Langevin Equation and Fokker-Planck Equation

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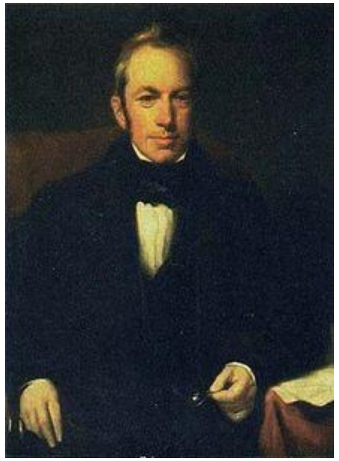
Motivation

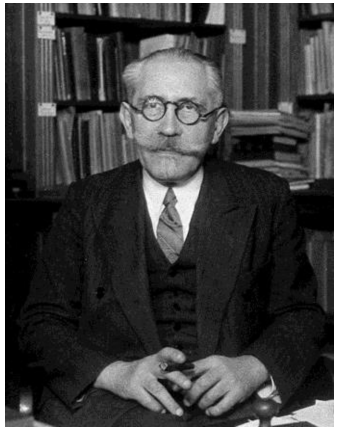
- Explains Brownian Motion
- One of the most fundamental stochastic differential equation
- Can show similarities between quantum mechanics and statistical physics
- Can also be used in economy

Historical Development

- Robert Brown, 1827: Discovery of Brownian motion
- Einstein, 1905: Kinetic Theory of Brownian motion
- Smoluchowski, 1906
- Langevin, 1908: Description of Langevin force
- Fokker, 1914
- Planck, 1917: Derivation of Fokker-Planck equation
- Kramers, 1940: Derivation of Fokker-Planck equation through Kramers-Moyal Expansion



Brown



Langevin

Mathematical Recap and Tools

Characteristic Function and Cumulants

- Fourier Transform of the probability density

$$C(u) = \left\langle e^{iux} \right\rangle = \int e^{iux} W(x) \mathrm{d}x \quad ; \quad W(x) = \frac{1}{2\pi} \int C(u) e^{-iux} \mathrm{d}u$$

- $M_n = \langle x^n \rangle = \frac{1}{i^n} \frac{\partial^n}{\partial u^n} C(u) \Big|_{u=0}$ n-th Moment of the distribution
-
- $C(u) = 1 + \sum_{n=1}^{\infty} \frac{(iu)^n M_n}{n!} =: \exp(\sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \ K_n)$
- $\ln C(u) = \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \ K_n$

Characteristic Functions: Examples

- $K_2 = K_3 = \cdots = 0$:

$$C(u) = \exp(iuK_1) \rightarrow W(u) = \delta(x - K_1)$$

- $K_3 = K_4 = \cdots = 0$:

$$C(u) = \exp\left(iuK_1 - \frac{1}{2}u^2K_2\right) \rightarrow W(x) = (2\pi K_2)^{\frac{1}{2}} \exp\left(-\frac{\frac{1}{2}(x - K_1)^2}{K_2}\right)$$

Multi-dim. Gaussian Distribution

- $C(u_1, \dots, u_r) = \exp\left(ia_j u_j - \frac{1}{2}\sigma_{jk} u_j u_i\right)$

where $\langle x_j \rangle = a_j$ and $\langle x_j x_k \rangle = \sigma_{jk} + a_j a_k$.

- $W(x_1, \dots, x_r) = (2\pi)^{-\frac{r}{2}} \operatorname{Det}(\sigma_{jk})^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\sigma_{jk}^{-1}(x_j - a_j)(x_k - a_k)\right)$

- $\eta_j \coloneqq x_j - a_j \;\Rightarrow \begin{cases} \langle \eta_{j_1}\eta_{j_2} \dots \eta_{j_{2n-1}} \rangle = 0 \\ \langle \eta_{j_1}\eta_{j_2} \dots \eta_{j_{2n}} \rangle = \sum_{P_d} \sigma_{k_1 \mathrm{k}_2} \sigma_{k_3 \mathrm{k}_4} \dots \sigma_{k_{2n-1} \mathrm{k}_{2n}} \end{cases}$

  where the Permutation P_d runs over all possible pairs from $j_1, \dots, j_{2n}$. The number of possible combinations in P_d are $\frac{(2n)!}{n! 2^n}$

Kramers-Moyal Forward Expansion

- An equation for the distribution function
- Expansion of distribution function
- Beginning with the Master Equation:

$$W(x, t+\tau) = \int P(x, t+\tau | x', t') W(x', t') \, \mathrm{d}x'$$

- For the moments of $P(x, t+\tau | x', t)$ we have:

$$M_n(x', t, \tau) = \langle((x(t+\tau) - x(t))^n\rangle\Big|_{x(t)=x\prime}$$

$$= \int (x - x')^n P(x, t+\tau | x', t) \, \mathrm{d}x$$

Kramers-Moyal Forward Expansion

$M_n(x',t,\tau) = = \int (x-x')^n P(x,t+\tau|x',t)\,\mathrm{d}x$

$W(x,t+\tau) = \int P(x,t+\tau|x',t')W(x',t')\,\mathrm{d}x'$

- $P(x,t+\tau|x',t) = \int \delta(y-x)\, P(y,t+\tau|x',t)\mathrm{d}y$
- $\delta(y-x) = \sum_{n=0}^{\infty} \frac{(y-x')^n}{n!}\left(-\frac{\partial}{\partial x}\right)^n \delta(x'-x)$
- $P(x,t+\tau|x',t) = \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\partial}{\partial x}\right)^n M_n(x,t,\tau)\right]\delta(x-x')$
- $(x,t+\tau) - W(x,t) \approx \frac{\partial W(x,t)}{\partial t}\tau = \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\partial}{\partial x}\right)^n M_n(x,t,\tau)\underbrace{\int \delta(x-x')W(x',t')\,\mathrm{d}x'}_{= W(x,t)}$

Kramers-Moyal Forward Expansion

- $\frac{\partial W(x,t)}{\partial t}\tau = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n \frac{M_n(x,t,\tau)}{n!} W(x,t) + \mathcal{O}(\tau^2)$

- $\Rightarrow \frac{\partial W(x,t)}{\partial t} = \lim_{\tau\to 0} \sum_{n=1}^{\infty}\frac{1}{\tau}\left(-\frac{\partial}{\partial x}\right)^n \frac{M_n(x,t,\tau)}{n!} W(x,t)$

- $\frac{\partial W(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x,t)\ W(x,t) = L_{KM}W$

$D^{(n)}(x,t) = \lim_{\tau\to 0}\frac{1}{\tau}\left[\frac{M_n(x,t,\tau)}{n!}\right]$ and $L_{KM}(x,t) = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x,t)$

Kramers-Moyal Forward Expansion: Solutions

- Assuming sharp position at t', i.e. $W(x,t') = \delta(x-x')$:
- $P(x,t|x',t') = W(x,t) \Rightarrow \frac{\partial}{\partial t} P(x,t|x',t') = L_{KM}(x,t) P(x,t|x',t')$
  with initial condition: $P(x,t'|x',t') = \delta(x-x')$
- $P(x,t|x',t') = \exp[L_{KM}(x)(t-t')] \,.\, \delta(x-x')$
- $P(x,t|x',t') = \left[1 + \sum_{n=1}^{\infty} \int_{t'}^{t} \mathrm{dt}_1 \int_{t'}^{t_1} \mathrm{dt}_2 \ldots \int_{t'}^{t_{n-1}} \mathrm{dt_n}\, L_{KM}(x,t_1) \ldots L_{KM}(x,t_n)\right] .\, \delta(x-x')$
- Dyson Series:
- $P(x,t|x',t') = \overleftarrow{T} \exp\left\{\int_{\mathrm{t'}}^{\mathrm{t}} L_{KM}(x,t'')dt''\right\} .\, \delta(x-x')$
- $\tau = t - t' \ll 1 \Rightarrow P(x,t|x',t') = [1 + L_{KM}(x,t).\tau + \mathcal{O}(\tau^2)].\, \delta(x-x')$

Kramers-Moyal Backward Expansion

- Differential Operators w.r.t. earlier values (x', t')
- Beginning with Chapman-Kolmogorov:
- $P(x, t|x', t') = \int P(x, t|x'', t+\tau).P(x'', t+\tau|x', t')\mathrm{d}x''$
- $P(x'', t+\tau|x', t') = \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} M_n(x', t', \tau)\left(\frac{\partial}{\partial x'}\right)^n\right].\delta(x' - x'')$

- $P(x, t|x', t') - P(x, t|x', t' + \tau)$

$$\approx -\frac{\partial P(x, t|x', t')}{\partial t'}.\tau = \sum_{n=1}^{\infty} \frac{1}{n!} M_n(x', t', \tau)\left(\frac{\partial}{\partial x'}\right)^n P(x, t, |x', t')$$

Kramers-Moyal Backward Expansion

$$-\frac{\partial P(x,t|x',t')}{\partial t'} = \sum_{n=1}^{\infty} \frac{1}{\tau n!} M_n(x',t',\tau) \left(\frac{\partial}{\partial x'}\right)^n P(x,t,|x',t')$$

- $\frac{\partial P(x,t|x',t')}{\partial t'} = -L_{KM}^{\dagger}(x',t')P(x,t|x',t')$
- $L_{KM}^{\dagger}(x',t') = \sum_{n=1}^{\infty} D^{(n)}(x',t') \left(\frac{\partial}{\partial x'}\right)^n$
- As for the forward case:
- $P(x,t|x',t') = \exp\left[L_{KM}^{\dagger}(x')(t-t')\right] \cdot \delta(x-x')$
- $P(x,t|x',t') = \vec{T} \exp\left\{\int_{t'}^{t} L_{KM}^{\dagger}(x',t'')dt''\right\} \cdot \delta(x-x')$
- $\tau = t - t' \ll 1 \Rightarrow P(x,t|x',t') = \left[1 + L_{KM}^{\dagger}(x',t') \cdot \tau + \mathcal{O}(\tau^2)\right] \cdot \delta(x-x')$
- Equivalence Because of $L_{KM}^{\dagger}(x')\,\delta(x-x') = L_{KM}(x)\,\delta(x-x')$

Pawula Theorem

- If there is a finite number of D^n such that $D^n \neq 0$ then $D^n = 0$ for all $n \geq 3$.

Langevin Equation

Langevin Equation

- Friction force for a particle in a fluid: $F \propto v$
- Collision with other particles
- No deterministic equation
- Stochastic differential equation with a random force
- $m\dot{v} = -\alpha v + F(t) \rightarrow \dot{v} = -\gamma v + \Gamma(t)$
- $\gamma = \frac{\mathrm{\alpha}}{\mathrm{m}}$ and $\Gamma(\mathrm{t})$ a random distribution with the properties:

$$\langle \Gamma(t) \rangle = 0 \quad \text{and} \quad \langle \Gamma(t)\Gamma(t') \rangle = q\delta(t - t')$$

- Spectral distribution of $\Gamma(t)$:

$$S(\omega) = 2 \int e^{-i\omega t} \langle \Gamma(t)\Gamma(0) \rangle \,\mathrm{d}t = 2 \int e^{-i\omega t} q\delta(t)\mathrm{d}t = 2q$$

$\Rightarrow$ white noise

Langevin Equation

$$\dot{v} = -\gamma v + \Gamma(t) \rightarrow v(t) = v_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-t')}\Gamma(t')\mathrm{d}t'$$

Calculating the correlations:

$$\langle v(t_1)v(t_2)\rangle = v_0^2 e^{-\gamma(t_1+t_2)} + \int_0^{t_1}\int_0^{t_2} e^{-\gamma(t_1-t_1'+t_2-t_2')} q\delta(t'_1 - t'_2)\mathrm{d}t_1'\mathrm{d}t_2'$$

$$= v_0^2 e^{-\gamma(t_1+t_2)} + q\int_0^{\min} e^{-\gamma(t_1+t_2-2t')}dt' = v_0^2 e^{-\gamma(t_1+t_2)} + \frac{q}{2\gamma}(e^{-\gamma|t_1-t_2|} + e^{-\gamma(t_1+t_2)})$$

$\approx \frac{q}{2\gamma} e^{-\gamma|t_1-t_2|}$ in stationary state, if $\gamma t_1, \gamma t_2 \gg 1$

$$\langle E\rangle = \frac{1}{2}m\langle v^2(t)\rangle = \frac{mq}{4\gamma} = \frac{1}{2}kT \Rightarrow q = \frac{2\gamma kT}{m}$$

Mean-Squared Displacement

- Assuming sharp initial velocity and position

$$\langle (x(t)-x_0)^2\rangle = \left\langle \left[\int_0^t v(t')\mathrm{d}t'\right]^2\right\rangle = \left\langle \int_0^t v(t_1)\mathrm{d}t_1 \int_0^t v(t_2)\mathrm{d}t_2 \right\rangle = \int \int \langle v(t_1)v(t_2)\rangle\, \mathrm{d}t_1 \mathrm{d}t_2.$$

$$\langle (x(t)-x_0)^2\rangle = \left(v_0^2 - \frac{q}{2\gamma}\right)\frac{(1-e^{-\gamma t})^2}{\gamma^2} + \frac{q}{\gamma^2}t - \frac{q}{\gamma^3}(1-e^{-\gamma t})$$

$$\rightarrow \frac{q}{\gamma^2}t = 2\frac{kT}{m\gamma}t \qquad \text{for } \gamma\text{t} \gg 1 \text{ and } \langle v_0^2\rangle = \frac{q}{2\gamma}$$

For 3-dim.

$$\langle (x(t)-x_0)^2\rangle \approx 6\frac{kT}{m\gamma}t \quad \text{and } E = \frac{3}{2}kT$$

Stationary State Distribution

$$\langle \eta_{j_1}\eta_{j_2}\dots\eta_{j_{2n}}\rangle = \sum_{P_d} \sigma_{k_1\mathrm{k}_2}\,\sigma_{k_3\mathrm{k}_4}\dots\sigma_{k_{2n-1}\mathrm{k}_{2n}}$$

- At the stationary state

$$\tilde{v}(t) = \lim_{t\to\infty} v(t) = \lim_{t\to\infty} v_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-t')}\Gamma(t')\mathrm{d}t' = \int_0^\infty e^{-\gamma\tau}\Gamma(\mathrm{t}-\tau)\mathrm{d}\tau$$

- Moments:

$$\langle \tilde{v}(t)^{2n+1}\rangle = 0$$

$$\langle \tilde{v}(t)^{2n}\rangle = \int_0^\infty \dots \int_0^\infty e^{-\gamma(\tau_1+\dots+\tau_{2n})}\langle \Gamma(t-\tau_1)\dots\Gamma(t-\tau_{2n})\rangle\,\mathrm{d}\tau$$

$$= \frac{(2n)!}{n!2^n}\left[\int_0^\infty \int_0^\infty e^{-\gamma(\tau_1+\tau_2)}\,q\delta(\tau_1-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2\right]^n = \frac{(2n)!}{n!2^n}\left(\frac{q}{2\gamma}\right)^n$$

Stationary State Distribution

$$\langle \tilde{v}(t)^{2n} \rangle = \frac{(2n)!}{n!\,2^n}\left(\frac{q}{2\gamma}\right)^n$$

- $C(u) = 1 + \sum_{n=1}^{\infty} (iu)^n \frac{\langle v(t)^n \rangle}{n!} = \sum_{n=0}^{\infty} (iu)^{2n} \frac{\langle v(t)^{2n} \rangle}{(2n)!} = \exp(-\frac{u^2 q}{4\gamma})$

- $W(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(u) e^{-iuv} du = \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv^2}{2kT}\right)$

- Stationary distribution is Maxwell distribution.

Markovian and non-Markovain Processes

- If $\Gamma(t)$is δ correlated then the process is Markovian
- By introducing additional variables Non-Markovian turn Markovian
- Example:

$$\dot{\xi} = h(\xi) + \tilde{\Gamma}(t) \quad \text{with } \left\langle \tilde{\Gamma}(t)\tilde{\Gamma}(t') \right\rangle = \frac{q}{2\gamma} e^{-\gamma|t-t'|}$$

By introducing $\eta \coloneqq \dot{\xi} - h(\xi)$

$$\dot{\xi} = h(\xi) + \eta$$

$$\dot{\eta} = -\gamma\eta + \Gamma(t) \quad \text{with } \langle \Gamma(t)\Gamma(t') \rangle = q\delta(t - t')$$

Non-linear Langevin Equation

- $\dot{\xi} = h(\xi,t) + g(\xi,t)\Gamma(t)$ with $\langle\Gamma(t)\rangle = 0$ and $\langle\Gamma(t)\Gamma(t')\rangle = 2\delta(x-x')$
- In particular for the Brownian motion $g = \sqrt{\frac{q}{2}}$ and $h = -\gamma v$
- Looking for $W(\xi,t)$ or an equation for it.
- Examining the moments w.r.t. the initial value $\xi(t) = x$,i.e. $\langle(\xi(t+\tau)-x)^n\rangle$

$$\xi(t+\tau) - x = \int_t^{t+\tau} [h(\xi(t'),t') + g(\xi(t'),t')\Gamma(t')]\mathrm{d}t'$$

$$h(\xi(t'),t') = h(x,t') + h'(x,t')(\xi(t')-x) + \cdots$$

$$g(\xi(t'),t') = g(x,t') + g'(x,t')(\xi(t')-x) + \cdots$$

Non-linear Langevin Equation

$$\xi(t+\tau) - x = \int_t^{t+\tau} h(x,t')dt' + \int_t^{t+\tau} h'(x,t') \int_t^{t'} h(x,t'')dt'\, dt''$$

$$+ \int_t^{t+\tau} h'(x,t') \int_t^{t'} g(x,t'')\Gamma(t'')dt''\, dt' + \int_t^{t+\tau} g(x,t')\Gamma(t')dt'$$

$$+ \int_t^{t+\tau} g'(x,t') \int_t^{t'} h(x,t'')\Gamma(t')dt''\, dt'$$

$$+ \int_t^{t+\tau} g'(x,t') \int_t^{t'} g(x,t'')\Gamma(\mathrm{t}')\Gamma(t'')dt''\, dt' + \cdots$$

Non-linear Langevin Equation

$$\xi(t+\tau) - x = \int_t^{t+\tau} h(x,t')dt' + \int_t^{t+\tau} h'(x,t') \int_t^{t'} h(x,t'')dt'\,dt''$$

$$+ \int_t^{t+\tau} h'(x,t') \int_t^{t'} g(x,t'')\Gamma(t'')dt''\,dt' + \int_t^{t+\tau} g(x,t')\Gamma(t')dt'$$

$$+ \int_t^{t+\tau} g'(x,t') \int_t^{t'} h(x,t'')\Gamma(t')dt''\,dt'$$

$$+ \int_t^{t+\tau} g'(x,t') \int_t^{t'} g(x,t'')\Gamma(\mathrm{t}')\Gamma(t'')dt''\,dt' + \cdots$$

Kramers-Moyal Coefficients

$$D^{(n)}(x,t) = \frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\left\langle\left(\xi(t+\tau)-\xi(t)\right)^n\right\rangle\Big|_{\xi(t)=x}$$

$$\langle\xi(t+\tau)-x\rangle = \int_t^{t+\tau} h(x,t')dt' + \int_t^{t+\tau} h'(x,t')\int_t^{t'} h(x,t'')dt'\,dt''$$

$$+\int_t^{t+\tau} g'(x,t')\underbrace{\int_t^{t'} g(x,t'')2\delta(t''-t')dt''\,dt'}_{=g(x,t')}$$

$\rightarrow [h(x,t)+g'(x,t)g(x,t)]\cdot\tau \quad$ for $\tau \ll 1$

$\Rightarrow D^{(1)}(x,t) = h(x,t)+g'(x,t)g(x,t)$ the second term describes a noise induced drift

- Similarly $D^{(2)}(x,t) = g^2(x,t)$ and $D^{(n)}(x,t) = 0$ for $n \geq 3$
- In particular for Brown. motion: $D^{(1)} = -\gamma v$ and $D^{(2)} = \frac{q}{2} = \frac{\gamma k T}{m}$

Generalization to many Variables

- Non-linear equation for many variables:

$$\dot{\xi}_i = h_i(\vec{\xi}, t) + g_{ij}(\vec{\xi}, t)\Gamma_{\mathrm{j}}(\mathrm{t})$$

$$D_i^{(1)}(\vec{x}, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle \xi_i(t+\tau) - \xi_i(t) \rangle \Big|_{\xi(t)=x}$$

$$D_{ij}^{(2)}(\vec{x}, t) = \frac{1}{2} \lim_{\tau \to 0} \frac{1}{\tau} \Big\langle (\xi_i(t+\tau) - \xi_i(t)) \big(\xi_j(t+\tau) - \xi_j(t)\big) \Big\rangle \Big|_{\xi(t)=x}$$

$$D_i^{(1)}(\vec{x}, t) = h_i(\vec{x}, t) + g_{kj}(\vec{x}, t). \frac{\partial}{\partial x_k} g_{ij}(\vec{x}, t)$$

$$D_{ij}^{(2)}(\vec{x}, t) = g_{ik}(\vec{x}, t) g_{jk}(\vec{x}, t)$$

Fokker-Planck Equation

Fokker-Planck Equation

- Kramers-Moyal expansion up to the second term
- $\dot{W}(x,t) = L_{FP}(x,t)W(x,t)$ where $\mathrm{L_{FP}} = -\frac{\partial}{\partial x}D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t)$
- $\frac{\partial W}{\partial t} + \frac{\partial S}{\partial x} = 0$ with $S(x,t) = \left[D^{(1)}(x,t) - \frac{\partial}{\partial x}D^{(2)}(x,t)\right]W(x,t)$
- Continuity Equation ⇒ S can be interpreted as a probability current
- stationary state means S is constant, vanishing boundaries means $S = 0$
- For the Brownian motion $D^{(1)} = -\gamma v$ and $D^{(2)} = \frac{q}{2} = \frac{\gamma kT}{m}$

$$\Rightarrow S = \left(-\gamma v - \frac{\gamma kT}{m}\frac{\partial}{\partial v}\right)W_{st}(v) = 0 \rightarrow W_{st}(v) = \sqrt{\frac{m}{2\pi kT}}\exp\left(-\frac{mv^2}{2kT}\right)$$

Fokker-Planck Equation

- Transition probability for small times τ:

$$P(x, t' + \tau | x', t') \approx \exp\left[-\frac{\partial}{\partial x} D^{(1)}(x,t)\tau + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t)\tau\right] . \delta(x' - x)$$

- Applying the Fourier-transformation we will obtain:

$$\rightarrow \frac{1}{2\pi} \int \exp\left[-iuD^{(1)}(x,t)\tau - u^2 D^{(2)}(x,t)\tau + iu(x - x')\right] du$$

$$= \frac{1}{2\sqrt{\pi D^{(2)}(x,t)\tau}} \exp\left(-\frac{\left[x - x' - D^{(1)}(x,t)\tau\right]^2}{4D^{(2)}(x,t)\tau}\right)$$

N-Dimensional Fokker-Planck Equation

- Every derivation is similar to the one dimensional case of FP and

$$W(\vec{x}, t' + \tau) = \int P\left(\vec{x}, t' + \tau \middle| \overrightarrow{x'}, t'\right) . W\left(\overrightarrow{x'}, t'\right) \mathrm{d}^N x'$$

$$\frac{\partial W(\vec{x}, t)}{\partial t} = \sum_{\nu=1}^{\infty} \frac{(-\partial)^{\nu}}{\partial x_{j_1} \dots \partial x_{j_\nu}} D^{(\nu)}_{j_1 \cdots j_\nu}(\vec{x}, t) \,. W(x, t)$$

with

$$D^{(\nu)}_{j_1 \cdots j_\nu}(\vec{x}, t) = \lim_{\tau \to 0} \frac{1}{\tau} \frac{1}{\nu!} \int \left(y_{j_1} - x_{j_1}\right) \dots \left(y_{j_\nu} - x_{j_\nu}\right) P(\vec{y}, t + \tau | \vec{x}, t) d^N y$$

N-Dim. FP-Equation: Transition Probability

- Initial condition as before: $W(x', t') = P(x, t'|x', t') = \delta(x - x')$

- $\frac{\partial}{\partial t} P(\vec{x}, t|\vec{x}', t') = L_{KM}(\vec{x}, t) P(\vec{x}, t|\vec{x}', t')$
- $\frac{\partial}{\partial t'} P(\vec{x}, t|\vec{x}', t') = -L_{KM}^{\dagger}(\vec{x}', t') P(\vec{x}, t|\vec{x}', t')$

- $L_{KM}(\vec{x}, t) = \sum_{\nu=1}^{\infty} \frac{(-\partial)^{\nu}}{\partial x_{j_1} \dots \partial x_{j_\nu}} D_{j_1 \dots j_\nu}^{(\nu)}(\vec{x}, t)$
- $L_{KM}^{\dagger}(\vec{x}', t') = \sum_{\nu=1}^{\infty} D_{j_1 \dots j_\nu}^{(\nu)}(\vec{x}, t) \frac{\partial^{\nu}}{\partial x_{j_1} \dots \partial x_{j_\nu}}$

- $\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i} S_i = 0$ where $S_i = D_i^{(1)} W - \frac{\partial}{\partial x_j} D_{ij}^{(2)} \mathrm{W}$

Examples of Fokker Planck Equation

- One-dim. Brownian motion in a potential

$$\frac{\partial W(x,v,t)}{\partial t} = \left\{-\frac{\partial}{\partial x}v + \frac{\partial}{\partial v}[\gamma v + \phi'(x)] + \frac{\gamma kT}{m}\frac{\partial^2}{\partial v^2}\right\} W(x,v,t)$$

- Three-dim. Brownian motion in a potential

$$\frac{\partial W(\vec{x},\vec{v},t)}{\partial t} = \left\{-\nabla_x v + \nabla_v \left[\gamma v - \frac{F}{m}\right] + \frac{\gamma kT}{m}\Delta_v\right\} W(\vec{x},\vec{v},t)$$

Wiener and Ornstein-Uhlenbeck Process

- Wiener Process: $D^{(1)} = 0$ and $D^{(2)} = D = const.$
- $\frac{\partial}{\partial t} P(x, t|x', t') = D \frac{\partial^2}{\partial x^2} P(x, t|x', t')$ and $P(x, t'|x', t') = \delta(x - x')$
- $P(x, t|x', t') = \frac{1}{\sqrt{4\pi D(t' - t)}} \exp\left(-\frac{(x - x')^2}{4D(t - t')}\right)$
- Ornstein-Uhlenbeck Process: $D^{(1)} = -\gamma x$ and $D^{(2)} = D = const.$
- $\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial x}(xP) + D \frac{\partial^2}{\partial x^2} P$
- Moving to Fourier space

Wiener and Ornstein-Uhlenbeck Process

- $\frac{\partial}{\partial t}\tilde{P} = -\gamma k \frac{\partial}{\partial k}\tilde{P} - Dk^2\tilde{P}$ with initial condition $\tilde{P}(k,t'|x',t') = e^{-ikx'}$
- With the help of methods of characteristics we get

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial}{\partial x}(xP) + D\frac{\partial^2}{\partial x^2} P$$

- $\tilde{P}(k,t'|x',t') = \exp\left[-ikx'e^{-\gamma(t-t')} - \frac{Dk^2\left(1-e^{-2\gamma(t-t')}\right)}{2\gamma}\right]$
- Transforming back:
- $P(x,t|x',t') = \sqrt{\frac{\gamma}{2\pi D\left(1-e^{-2\gamma(t-t')}\right)}}\exp\left[-\frac{\gamma\left(x-e^{-\gamma(t-t')}x'\right)^2}{2D\left(1-e^{-\gamma(t-t')}\right)}\right]$ valid for all γ.
- $W_{st}(x) = \sqrt{\frac{\gamma}{2\pi D}}\exp\left(-\frac{\gamma x^2}{2D}\right)$ valid for positive γ.

Stationary Solutions and Asymptotic Behavior

Stationary Solution

- $\frac{\partial W}{\partial t} = 0 \Rightarrow \frac{\partial S}{\partial x} = 0 \Rightarrow S = \text{const.}$
- $\Rightarrow D^{(1)}(x) W_{st}(x) = \frac{D^1(x)}{D^2(x)} D^{(2)}(x) W_{st}(x) = \frac{\partial}{\partial x} D^{(2)}(x) W_{st}(x) - S$
- $W_{st}^{h}(x) = \frac{N}{D^2} \exp\left(\int \frac{D^{(1)}(x')}{D^{(2)}(x')} dx'\right) = N e^{-\Phi(x)}$

  with $\Phi(x) = \ln\left(D^{(2)}(x)\right) - \int \frac{D^{(1)}(x')}{D^{(2)}(x')} dx'$
- $W_{st}^{inh}(x) = -S e^{-\Phi(x)} \int \frac{e^{\Phi(x')}}{D^2(x')} \mathrm{d}x' \rightarrow W_{st} = N e^{-\Phi(x)} - S e^{-\Phi(x)} \int \frac{e^{\Phi(x')}}{D^2(x')} \mathrm{d}x'$

Master Equation and Detailed Balance

- We begin with the master equation:

$$\dot{W}_n = \sum_m [w(m \to n) W_m - w(n \to m) W_n]$$

- For stationary state:

$$\sum_m w(m \to n) W_m = \sum_m w(n \to m) W_n$$

- Detailed Balance (sufficient condition):

$$w(m \to n) W_m = w(n \to m) W_n$$

- Odd and even variables: If a variable changes its sign under time reversal it is called odd and if it remains the same it is called even.

Detailed Balance for even variables

$$\frac{\partial W(x,t)}{\partial t} = \int [w(x' \to x)W(x',t) - w(x \to x')W(x,t)]d^N x'$$

the transition rate and the transition probability have the relation:

$$w(x' \to x) = \frac{\mathrm{d}}{\mathrm{d}\tau} P(x,\tau|x',0)|_{\tau=0} = L_{FP}(x)\delta(x-x')$$

The detailed Balance for the continuous case can be written as:

$$w(x' \to x)W_{st}(x') = w(x \to x')W_{st}(x)$$

$$L_{FP}(x)\delta(x-x')W_{st}(x') = L_{FP}(x')\delta(x-x')W_{st}(x)$$

$$L_{FP}(x)W_{st}(x)\delta(x-x') = W_{st}(x)L_{FP}^{\dagger}(x)\delta(x-x')$$

$$L_{FP}(x)W_{st}(x) = W_{st}(x)L_{FP}^{\dagger}(x)$$ Operator equation

$L_{KM}^{\dagger}(x')\,\delta(x-x')$
$= L_{KM}(x)\,\delta(x-x')$

Detailed Balance for odd variables

- Variables that transform as $x \rightarrow -x$ under time reversal
- Gain and Loss term as Forward and Backward in time
- Following Detailed Balance:

$w(x' \rightarrow x)W_{st}(x') = w(-x \rightarrow -x')W_{st}(-x)$ and $W_{st}(x) = W_{st}(-x)$

- Similar to even variables: $L_{FP}(x)W_{st}(x) = W_{st}(-x)L_{FP}^{\dagger}(-x)$

Fokker-Planck Equation as Schrödinger Equation

$$\Phi(x) = \ln(D^2(x)) - \int \frac{D^1(x')}{D^2(x')} dx'$$

- Separation ansatz for the probability:

$$W(x,t) = \varphi(x) e^{-\lambda t} \text{ with } L_{FP}\varphi(x) = -\lambda\varphi(x)$$

- Transform L_{FP} to a Hermitian operator:

$$L_{FP} \rightarrow L = e^{\Phi(x)/2} L_{FP} e^{-\Phi(x)/2}$$

with eigenfuntions $\psi(x) = e^{\Phi(x)/2}\varphi(x)$ and same eigenvalues

$$L = \frac{\partial}{\partial x} D^{(2)} \frac{\partial}{\partial x} - V \quad \text{with } V = \frac{1}{4} \frac{\left(\frac{\partial D^{(2)}}{\partial x} - D^{(1)}\right)^2}{D^{(2)}} + \frac{1}{2} \frac{\partial D^{(1)}(x)}{\partial x} - \frac{1}{2} \frac{\partial^2 D^{(2)}(x)}{\partial^2 x}$$

Fokker-Planck Equation as Schrödinger Equation

- There is always a transformation such that $D^{(2)} = D = const$.

$$L \to L' = D\frac{\partial}{\partial x^2} - V' \quad \text{with V} = \frac{1}{4D}\left((D^{(1)}(x)\right)^2 - \frac{1}{2}\frac{\partial D^{(1)}(x)}{\partial x}$$

- With $t_{schr} = -i\hbar t$ and $m_{schr} = \frac{\hbar^2}{2D}$ Fokker-Planck would be equivalent to Schrödinger.

References

References

- Lemons, Don S., and Anthony Gythiel. "Paul langevin’s 1908 paper “on the theory of brownian motion”[“sur la théorie du mouvement brownien,” cr acad. sci.(paris) 146, 530–533 (1908)]." *American Journal of Physics* 65.11 (1997): 1079-1081.
- Von Smoluchowski, Marian. "Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen." *Annalen der Physik* 326.14 (1906): 756-780.
- Einstein, Albert. "Zur Theorie der Brownschen Bewegung." *Annalen der Physik* 324.2 (1906): 371-381.
- Fokker, Adriaan Daniël. "Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld." *Annalen der Physik* 348.5 (1914): 810-820.
- Planck, V. M. "Über einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie." *Sitzungsberichte der* (1917).
- Konno, Hidetoshi, and Keiichi Saito. "Derivation of a Generalized Fokker-Planck Equation for a Non-Markoffian System with the Time Reversal Asymmetry." *Progress of Theoretical Physics* 60.5 (1978): 1593-1595.
- Kramers, Hendrik Anthony. "Brownian motion in a field of force and the diffusion model of chemical reactions." *Physica* 7.4 (1940): 284-304.
- Risken, Hannes. "Fokker-planck equation." *The Fokker-Planck Equation*. Springer, Berlin, Heidelberg, 1996. 63-95.