# Langevin Equation and Fokker-Planck Equation

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**Abstract**

The original aim of the Fokker-Planck and Langevin equations is to describe the Brownian motion. But in general, they could also explain the behavior of a system in presence of a random noise and its evolution toward a stationary state. Therefore, they could also be applied to thermalization processes of non-equilibrium systems.

In this report we will introduce these equations, demonstrate the analytical solutions to some simple examples, present some solution methods and show some modern developments of them at the end.

# 1 Introduction

Langevin equation and Fokker-Planck equation were developed in the beginning of the twentieth century as tools to study the Brownian motion. But the history of the Brownian motion began in the nineteenth century, in 1827, when Robert Brown, a botanist, observed the random movement of little particles in water. In 1905 and 1906 Einstein and Von Smoluchowski explained the physical meaning behind Brown's observations and could calculate the mean squared displacement of the Brownian particles in the stationary state. Einstein also showed that the value of the mean squared displacement is related to a diffusion coefficient and so was able to set a diffusion equation for the probability density of the Brownian particle. In 1908 Langevin constructed a random Markovian force, the Langevin force, that described the random collisions and interaction of the particle and with its help could set up an stochastic differential equation that could serve as the equation of motion for the Brownian particle. Fokker in 1914 and Planck in 1917 introduced a partial differential equation that governs the evolution of the probability density for the velocity of the Brownian particle under the influence of a diffusion generated by the interactions with the particles of the fluid and under the influence of a drift generated by the friction in the fluid. Also in 1930 Ornstein and Uhlenbeck generalized the Brownian process into a process, which is named the Ornstein-Uhlenbeck process after them, with a constant diffusion coefficient and a linear drift term. Kramers' paper from 1940 and Moyal's paper from 1949 introduced the Kramers-Moyal expansion as a method to describe the time evolution of a time-dependent probability distribution and that is the approach demonstrated in this report. Finally, in 1996 Debbasch, Mallick and Rivet constructed the relativistic form of the Ornstein-Uhlenbeck

process with applications in examination of plasma thermalization or thermalization of astrophysical systems.

In this report we begin by introducing the Kramers-Moyal expansion and the mathematical tools, that we will need in the report. The next sections will introduce the Langevin and Fokker-Planck equations and some of the physical results of the papers mentioned above will be shown. In the last section, as an example of more recent developments of the field, the relativistic Ornstein-Uhlenbeck will be depicted.

In this report we have used the Stratonovich-Fisk discretization rule for random forces. A comprehensive overview of different discretization rules could be found in the appendix A of [1].

## 2 Mathematical Tools

A very useful mathematical tool for investigating the Fokker-Planck equations was introduced by Kramers and Moyal. This method, named Kramers-Moyal expansion, is an expansion of the time development of a time-dependent probability distribution in terms of the moments of the transition probability. In the first step the definition of the transition probability will be used to express the relation between the probability density at the time $t$ and the probability density at the time $t + \tau$.

$$W(x, t + \tau) = \int P(x, t + \tau | x', t) \cdot W(x', t) dx' \tag{1}$$

where the integral is carried over the space of all possible $x$. As for every probability, one can also define moments for the transition probability, with the only difference that the moments for a transition possibility would depend on three arguments, namely the initial time, $t$, the initial value of the random variable $x'$ and the passed time $\tau$:

$$M_n(x', t, \tau) = \langle (x(t + \tau) - x(t)) \rangle|_{x'=x(t)} = \int (x - x')^n P(x, t + \tau | x', t) \mathrm{d}x' \tag{2}$$

In the next step the transition probability will be expressed with the help of the Dirac delta which substitutes $x$ for $y$.

$$P(x, t + \tau | x', t) = \int \delta(x - y) P(y, t + \tau | x', t) \mathrm{d}x \tag{3}$$

Next, one can substitute the Dirac delta in the above equation with its Taylor expansion around the point $x' - x$:

$$P(x, t + \tau | x', t) = \sum_{n=0}^{\infty} \int \frac{(y - x')^n}{n!} \left( -\frac{\partial}{\partial x} \right)^n \delta(x' - x) P(x, t + \tau | x', t) \mathrm{d}x' \tag{4}$$

One is here allowed to swap the integral and the sum because the result is convergent and equal to $P(x, t + \tau | x', t)$. Computing the integral for each $n$ results in:

$$P(x, t + \tau | x', t) = \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n M_n(x, t, \tau) \right] \delta(x' - x)$$

$$= \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n M_n(x, t, \tau) \right] \delta(x' - x) \tag{5}$$

Inserting this into the integral in (1) and bringing the first term to the left hand side of the equation results in,

$$W(x, t + \tau) - W(x, t) = \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n M_n(x, t, \tau) W(x, t) \tag{6}$$

Also for $\tau \ll 1$ the right hand side could be written as:

$$W(x, t + \tau) - W(x, t) = \frac{\partial W(x, t)}{\partial t} \tau + O(\tau^2) \tag{7}$$

Putting the two right hand sides of equations (6) and (7) in one equation, dividing both sides by $\tau$ and taking the limit of $\tau \to 0$ gives:

$$\frac{\partial W(x, t)}{\partial t} = \lim_{\tau \to 0} \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial x} \right)^n \frac{M_n(x, t, \tau)}{\tau \cdot n!} W(x, t)$$

$$= \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial x} \right)^n D^{(n)}(x, t) W(x, t) = L_{KM} W(x, t) \tag{8}$$

where the Kramers-Moyal coefficients and the Kramer-Moyal operator are defined as:

$$D^{(n)}(x, t) = \lim_{\tau \to 0} \frac{M_n(x, t, \tau)}{\tau \cdot n!} \tag{9}$$

$$L_{KM} = \sum_{n=1}^{\infty} \left( -\frac{\partial}{\partial x} \right)^n D^{(n)}(x, t) \tag{10}$$

Equation (8) is a differential equation, that governs the time development of the probability distribution.

Now some possible solutions of the equation for different cases of $L_{KM}$ will be discussed. Before introducing the examples one can make the assumption of a sharp position, $x'$, at the initial time which means:

$$W(x, t') = \delta(x - x') \Rightarrow W(x, t' + \tau) = P(x, t' + \tau | x', t') \tag{11}$$

1. The first simple case is the case of a time-independent Kramers-Moyal operator. In this case the solution is given by the following expression:

$$P(x, t | x', t') = \exp \left[ L_{KM}(x)(t - t') \right] \delta(x - x') \tag{12}$$

where the Dirac delta comes from the initial condition at the time $t'$.

2. Another case, which is easily solvable is the case of $\tau = t - t' \ll 1$. In this case the transition probability can be written as:

$$P(x, t | x', t') = \left[ 1 + L_{KM}(x, t) \cdot \tau + O(\tau^2) \right] \delta(x - x') \tag{13}$$

## 2.1 Backward Kramers-Moyal Expansion

In this expansion so far, the Kramers-Moyal operator was constructed only through derivatives with respect to the later time $t$ and the values of the random variable at that time, i.e. $x$. Because of that this expansion is called the forward Kramers-Moyal expansion. One can also imagine a similar expansion of the transition probability in a later time with the help of the derivatives with respect to the earlier time $t'$ and the initial position $x'$, which is called the backward expansion. The ansatz for this expansion would be:

$$P(x, t | x', t') = \int P(x, t | x'', t' + \tau) \cdot P(x'', t' + \tau | x, t') dx'' \tag{14}$$

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The derivation of the backward expansion is similar to the forward expansion. The transition probability in terms of its moments can be expressed as

$$P(x'', t' + \tau | x', t') = \left[ 1 + \sum_{n=1}^{\infty} \frac{M_n(x', t', \tau)}{n!} \left( \frac{\partial}{\partial x'} \right)^n \right] \delta(x - x'') \tag{15}$$

The only difference between this expression and the expression of (5) are derivatives with respect to $x$ that have been substituted with derivatives with respect to $x'$. The difference in the minus sign comes from the inner derivative of the function inside the Dirac delta. Similar to the forward case one arrives at the following equation for the transition probability:

$$\frac{\partial}{\partial t'} P(x, t | x', t') = -L_{KM}^{\dagger}(x', t') P(x, t | x', t') \tag{16}$$

with the backward Kramers-Moyal operator or conjugate Kramers-Moyal operator being defind as:

$$L_{KM}^{\dagger}(x', t') = \sum_{n=1}^{\infty} D^{(n)}(x', t') \left( \frac{\partial}{\partial x'} \right)^n \tag{17}$$

And just like the forward case two of the simple solutions of the equation would be:

1. For the case of time-independent Kramers-Moyal operator:

$$P(x, t | x', t') = \exp \left[ L_{KM}^{\dagger}(x')(t - t') \right] \delta(x - x') \tag{18}$$

2. And for the case of $\tau = t - t' \ll 1$:

$$P(x, t | x', t') = \left[ 1 + L_{KM}^{\dagger}(x', t') \cdot \tau \right] \delta(x - x') \tag{19}$$

The equivalence of two expressions for the transition probability from (12) and (13) to (18) and (19) respectively follows from the fact that:

$$L_{KM}^{\dagger}(x') \delta(x - x') = L_{KM}(x) \delta(x - x') \tag{20}$$

In the next section the Langevin equation will be introduced and at the end of the section the tools developed here will be applied to the langevin equation. [2, 3, 4]

# 3   Langevin Equation

## 3.1   Brownian Motion

The main aim of the Langevin equation is to explain the Brownian motion and its random behavior. Without taking the thermal fluctuation of the particles into account a particle in a fluid is subjected only to a friction force that is proportional to the velocity of the particle. The solution to the equation of motion is an exponential fall of the velocity toward zero.

But in the nature due to the collisions and interactions with other particles that would not be the case. Investigating each single interaction and writing a proper equation of motion for each interaction is an impossible act due to the huge number of the variables involved in it. This means one has to introduce a force called the Langevin force to take care of all of these interactions and their random nature. After introducing the random force the equation of motion of the particle would become the following stochastic differential equation:

$$m\dot{v} = -\alpha v + F(t) \tag{21}$$

where $F(t)$ is the random force, $m$ and $v$ the mass and the velocity of the particle and $\alpha$ the constant of friction.

Dividing both side by $m$ results in:

$$\dot{v} = -\gamma v + \Gamma(t) \tag{22}$$

with $\gamma = \frac{\alpha}{m}$ and $\Gamma(t) = \frac{F(t)}{m}$.

Two further additional assumptions can be imposed on the random Langevin force. The first one follows from assuming that the Langevin force does not have any spatial preferences and the second one follows from assuming that the process described by the Langevin force is a Markovian one, i.e. the force at each point in time depends only on the immediate point before that time. Mathematically these two assumptions can be expressed as:

$$\langle \Gamma(t) \rangle = 0 \tag{23}$$

$$\langle \Gamma(t)\Gamma(t') \rangle = q\delta(t - t') \tag{24}$$

where $q$ is a constant that will be determined later. It is worth mentioning that the spectral distribution of $\Gamma(t)$ is given as:

$$S(\omega) = 2 \int e^{-i\omega t} \langle \Gamma(t)\Gamma(0) \rangle = 2q \int e^{-i\omega t} \delta(t)\mathrm{d}t = 2q \tag{25}$$

As one can see here the distribution does not depend on the frequency $\omega$ and that is why the background noise generated by such a Langevin force is called the white noise.

Now back to (22), the equation can be solved by:

$$v(t) = v_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-t')}\Gamma(t')\mathrm{d}t' \tag{26}$$

where $v_0$ is the sharp velocity at the initial time. But this expression is not deterministic for the velocity and gives it rather as a random variable.

The stochastic properties of the velocity can be calculated from the correlation functions of the velocity.

$$\langle v(t) \rangle = v_0 e^{-\gamma t} \tag{27}$$

$$\langle v(t_1)v(t_2) \rangle = v_0 e^{-\gamma(t_1+t_2)} + \int_0^{t_1} \int_0^{t_2} e^{-\gamma(t_1-t_1'+t_2-t_2')}$$

$$= v_0^2 e^{-\gamma(t_1+t_2)} + \frac{q}{2\gamma} \left( e^{-\gamma|t_1-t_2|} + e^{-\gamma(t_1+t_2)} \right) \tag{28}$$

Taking $\gamma t_1, \gamma t_2 \gg 1$ in order to calculate the correlation function near the stationary state, one arrives at

$$\langle v(t_1)v(t_2) \rangle \approx \frac{q}{2\gamma} e^{-\gamma|t_1-t_2|} \tag{29}$$

Finally, the mean kinetic energy at the stationary state can be calculated as

$$\langle E \rangle = \frac{1}{2}m\langle v^2(t) \rangle = \frac{mq}{4\gamma} \tag{30}$$

Comparing the above expression with the known equation $\langle E \rangle = \frac{1}{2}k_B T$, the constant $q$ could be written as:

$$q = \frac{2\gamma k_B T}{m} \tag{31}$$

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Another important quantity in studying the Brownian movement is the mean squared displacement of the particle. This quantity could be derived from the correlation function of the velocity distribution. Assuming a sharp velocity $v_0$ and sharp position $x_0$ at the initial time the mean squared displacement can be expressed as:

$$\langle (x(t) - x_0)^2 \rangle = \int_0^t \int_0^t \langle v(t_1)v(t_2) \rangle dt_1 dt_2$$
$$= \left( v_0 - \frac{q}{2\gamma} \right) \frac{(1 - e^{-\gamma t})^2}{\gamma^2} + \frac{q}{\gamma^2} t - \frac{q}{\gamma^3}(1 - e^{-\gamma t}) \tag{32}$$

The mean squared displacement around the stationary state can be derived by taking the limit of large $t$ after which the leading order would be the second term of (32).

$$\langle (x(t) - x_0)^2 \rangle \approx \frac{q}{\gamma^2} t = 2\frac{k_B T}{m\gamma} t \tag{33}$$

This result is in accordance with the result obtained by Einstein in his paper of 1906. The value of the velocity in the stationary state distribution is given by:

$$\tilde{v}(t) = \lim_{t \to \infty} v(t) = \int_0^\infty e^{-\gamma \tau} \Gamma(t - \tau) \tag{34}$$

From this the moments of the velocity distribution can be given as:

$$\langle \tilde{v}(t)^{2n+1} \rangle = 0 \tag{35}$$

$$\langle \tilde{v}(t)^{2n} \rangle = \frac{(2n)!}{n!2^n} \left( \frac{q}{2\gamma} \right)^n \tag{36}$$

Now that all of the moments are given the characteristic function can be calculated through:

$$C(u) = 1 + \sum_{n=1}^{\infty} (iu)^n \frac{\langle v(t)^n \rangle}{n!} = \exp\left( -\frac{u^2 q}{4\gamma} \right) \tag{37}$$

The probability density is given as the inverse Fourier transformation of the characteristic function which means:

$$W(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(u) e^{-iuv} du = \sqrt{\frac{m}{2\pi k_B T}} \exp\left( -\frac{mv^2}{2k_B T} \right) \tag{38}$$

And this is as expected the one dimensional Maxwellian distribution.[4, 5, 6, 7]

## 3.2 Non-linear Langevin Equation

The next important generalization of the Langevin equation is the generalization to a non-linear equation. In this case the friction term and the prefactor of the Langevin force become arbitrary functions of $v$ and $t$. Renaming $v$ to $\xi$, the equation is then given as:

$$\dot{\xi} = h(\xi, t) + g(\xi, t)\Gamma(t) \tag{39}$$

with $\langle \Gamma(t) \rangle = 0$ and $\langle \Gamma(t)\Gamma(t') \rangle = 2\delta(t - t')$. In the case of the Brownian motion $g = \sqrt{\frac{q}{2}}$ is a constant and $h = -\gamma v$ is a linear function of $v$. In order to calculate the time-development of the velocity distribution density one needs to calculate its Kramers-Moyal equation, for which the Kramers-Moyal

coefficients could to be calculated with the help of (9).

Through complicated calculation the Kramers-Moyal coefficients would be given as:

$$D^{(1)}(\xi, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle (\xi(t + \tau) - \xi(t)) \rangle \Big|_{\xi(t) = \xi} = h(\xi, t) + g'(\xi, t)g(\xi, t) \tag{40}$$

$$D^{(2)}(\xi, t) = \lim_{\tau \to 0} \frac{1}{2\tau} \langle (\xi(t + \tau) - \xi(t))^2 \rangle \Big|_{\xi(t) = \xi} = g^2(\xi, t) \tag{41}$$

$$D^{(n)}(\xi, t) = 0 \quad \text{for all} \quad n \geq 3 \tag{42}$$

where the prime denotes the derivative with respect to $\xi$. In particular for the Brownian motion, $D^{(1)}(v, t) = -\gamma v$ and $D^{(2)}(v, t) = \frac{q}{2} = \frac{\gamma k_B T}{m}$.

It is worth noting that a $\xi$-dependent noise term induces a drift in the probability distribution of the velocity even though $\langle \Gamma(t) \rangle = 0$. This contribution to the drift coefficient is related to the diffusion coefficient by:

$$D^{(1)}_{noise}(\xi, t) = \frac{1}{2} \frac{\partial}{\partial \xi} D^{(2)}(\xi, t) \tag{43}$$

With the help of these coefficients the equation of motion for the probability distribution can be written as:

$$\frac{\partial}{\partial t} W(\xi, t) = -\frac{\partial}{\partial \xi} \{[h(\xi, t) + g'(\xi, t)g(\xi, t)] W(\xi, t)\} + \frac{\partial^2}{\partial \xi^2} \{g^2(\xi, t)W(\xi, t)\} \tag{44}$$

This equation is a special form of Fokker-Planck equations, which will be introduced in the next section.

Lastly, the generalization of the Langevin equation to the multi-dimensional case will be introduced. This case will be used later in the last section to derive the relativistic Langevin equation.

The non-linear multi-dimensional Langevin equation has the form:

$$\dot{\xi}_i = h_i(\vec{\xi}, t) + g_{ij}(\vec{\xi}, t)\Gamma_j(t) \tag{45}$$

where the random force has the properties:

$$\langle \Gamma_i(t) \rangle = 0 \tag{46}$$

$$\langle \Gamma_i(t_1)\Gamma_j(t_2) \rangle = \delta(t_1 - t_2)\delta_{ij} \tag{47}$$

The corresponding Kramers-Moyal coefficients are then defined and computed analogously to the one-dimensional case as:

$$D^{(1)}_i(\xi, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle (\xi_i(t + \tau) - \xi_i(t)) \rangle \Big|_{\xi(t) = \xi} = h_i(\xi, t) + g_{kj}(\xi, t)\frac{\partial}{\partial \xi_k}g_{ij}(\xi, t) \tag{48}$$

$$D^{(2)}(\xi, t) = \lim_{\tau \to 0} \frac{1}{2\tau} \langle (\xi_i(t + \tau) - \xi_i(t)) \cdot (\xi_j(t + \tau) - \xi_j(t)) \rangle \Big|_{\xi(t) = \xi} = g_{ik}(\xi, t)g_{jk}(\xi, t) \tag{49}$$

$$D^{(n)}_{i_1, \dots, i_n}(\xi, t) = 0 \quad \text{for all} \quad n \geq 3 \tag{50}$$

The corresponding Kramers-Moyal equation for this coefficients will be given in equation (55) and they will both be applied in order to derive the relativistic Ornstein-Uhlenbeck equation in the last section.[4, 6]

# 4  Fokker-Planck Equation

Fokker-Planck equation is a special case of the Kramers-Moyal expansion, in which only the first two terms appear in the equation of motion of the probability density.

$$\dot{W}(x,t) = L_{FP}W(x,t) \tag{51}$$

where the Fokker-Planck operator is given by:

$$L_{FP} = -\frac{\partial}{\partial x}D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t) \tag{52}$$

The first term is called the drift term with $D^{(1)}$ being the drift coefficient and the second term is called the diffusion term which makes $D^{(2)}$ the diffusion coefficient.

It may seem that the Fokker-Planck equations are a very particular case of the Kramers-Moyal expansions, but they are in fact the only finite cases of them. According to Pawula theorem, there are either infinite number of Kramers-Moyal coefficients such that $D^{(n)} \neq 0$ or otherwise $D^{(n)} = 0$ for all $n \geq 3$, which makes the Fokker-Planck equation the only finite case.

Another way of expressing the Fokker-Planck equations is through the following identity:

$$\frac{\partial W}{\partial t} + \frac{\partial S}{\partial x} = 0 \tag{53}$$

with $S$ being defined as:

$$S(x,t) = \left[ D^{(1)}(x,t) - \frac{\partial}{\partial x}D^{(2)}(x,t) \right] W(x,t) \tag{54}$$

The equation (53) looks very similar to the continuity equation for the probability density $W$. This makes the interpretation of $S$ as the corresponding probability current a reasonable one.

From (54) one can also read that at the stationary state the probability current $S$ should be a constant. The value of the constant depends on the boundary conditions imposed on the probability distribution. If the boundaries are moved to infinity $x = \pm\infty$ or the probability density is bounded between two walls such that the probability density and its derivative are zero at the walls, then the probability current is uniformly zero at the stationary state, $S_{st} = 0$.

For the Brownian motion and by setting the boundaries such that $S_{st} = 0$, the probability density at the stationary state can be calculated from:

$$S = \left( -\gamma v - \frac{\gamma k_B T}{m}\frac{\partial}{\partial v} \right)W_{st}(v) = 0 \Rightarrow W_{st}(v) = \sqrt{\frac{m}{2\pi k_B T}}\exp\left( -\frac{mv^2}{2k_B T} \right) \tag{55}$$

The prefactor is obtained from the normalization condition on the probability distribution. This is exactly the same Maxwellian distribution that was calculated in the last section, but this derivation did not need a calculation of all moments of the probability distribution and was much easier thanks to the Pawula theorem and the Fokker-Planck equation.

The generalization of the Fokker-Planck equation to many-dimensional case is similar to the one-dimensional case:

$$\frac{\partial W(\vec{x},t)}{\partial t} = \left[ -\frac{\partial}{\partial x_i}D_i^{(1)}(\vec{x},t) + \frac{\partial^2}{\partial x_i \partial x_j}D_{ij}^{(2)}(\vec{x},t) \right] W(\vec{x},t) \tag{56}$$

where the Einstein summation convention is implied. The drift coefficients for the multi-dimensional case will form a vector and the diffusion coefficients will form a matrix. Aside from the three-dimensional Brownian motion the multi-dimensional case will find application in studying the one-dimensional Brownian motion in presence of a potential, in which case the equation will be given

as:

$$\frac{\partial W(v, x, t)}{\partial t} = \left\{ -\frac{\partial}{\partial x} v + \frac{\partial}{\partial v} [\gamma v + \phi'(x)] + \frac{\gamma k_B T}{m} \frac{\partial^2}{\partial v^2} \right\} W(v, x, t) \tag{57}$$

where the potential is given by $\phi(x)$.

The above equation can be easily generalized to the three dimensional case upon which one will obtain:

$$\frac{\partial W(\vec{x}, \vec{v}, t)}{\partial t} = \left\{ -\nabla_x v + \nabla_v \left[ \gamma v - \frac{F}{m} \right] + \frac{\gamma k_B T}{m} \Delta v \right\} W(\vec{x}, \vec{v}, t) \tag{58}$$

where F is the force acting upon the particle by the potential. [8, 9, 4]

## 4.1 Wiener and Ornstein-Uhlenbeck Processes

At the end some of the most simple examples of the Fokker-Planck equation and their solutions will be presented. The simplest case is called the Wiener process with the following condition:

$$D^{(1)} = 0 \quad \text{and} \quad D^{(2)} = D = \text{const.}$$

$$\Rightarrow \frac{\partial}{\partial t} P(x, t | x', t') = D \frac{\partial^2}{\partial t^2} P(x, t | x', t') \tag{59}$$

with the Dirac delta distribution as the initial condition:

$$P(x, t' | x', t') = \delta(x - x') \tag{60}$$

The solution will be given by

$$P(x, t | x', t') = \frac{1}{\sqrt{4\pi D(t' - t)}} \exp\left( -\frac{(x - x')^2}{4D(t - t')} \right) \tag{61}$$

Another more complicated process is the Ornstein-Uhlenbeck process, which can also be used to describe the Brownian motion. In this case the coefficients are given by:

$$D^{(1)} = -\gamma x \quad D^{(2)} = D = \text{const.}$$

$$\Rightarrow \frac{\partial}{\partial t} P(x, t | x', t') = \gamma \frac{\partial}{\partial x} (x \cdot P(x, t | x', t')) D \frac{\partial^2}{\partial x^2} P(x, t | x', t') \tag{62}$$

In order to be solved, the equation should be moved to the Fourier space. In the Fourier space the equation and its Dirac delta initial condition will take the form:

$$\frac{\partial}{\partial t} \tilde{P}(k, t | x', t') = -\gamma k \frac{\partial}{\partial k} \tilde{P}(k, t | x', t') - Dk^2 \tilde{P}(k, t | x', t') \tag{63}$$

$$\tilde{P}(k, t' | x', t') = e^{-ikx'} \tag{64}$$

The solution of these types of partial differential equations is given through the methods of characteristics:

$$\tilde{P}(k, t | x', t') = \exp\left[ -ikx' e^{-\gamma(t-t')} - \frac{Dk^2 \left( 1 - e^{2\gamma(t-t')} \right)}{2\gamma} \right] \tag{65}$$

transforming it back to the original space with the help of a Gaussian type integral the transition probability will be given as:

$$P(x, t|x', t') = \sqrt{\frac{\gamma}{2\pi D\left(1 - e^{-2\gamma(t-t')}\right)}} \exp\left[-\frac{\gamma\left(x - e^{-\gamma(t-t')}x'\right)^2}{2D\left(1 - e^{-\gamma(t-t')}\right)}\right] \tag{66}$$

In order to derive the probability distribution at the stationary state, the limit of $t \to \infty$ needs to be taken after which the probability distribution will be:

$$W_{st}(x) = \sqrt{\frac{\gamma}{2\pi D}} \exp\left(-\frac{\gamma x^2}{2D}\right) \tag{67}$$

It is worth noting that the general expression for the transition probability is valid for all possible $\gamma$, but the expression for the stationary state is valid only for positive $\gamma$ which can be explained with the fact, that a stationary solution only exists for the case in which $\gamma$ is positive.

In the next section the behavior of the solutions at the stationary state and the conditions for the existence of the stationary solution will be investigated in more details.[4, 10]

# 5   Solution Methods and Asymptotic Behavior

The stationary state could only be considered in the case that the Fokker-Planck operator is time-independent or there is a unique limit of the Fokker-Planck operator for $t \to \infty$.

One of the most useful properties of the probability distributions at the stationary state that has also been mentioned before is the constant probability current. From the constant probability current the following equation would be obtained:

$$\frac{\partial}{\partial x} D^{(2)}(x)W_{st}(x) - S = D^{(1)}(x)W_{st}(x) = \frac{D^{(1)}(x)}{D^{(2)}(x)}D^{(2)}(x)W_{st}(x) \tag{68}$$

The solution to the above equation could be given by:

$$W_{st}(x) = Ne^{-\Phi(x)} - Se^{-\Phi(x)} \int \frac{e^{\Phi(x')}}{D^{(2)}(x')}\mathrm{d}x' \tag{69}$$

with the potential $\Phi(x)$ defined as:

$$\Phi(x) = \ln\left(D^{(2)}(x)\right) - \int_0^x \frac{D^{(1)}(x')}{D^{(2)}(x')}\mathrm{d}x' \tag{70}$$

The two prefactors $N$ and $S$ could be determined from the boundary conditions and the normalization requirement imposed on the probability density. But as we have already seen in the case of the Ornstein-Uhlenbeck with negative $\gamma$, this does not imply that the stationary state always exists. For instance a problem can arise when the expression for stationary probability density given at (69) is not normalizable.

That is the reason why the next part would be concerning the existence of such a stationary distribution. A sufficient condition for the existence of the stationary state is given by the detailed balance. Detailed balance can be introduced from the master equation among discrete states. The master equation is given as:

$$\dot{W}_n = \sum_m \left[w(m \to n)W_m - w(n \to m)W_n\right] \tag{71}$$

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with $w(m \rightarrow n)$ being the transition rate from the state $m$ to state $n$. This means, for the stationary state holds:

$$\sum_m w(m \rightarrow n)W_m = \sum_m w(n \rightarrow m)W_n \tag{72}$$

A sufficient condition for equation (72) to hold is that every single term from the two sums would be equal to each other.

$$w(m \rightarrow n)W_m = w(n \rightarrow m)W_n \tag{73}$$

This condition is called the detailed balance in the discrete case. In order to move on to the continuous case with the Fokker-Planck equation two new definitions should be given.

A variable is called even when it is invariant under time reversal and it is called odd when it changes its sign under the time reversal. For example the position is an even variable while the velocity is an odd one. The continuous detailed balance would be different for each of these terms. First the detailed balance for even variables would be formulated.

The master equation for continuous states could be written as:

$$\dot{W} = \int \left[ w(x' \rightarrow x)W(x',t) - w(x \rightarrow x')W(x,t) \right] dx' \tag{74}$$

The relation between the transition rate and the transition probability is given by:

$$w(x' \rightarrow x) = \left. \frac{dP(x, \tau | x' 0)}{d\tau} \right|_{\tau=0} = L_{FP}(x)\delta(x - x') \tag{75}$$

Similar to the discrete case the detailed balance for the continuous case could be written as the equality of the loss and gain term for a certain point $x$ with respect to another certain point $x'$:

$$w(x' \rightarrow x)W_{st}(x') = w(x \rightarrow x')W_{st}(x) \tag{76}$$

Inserting (75) into (76) one obtains:

$$L_{FP}(x)\delta(x - x')W_{st}(x') = L_{FP}(x')\delta(x - x')W_{st}(x) \tag{77}$$

On the left hand side the existence of the Dirac delta allows changing the argument in the probability from $x'$ to $x$ and swapping the probability density and the Dirac delta. On the right hand side since the Fokker-Planck operator only acts on $x'$ and $W_{st}(x)$ depends on $x$ one can use the linearity of the operator and move $W(x)$ to the left of the operator.

$$L_{FP}(x)W_{st}(x)\delta(x - x') = W_{st}(x)L_{FP}(x')\delta(x - x') \tag{78}$$

After that by applying (20) to the right hand side one obtains:

$$L_{FP}(x)W_{st}(x)\delta(x - x') = W_{st}(x)L_{FP}^{\dagger}(x)\delta(x - x') \tag{79}$$

which is only true if

$$L_{FP}(x)W_{st}(x) = W_{st}(x)L_{FP}^{\dagger}(x) \tag{80}$$

This is the formulation of the detailed balance condition for an even variable in the continuous case. If the Fokker-Planck operator fulfills the above condition for a probability distribution then the probability distribution describes a stationary state of the Fokker-Planck equation. This distribution is not necessarily the unique stationary distribution of the Fokker-Planck equation.

For the odd case one should make a somewhat different interpretation of the loss and gain terms in the master equation. While the two term could be interpreted as gain of and loss from a state, the loss term can also be interpreted as gain of the state if time moved backwards. Keeping in mind that an odd variable transforms as $x \to -x$ under time reversal, the continuous detailed balance could be written as:

$$w(x' \to x)W_{st}(x') = w(-x \to -x')W_{st}(-x) \tag{81}$$

This follows from the invariance of the stationary state under the time reversal which would also mean that:

$$W_{st}(-x) = W_{st}(x) \tag{82}$$

With similar steps as in the case of even variables the detailed balance condition for the odd variables would be:

$$L_{FP}(x)W_{st}(x) = W_{st}(-x)L_{FP}^{\dagger}(-x) \tag{83}$$

These two give sufficient conditions for the existence of a stationary distribution for a given Fokker-Planck operator.

Lastly, two solution methods of the Fokker-Planck equations will be conceptually introduced. The first method uses the approximation for evolutions in small time intervals $\tau = t - t' \ll 1$ as it was introduced in the case of (13). In this case the transition probability can be written as:

$$P(x, t' + \tau | x', t') \approx \exp\left[-\frac{\partial}{\partial x}D^{(1)}(x, t) \cdot \tau + \frac{\partial^2}{\partial x^2}D^{(2)}(x, t) \cdot \tau\right] \cdot \delta(x' - x) \tag{84}$$

Applying a Fourier transform and calculating the resulting Gaussian type integral results in

$$P(x, t' + \tau | x', t') \approx \frac{1}{2\sqrt{\pi D^{(2)}(x, t)\tau}} \exp\left(-\frac{\left[x - x'D^{(1)}(x, t)\tau\right]^2}{4D^{(2)}(x, t)\tau}\right) \tag{85}$$

This approximation can be used by splitting any interval between two arbitrary points in time in many smaller intervals, such that the above approximation could be applicable. And then applying the above expression for those intervals step by step in order to obtain the full evolution.

The other method of solving the Fokker-Planck equations is through finding the eigenfunctions of the Fokker-Planck equation. For that purpose a separation ansatz for the probability distribution could be used.

$$W(x, t) = \varphi(x)e^{-\lambda t} \tag{86}$$

Inserting this in the Fokker-Planck equations results in:

$$L_{FP}\varphi(x) = -\lambda\varphi(x) \tag{87}$$

This is a eigenfunction equation with $\varphi(x)$ being the eigenfunction of the Fokker-Planck operator and $-\lambda$ being the corresponding eigenvalue.

One requirement for this equation to make sense physically is that the Fokker-Planck should be Hermitian, such that the eigenvalues are always real. Although the Fokker-Planck operator is not always Hermitian, there exists a transformation of the operator, which is a Hermitian operator. The Hermitian operator is constructed through:

$$L := e^{\Phi(x)/2}L_{FP}e^{-\Phi(x)/2} \tag{88}$$

where $\Phi(x)$ are the potentials introduced in (70). The eigenfunctions of this operator $\psi(x)$ could be obtained form the eigenfunctions of the Fokker-Planck operator $\varphi(x)$ through:

$$\psi(x) = e^{\Phi(x)/2}\varphi(x) \tag{89}$$

with the same eigenvalues.[4]

# 6 Relativistic Ornstein-Uhlenbeck Process

One of the more modern applications of the Fokker-Planck equations is in studying the relativistic Ornstein-Uhlenbeck processes. The first problem that one encounters in the relativistic formulation of the problem is that there is no Markovian process in the Minkowski space that is invariant under Lorentz transformation, i.e. the introduced expression in the last sections for the Langevin force $\Gamma(t)$ is not valid any more in the Minkowski space. The problem is solved through moving to the phase space of $(\vec{p}, \vec{x})$. It has been shown in the non-relativistic case that in the presence of an external force the position coordinates are needed. In the relativistic case these coordinate are needed even in the absence of external forces. But there are also some other conditions that should be fulfilled in the relativistic case. The first one of them is the upper limit of the particle velocity. This is ensured through the relativistic momentum-velocity relation.

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} v^i(t) = \frac{p^i(t)}{m\gamma} = \frac{p^i(t)}{p^0} = \frac{p^i(t)}{\sqrt{m^2 + |\vec{p}|^2}} \tag{90}$$

where $c = 1$ has been assumed and $m$ is the rest mass of the Brownian particle, $\gamma = \sqrt{1 + \left(\frac{\vec{p}}{m}\right)^2}$ the Lorentz factor, $p^0$ the energy of the of the particle and $\vec{p}$ the spatial components of the 4-momentum of the particle. This means for the norm of the velocity:

$$|\vec{v}| = \frac{|\vec{p}|}{\sqrt{m^2 + |\vec{p}|^2}} \leq 1 \tag{91}$$

And so the problem of the upper limit of the velocity has been taken care of.
The other question is the question of the time component. There are two reasonable choices of the time component in a system consisting of a particle moving in a fluid, the time component of the fluid $t$ and the proper time of the particle $\tau$. Here the time component of the fluid would be taken as the time parameter used in the equation, i.e. the equations will be formulated in the rest frame of the fluid. The relation between the two time parameters is given by $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\gamma}$, with the help of which the Lorentz-invariant equations can be transformed from one rest frame to the other.
In general, the multi-dimensional Langevin equations has the form of equation (45), writing it in particular for the momentum vector will give:

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = -a^i_j p^j + c^i_j \Gamma^j(t) \tag{92}$$

where $a^i_j$ and $c^i_j$ are $(1,1)$ tensors depending on $\vec{x}, \vec{p}$ and $t$ in general, $\Gamma^j(t)$ are random functions that fulfill the relations (46) and (47). Using the derived Kramers-Moyal coefficients in (48) - (50) and inserting them in the Fokker-Planck equation and by using $\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{p^i}{m\gamma}$ one arrives at:

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m\gamma}\frac{\partial}{\partial x^i}\right) W(t, \vec{x}, \vec{p}) = \frac{\partial}{\partial p^i}\left[\left(a^i_j p^j - c^k_r \frac{\partial c^i_r}{\partial p^k}\right) W(t, \vec{x}, \vec{p}) + \frac{\partial}{\partial p^k}\left(c^i_r c^k_r W(t, \vec{x}, \vec{p})\right)\right]$$

$$= \frac{\partial}{\partial p^i}\left[a^i_j p^j W(t, \vec{x}, \vec{p}) + c^i_r \frac{\partial}{\partial p^k}\left(c^k_r W(t, \vec{x}, \vec{p})\right)\right] \tag{93}$$

where the sum over the lower indices $r$ is implied.
But one has not yet a free choice for the tensors, $a^i_j$ and $c^i_j$. There are still some conditions that they have to fulfill, such as equivalence of the non-relativistic limit of the above equations with the non-relativistic Brownian motion and the fact that the norm of the 4-velocity vector of the particle

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$v^\mu v_\mu = 1$ should be conserved under the action of the friction force. These conditions are fulfilled, without going into the detail of the calculation, by the following functions:

$$a^i_{\ j} = \alpha\left(p^0\right)\delta^i_{\ j} \tag{94}$$

$$c^i_{\ j} = \sqrt{D\left(p^0\right)}\delta^i_{\ j} \tag{95}$$

with $\alpha$ and $D$ depending only on the energy of the particle. Plugging it back into equation (93), one obtains:

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m\gamma}\frac{\partial}{\partial x^i}\right)W(t,\vec{x},\vec{p}) = \frac{\partial}{\partial p^i}\left[\alpha\left(p^0\right)\cdot p^i W(t,\vec{x},\vec{p}) + \sqrt{D\left(p^0\right)}\frac{\partial}{\partial p^i}\left(\sqrt{D\left(p^0\right)}W(t,\vec{x},\vec{p})\right)\right] \tag{96}$$

Requiring the stationary solution to be the relativistic Maxwell-Boltzmann equation given by:

$$W_{st}(\vec{x},\vec{p}) = \mathcal{N}\exp\left(\frac{p^0}{k_B T}\right) \tag{97}$$

and inserting it into the right hand side of (96) and putting the left hand side to zero results in the following relation between the functions $\alpha(p^0)$ and $D(p^0)$, called the relativistic Einstein relation:

$$\alpha\left(p^0\right)p^0 - \frac{D\left(p^0\right)}{k_B T} + \frac{D'\left(p^0\right)}{2} = 0 \tag{98}$$

Here the prime denotes the derivative with respect to $p^0$.
The Ornstein-Uhlenbeck relativistic process is defined by setting the diffusion coefficient $D\left(p^0\right)$ to a constant $\tilde{D}$. From the Einstein relation the friction coefficient $\alpha(p^0)$ is set to:

$$\alpha\left(p^0\right) = \frac{\tilde{D}}{\gamma m k_B T} = \frac{\tilde{\alpha}}{\gamma} \tag{99}$$

Since a further analytic examination of the relativistic would be too complicated if not impossible, the mean squared displacement has been determined numerically. While taking sharp momentum and sharp position as the initial conditions
In the following diagrams Dunkel and Hänggi [1] have plotted the mean squared displacement divided by time

$$D_t = \frac{\langle(\vec{x}(t) - \vec{x}(0))^2\rangle}{2t} \tag{100}$$

$$D_\infty = \lim_{t\to\infty} D_t = \lim_{t\to\infty}\int_0^t \langle\vec{v}(t)\cdot\vec{v}(s)\rangle ds \tag{101}$$
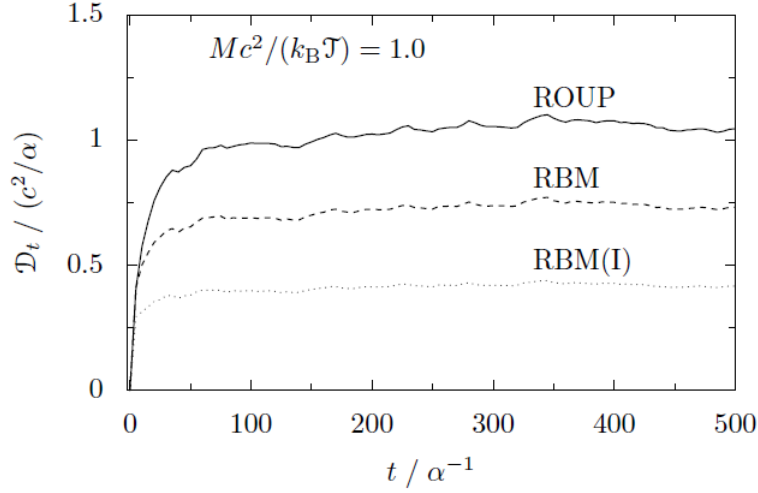
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Figure 1: The evolution of the mean squared displacement in time

The RUOP line shows the evolution of $D_t$ for the relativistic Ornstein-Uhlenbeck process. The other plots show the evolution of $D_t$ in the case of constant friction coefficient for two different discretization rules. The choice of the discretization rule does not make any difference for the Ornstein-Uhlenbeck processes, because of the constant noise coefficient. But for the constant friction coefficient as it is easily recognizable from the Einstein relation the noise coefficient would be momentum-dependent which means time-dependent. This makes the mean squared displacement different for different discretization rules.
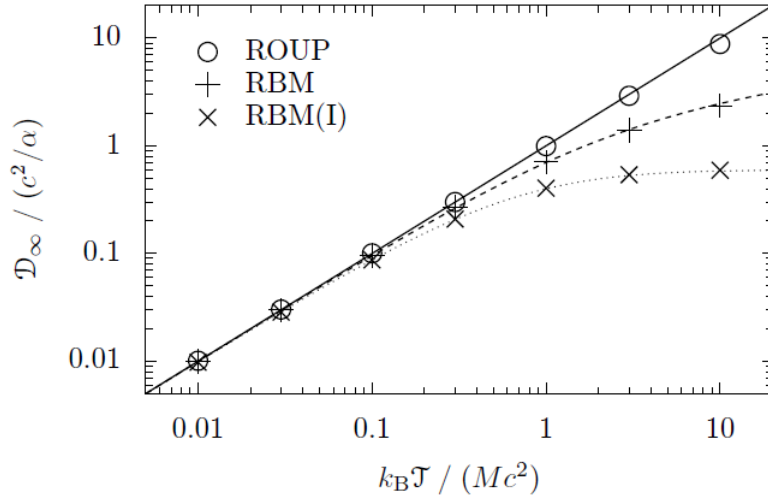


Figure 2: The mean squared displacement at the stationary state as a function of temperature

A more interesting fact about the relativistic Ornstein-Uhlenbeck processes could be read from the next diagram. The next diagram shows that in fact in the Ornstein-Uhlenbeck processes the asymptotic mean squared displacement is the same as in the non-relativistic case even for high temperature. Again the two other plots demonstrate the constant friction coefficient and one can see that they deviate from the non-relativistic limit in high temperatures.[11, 12, 1, 13]

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# 7  Conclusion and Outlook

In this report we introduced the Langevin and Fokker-Planck equation and calculated some properties of the Brownian motion. We also presented some solutions to the simpler cases of these equations that could be solved analytically. We also briefly mentioned some properties of the Fokker-Planck equations that could be used to set up a numerical solution. At the end we presented the relativistic form of the Ornstein-Uhlenbeck process.

These two equations have applications beyond studying the Brownian motion. One of the most considerable applications occurs in studying the non-equilibrium states and thermalization and relaxation behaviors of different systems. The relativistic form of these equations as mentioned before can have application in thermalization processes of high-energy collision experiments, such as thermalization of heavy quarks in quark-gluon plasma or studying the thermalization processes in astrophysics such as thermal decoupling of particles in the early universe.

But the fields of application of these equations go even beyond physics. In fact they are two of the most useful equations in describing the time evolution under the influence of a random noise. That is the reason why one can find some problems in the neuroscience, for example explaining the behavior of neuron under influence of noisy input, where the Fokker-Planck equation could be helpful. There are also some problem in finance that could be solved with the help of Fokker-Planck equation or the inversion of it, i.e. derivation of a noise input from a known probability density.

All in all, these equations are two of the most fundamental equations in statistical physics with applications beyond physics.

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