consider a black hole which starts with some mass and charge obeying the extremality bound (5). The black hole is always in a position to lose mass due to Hawking radiation of massless particles. But below a certain threshold the discharge of the black hole is exponentially suppressed. Demanding that at the threshold the black hole has still enough mass to be able to radiate away all of its charge, we get the relation

$$M_{\text{discharge}} \geq Qm$$  \hspace{1cm} (9)

where $m$ denotes the mass of the lightest charged particle. Due to the extremality bound, the black hole can merely evaporate down to some mass of order $M_P$. If (9) is not satisfied the black hole is not able to decay and we end up with a large number of Planck scale remnants which violate the Bekenstein bound.

To make this argument more clear, let us illustrate this by an explicit example: Consider a black hole of mass $M \sim 10M_P$ and a gauge coupling $g \sim 10^{-100}$. To satisfy the bound (5) the black hole can have any charge between 0 and $\sim 10^{100}$. If there were no very light charged particles, it turns out that none of this charge could be radiated away as the black hole radiates down to Planck scale. But this would lead to $10^{100}$ different Planck scale remnants with charge from 0 to $10^{100}$. But it is expected that a large number of exactly stable objects should not exist if it is not protected by any symmetries.
Therefore we demand that also extremal black holes can decay, i.e. also the extremal mass should obey (9). Using this we can directly recover the first claim of Arkani-Hamed et al. that the mass of the lightest charged particle obeys
\[ m \leq gM_P \]  

Rearranging this equation as \( g \geq \frac{m}{M_P} \) we see that there is a lower bound for the gauge coupling which prevents us from the problems discussed in the last section.

The conjecture can me reformulated as the familiar statement “gravity is the weakest force”. To see this consider the ratio of a \( U(1) \) force and the gravitational force:
\[
\frac{F_{el}}{F_G} = \frac{g^2}{r^2} \cdot \frac{r^2}{m^2G} = \frac{g}{m^2G} = \frac{M_P^2}{g^2} \geq \frac{g}{gM_P^2} = 1
\]

One should mention that this argument for the statement “gravity is the weakest force” is only valid for a pair of particles obeying (10).

We have formulated the conjecture in terms of the \( U(1) \) gauge coupling. Since this is a running coupling we should clarify at which scale we have to evaluate the coupling: For the statement of the existence of a light charged particle with mass \( m \leq gM_P \) it is natural to evaluate the coupling at the mass of the lightest charged particle. For the statement about the new cutoff \( \Lambda \leq gM_P \), which we will discuss later, it is reasonable to evaluate the coupling near the scale \( \Lambda \).

5 Magnetic Monopoles [5]

We will soon argue that the first part of the conjecture leads to the prediction that a \( U(1) \) gauge theory must have a cutoff below the Planck scale. To predict anything about a cutoff, we need to study non-perturbative effects. A particular example of such a non-perturbative effect are magnetic monopoles. Magnetic monopoles arise quite universally in GUTs. The defining property of a magnetic monopole with charge \( g_{mag} \) is its B field which should satisfy, at least in the far field limit:
\[ \tilde{B} = \frac{g_{mag}}{r^2} e_r \]

Suppose now that we deal with a \( U(1) \) theory that has a cutoff \( \Lambda \). In principle, neglecting gravity, this could be arbitrarily large, even exceeding the Planck scale but not the Landau pole. The mass of a magnetic monopole should be of order it’s field energy:
\[
m_{mon} \approx E_{mon} = \int_{r \leq 1/\Lambda} dV \epsilon(x) = \int_{r \leq 1/\Lambda} dV \tilde{B}^2 \propto g_{mag}^2 \int_{1/\Lambda}^{\infty} \frac{1}{r^2} dr = g_{mag}^2 \Lambda
\]

So we see explicitly that monopoles can be useful to analyse the UV structure of a gauge theory. Monopoles usually arise as topologically stable classical solutions to
non-abelian gauge field equations. Contrast this with charged elementary fermions: their mass is logarithmically divergent and it’s low energy value is a freely adjustable parameter of the theory, fixed by renormalisation.

Another ingredient that we will need is the fact that in presence of magnetic monopoles, electric charge is quantized. This important fact is due to Dirac and was later interpreted in the framework of fiber bundles by Wu and Yang.

It turns out that for topological reasons, a vector potential that is regular on the whole $\mathbb{R}^3 \backslash \{0\}$ cannot be defined. The way to go is to define $\vec{A}$ only patchwise and demand that the differences on overlaps are gauge transformations, hence unphysical. Explicitly, in spherical coordinates:

$$\vec{A}^N := \frac{g_{\text{mag}}(1 - \cos \theta)}{r \sin \theta} \vec{e}_\phi$$

$$\vec{A}^S := -\frac{g_{\text{mag}}(1 + \cos \theta)}{r \sin \theta} \vec{e}_\phi$$

are vector potentials for (12) on the northern and southern hemisphere satisfying

$$\vec{A}^N - \vec{A}^S = \vec{\nabla}(2g_{\text{mag}}\phi) = \vec{\nabla}\lambda$$

on the equator. An electrically charged particle with charge $g_{\text{el}}$ will then have wave functions $\psi^N$ and $\psi^S$ which are related by the corresponding gauge transformation

$$\psi^S = e^{-ig_{\text{el}}\lambda}\psi^N = e^{-2ig_{\text{el}}g_{\text{mag}}\phi}\psi^N$$

which must be single valued on the equator, so we find:

$$2g_{\text{el}}g_{\text{mag}} \in \mathbb{Z}$$

This is the celebrated Dirac quantization condition.

6 Existence of a UV cutoff [2]

We can now finally derive the existence of a UV cutoff for the U(1) gauge theory satisfying the first part of the conjecture. Observe first, that the argument leading to the bound

$$m_{\text{el}} \leq g_{\text{el}}M_P$$

does not depend on the label “electric”. In particular we could also consider magnetically charged black holes and magnetically charged test particles. We would conclude that

$$m_{\text{mon}} \leq g_{\text{mag}}M_P$$

if we now insert our monopole mass estimate and the Dirac quantisation condition we find

$$\frac{\Lambda}{g_{\text{el}}^2} \sim g_{\text{mag}}^2\Lambda \sim m_{\text{mon}} \leq g_{\text{mag}}M_P \sim \frac{M_P}{g_{\text{el}}}$$
and after rearranging the terms
\[ \Lambda \leq g_\alpha M_P \]  
(22)

So we see that tiny gauge couplings lead to tiny cutoff scales and in particular we see that global symmetries are forbidden because in this case \( \Lambda = 0 \).

7 Variants of the Conjecture and Generalizations

In section 4 we already discussed that our claim can be rephrased as the statement that we want to allow extremal black holes to decay. A necessary condition for this is that there exists a particle with smaller mass/charge ratio. This can be seen as follows: Consider a black hole with mass/charge ratio \( \gamma = \frac{M}{Q} \) which is able to decay into a bulk of particles with charge \( \sum q_i = Q \) and mass \( \sum m_i < M \) (The “\(<\)” appears due to the kinetic energy of the particles). Assuming that there is no particle with a smaller mass/charge ratio, i.e. \( \gamma_i = \frac{m_i}{q_i} > \gamma \forall i \), leads to a contradiction

\[ \sum_i m_i = \sum_i \gamma_i q_i > \gamma \sum_i q_i = \gamma Q = M . \]  
(23)

For the following discussion we set the Planck mass as well as the gauge coupling to one. Due to the extremality bound (5) we get \( \frac{M}{Q} = 1 \) for extremal black holes. Thus the conjecture translates into the existence of a particle with mass/charge ratio

\[ \frac{m}{q} \leq 1 \]  
(24)

Now, to formulate a sharper claim, we may ask what states should satisfy this bound. There are three natural possibilities:

(i) \( \left( \frac{m_{\min}}{q_{\min}} \right) \leq 1 \), for the state of minimal charge

(ii) \( \left( \frac{m_{\min}}{q_{\min}} \right) \leq 1 \), for the lightest charged particle

(iii) \( \left( \frac{m}{q} \right)_{\min} \leq 1 \), for the state with smallest mass/charge ratio

To get a meaningful statement, the appropriate state must be exactly stable. The particle with the smallest charged is not necessarily stable. To see this, consider a heavy particle of charge +1. This could decay in lighter particles of charges −2 and +3, which will subsequently form a Kepler/Coulomb bound state of charge +1. Such a bound state is stable but its mass/charge ratio may be larger than for the single particles and in particular it may be larger than one. Furthermore, there are counterexamples in string theory in which the minimally charged particles are stable but violate the bound. So we conclude that (i) cannot be the right assumption.

Obviously the lightest particle is stable, as well as the particle with the smallest mass/charge ratio as we showed to the beginning of the section, and so (ii) and
(iii) are well-defined. Clearly (ii) induces (iii), thus (ii) is the stronger assumption. Assumption (ii) implies that there must exist a light charged particle in the theory while (iii) can also be satisfied by a heavy state with appropriate charge. Although, there are no counterexamples for (ii) until now, it is conjectured that only the weaker claim (iii) is true.

So far we only focused on the discussion in context of a single $U(1)$ symmetry. But we should mention that it is possible to generalize this argument to other gauge groups. Consider for example a theory where we have several $U(1)$ groups. In this case, we will get several black hole solutions in each direction in charge space. Therefore, the conjectures (ii) and (iii) imply the existence of light charge particles with $(m/q) < (M/Q)_{\text{extremal}}$ in certain direction of the charge space. More precisely, the set of directions where the inequality is satisfied should form a basis of the full charge space. The full generalization can be found in [3].

8 Conclusion

In this paper we discussed the interplay between $U(1)$ gauge symmetries and quantum gravity, and “derived” the existence of a light charged particle with $m/q$ smaller than the corresponding ratio for extremal black holes, as first conjectured by Arkani-Hamed et al. The existence of such a particle can be seen to bound the gauge coupling away from zero, consistent with the folk theorem that quantum gravity should have no global symmetries. Similar considerations involving magnetic monopoles constrain the scale of the $U(1)$’s cutoff. In particular one cannot send the gauge coupling to zero without rendering the theory trivial. In section 7 we briefly considered attempts to make the conjecture more precise and also generalizations to product gauge groups $U(1)^n$. People have also considered generalizing to non-abelian gauge groups. As string theory is at the moment the only consistent and unified quantum theory of gravity and gauge forces, evidence for the conjecture mainly comes from stringy constructions. The result should hold in complete generality though, since the derivation did not depend on any string physics. It does though heavily rely on the absence of black hole remnants. The main conclusion is that the presence of quantum gravity effects like Hawking radiation constrains the possible matter and gauge field content of consistent physical theories.
References


