The Unruh Effect
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June 17, 2015

Consider an observer with constant proper acceleration (relativistic uniform acceleration) in Minkowski space. The Unruh effect is the phenomenon that this observer will measure a density of particles following a Bose-Einstein distribution, where the temperature is identified with the acceleration. In the following, we will first set up suitable coordinates and then demonstrate the effect for a 1+1-dimensional massless scalar quantum field.

The sign convention is $\eta = \text{diag}(1, -1, -1, -1)$.

I. Trajectory of a Uniformly Accelerated Observer

We refer to the Minkowski coordinates $t, x, y, z$ as the lab frame. It is chosen such that the observer is accelerated only in positive $x$-direction. Her proper coordinate system (the one in which she is at always at rest) is given by $\tau, \xi, y, z$. The world line is parametrized by the proper time $\tau$ and the observer has a 4-velocity vector $u^\mu = \frac{dx^\mu}{d\tau}$, with $u^2 = 1$.

Hence, in the proper frame the 4-acceleration

$$a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau}$$

assumes the simple form

$$a = (0, a, 0, 0).$$

Conveniently, this implies that

$$d^\mu a_\mu = -a^2$$

in all frames. From now on we abandon the constant coordinates $y$ and $z$ and work in 1+1-dimensional Minkowski space.

The differential equation in (1) is hyperbolic and has the solutions

$$u^0(\tau) = \cosh(F(\tau)), \quad u^1(\tau) = \sinh(F(\tau)),$$

where $F(\tau)$ is a differentiable function of $\tau$ and we assume that proper time runs into the same direction as coordinate time, $u^0 > 0$. Deriving and comparing with (2) yields

$$a(\tau) = \dot{F}(\tau) \sinh(F(\tau)) + \cosh(F(\tau)),$$

$$a^2 = \ddot{F}^2 = \text{const.},$$

$$F(\tau) = a \tau,$$

where we choose the initial condition $u^1(0) = 0$. By integration we obtain the world line (setting $x(0) = a^{-1}$)

$$x(\tau) = (a^{-1} \sinh(a\tau), a^{-1} \cosh(a\tau)).$$

II. Proper Coordinates and Rindler Space

To compare a quantum field in lab frame and proper frame we need a coordinate transformation $t(\tau, \xi), x(\tau, \xi)$. Since the accelerated frame is not inertial it cannot be a Lorentz transformation. However, we can define at each $\tau_0$ a momentarily comoving inertial frame with 4-velocity $u(\tau_0)$ whose coordinates match the proper coordinates (only at $\tau = \tau_0$).

We express the vector of each proper coordinate $(\tau, \xi)$ as a sum $(\tau, 0) + (0, \xi)$. The former is in all frames given by the trajectory $x(\tau)$, while the latter now can be obtained by a Lorentz boost between the lab frame and the comoving frame

$$\gamma \cdot \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} u^0 & u^1 \\ u^1 & u^0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \xi \end{pmatrix} = \begin{pmatrix} u^0 \xi \\ u^1 \xi \end{pmatrix}.$$ 

Summing up both vectors in the lab frame, we obtain the coordinate transformations

$$t(\tau, \xi) = \frac{1 + a\xi}{a} \sinh(a\tau),$$

$$x(\tau, \xi) = \frac{1 + a\xi}{a} \cosh(a\tau).$$

The inverse transformations are then:

$$\tau(t, x) = \frac{1}{2a} \ln \frac{x + t}{x - t},$$

$$\xi(t, x) = -a^{-1} + \sqrt{x^2 - t^2}.$$ 

Note that $\tau$ takes values in the interval $(-\infty, \infty)$, while $\xi$ is restricted to $(-a^{-1}, \infty)$. In particular, only the right Rindler wedge (c.f. Figure 1) given by $0 < x \leq |t|$ is covered by the proper coordinates. The boundary $x = |t|$ marks the asymptotes of $\tau = \pm \infty$. Therefore every signal reaching the accelerated observer from outside will be interpreted as originating from $\tau = -\infty$.

Insertion of the proper coordinates into the metric yields the 1+1 dimensional Rindler metric:

$$ds^2 = dt^2 - dx^2 = (1 + a\xi)^2 dr^2 - d\xi^2.$$  

It can be written in a conformally flat form by introducing $\tilde{\xi} = \frac{1}{a} \ln(1 + a\xi) \in (-\infty, \infty)$:

$$ds^2 = e^{2\tilde{\xi}}(d\tau^2 - d\tilde{\xi}^2).$$

The transformation equations become:

$$t(\tau, \tilde{\xi}) = a^{-1} e^{a\tilde{\xi}} \sinh(a \tau),$$

$$x(\tau, \tilde{\xi}) = a^{-1} e^{a\tilde{\xi}} \cosh(a \tau).$$

Observe how (8) resembles the FRW metric with conformal time $\eta$ and scale factor $a(\eta)$:

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2).$$
III. MASSLESS SCALAR FIELD IN RINDLER SPACE

The action of a massless scalar field $\phi(t, x)$ is

$$S[\phi] = \frac{1}{2} \int g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \sqrt{-g} \, dt \, dx,$$

where $\sqrt{-g} \, dt \, dx$ is the invariant measure. Note that the action (12) is conformally invariant, i.e. on replacing

$$g_{\alpha\beta} \rightarrow \tilde{g}_{\alpha\beta} = \Omega^2 (t, x) g_{\alpha\beta}$$

the determinant $\sqrt{-g}$ picks up a factor $\Omega^2 (t, x)$ which cancels the factor of $\Omega^{-2}$ from the inverse metric $g^{\alpha\beta}$. Apart from the prefactor $\Omega^2 = e^{2\alpha \xi}$, the conformal Rindler metric (8) is indeed Minkowskian, i.e. $\sqrt{-g} = \Omega^2$. The actions in lab and conformal Rindler coordinates then read

$$S[\phi] = \frac{1}{2} \int [(\partial_t \phi)^2 + (\partial_x \phi)^2] \, dt \, dx,$$

$$S[\phi] = \frac{1}{2} \int [(\partial_{\tilde{t}} \phi)^2 + (\partial_{\tilde{x}} \phi)^2] \, d\tilde{t} \, d\tilde{x}$$

respectively.

The corresponding equations of motion can be derived by forming the Euler-Lagrange equation of (13) and (14). One obtains:

$$\partial_{\tilde{t}}^2 \phi - \partial_{\tilde{x}}^2 \phi = 0, \quad \partial_t^2 \phi - \partial_x^2 \phi = 0.$$

The general solutions are given by

$$\phi(t, x) = A(t - x) + B(t + x),$$

$$\phi(\tau, \xi) = P(\tau - \xi) + Q(\tau + \xi),$$

where $A, B, P$, and $Q$ are arbitrary smooth functions.

Since (16) and (17) solve the Klein-Gordon equations (15), one can formulate the mode expansions in both sets of coordinates. Using the dispersion relation $\omega_k = |k|$ (for the 1-D spatial momentum $k^1 = k$), one obtains

$$\hat{\phi}(t, x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} [e^{-ik[t + ikx]} \hat{a}_k^- + e^{ik[t - ikx]} \hat{a}_k^+]$$

$$\hat{\phi}(\tilde{\tau}, \tilde{\xi}) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} [e^{-ik[\tau + ik\xi]} \hat{b}_k^- + e^{ik[\tau - ik\xi]} \hat{b}_k^+]$$

where the mode operators $\hat{a}_k^\pm$ and $\hat{b}_k^\pm$ do not agree in general. This result is similar to that derived in the first talk [1] (cf. pp. 67-70 of [2]), which is not surprising given how (8) resembles (11)². Consequently, the vacua of Rindler and Minkowski observers differ:

$$|0_M \neq |0_R\rangle,$$

An accelerating observer will measure that the corresponding vacuum state $|0_R\rangle$ has the lowest possible energy—lower than that of the Minkowski vacuum state $|0_M\rangle$. Particularly, a particle detector at rest in the accelerated frame will detect particles when the scalar field is in $|0_M\rangle$. Conversely, the Rindler vacuum $|0_R\rangle$ will appear excited to an observer in the lab frame. This is the Unruh effect.

In the following sections, the mode operators $\hat{a}_k^\pm$ and $\hat{b}_k^\pm$ will be shown to be related by a generalized Bogolyubov transformation after introducing a convenient set of coordinates, which would enable us to express $|0_M\rangle$ as a superposition of excited states (with respect to $|0_R\rangle$) and compute the particle occupation number distribution as observed in the accelerated frame. The particle occupation density will resemble a Bose-Einstein distribution with Unruh temperature $T$.

IV. BOGOLYUBOV TRANSFORMATIONS

For further computations, it is convenient to introduce the lightcone coordinates:

$$\bar{u} = t - x, \quad \bar{v} = t + x, \quad (20)$$

$$u = \tau - \xi, \quad v = \tau + \xi. \quad (21)$$

The transformation equations (4) and (3) assume the form:

$$\bar{u} = -a^{-1} e^{-av}, \quad \bar{v} = a^{-1} e^{-av},$$

as can be found by using that $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. The metric (7) becomes

$$ds^2 = dt^2 - dx^2 = (dt + dx)(dt + dx) = du dv = e^{a(v - u)} du dv$$

and the field equations (15) can be expressed as

$$\partial_u \partial_v \phi(\bar{u}, \bar{v}) = 0, \quad \partial_u \partial_v \phi(u, v) = 0,$$

with

$$\phi(\bar{u}, \bar{v}) = A(\bar{u}) + B(\bar{v}), \quad (24)$$

$$\phi(u, v) = P(u) + Q(v). \quad (25)$$

²However, unlike (11), (8) is static (i.e. there exists a time-like Killing vector field), which justifies the choice of positive frequency solutions $\sim e^{-i\omega t}$ to the Klein-Gordon equation.
To obtain the lightcone mode expansion of \( \phi(\bar{u}, \bar{v}) \), (18) must first be split in two integrals:

\[
\hat{\phi}(t, x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi \sqrt{2k}} \left[ e^{-ikx} \hat{a}_k + e^{ikx} \hat{a}_k^+ \right] + \int_{-\infty}^{0} \frac{dk}{\sqrt{2\pi}} \left[ e^{-ikx} \hat{a}_k + e^{ikx} \hat{a}_k^+ \right],
\]

(26)

On introducing \( \omega = |k| \) as the integration variable and substituting from (20), (26) becomes:

\[
\hat{\phi}(\bar{u}, \bar{v}) = \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[ e^{-i\omega \bar{u}} \hat{a}_\omega + e^{i\omega \bar{u}} \hat{a}_\omega^+ \right] + e^{-i\omega \bar{v}} \hat{a}_- - e^{i\omega \bar{v}} \hat{a}_-^+ \left( \hat{a}_\omega \right] \right]
\]

(27)

Compare this with the general solution (24) of the field equations (23) to find:

\[
\hat{A}(u) = \int_{0}^{\infty} \frac{d\Omega}{2\pi} \frac{1}{\sqrt{2\Omega}} \left[ e^{-i\Omega u} \hat{a}_\Omega + e^{i\Omega u} \hat{a}_\Omega^+ \right]
\]

(28)

\[
\hat{B}(v) = \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[ e^{-i\omega v} \hat{a}_- - e^{i\omega v} \hat{a}_-^+ \right]
\]

(29)

The exact procedure gets through for \( \hat{\phi}(u, v) = \hat{P}(u) + \hat{Q}(u, v) \) and one finds

\[
\hat{P}(u) = \int_{0}^{\infty} \frac{d\Omega}{2\pi} \frac{1}{\sqrt{2\Omega}} \left[ e^{-i\Omega u} \hat{b}_\Omega + e^{i\Omega u} \hat{b}_\Omega^+ \right]
\]

(30)

\[
\hat{Q}(v) = \int_{0}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[ e^{-i\omega v} \hat{b}_- - e^{i\omega v} \hat{b}_-^+ \right]
\]

(31)

Now, observe that the coordinate transformations (20) and (21) do not mix u’s and v’s. One can therefore write

\[
\hat{A}(\bar{u}(u)) = \hat{P}(u),
\]

(32)

\[
\hat{B}(\bar{v}(v)) = \hat{Q}(v).
\]

(33)

Remember that the goal is to express mode operators \( \hat{a}_\omega^\pm \) as linear combinations of \( \hat{b}_\omega^\pm \). The obvious next step would therefore be to Fourier transform both sides of equations (32) and (33). The transformed RHS of (32) reads

\[
\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{P}(u) = \int_{0}^{\infty} \frac{d\Omega}{2\pi} \frac{1}{\sqrt{2\Omega}} \left[ e^{-i\Omega u} \hat{b}_\Omega + e^{i\Omega u} \hat{b}_\Omega^+ \right]
\]

\[
= \int_{-\infty}^{+\infty} \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[ e^{-i\Omega u} \hat{b}_\Omega + e^{i\Omega u} \hat{b}_\Omega^+ \right]
\]

\[
= \int_{0}^{+\infty} \frac{d\Omega}{\sqrt{2\pi}} \left[ e^{-i\Omega |\Omega|} \hat{b}_\Omega + e^{i\Omega |\Omega|} \hat{b}_\Omega^+ \right]
\]

\[
= \frac{1}{\sqrt{2|\Omega|}} \left\{ \begin{array}{ll}
\hat{b}_{\Omega}, & \Omega > 0 \\
\hat{b}_{\Omega}, & \Omega < 0
\end{array} \right.
\]

(34)

while the LHS is given by

\[
\int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} \hat{A}(u) = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} \left[ e^{i\Omega u} \hat{a}_\Omega + e^{-i\Omega u} \hat{a}_\Omega^+ \right]
\]

(35)

\[
= \int_{0}^{+\infty} \frac{d\omega}{\sqrt{2\omega}} \left[ F(\omega, \Omega) \hat{a}_\omega^- + F(-\omega, \Omega) \hat{a}_\omega^- \right],
\]

where the apparently divergent auxiliary function

\[
F(\omega, \Omega) = \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{i\Omega u - i\omega u}
\]

\[
= \int_{-\infty}^{+\infty} \frac{du}{2\pi} \exp \left[ i\Omega u + i\omega \frac{a}{a} e^{-au} \right]
\]

(36)

is defined in the distributional sense\(^3\). Compare (34) and (35) to find

\[
\hat{b}_\Omega^- = \int_{-\infty}^{+\infty} d\omega \left[ \alpha_{\Omega\Omega} \hat{a}_\omega^- + \beta_{\Omega\Omega} \hat{a}_\omega^+ \right], \quad \Omega > 0,
\]

(37)

with

\[
\alpha_{\Omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(\omega, \Omega),
\]

\[
\beta_{\Omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(-\omega, \Omega),
\]

\(\hat{b}_\Omega^-\) can be computed by hermitian conjugating (37). The transformations relating \( \hat{a}_\omega^\pm \) and \( \hat{b}_\Omega^\pm \) follow analogously from (33).

Observe that the Bogolyubov transformation (37) mixes mode operators at different momenta \( \omega \) and \( \Omega \) and is therefore more general than that introduced in [1]. In fact, the general Bogolyubov transformation is given by

\[
\hat{b}_\Omega^- = \int_{-\infty}^{+\infty} d\omega \left[ \alpha_{\omega\Omega} \hat{a}_\omega^- + \beta_{\omega\Omega} \hat{a}_\omega^+ \right]
\]

(38)

of which (37) is a special case, for \( \alpha_{|\omega|, \Omega} = \beta_{-|\omega|, \Omega} \equiv 0 \). To derive the corresponding normalization conditions, we use the commutation relations \( [\hat{a}_\omega^-, \hat{a}_\omega^+] = \delta(\omega - \omega') \) and \( [\hat{b}_\Omega^-, \hat{b}_\Omega^+] = \delta(\Omega - \Omega') \) together with (38) to find

\[
\int_{-\infty}^{+\infty} d\omega (\alpha_{\omega\Omega} \alpha_{\omega'\Omega'}^* - \beta_{\omega\Omega} \beta_{\omega'\Omega'}^*) = \delta(\Omega - \Omega')
\]

(39)

(cf. equation (21) of [1]).

V. PARTICLE DENSITY AND THE UNRUH TEMPERATURE

The mean number of particles the accelerated observer detects is given by the Minkowski vacuum expectation value of the \( \beta \)-particle number operator \( \hat{N}_\Omega = \hat{b}_\Omega^\dagger \hat{b}_\Omega \) as

\[
\langle \hat{N}_\Omega \rangle = \langle 0 | \hat{b}_{\Omega}^\dagger \hat{b}_\Omega | 0 \rangle
\]

\[
= \langle 0 | \int d\omega [\alpha_{\omega\Omega} \hat{a}_\omega^- + \beta_{\omega\Omega} \hat{a}_\omega^+] \hat{b}_\Omega^* \hat{b}_\Omega | 0 \rangle
\]

\[
\times \int d\omega' [\alpha_{\omega'\Omega'} \hat{a}_\omega^- + \beta_{\omega'\Omega'} \hat{a}_\omega^+] | 0 \rangle
\]

\[
= \int d\omega |\beta_{\omega\Omega}|^2 = \int d\omega \frac{\Omega}{\omega} |F(-\omega, \Omega)|^2,
\]

using (37) and its hermitian conjugate. To proceed, we must have a closer look at the auxiliary function (36). We claim that

\[
F(\omega, \Omega) = F(-\omega, \Omega) \exp \left( \frac{\pi \Omega}{a} \right), \quad \text{for } \omega, \Omega, a > 0.
\]

(41)

\(^3\)Much like how \( \delta(x) = \int_{-\infty}^{+\infty} dk \right. \exp(ikx). \)
To see this, substitute \( x = \exp(-au) \) to find

\[
F(\omega, \Omega) = \frac{1}{2\pi a} \int_0^\infty dx \, x^{s+1} e^{-bx}.
\]

where \( s = -i\Omega/a, \ b = -i\omega/a \). If we impose a shift of \( s \) and \( b \) by a small, positive parameter \( \varepsilon > 0 \), we may use a \( \Gamma \)-function identity \(^4\) to obtain the distributional limit as

\[
F(\omega, \Omega) = \lim_{\varepsilon \to 0} \frac{1}{2\pi a} \int_0^\infty dx \, x^{s+\varepsilon-1} e^{-(b+\varepsilon)x}.
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{2\pi a} e^{-(s+\varepsilon)\ln(b+\varepsilon)} \Gamma(s + \varepsilon).
\]

The relevant observation is that the sign function leads to \(+1\), since \( \omega \) is positive and under change to \( F(-\omega, \Omega) \) this changes to \(-1\). Hence, we obtain (41).

The last step is then to take (39) while setting \( \Omega' = \Omega \) and insert (41) to find that

\[
\int_0^\infty d\omega |F(-\omega, \Omega)|^2 = \delta(0) \left[ \exp \left( \frac{2\pi \Omega}{a} \right) - 1 \right]^{-1}.
\]

If we divide out the volume factor \( \delta(0) \) we thus obtain the number density

\[
n_\Omega = \left[ \exp \left( \frac{2\pi \Omega}{a} \right) - 1 \right]^{-1}, \quad \Omega > 0.
\]

A completely analogous treatment of the negative frequency modes operators leads to

\[
n_{\Omega} = \left[ \exp \left( \frac{2\pi |\Omega|}{a} \right) - 1 \right]^{-1}, \quad \Omega < 0.
\]

Now note that for massless 2-dimensional scalar fields \( |\Omega| \equiv E \). We formulate the final result as a Bose-Einstein distribution

\[
n(E) = \frac{1}{\exp \frac{E}{T} - 1},
\]

where \( T = a/2\pi \) defines the Unruh temperature.

VI. CONCLUSION

We found that in a Minkowski spacetime (reduced to 1+1 dimensions) hosting a massless scalar field a uniformly accelerated observer finds a thermal spectrum of field excitations when travelling through Minkowski vacuum, where temperature is identified with proper acceleration.

This effect generalizes to 1+3-dimensional scalars (and other quantum field theories) with much more effort [3]. It is very small in practical terms, as becomes clear from reinstating SI-Units: a temperature of 1K corresponds to \( a \approx 10^{30}\)m/s\(^2\).

\(^4\)The \( \Gamma \)-function is defined on \( \mathbb{C} \) except non-positive integers as the analytic continuation of the integral

\[
\Gamma(t) = \int_0^\infty dx \, x^{t-1} e^{-x}, \quad \text{for} \quad t \in \mathbb{C} \text{ with Re } t > 0.
\]

One can show that for \( \text{Re } b > 0, \ 1 > \text{Re } s > 0 \)

\[
\int_0^\infty dx \, x^{s-1} e^{-bx} = e^{-s \ln b} \Gamma(s),
\]

where the logarithm here is defined in the right complex half-plane as \( \ln(A + iB) = \ln(|A + iB|) + i \text{sgn}(B) \arctan(|B|/A)) \).

REFERENCES