Thermalization of Gluons in Relativistic Collisions

Mathieu Kaltschmidt

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Institute for Theoretical Physics
Heidelberg University

This report summarizes my talk given at the Statistical Physics Seminar organized by Prof. Georg Wolschin at the Institute for Theoretical Physics in Heidelberg during the summer term 2020.
We study the thermalization process of gluons at the initial stages of relativistic heavy-ion collisions at energies realized for example at the RHIC at the Brookhaven National Laboratory or at the LHC at CERN. After shortly introducing the experimental and theoretical situation we make use of kinetic theory to understand the complex thermalization process as a dynamical interplay of elastic and inelastic collisions. The second part focuses on the derivation and solution of a Nonlinear Boson Diffusion equation (NBDE) which provides a suitable and analytically solvable model describing thermalization based on a Boltzmann-type equation.

1 Introduction

The study of relativistic heavy-ion collisions provides a fruitful playground for physicists to understand many different aspects of high-energy nuclear physics ranging from the resolution of the complex nuclear structure to the highly interesting physics of the Quark-Gluon-Plasma (QGP) providing further insights into to state of the art of our Universe shortly after the Big Bang.
A complete treatment of these collision events would need concepts from quantum chromodynamics (QCD), relativistic hydrodynamics and also methods from statistical physics. Here we will focus on the latter since the aim of this seminar is to get insights into the various applications of the concepts presented in the introductory lecture on statistical physics and the previous talks to modern research.
For a visualization of the complexity of the physical processes relevant for the study of relativistic heavy-ion collisions have a look at figure 1 at the beginning of the next page. In this talk we will have a closer look at the initial stages of such events since only a detailed understanding of the relaxation of the out-of-equilibrium initial state into a (hopefully) known equilibrium final state allows further analysis using the well-known techniques from e. g. hydrodynamics mentioned before (and displayed in the upper part of the corresponding figure).

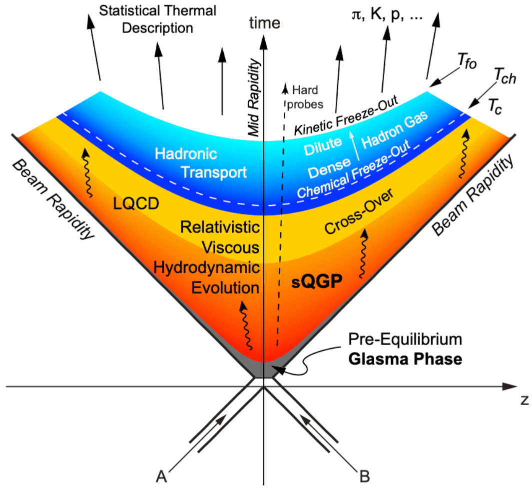


Figure 1: Visualization of the spacetime evolution of the system created in relativistic heavy-ion collisions. We will focus on the pre-equilibrium phase which is displayed here in the gray area near the origin.[1]

2 Experimental and Theoretical Setup

The experimental realization of such collision events is a highly challenging problem itself. The two most known colliders focusing on different aspects of relativistic heavy-ion collisions are on the one hand the "**R**elativistic **H**eavy-**I**on **C**ollider" (RHIC) at the Brookhaven National Lab in the USA studying various areas of high-energy nuclear physics for example the spin structure of the proton and as mentioned before the physics of the QGP at center-of-mass energies of about $\sqrt{s} = 500$ GeV and on the other hand the "**L**arge **H**adron **C**ollider" (LHC) at CERN in Geneva, or more precisely the ALICE experiment at the LHC which is specialized on the analysis of Pb-Pb collision events at even higher center-of-mass energies of $\sqrt{s} = 5.02$ TeV. We won't go into detail here concerning the exact experimental setup, the interested reader is invited to visit the corresponding websites of the experiments for more information and pictures.

The central goal of our discussion is to understand how the partons freed in relativistic heavy-ion collisions thermalize. The relevant quantity serving as an input for the subsequent hydrodynamical description is the energy-momentum tensor $T^{\mu\nu}$. We are therefore interested in its exact form and evolution. Since we are dealing with a high-energy experiment the relevant parton contribution is dominated by gluon saturation and occupation numbers of the order of $\sim 1/\alpha_s$, where α_s denotes the strong coupling constant. One refers to this state characterized by the fact that the chromo-electric and chromo-magnetic fields are collinear to the collision axis as "Glasma". The theoretical model suitable for a description of this situation immediately after the collision is the so-called "**C**olor-**G**lass **C**ondensate" effective field theory (CGC).

1. The figure was taken from B. Hippolyte's slides: http://www.nupecc.org/presentations/hippo_mar17.pdf (23.06.2020)

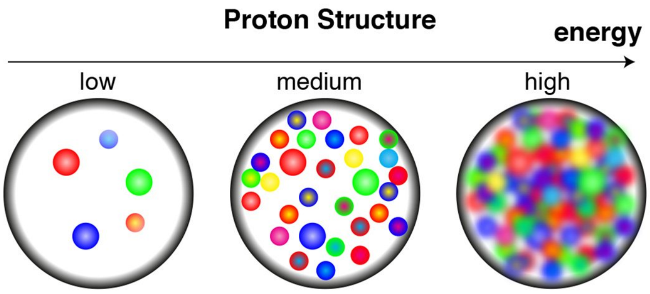


Figure 2: Visualization of the Color-Glass Condensate model.[2]

As visualized in figure 2 above, the gluon density increases rapidly for high energies until the resolution is saturated at some energy scale Q_s providing a natural cutoff for the validity of the CGC model. The observation that we are dealing with very high gluon densities and therefore a highly packed phase space in this limit leads us to the conclusion that the high-p states are also occupied resulting in a small coupling constant $\alpha_s \ll 1$ due to asymptotic freedom as characteristic property of QCD.
The corresponding energy-momentum tensor takes the rather simple diagonal form

$$T^{\mu\nu}_{\text{Glasma}} = \text{diag}(\varepsilon, \varepsilon, \varepsilon, -\varepsilon). \tag{1}$$

The minus sign in the last component signals a negative longitudinal pressure at early times which is causing problems with the usability of this explicit energy-momentum tensor as an initial parameter of the hydrodynamical evolution. One expects this situation to change rapidly on a time scale $\sim 1/Q_s$.
But even if this change occurs we cannot be completely sure if at all and under which conditions the expected equilibrium distribution, given by the well-known Bose-Einstein distribution

$$f_{\text{eq}}(\mathbf{k}) = \frac{1}{\exp\left(\frac{\omega_{\mathbf{k}}-\mu}{T}\right) - 1}, \tag{2}$$

is reached at the end of the thermalization process. Here ω_k denotes the energy of a gluon with momentum k, μ is the chemical potential and T is the equilibrium temperature.

To conclude our study of the initial situation of the collision event we have to take a look at the population of the QGP based on the CGC description presented before. The gluons have typical transverse momenta of the order of Q_s and an energy density of

$$\varepsilon_0 = \varepsilon(\tau = Q_s^{-1}) \sim \frac{Q_s^4}{\alpha_s}, \tag{3}$$

2. Figure taken from: https://www.uu.nl/en/research/institute-for-subatomic-physics/research/color-glass-condensate (29.06.2020)

the number of gluons per unit volume is

$$n_0 = n(\tau = Q_s^{-1}) \sim \frac{Q_s^3}{\alpha_s}, \tag{4}$$

and therefore we indeed find $\varepsilon_0/n_0 \sim Q_s$ for the average energy per gluon. The initial distribution may be characterized by the dimensionless combination

$$n_0 \cdot \varepsilon_0^{-3/4} \sim \frac{1}{\alpha_s^{1/4}}. \tag{5}$$

For the expected equilibrium distribution at temperature T we know

$$\varepsilon_{eq} \sim T^4, \qquad n_{eq} \sim T^3 \tag{6}$$

and therefore

$$n_{eq} \cdot \varepsilon_{eq}^{-3/4} \sim 1, \tag{7}$$

leaving a mismatch by a large[3] factor of $\alpha_s^{-1/4}$ corresponding to an "overpopulation" of the initial distributions.

In the following we want to make use of a bottom-up approach to resolve this problem and to understand the thermalization process, i. e. we assume that the relaxation occurs as a result of hard elastic and inelastic collisions based on the work published in [1, 2].

3 Thermalization: The Elastic Case

Let us first consider the case of thermalization via elastic collisions. The characteristic property of elastic collisions is particle number conservation which is directly related to the introduction of a chemical potential μ as already seen in eqn. (2), representing the assumed equilibrium phase space distribution function.

For a given distribution $f_{eq}(\mathbf{p})$ the expressions for the energy density and the number density read

$$\varepsilon_{eq} = \int_{\mathbf{p}} \omega_{\mathbf{p}} \cdot f_{eq}(\mathbf{p}), \tag{8}$$

and

$$n_{eq} = \int_{\mathbf{p}} f_{eq}(\mathbf{p}). \tag{9}$$

Tuning the temperature T and the chemical potential μ allows us to reproduce the initial values of ε_0 and n_0.

3. Remember we are working in small coupling asymptotics ($\alpha_s \ll 1$) due to the occupation of the high-momentum states and asymptotic freedom.

Due to complex many-body interactions the gluons acquire a medium-dependent effective mass, i. e. $\omega_{\mathbf{p}=0} \neq 0$, which can be obtained for example using the so called "Hard Thermal Loop approximation" assuming $m \sim \alpha_s^{1/2} T$. In our case this estimation yields

$$m_0^2 \sim \alpha_s \int_{\mathbf{p}} \frac{\mathrm{d}f_0}{\mathrm{d}\omega_{\mathbf{p}}} \sim Q_s^2, \tag{10}$$

which can be compared to the equilibrium case

$$m_{\mathrm{eq}} \sim \alpha_s^{1/2} T \sim \alpha_s^{1/4} Q_s. \tag{11}$$

This effective mass allows us to set an upper bound on the allowed particle number density by observing that $f_{\mathrm{eq}}(\mathbf{p})$ grows with μ and demanding that the chemical potential cannot grow larger than the m without changing the sign of $f_{\mathrm{eq}}(\mathbf{p})$. The maximum number density arising from this consideration is given by

$$n_{\max} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{\exp\left(\frac{\omega_{\mathbf{k}} - m_0}{T}\right) - 1} \sim T^3 \sim \frac{Q_s^3}{\alpha_s^{3/4}}, \tag{12}$$

where $m \ll T$ is assumed. This shows again, as already discussed previously, that the maximum number density is always smaller than the initial one. If we only consider elastic collisions one possible explanation of this apparent excess of gluons is the formation of a Bose-Einstein condensate (BEC) resulting in a modification of the equilibrium distribution:

$$f_{\mathrm{eq}}(\mathbf{k}) = n_c \cdot \delta(\mathbf{k}) + \frac{1}{\exp\left(\frac{\omega_{\mathbf{k}} - m_0}{T}\right) - 1}, \tag{13}$$

where n_c is number density of the condensate with $n_{\mathrm{tot}} = n_c + n_g$. This also implies that most of the gluons are part of the condensate, i. e.

$$n_c \sim \frac{Q_s^3}{\alpha_s} \left(1 - \alpha_s^{1/4}\right) \tag{14}$$

but the condensate carries only a small fraction of the total energy density

$$\varepsilon_c = n_c \cdot m \sim \alpha_s^{1/4} T^4 \ll \varepsilon_0. \tag{15}$$

At this point we conclude that the excess of gluons in the case of only particle-number conserving elastic processes can be explained by the formation of a BEC or by the importance of inelastic processes, which is the nowadays favored explanation. We therefore have two possible equilibrium states, one with a condensate only considering elastic collisions and one with fewer gluons in the final state due to mostly $2 \to 1$ inelastic gluon scattering which is a special feature of the non-abelian SU(3) structure of QCD. The latter is a difficult dynamical issue depending on many factors such as production and annihilation rates.

Our next goal is to get an estimate for the timescale of the thermalization process. In a first approximation we stick to the case of only elastic collisions driving the system towards equilibrium. This can be understood via the transport equation

$$\partial_t f(\mathbf{k}, X) = C_{\mathbf{k}}[f], \tag{16}$$

which is a simplified version of the Boltzmann equation (introduced in Pavel's talk) omitting the drift term. Here the right hand side is given by the collision integral, which reads

$$\partial_t f \Big|_{\text{coll}} \sim \frac{\Lambda_\text{s}\Lambda}{p^2}\partial_p \left\{ p^2 \left[\frac{\partial f}{\partial p} + \frac{\alpha_\text{s}}{\Lambda_\text{s}} f(p)(1+f(p)) \right] \right\} \tag{17}$$

in the small angle approximation for elastic $2 \to 2$ scattering. The two relevant scales Λ and Λ_s are assumed to dominate the evolution. To be more explicit, Λ is a hard scale above which the Glasma is assumed to be dilute and Λ_s is the so called "coherence scale" meaning $f(p) \sim 1/\alpha_\text{s}$ below Λ_s. In between we have $f(p) \sim \frac{1}{\alpha_\text{s}} \frac{\Lambda_\text{s}}{\omega_p}$. We want to use the evolution of these scales to derive an expression for the equilibration time τ_eq based on energy conservation and the fact that after a time $t \sim 1/Q_\text{s}$ both scales coincide with Q_s i. e. $\Lambda_\text{s}/\Lambda \sim \alpha_\text{s}$. The Bose-Einstein distribution (2) with temperature $T = \Lambda_\text{s}/\alpha_\text{s}$ provides a stationary solution to the transport equation (17). By taking moments of the collision integral one finds

$$t_\text{scat} = \frac{\Lambda}{\Lambda_\text{s}^2} \sim t \tag{18}$$

as an estimate for the timescale of the scattering processes which should not be confused with the equilibration time we want to derive now. For all the following steps we assume the integrals to be dominated by the contribution from the largest momenta $\sim \Lambda$ which allows us to simplify the distribution $f(p) \sim \Lambda_\text{s}/(\alpha_\text{s} p)$ up to the cutoff Λ. This gives us the following dependencies for the gluon number density, the gluon energy density and the energy density of the condensate:

$$n_\text{g} \sim \frac{1}{\alpha_\text{s}}\Lambda^2\Lambda_\text{s} \tag{19}$$

$$\varepsilon_\text{g} \sim \frac{1}{\alpha_\text{s}}\Lambda^3\Lambda_\text{s} \tag{20}$$

$$\varepsilon_\text{c} \sim n_\text{c} \cdot m \sim n_\text{c} \cdot \sqrt{\Lambda_\text{s}\Lambda}. \tag{21}$$

As mentioned before, additionally we want to assume energy conservation which can be obtained by stating

$$\Lambda_\text{s}\Lambda^3 \sim \text{const.} \tag{22}$$

In total we arrive at the following evolution equations for the coherence scale

$$\Lambda_\text{s} \sim Q_\text{s}\left(\frac{t_0}{t}\right)^{\frac{3}{7}}, \tag{23}$$

and for the cutoff

$$\Lambda \sim Q_{\rm s} \left(\frac{t}{t_0}\right)^{\frac{1}{7}}. \tag{24}$$

From those equations it is clearly visible that the gluon number density $n_{\rm g}$ decreases with time whereas the corresponding energy density remains approximately constant. The energy fraction carried by the gluons in the condensate remains small $\forall t$, i. e.

$$\frac{\varepsilon_{\rm c}}{\varepsilon_{\rm g}} \sim \left(\frac{t_0}{t}\right)^{\frac{1}{7}} \ll 1. \tag{25}$$

Now we are able to estimate the thermalization time from the condition $\Lambda_{\rm s}/\Lambda \sim \alpha_{\rm s}$ as

$$\tau_{\rm eq} \sim \frac{1}{Q_{\rm s}} \left(\frac{1}{\alpha_{\rm s}}\right)^{\frac{7}{4}}. \tag{26}$$

In our case the relevant scale, i. e. the saturation scale is of the order of $Q_{\rm s} \simeq 1$ GeV.

A similar analysis for the fermionic quarks leaves us with the observation that $n_{\rm quarks} \sim \Lambda^3$ which is of the order of $\alpha_{\rm s}$ smaller than $n_{\rm g}$ and therefore the quark contribution can be neglected until we reach $\tau_{\rm eq}$.

4 The Importance of Inelastic Collisions

The treatment of the inelastic collisions will not be as detailed as for the elastic case as we will focus on the qualitative implications of allowing inelastic annihilation and creation processes during thermalization. A full treatment of all inelastic effects is nearly impossible and even today it is not yet fully understood which processes are the most relevant ones driving the system towards the equilibrium state. I will try to highlight the most important observations and refer to the respective research papers for further details.
First of all we observe that the inelastic processes will modify the collision integral on the right hand side of eqn. (16). Interestingly it can be shown that this modification leaves the relation for the scattering time t unchanged and therefore also the evolution equation for the defining scales Λ and $\Lambda_{\rm s}$. Implications on the condensate formation can be obtained via an extensive numerical study of solutions of the complete transport equation. The elastic contributions to the collision integral provide a source term for the condensate whereas the inelastic contributions may be interpreted as sink term. The task is to understand the balancing of the source and sink contributions which may allow a condensate to exist during most of the thermalization process.
One could consider for example the effect of the strong longitudinal expansion of matter involved in relativistic heavy-ion collisions, i. e. the flattening of an isotropic initial distribution in the direction of the collision axis, by introducing a suitable drift term on the left hand side of the transport equation or instabilities in the isotropy of p_z and the

transversal momentum p_{T} (cf. [2]). Another important effect is the influence of gluon radiation allowing transport of momenta over a wide range of momenta on a very short time scale. This can be understood as a certain feature of gauge theories. For more details the interested reader is referred to publication [1].

5 Thermalization via a Nonlinear Boson Diffusion Equation

The second part of our discussion of the thermalization process of gluons in relativistic heavy-ion collisions is based on a Nonlinear Boson Diffusion equation (NBDE) derived from the Boltzmann equation using a gradient expansion of the respective collision integral.

5.1 Derivation of the NBDE

Our central assumptions are spatial homogeneity of the boson distribution function $f(\mathbf{x}, \mathbf{p}, t)$ and spherical symmetry in the momentum dependence. These assumptions allow us to simplify the kinetic equations by performing the angular integration. We arrive at the following expression for the single-particle occupation numbers $n_j = n(\varepsilon_j, t)$ in a finite Bose system:

$$\begin{aligned}\frac{\partial n_1}{\partial t} &= \sum_{\varepsilon_2,\varepsilon_3,\varepsilon_4} \langle V^2\rangle \cdot G(\varepsilon_1+\varepsilon_2, \varepsilon_3+\varepsilon_4) \\ &\quad \times \left[(1+n_1)(1+n_2)n_3n_4 - (1+n_3)(1+n_4)n_1n_2\right]\end{aligned} \tag{27}$$

Here $\langle V^2\rangle$ is the second moment of the interaction potential and the function G encodes energy conservation[4]. The different indices 1 to 4 indicate the respective particles in the elastic $2 \to 2$ scattering process.
The collision term on the right hand side can be rewritten in the form of a Master equation (cf. Talha's talk):

$$\frac{\partial n_1}{\partial t} = (1+n_1)\sum_{\varepsilon_4} W_{4\to 1} n_4 - \sum_{\varepsilon_4} W_{1\to 4}(1+n_4), \tag{28}$$

with the transition probability

$$W_{4\to 1} = W_{41} g_1 = \sum_{\varepsilon_2,\varepsilon_3} \langle V^{\,2}\rangle G(\varepsilon_1+\varepsilon_2, \varepsilon_3+\varepsilon_4)(1+n_2)n_3 \tag{29}$$

and $W_{1\to 4}$ analogously. Here we already introduced the density of states $g_j \equiv g(\varepsilon_j)$ which occur when taking the continuum limit. This also leads us to a replacement of the sum-

4. For an infinite system this is simply a delta function $G(\varepsilon_1+\varepsilon_2, \varepsilon_3+\varepsilon_4) \to \pi \cdot \delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)$ but in a finite system, as we want to have a look at here, the function may acquire a width due to off-shell scatterings between single-particle states which lie apart in the space of possible energies.

mations by integrations. Since bosons are interchangeable, $W_{14} = W_{41}$ and

$$W_{14} = W_{41} = W \left[\frac{1}{2}(\varepsilon_4 + \varepsilon_1), \underbrace{|\varepsilon_4 - \varepsilon_1|}_{=:x} \right] \tag{30}$$

for the case of a finite system where G acquires a width (cf. footnote on the last page). This means that the transition probabilities depend only on the absolute energy difference x and they are peaked around $x \simeq 0$.

Performing a gradient expansion of n_4 and $g_4 n_4$ around $x \simeq 0$ and introducing the transport coefficients

$$D(\varepsilon, t) = \frac{g_1}{2} \int_0^\infty \mathrm{d}x \, W(\varepsilon_1, x) \, x^2 \tag{31}$$

and

$$v(\varepsilon, t) = g_1^{-1} \frac{d}{d\varepsilon_1}(g_1 D) \tag{32}$$

via moments of the transition probability we arrive at a nonlinear partial differential equation for the number density:

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \varepsilon} \left[v \cdot n(1+n) + n \frac{\partial D}{\partial \varepsilon} \right] + \frac{\partial^2}{\partial \varepsilon^2} [Dn] \,. \tag{33}$$

Here $D(\varepsilon, t)$ is referred to as diffusion term and $v(\varepsilon, t)$ as drift term taking dissipative effects into account. Their explicit values are important since they define the equilibrium temperature T via the relation

$$T = -\frac{D}{v}. \tag{34}$$

The minus sign is explained by the fact that the drift is always towards the infrared i. e. the low-energy regime.

For our case of relativistic heavy-ion collisions it is well motivated to assume constant transport coefficients which allows us to simplify our equation to

$$\frac{\partial n}{\partial t} = -v \frac{\partial}{\partial \varepsilon} [n(1+n)] + D \frac{\partial^2 n}{\partial \varepsilon^2}, \tag{35}$$

with the well-known Bose-Einstein distribution

$$n_{\text{eq}}(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \mu}{T}\right) - 1} \tag{36}$$

as a stationary solution.

At this point we need to discuss some general features of this model. It does not resolve the second order phase transition to the condensate discussed earlier explicitly but nevertheless the kinetics of Bose condensation before and after the transition are taken into ac-

count. After some time t a certain fraction of bosons is pushed into the condensate which is a feature that may not be realized in nature. Of course it is based on particle-number conserving elastic scattering which may not provide the dominant contribution to the thermalization process as emphasized before. Another interesting feature of this certain model is that the particle number is only conserved when the integration is performed over the whole x-range opposed to the Boltzmann equation with energy-conserving delta function. We will see this in a moment when we analyze some plots of the solutions of the NBDE for different integration ranges.
To get a first analytical solution of the NBDE in the form of eqn. (35) we first want to study a linear approximation, the so-called linear relaxation ansatz (RTA).

5.2 Linear Relaxation Ansatz

For a given initial distribution $n_{\mathrm{t}}(\varepsilon)$ the linear relaxation-time ansatz reads

$$\frac{\partial n_{\mathrm{rel}}}{\partial t} = \frac{(n_{\mathrm{eq}} - n_{\mathrm{rel}})}{\tau_{\mathrm{eq}}}, \tag{37}$$

where we introduced the equilibration time $\tau_{\mathrm{eq}} = 4D/(9v^2)$ again as a certain ratio of the transport coefficients. General solutions to this simplified model are of the form of

$$n_{\mathrm{rel}}(\varepsilon, t) = n_{\mathrm{i}}(\varepsilon) \cdot \exp\left(-\frac{t}{\tau_{\mathrm{eq}}}\right) + n_{\mathrm{eq}}(\varepsilon)\left(1 - \exp\left(-\frac{t}{\tau_{\mathrm{eq}}}\right)\right). \tag{38}$$

Following ref. [3] we may use an initial distribution of the subsequent form in order to concentrate on a suitable distribution in the context of heavy-ion collisions, i. e.

$$n_{\mathrm{i}}(\varepsilon) = N_{\mathrm{i}} \cdot \theta\left(1 - \varepsilon/Q_{\mathrm{s}}\right) \cdot \theta(\varepsilon). \tag{39}$$

Here we find again the dependency on the saturation $Q_{\mathrm{s}} \simeq 1$ GeV as upper limit of the box-shaped initial gluon distribution.
The results for the thermalization process from the NBDE following from the RTA are displayed in figure 3 at the beginning of the next page. We observe that the thermal equilibrium is reached fast after a time of approximately $t \simeq 2\tau_{\mathrm{eq}}$ approaching the Bose-Einstein limit from above in the IR and from below in the UV. At the boundary $p \simeq Q_{\mathrm{s}}$ we encounter unphysical discontinuities. Note that here the total gluon number is set to $N_{\mathrm{i}} = 1$ to facilitate the comparison with the exact analytical solutions later on. Additionally we observe that we we are again dealing with an overoccupied system since

$$N_{\mathrm{i}} = (4/3)\pi \cdot V \cdot Q_{\mathrm{s}}^3 \cdot n_{\mathrm{i}} \quad > \quad N_{\mathrm{eq}} = 4\pi \cdot V \int_0^\infty n_{\mathrm{eq}} \cdot \varepsilon^2 \,\mathrm{d}\varepsilon \tag{40}$$

If we would assume overall particle number conservation we would come again to the conclusion that the excess particles are driven into a condensate such that not only the

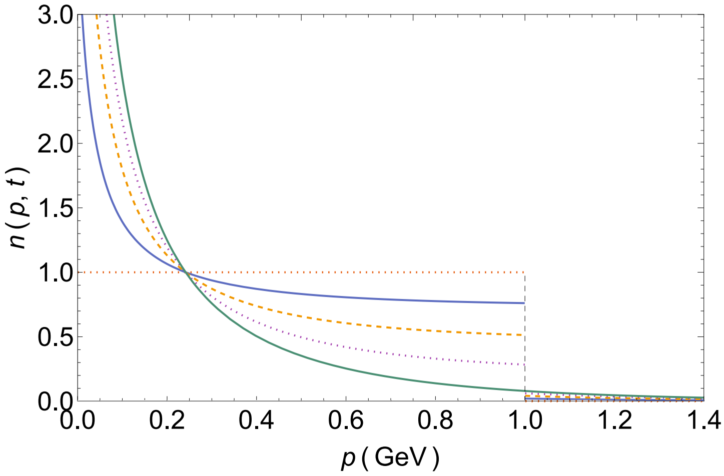


Figure 3: Linear relaxation of a finite Bose system towards the equilibrium [5]. Here $T = -D/v \simeq 0.4$ GeV, $\tau_{\text{eq}} = 4D/(9v^2) = 0.33 \cdot 10^{-23}\text{s} \simeq 1$ fm/c and the timesteps are $\{0.1, 0.25, 0.5, \infty\}$ (in units of $10^{-23}s$) from top to bottom.

gluon number in the thermal spectrum but rather the sum of the thermal gluons and the ones in the condensate is conserved.

5.3 Exact solution of the NBDE

To be able to find a solution to the full NBDE we perform the nonlinear transformation[5]

$$n(\varepsilon, t) = -\frac{D}{v}\frac{\partial \ln \mathcal{Z}(\varepsilon, t)}{\partial \varepsilon} \tag{41}$$

with the usual partition sum $\mathcal{Z}(\varepsilon, t)$ reducing our problem to a linear diffusion equation for the partial sum, i. e.

$$\frac{\partial \mathcal{Z}}{\partial t} = -v\frac{\partial \mathcal{Z}}{\partial \varepsilon} + D\frac{\partial^2 \mathcal{Z}}{\partial \varepsilon^2}. \tag{42}$$

General solutions of the NBDE are then of the form

$$n(\varepsilon, t) = \frac{1}{2v}\frac{\int_{-\infty}^{+\infty} \frac{\varepsilon - x}{t} F(x) \cdot G_{\text{free}}(\varepsilon - x, t)\, \mathrm{d}x}{\int_{-\infty}^{+\infty} F(x) \cdot G_{\text{free}}(\varepsilon - x, t)\, \mathrm{d}x} - \frac{1}{2}, \tag{43}$$

with the free Green's function, being an usual Gaussian,

$$G_{\text{free}}(\varepsilon - x, t) = \exp\left[-\frac{(\varepsilon - x)^2}{4Dt}\right], \tag{44}$$

and the function $F(x)$ taking the initial conditions into account, i. e.

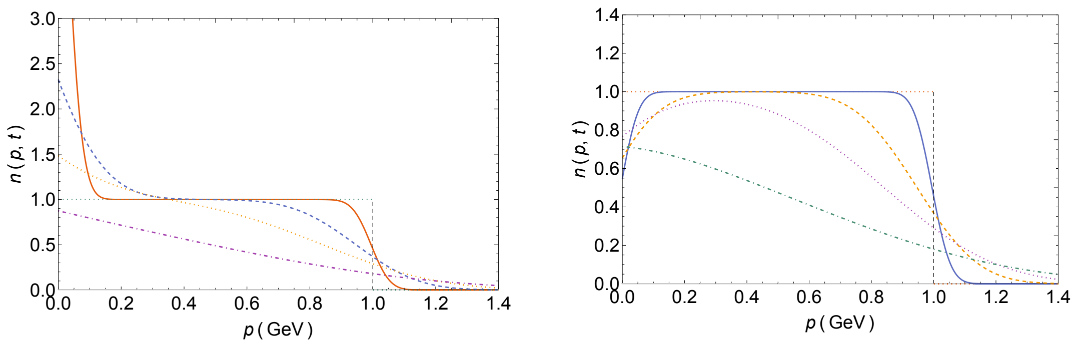
$$F(x) = \exp\left[-\frac{1}{2D}\left(vx + 2v\int_0^x n_{\text{i}}(y)\mathrm{d}y\right)\right]. \tag{45}$$

5. Another possible solution is given by Burger's equation $\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial \varepsilon} = D\frac{\partial^2 w}{\partial \varepsilon^2}$ following from the linear transformation $n(\varepsilon, t) = \frac{1}{2v}w(\varepsilon, t) - \frac{1}{2}$ but we will not discuss the details of this approach here.

They define the partition sum via the relation

$$\mathcal{Z}(\varepsilon, t) = a(t) \cdot \int_{-\infty}^{\infty} G_{\text{free}}(\varepsilon, x, t) \cdot F(x)\, \mathrm{d}x, \tag{46}$$

with some energy-independent scaling factor $a(t)$ which is in our case not important since it drops out due to the $\log$-derivative. As already mentioned before one may check that the particle number is not conserved for an integration over only the positive x-range. With these definitions in mind we are now set to present the results of the NBDE. At this point I should mention that I will not display the full lengthy expressions for the full analytical solution but rather focus on the plots and the implications arising from these different solutions. In case the reader may be interested in the exact expressions I refer to ref. [5] for the first part and to ref. [6] for the second part.



(a) Integration range restricted to $x \geq 0$. The timesteps are $\{0.005, 0.05, 0.15, 0.5\}$ (in units of $10^{-23}s$) from top to bottom.

(b) Integration range extended to $-\infty \leq x \leq \infty$. The timesteps are $\{0.005, 0.05, 0.15, 0.5\}$ (in units of $10^{-23}s$) from top to bottom.

Figure 4: Equilibration of a finite Bose system from the NBDE for $T \simeq 0.4$ GeV and $\tau_{\text{eq}} = 0.33 \cdot 10^{-23}$s [5].

In the above figure 4 we present the results for the equilibration following from the exact solutions for different integration ranges to highlight the importance of integrating over the whole x-range. In the left figure we find the occupation rising above the thermal limit for very short times but due to redistribution into the condensate it depletes and we have $n(0, t) < 1\ \forall t > \tau_{\text{eq}}$ which encodes the explicit violation of particle number conservation. In the right plot, representing the solution with integration over the whole range, we directly see the redistribution into the condensate in the IR already at early times and find a new thermal tail developed in the UV. We therefore conclude that these solutions do not reproduce the expected features of the Bose-Einstein equilibrium distribution, yet. This can be explained by the fact that we did not yet include the singularity of the Bose-Einstein distribution at $\varepsilon = \mu < 0$. We will see how to resolve this problem in the following.

5.4 Including the Singularity

To account for the singularity at $\varepsilon = \mu < 0$ we have to modify the initial distribution (39) in the following way:

$$\tilde{n}_{\mathrm{i}}(\varepsilon) = n_{\mathrm{i}}(\varepsilon) + \frac{1}{\exp\left(\frac{\varepsilon-\mu}{T}\right) - 1}. \tag{47}$$

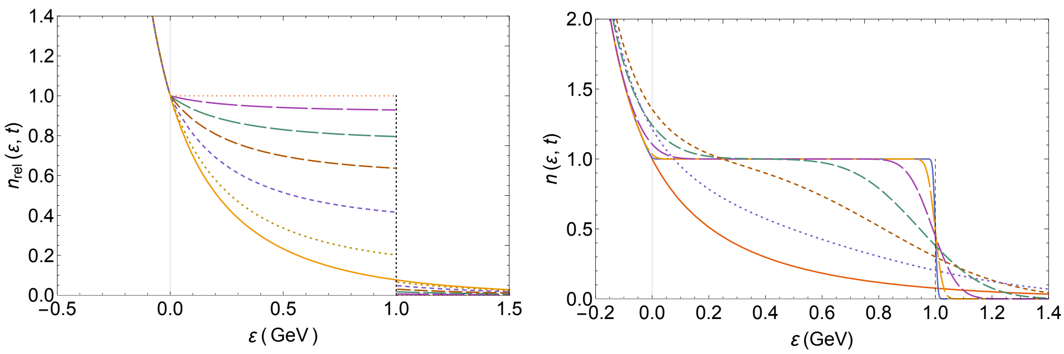
For the subsequent analysis we treat the chemical potential μ as a fixed parameter and consider the asymptotics of our distribution at the singularity

$$\lim_{\varepsilon \to \mu^+} n(\varepsilon, t) = \infty \; \forall t \tag{48}$$

yielding $\mathcal{Z}(\mu, t) = 0$. This results in a modified expression for the Green's function

$$G(\varepsilon, x, t) = G_{\mathrm{free}}(\varepsilon - \mu, x, t) - G_{\mathrm{free}}(\varepsilon - \mu, -x, t). \tag{49}$$

This means that in general our expressions for the partition sum and the function F stay almost the same except for a shift in the respective argument by the chemical potential μ. The techniques to solve the NBDE with this boundary condition remain the same and we can therefore have a look at the solutions displayed below in figure (5).



(a) Local thermalization of gluons in the linear RTA. The timesteps are $\{0.02, 0.08, 0.15, 0.3, 0.6\}$ (fm$/c$) for decreasing dash length.

(b) Local thermalization of gluons from the time-dependent solutions of the NBDE. Here $\{6 \cdot 10^{-5}, 6 \cdot 10^{-4}, 6 \cdot 10^{-3}, 0.12, 0.36\}$ (fm$/c$) for decreasing dash length.

Figure 5: Results for the full solution of the NBDE including the singularity at $\varepsilon = \mu < 0$ at $T \simeq 513$ MeV [6].

In the left figure we see again the simplified solution arising from the RTA where we can already observe that the solution entails the correct physical properties of the Bose-Einstein equilibrium distribution in the UV and finally also in the IR except the discontinuities at $p \simeq Q_{\mathrm{s}}$. Note that here the thermalization occurs much slower than in the nonlinear case on the right hand side.

The full nonlinear solution presented in the right figure now appropriately describes the thermalization process of a finite gluon system encoding all expected properties in the UV and also in the IR opposed to the first solutions omitting the important boundary condition at $\varepsilon = \mu$. Note that in our concrete example the numerical values of the transport coefficients have been chosen such that they represent the situation of a Pb-Pb collision at the LHC with a center-of-mass energy of $\sqrt{s} = 5$ TeV leading to a thermalization with $T \simeq 513$ MeV.

6 Conclusion and Outlook

The understanding of the thermalization process of gluons plays a central role in finding an appropriate theoretical description of the complex interplay of physical processes during relativistic heavy-ion collisions.
Using kinetic theory and statistical transport equations we were able to estimate important quantities such as the equilibration time τ_{eq} by understanding thermalization as a dynamical interplay of elastic and inelastic scatterings.
One explanation for the apparent excess of particles in the thermal spectrum compared to the expected equilibrium distribution is the formation of a Bose-Einstein condensate which may survive during most of the thermalization process. This option seems however not to be the favored interpretation amongst the researchers nowadays as the understanding of the various inelastic contributions to the gluon scattering is understood better than back when the idea of the condensate came up first.
In the second part we focused on the derivation and solution of a nonlinear boson diffusion equation providing further insights into the thermalization process and an analytically accessible model to study different aspects of thermalization in more detail.
In the future one may elaborate on the different approximation schemes and solution techniques for example by considering the time- and energy-dependence of the values of the transport coefficients or by extending the model in more than $1+1$ spacetime dimensions to be able to account for example for possible anisotropies.

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