# Thermalization of Gluons in Relativistic Collisions

**Mathieu Kaltschmidt**
ITP Heidelberg

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supervised by
Prof. Georg Wolschin

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# Outline

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# Relativistic Heavy-Ion Collisions

Why do we study high-energy nuclear physics?

- We want to resolve the nuclear structure.
- The Quark-Gluon Plasma (QGP) provides insights into the physical processes relevant shortly after the Big Bang.
- Particle colliders such as the LHC or the RHIC are built to reach high energies.
- Collision events offer a fruitful playground for testing QCD and statistical models (focus of this talk/seminar).

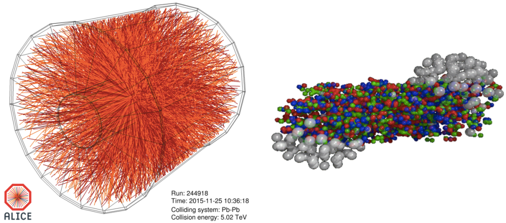


**Figure:** Visualization of a Pb-Pb collision event in the ALICE detector at the LHC.[1]

[1]Source: https://www.physi.uni-heidelberg.de/~reygers/lectures/2019/qgp/qgp_lecture_ss2019.html (23.06.2020)

# The different Phases of RHICs

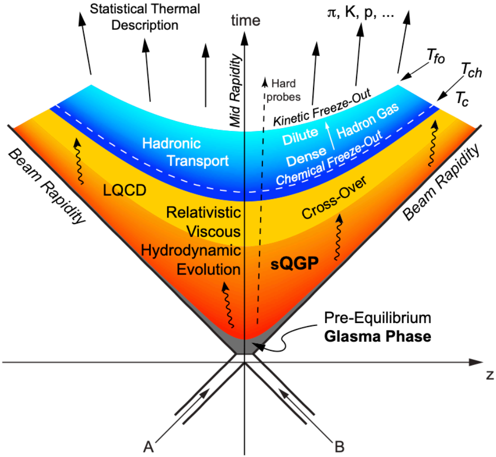


**Figure:** Visualization of the spacetime evolution of the system created in RHICs.[2]
In this talk, we will have a closer look at the pre-equilibrirum phase (gray area).

[2]Figure taken from B. Hippolyte's slides: http://www.nupecc.org/presentations/hippo_mar17.pdf (23.06.2020)

# Outline

# The ALICE Experiment at the LHC

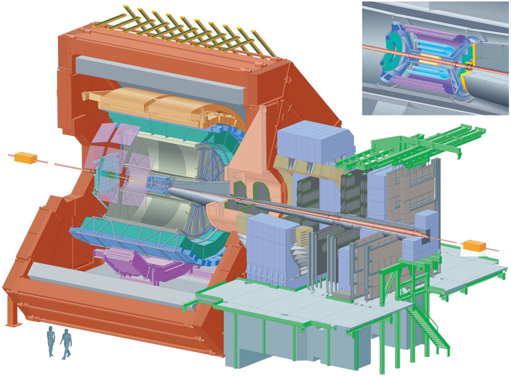

**Figure:** Schematic picture of the ALICE detector at the LHC at CERN in Geneva.[3] The experiment is specialized on heavy-ion collisions (mostly Pb-Pb) and reaches center-of-mass energies of $\sqrt{s} = 5.02\ \mathrm{TeV}$.

[3]Figure taken from ALICEinfo: http://aliceinfo.cern.ch/Public/en/Chapter2/Chap2Experiment-en.html (26.06.2020)

# The RHIC at the Brookhaven National Lab

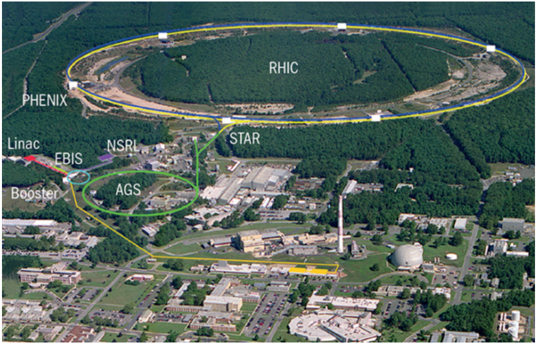


**Figure:** The RHIC at the Brookhaven National Lab.[4]

The different experiments (STAR, sPHENIX[5]) study different aspects of the QGP and the spin structure of the proton. Center-of-mass energies of $\sqrt{s} = 500\ \mathrm{GeV}$ are reached.

[2]Figure taken from CernCourier: https://cerncourier.com/a/rhics-new-gold-record/ (26.06.2020)

[5]Replaces PHENIX (operated until 2016). Preliminary starts operating in 2023.

# The Situation immediately after the Collision I

Question: How do the partons freed by a RHIC thermalize?

- The thermalization process provides a starting point for hydrodynamical evolution in terms of the energy-momentum tensor $T^{\mu\nu}$.
- The dominant parton contribution is dominated by gluon saturation and occupation numbers $\sim 1/\alpha_s$.
- Theoretical model: Color-Glass condensate effective field theory (CQC).

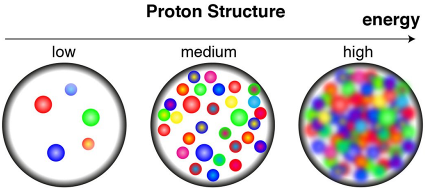


**Figure:** Visualization of the Color-Glass Condensate model.[6]

7

[7]Source: https://www.uu.nl/en/research/institute-for-subatomic-physics/research/color-glass-condensate (29.06.2020)

# The Situation immediately after the Collision II

- Problem: The initial situation $T^{\mu\nu}_{\text{Glasma}} = \text{diag}(\varepsilon, \varepsilon, \varepsilon, -\varepsilon)$, does not serve as starting point!
- Expectation: Situation changes rapidly on a time scale $\sim 1/Q_{\text{s}}$.

But does the phase-space distribution function relax towards the expected equilibrium Bose-Einstein distribution?

- Bottom-Up approach: Relaxation as a result of hard elastic and inelastic collisions.

# The overpopulated Quark-Gluon-Plasma

The following discussion is based on the publications [1] and [2].

- Typical gluon energy densities: $\varepsilon_0 = \varepsilon(\tau = Q_s^{-1}) \sim \frac{Q_s^4}{\alpha_s}$
- Gluons produced per unit volume: $n_0 = n(\tau = Q_s^{-1}) \sim \frac{Q_s^3}{\alpha_s}$
- This implies that the average energy per gluon is $\varepsilon_0 / n_0 \sim Q_s$.

Comparison with the equilibrated system at temperature $T$ leaves a mismatch:

- Assume an initial distribution of the form $n_0 \cdot \varepsilon_0^{-3/4} \sim 1/\alpha_s^{1/4}$.
- In equilibrium we know $\varepsilon_{eq} \sim T^4$, $n_{eq} \sim T^3$ and $n_{eq} \cdot \varepsilon_{eq}^{-3/4} \sim 1$.

Mismatch by a large factor of $\alpha_s^{-1/4}$ corresponding to an overpopulation of the initial distribution. ($\alpha_s \ll 1$ in weak coupling asymptotics)

# Outline

# Kinetic Evolution dominated by Elastic Collisions I

Elastic collisions conserve particle number $\longrightarrow$ Introduce chemical potential $\mu$

- Phase space distribution function given by Bose-Einstein distribution:

$$f_{\mathrm{eq}}(\mathbf{k}) = \frac{1}{\exp\left(\frac{\omega_{\mathbf{k}}-\mu}{T}\right) - 1} \tag{1}$$

- The energy density and the number density then read

$$\varepsilon_{\mathrm{eq}} = \int_{\mathbf{p}} \omega_{\mathbf{p}} \cdot f_{\mathrm{eq}}(\mathbf{p}) \tag{2}$$

$$n_{\mathrm{eq}} = \int_{\mathbf{p}} f_{\mathrm{eq}}(\mathbf{p}) \tag{3}$$

- Remark: Due to many-body interactions, the gluons can develop an effective medium dependent mass with

$$m_0^2 \sim \alpha_{\mathrm{s}} \int_{\mathbf{p}} \frac{\mathrm{d}f_0}{\mathrm{d}\omega_{\mathbf{p}}} \sim Q_{\mathrm{s}}^2 \qquad (\text{cf. } m_{\mathrm{eq}} \sim \alpha_{\mathrm{s}}^{1/2} T \sim \alpha_{\mathrm{s}}^{1/4} Q_{\mathrm{s}}) \tag{4}$$

# Kinetic Evolution dominated by Elastic Collisions II

- The mass $m$ defines an upper bound on the number density:

$$n_{\max} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{\exp\left(\frac{\omega_{\mathbf{k}} - m_0}{T}\right) - 1} \sim T^3 \qquad (m \ll T) \tag{5}$$

- This observation yields the statement, that $n_{\max} \sim Q_s^3/\alpha_s^{3/4}$ is smaller than the initial density $n_0 \sim Q_s^3/\alpha_s$.
- Interpretation: When we consider only elastic collisions, the gluons form a Bose-Einstein condensate (BEC) with distribution function

$$f_{\mathrm{eq}}(\mathbf{k}) = n_c \cdot \delta(\mathbf{k}) + \frac{1}{\exp\left(\frac{\omega_{\mathbf{k}} - m_0}{T}\right) - 1} \tag{6}$$

with

$$n_c \sim \frac{Q_s^3}{\alpha_s}\left(1 - \alpha_s^{1/4}\right) \qquad (\text{note } n_c \cdot m \sim \alpha_s^{1/4} T^4 \ll \varepsilon_0) \tag{7}$$

# BEC: A short Reminder (and Teaser)

What is a Bose-Einstein Condensate?

- Bosons are allowed to share the same quantum state.
- At very low temperatures the occupation of the lowest quantum state rises extremely fast.
- New "state of matter" has extremely interesting properties.

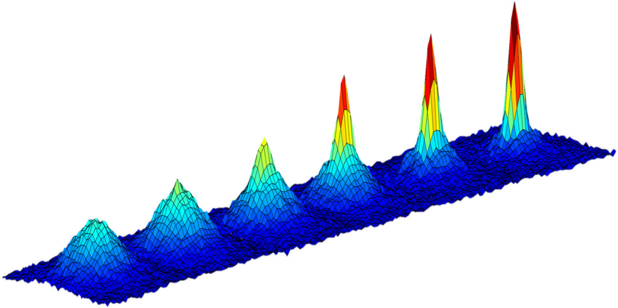

**Figure:** Velocity distribution for a gas of rubidium atoms.[8]
This demonstrates the formation of a BEC in great detail.

[8] Source: https://www.jpl.nasa.gov/spaceimages/details.php?id=PIA22561 (23.06.2020)

# Implications

- In order to reach the expected B-E equilibrium distribution, particle-number decreasing inelastic processes must occur.
- Two possible equilibrium states: Either a system with a condensate (only elastic collisions) or a system with fewer particles (affected by inelastic collisions).
- Dynamical issue depending on many factors, e. g. production/annihilation rates.

# Kinetic Evolution dominated by Elastic Collisions III

- Consider the transport eqn.

$$\partial_t f(\mathbf{k}, X) = C_{\mathbf{k}}[f], \tag{8}$$

a simplified version of the Boltzmann eqn. (cf. Pavel's talk) without drift terms and the collision integral $C_{\mathbf{k}}[f]$ which reads

$$\partial_t f\Big|_{\mathrm{coll}} \sim \frac{\Lambda_{\mathrm{s}}\Lambda}{p^2}\partial_p \left\{ p^2 \left[ \frac{\partial f}{\partial p} + \frac{\alpha_{\mathrm{s}}}{\Lambda_{\mathrm{s}}} f(p)(1+f(p)) \right] \right\} \tag{9}$$

in the small-angle approximation.

- The two relevant scales $\Lambda_{\mathrm{s}}$ and $\Lambda$ are used to compute the thermalization time defined by the relation $\Lambda_{\mathrm{s}}/\Lambda \sim \alpha_{\mathrm{s}}$.
- Taking moments of the collision integral one finds:

$$t_{\mathrm{scat}} = \frac{\Lambda}{\Lambda_{\mathrm{s}}^2} \sim t \tag{10}$$

# Kinetic Evolution dominated by Elastic Collisions IV

- The integrals are dominated by the largest momenta $\sim \Lambda$. This allows us to approximate the distribution function $f(p) \sim \Lambda_\mathrm{s}/(\alpha_\mathrm{s} p)$ up to a cutoff $\Lambda$.
- This leaves us with:

$$n_\mathrm{g} \sim \frac{1}{\alpha_\mathrm{s}} \Lambda^2 \Lambda_\mathrm{s} \tag{11}$$

$$\varepsilon_\mathrm{g} \sim \frac{1}{\alpha_\mathrm{s}} \Lambda^3 \Lambda_\mathrm{s} \tag{12}$$

$$\varepsilon_\mathrm{c} \sim n_\mathrm{c} \cdot m \sim n_\mathrm{c} \cdot \sqrt{\Lambda_\mathrm{s} \Lambda} \tag{13}$$

  with the total number density $n = n_\mathrm{g} + n_\mathrm{c}$.
- Assuming energy conservation, i. e. $\Lambda_\mathrm{s} \Lambda^3 \sim \mathrm{const.}$ we can compute the time-dependence of the two scales and therefore the thermalization time.

# Kinetic Evolution dominated by Elastic Collisions V

From the considerations made before, we determine the time evolution of the scales:

$$\Lambda_{\mathrm{s}} \sim Q_{\mathrm{s}} \left(\frac{t_0}{t}\right)^{\frac{3}{7}} \tag{14}$$

$$\Lambda \sim Q_{\mathrm{s}} \left(\frac{t}{t_0}\right)^{\frac{1}{7}} \tag{15}$$

and we can confirm that the energy carried by the condensate remains negligible:

$$\frac{\varepsilon_{\mathrm{c}}}{\varepsilon_{\mathrm{g}}} \sim \left(\frac{t_0}{t}\right)^{\frac{1}{7}} \tag{16}$$

Now, we have computed all dependencies to find the estimated thermalization time for $\Lambda_{\mathrm{s}} \sim \alpha_{\mathrm{s}}\Lambda$:

$$t_{\mathrm{th}} \sim \frac{1}{Q_{\mathrm{s}}} \left(\frac{1}{\alpha_{\mathrm{s}}}\right)^{\frac{7}{4}} \tag{17}$$

# Outline

# The Importance of Inelastic Processes

- Interesting: The modification of the collision integral on the RHS due to inleastic effects, leaves the time evolution of the scales invariant!
- Implications on the condensate formation can be obtained from numerical analysis of the modified transport equation.
- The inelastic contribution to the collision integral gives a sink term.
- Balancing source (elastic) und sink (inelastic) contributions may result in a condensate surviving during most of the thermalization process.

Further insights can be gained by considering e. g. the effect of longitudinal expansion (cf. [1]) or studying in more detail the effects of radiation (cf. [2])

# Outline

# Deriving the Nonlinear Boson Diffusion Equation I

The following derivation follows reference [5].

- The starting point for our investigation is the Boltzmann eqn. (cf. Pavel's talk).
- For spatial homogeneity of the the boson distribution function $f(\mathbf{x}, \mathbf{p}, t)$ and a spherically symmetric momentum dependence the equation for the single-particle occupation numbers $n_j \equiv n_{\text{th}}(\varepsilon_j, t)$ reads:

$$\frac{\partial n_1}{\partial t} = \sum_{\varepsilon_2, \varepsilon_3, \varepsilon_4} \langle V^2 \rangle G(\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4) \times [(1 + n_1)(1 + n_2) n_3 n_4 - (1 + n_3)(1 + n_4) n_1 n_2] \tag{18}$$

- The collision term can be written in the form of a Master eqn.:

$$\frac{\partial n_1}{\partial_t} = (1 + n_1) \sum_{\varepsilon_4} W_{4\to1} n_4 - \sum_{\varepsilon_4} W_{1\to4}(1 + n_4) \tag{19}$$

with

$$W_{4\to1} = W_{41} g_1 = \sum_{\varepsilon_2, \varepsilon_3} \langle V^2 \rangle G(\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4)(1 + n_2) n_3 \tag{20}$$

# Deriving the Nonlinear Boson Diffusion Equation II

- In continuum $\sum \to \int$ and introduce density of states $g_j \equiv g(\varepsilon_j)$.
- If $G$ acquires a width in a finite system:

$$W_{14} = W_{41} = W\left[\frac{1}{2}(\varepsilon_4 + \varepsilon_1), \underbrace{|\varepsilon_4 - \varepsilon_1|}_{=:x}\right] \tag{21}$$

- Perform a gradient expansion of $n_4$ and $g_4 n_4$ around $x \approx 0$.
- Introduce transport coefficients via moments of the transition probability:

$$D = \frac{g_1}{2} \int_0^\infty \mathrm{d}x\ W(\varepsilon_1, x)\ x^2 \tag{22}$$

$$v = g_1^{-1} \frac{d}{d\varepsilon_1}(g_1 D) \tag{23}$$

# Deriving the Nonlinear Boson Diffusion Equation III

- Nonlinear partial differential equation for $n \equiv n(\varepsilon_1, t) = n(\varepsilon, t)$:

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \varepsilon}\left[v \cdot n(1+n) + n\frac{\partial D}{\partial \varepsilon}\right] + \frac{\partial^2}{\partial \varepsilon^2}[Dn] \qquad (24)$$

- Consider the limit of constant transport coefficients:

$$\frac{\partial n}{\partial t} = -v\frac{\partial}{\partial \varepsilon}[n(1+n)] + D\frac{\partial^2 n}{\partial \varepsilon^2} \qquad (25)$$

- Thermal Bose-Einstein distribution provides stationary solution:

$$n_{\mathrm{eq}}(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon-\mu}{T}\right) - 1} \qquad (26)$$

# Some Remarks

- The present model does not resolve the 2nd-order phase transition.
- The effects of condensation are included (cf. the following figures).
- A treatment resolving the singularity at $\epsilon = \mu$ is presented later.

# Linear Relaxation-Time Approximation (RTA)

- Given some initial distribution $n_{\mathrm{i}}(\varepsilon)$ we find an approximated solution for the thermalization process via the RTA:

$$\frac{\partial n_{\mathrm{rel}}}{\partial t} = \frac{(n_{\mathrm{eq}} - n_{\mathrm{rel}})}{\tau_{\mathrm{eq}}} \tag{27}$$

with solution:

$$n_{\mathrm{rel}}(\varepsilon, t) = n_{\mathrm{i}}(\varepsilon) \cdot \exp\left(-\frac{t}{\tau_{\mathrm{eq}}}\right) + n_{\mathrm{eq}}(\varepsilon)\left(1 - \exp\left(-\frac{t}{\tau_{\mathrm{eq}}}\right)\right) \tag{28}$$

where $\tau_{\mathrm{eq}} = 4D/(9v^2)$.

- Motivated by the study of early stages of RHICs, the initial distribution is chosen such that:

$$n_{\mathrm{i}}(\varepsilon) = N_{\mathrm{i}} \cdot \theta\left(1 - \varepsilon/Q_{\mathrm{s}}\right) \cdot \theta(\varepsilon) \tag{29}$$

with limiting momentum $Q_{\mathrm{s}} \sim \tau_0^{-1} \approx 1\ \mathrm{GeV}$.

Mueller (2000)

# Results for the RTA

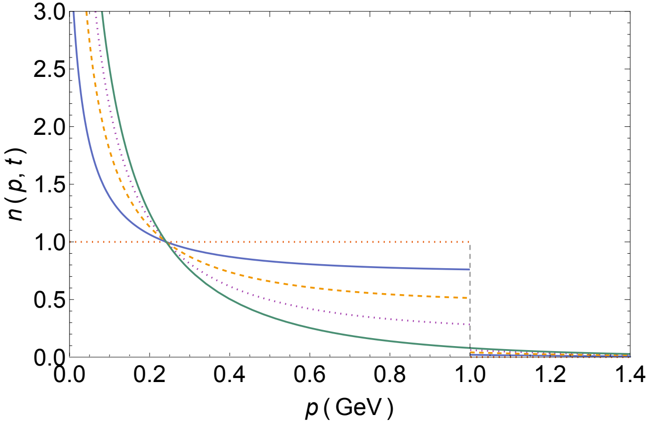


**Figure:** Relaxation of a finite Bose system towards the equilibrium. [5]
Here $T = -D/v \simeq 0.4$ GeV, $\tau_{\rm eq} = 4D/(9v^2) = 0.33 \cdot 10^{-23}{\rm s} \simeq 1$ fm/c and the timesteps are $\{0.1, 0.25, 0.5, \infty\}$ (in units of $10^{-23}s$) from top to bottom.

# Exact Solution of the Nonlinear Boson Diffusion Equation

- To solve eqn. (24) analytically, we perform the following nonlinear transformation:

$$n(\varepsilon, t) = -\frac{D}{v}\frac{\partial \ln \mathcal{Z}(\varepsilon, t)}{\partial \varepsilon} \tag{30}$$

which reduces our problem to a linear diffusion eqn. for $\mathcal{Z}(\varepsilon, t)$:

$$\frac{\partial \mathcal{Z}}{\partial t} = -v\frac{\partial \mathcal{Z}}{\partial \varepsilon} + D\frac{\partial^2 \mathcal{Z}}{\partial \varepsilon^2} \tag{31}$$

- Solutions to this equation can be written as:

$$n(\varepsilon, t) = \frac{1}{2v}\frac{\int_{-\infty}^{+\infty} \frac{\varepsilon - x}{t} F(x) \cdot G_{\text{free}}(\varepsilon - x, t)\ \mathrm{d}x}{\int_{-\infty}^{+\infty} F(x) \cdot G_{\text{free}}(\varepsilon - x, t)\ \mathrm{d}x} - \frac{1}{2} \tag{32}$$

# Additional Definitions

- The quantities appearing in the solution are the free Green's function

$$G_{\mathrm{free}}(\varepsilon - x, t) = \exp\left[-\frac{(\varepsilon - x)^2}{4Dt}\right], \tag{33}$$

$$\tag{34}$$

and the implementation of the initial conditions

$$F(x) = \exp\left[-\frac{1}{2D}(vx + 2v\int_0^x n_{\mathrm{i}}(y)\mathrm{d}y)\right]. \tag{35}$$

- They define the free partition function via:

$$\mathcal{Z}(\varepsilon, t) = a(t) \cdot \int_{\infty}^{\infty} G_{\mathrm{free}}(\varepsilon, x, t) \cdot F(x)\ \mathrm{d}x \tag{36}$$

# Results for the Solution of the NBDE I

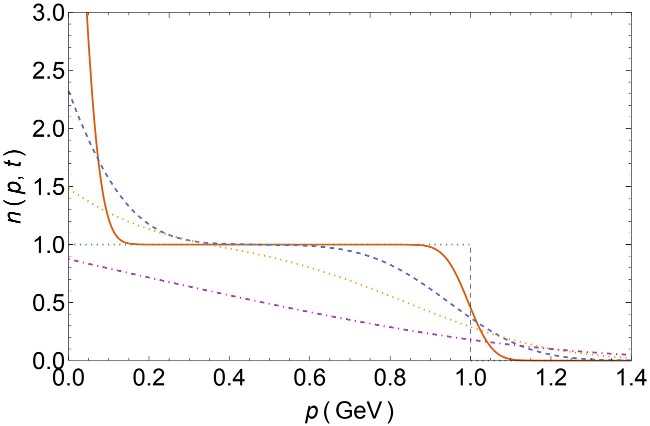


**Figure:** Equilibration of a finite Bose system from the NBDE. [5]
The integration range is restricted to $x \geq 0$. Here $T \simeq 0.4$ GeV, $\tau_{\text{eq}} = 0.33 \cdot 10^{-23}$s and the timesteps are $\{0.005, 0.05, 0.15, 0.5\}$ (in units of $10^{-23}s$) from top to bottom.

# Results for the Solution of the NBDE II

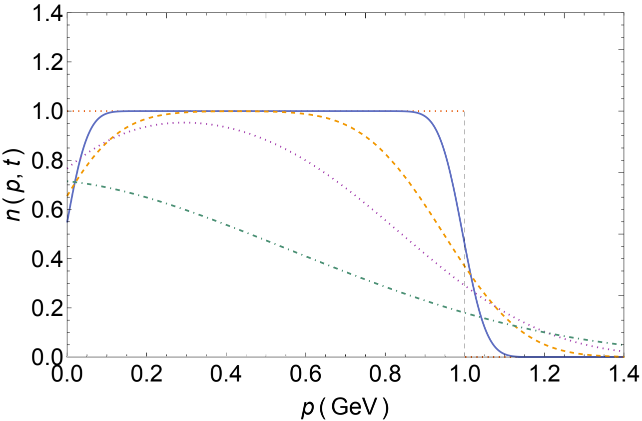


**Figure:** Equilibration of a finite Bose system from the NBDE. [5]
The integration range is extended to $-\infty \leq x \leq \infty$. Here $T \simeq 0.4$ GeV, $\tau_{\mathrm{eq}} = 0.33 \cdot 10^{-23}\mathrm{s}$ and the timesteps are $\{0.005, 0.05, 0.15, 0.5\}$ (in units of $10^{-23}s$) from top to bottom.

# Results for the Solution of the NBDE III

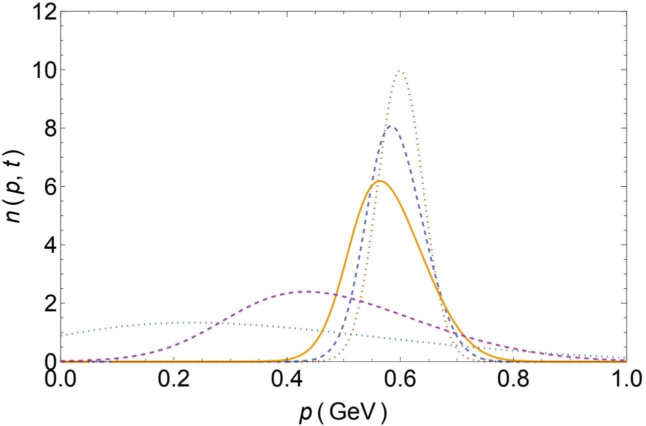


**Figure:** Equilibration of a finite Bose system from the NBDE for Gaussian initial conditions $n_\mathrm{i}(\varepsilon) = N_\mathrm{i}\left(\sqrt{2\pi}\sigma\right)^{-1} \exp\left((\varepsilon - \langle\varepsilon\rangle)/(2\sigma^2)\right)$ with $\sigma = 0.04$ GeV. [5]
Here $T \simeq 0.4$ GeV, $\tau_\mathrm{eq} = 0.33 \cdot 10^{-23}\mathrm{s}$ and the timesteps are $\{0.002, 0.006, 0.02, 0.2\}$ (in units of $10^{-23}s$) from top to bottom.

# Treating the Singularity

This part is based on the publication [6] which provides an extension of [5] and was published just recently.

- To account for the singularity at $\varepsilon = \mu < 0$ we have to modify the initial distribution given before (eqn. (29)) as follows:

$$\tilde{n}_{\mathrm{i}}(\varepsilon) = n_{\mathrm{i}}(\varepsilon) + \frac{1}{\exp\left(\frac{\varepsilon-\mu}{T}\right) - 1} \tag{37}$$

- The chemical potential $\mu$ has to be treated as a fixed parameter.
- Considering the limit $\lim_{\varepsilon\to\mu^+} \; n(\varepsilon, t) = \infty \; \forall t$ yields $\mathcal{Z}(\mu, t) = 0$.
- This results in a modified expression for the Green's function

$$G(\varepsilon, x, t) = G_{\mathrm{free}}(\varepsilon - \mu, x, t) - G_{\mathrm{free}}(\varepsilon - \mu, -x, t) \tag{38}$$

# Results for the RTA for the modified Initial Conditions

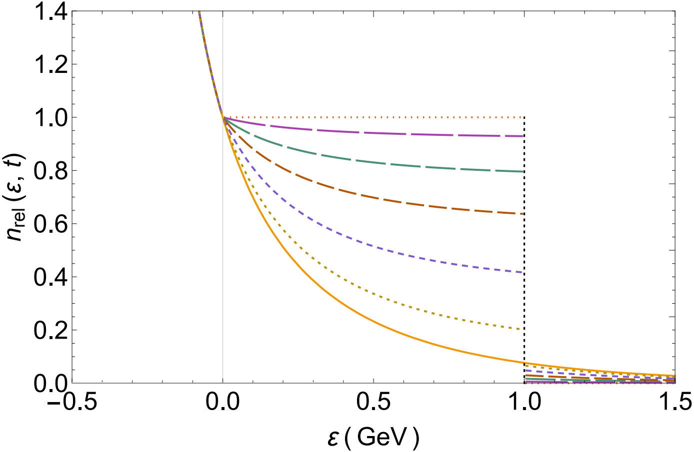


**Figure:** Local thermalization of gluons in the linear RTA for $\mu < 0$. [6]
Here $T \simeq 513$ MeV and the timesteps are $\{0.02, 0.08, 0.15, 0.3, 0.6\}$ (in units of $\mathrm{fm}/c$) from top to bottom.

# Results for the full Solution of the NBDE

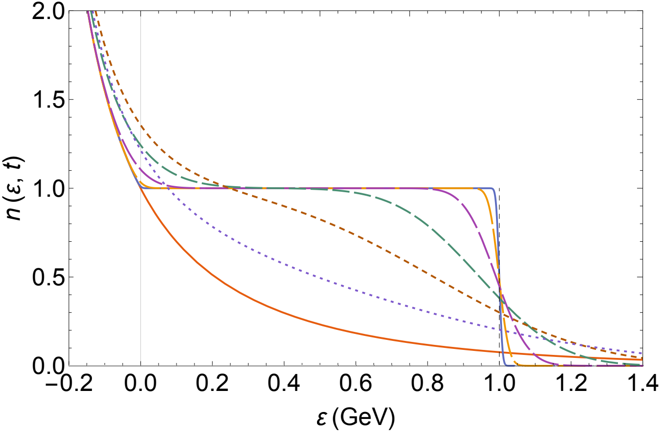


**Figure:** Local thermalization of gluons from the time-dependent solutions of the NBDE for $\mu < 0$. [6]
Here $T \simeq 513\ \mathrm{MeV}$ and the timesteps are $\{6 \cdot 10^{-5}, 6 \cdot 10^{-4}, 6 \cdot 10^{-3}, 0.12, 0.36\}$ (in units of $\mathrm{fm}/c$) from top to bottom.

# Outline

# Conclusion

- The understanding of the thermalization process of gluons is key to find an appropriate description of the complex physical processes during RHICs.
- Using kinetic theory and statistical transport equations we can estimate important quantities such as the equilibration time and understand the importance and differences of elastic and inelastic collisions.
- The role of Bose-Einstein condensation during the thermalization process
- It is possible to find analytic solutions for a Nonlinear Boson Diffusion equation providing further insights into the thermalization process.

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