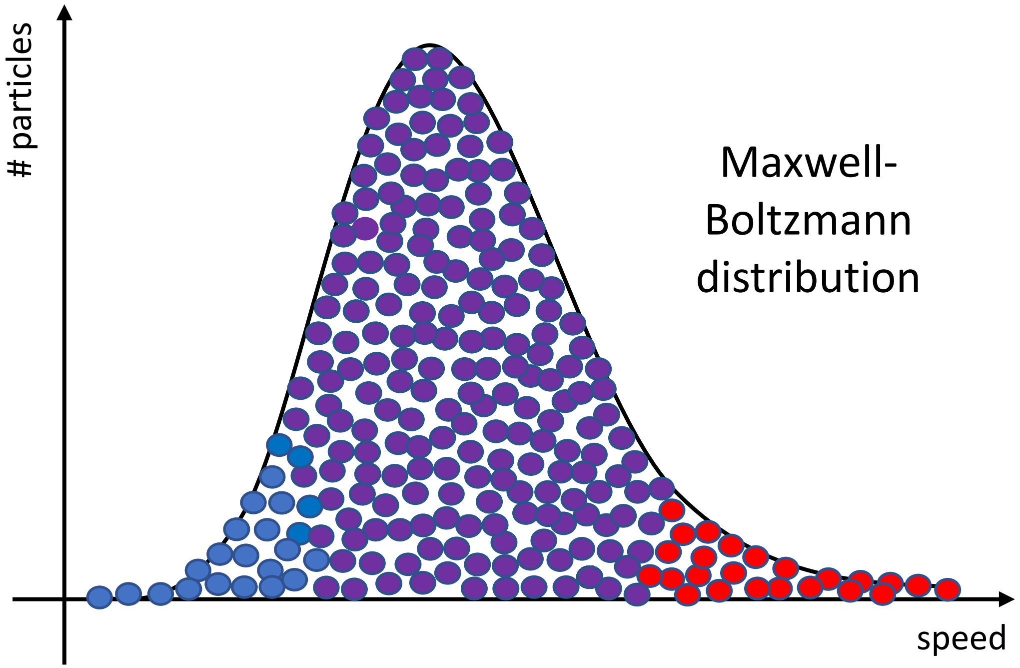
Evaporative Cooling & Thermalization of bosonic cold quantum gases

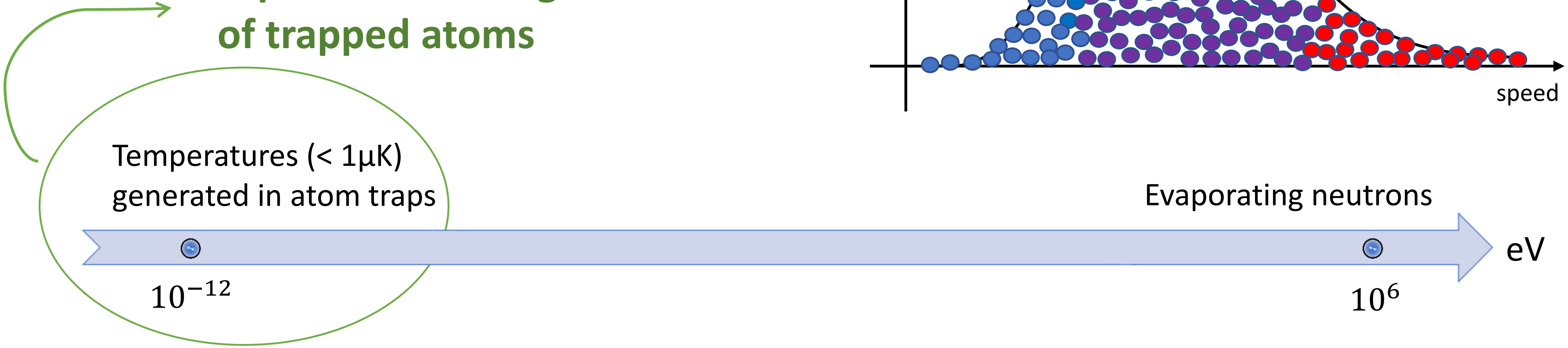
Tara Butler

Content

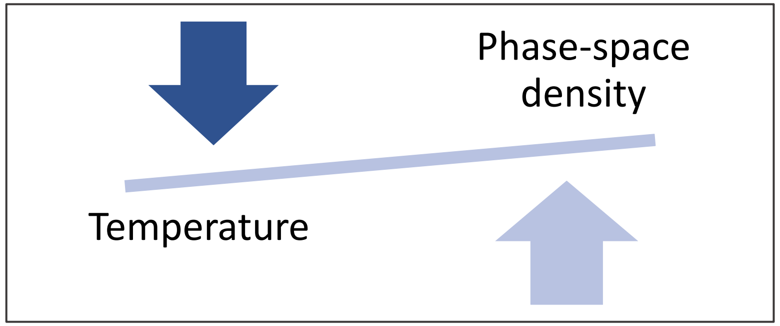
What is Evaporative Cooling?



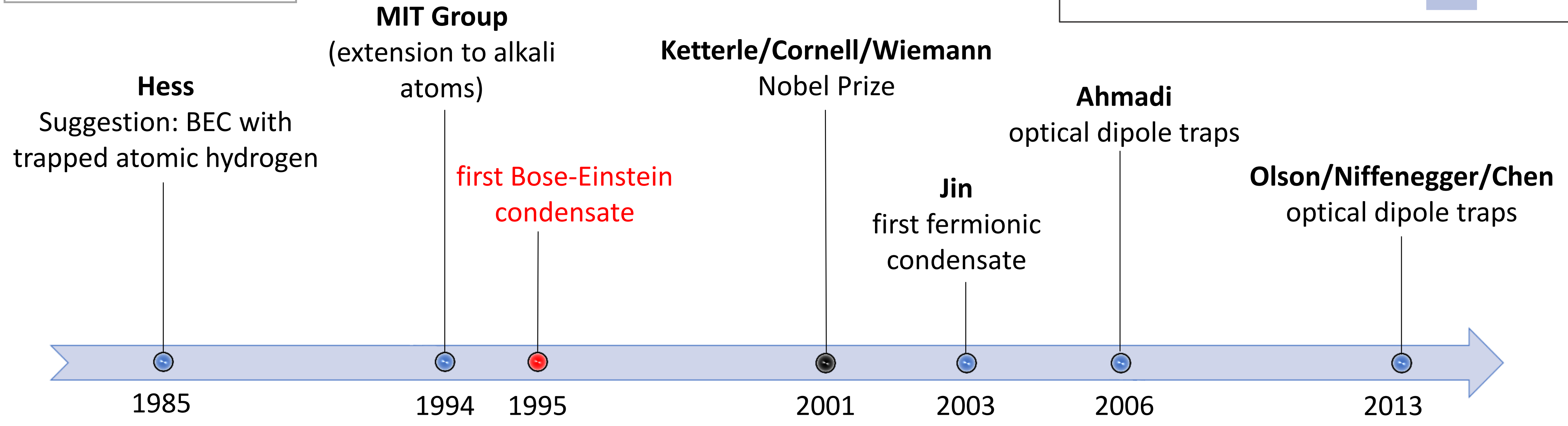
Evaporative Cooling of trapped atoms



Why Evaporative Cooling and **what** is it good for?



A cool history . . .



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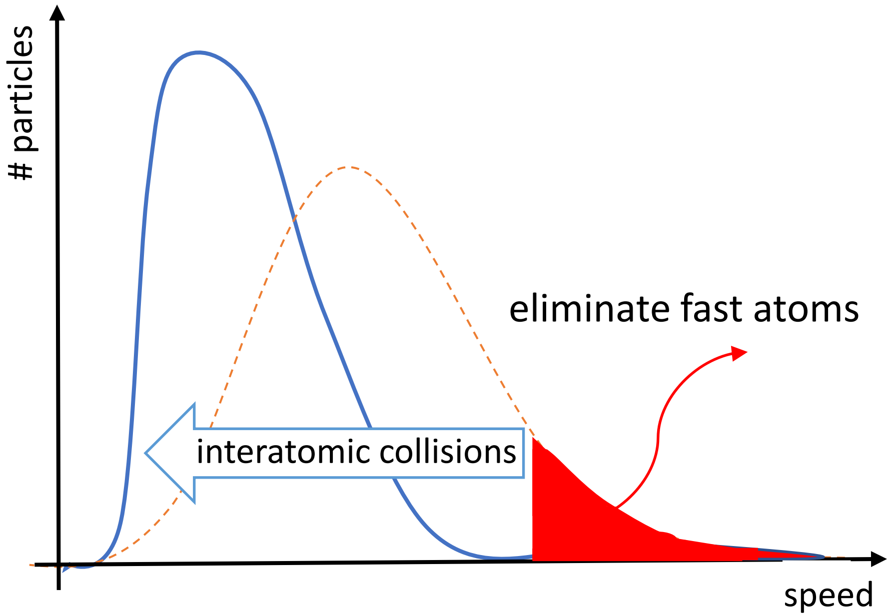
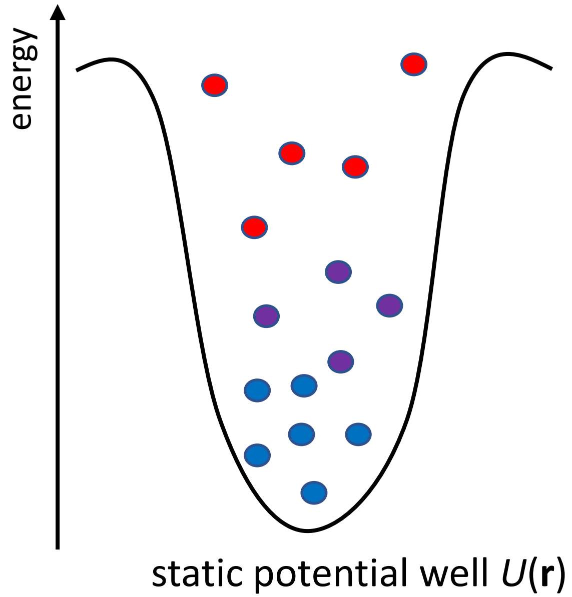
Basic assumption:
Sufficient ergodicity
Example:
ergodic single-particle motion





$$\rho(\varepsilon) \equiv (2\pi\hbar)^{-3} \int d\boldsymbol{r}\, d\boldsymbol{p}\;\; \delta[\varepsilon - U(\boldsymbol{r}) - p^2/2m] \;=\; \frac{2\pi(2m)^{3/2}}{(2\pi\hbar)^3} \int_{U(\boldsymbol{r})\le\epsilon} d\boldsymbol{r}\,\sqrt{\varepsilon - U(\mathbf{r})}$$

$$\rho(\varepsilon) = A_{PL}\varepsilon^{1/2+\delta}$$

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Truncated energy distribution

$$f(\varepsilon) = \sum_{l=1}^{\infty} z^l \exp(-l\,\varepsilon/k\,T)\theta(\varepsilon_t - \varepsilon)$$

$$f(\varepsilon) \;=\; z\mathrm{e}^{-\varepsilon/kT}\theta(\varepsilon_t - \varepsilon)$$

with $N = z\int_0^{\varepsilon_t} d\varepsilon\,\rho(\varepsilon)\,\mathrm{e}^{-\varepsilon/kT}$

Reminder: distribution function

$$f(\varepsilon) = \frac{1}{z^{-1}\exp(-\varepsilon/k\,T) - 1}$$

$$= \sum_{l=1}^{\infty} z^l \exp(-l\,\varepsilon/k\,T)$$

fugacity: $z = \exp(\mu/k\,T)$

$$N = \int \mathrm{d}\varepsilon\,\rho(\varepsilon) f(\varepsilon)$$

Canonical partition function

$$N = z \int_0^{\varepsilon_t} d\varepsilon \, \rho(\varepsilon) \, e^{-\varepsilon/kT} = n_0 \Lambda^3 \varsigma$$

$$\varsigma = (2\pi\hbar)^{-3} \int d\boldsymbol{p} \, d\boldsymbol{r} \, e^{-H(\boldsymbol{r},\boldsymbol{p})/kT} \, \theta[\varepsilon_t - H(\boldsymbol{r},\boldsymbol{p})]$$

$$z = n_0 \Lambda^3$$

$$\Lambda = [2\pi\hbar^2/mkT]^{1/2}$$

$$\varsigma = \int_0^{\varepsilon_t} d\varepsilon \, \rho(\varepsilon) \, e^{-\varepsilon/kT}$$

$$H(\boldsymbol{r},\boldsymbol{p}) = U(\boldsymbol{r}) + p^2/2\,m$$

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Density distribution

$$n(\boldsymbol{r}) = \underbrace{n_0 e^{-U(\boldsymbol{r})/KT}}_{= n_\infty(\boldsymbol{r})} P\left(\frac{3}{2}, \eta_t(\boldsymbol{r})\right)$$

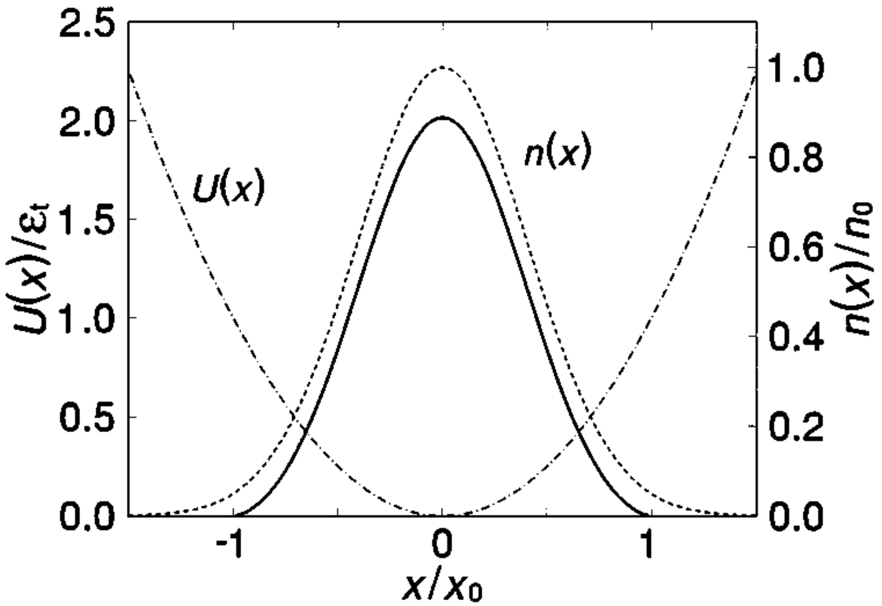

Figure: Density distribution, [4]

$$N = \int dr\, n(\boldsymbol{r})$$

$$\eta_t(\boldsymbol{r}) \equiv (\varepsilon_t - U(\boldsymbol{r}))/k\,T$$

$$P\left(\frac{3}{2}, \eta_t(\boldsymbol{r})\right) = \sqrt{\eta_t} - 2\sqrt{\eta_t/\pi}\exp(-\eta_t)$$

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Reference Volume

$$V_e = N/n_0 = \Lambda^3 \varsigma = \Lambda^3 \int_0^{\varepsilon_t} d\varepsilon\, \rho(\varepsilon)\, \mathrm{e}^{-\varepsilon/kT}$$

Power-law traps:

$$\varsigma = A_{PL} \int_0^{\varepsilon_t} d\varepsilon\, \varepsilon^{1/2+\delta} \mathrm{e}^{-\varepsilon/kT} = \varsigma_\infty P\left(\frac{3}{2}+\delta, \eta\right)$$

$$\varsigma_\infty = A_{PL}[kT]^{3/2+\delta}\Gamma\left(\frac{3}{2}+\delta\right)$$

$$V_e = \Lambda^3 \varsigma_\infty P\left(\frac{3}{2}+\delta, \eta\right)$$

$$V_e \propto T^\delta \quad \text{for } \eta \to \infty$$

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Internal energy

$$E = n_0 \Lambda^3 \int_0^{\varepsilon_t} d\varepsilon\, \rho(\varepsilon)\, e^{-\varepsilon/kT} = NkT^2 \frac{1}{\varsigma}\frac{\partial \varsigma}{\partial T} = \left(\frac{3}{2} + \tilde{\gamma}\right) NkT$$

Power-law traps: $E = E_\infty R\left(\frac{3}{2} + \delta, \eta\right)$

$$dE = C_{\varepsilon_t}\, dT + \mu_{\varepsilon_t}\, dN$$

$$C_{\varepsilon_t} = \left(\frac{3}{2} + \tilde{\gamma} + T(\partial\,\tilde{\gamma}/\partial\, T)_{\varepsilon_t}\right) Nk \qquad \mu_{\varepsilon_t} = \left(\frac{3}{2} + \tilde{\gamma}\right) kT$$

$$\varsigma = \int_0^{\varepsilon_t} \mathrm{d}\varepsilon\, \rho(\varepsilon) e^{-\varepsilon/kT} = V_e/\Lambda^3$$

$$\tilde{\gamma} = \left(\frac{\partial \ln V_e}{\partial \ln T}\right)_{\varepsilon_t}$$

$$R(a,\eta) \equiv P\,(a+1,\eta)/P\,(a,\eta)$$

$$E_\infty = \left(\frac{3}{2} + \delta\right) NkT$$

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Kinetic equation

Boltzmann equation: $\left(\frac{\boldsymbol{p}}{m} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} - \boldsymbol{\nabla}_{\boldsymbol{r}} U \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} + \frac{\partial}{\partial t}\right) f(\boldsymbol{r}, \boldsymbol{p}) = \boldsymbol{c}(\boldsymbol{r}, \boldsymbol{p})$

collision integral: $\boldsymbol{c}(\boldsymbol{r}, \mathbf{p_4}) = \frac{\sigma}{(2\pi\hbar)^3 2\pi m} \int d^3p_3 \, d\Omega' q \{(f(\boldsymbol{r}, \boldsymbol{p_1}) f(\mathbf{r}, \boldsymbol{p_2}) - f(\boldsymbol{r}, \boldsymbol{p_3}) f(\boldsymbol{r}, \boldsymbol{p_4})\}$

sufficient ergodicity

$$f(\boldsymbol{r}, \boldsymbol{p}) = \int d\varepsilon \, \delta(U(\boldsymbol{r}) + p^2/2\,m - \varepsilon) f(\varepsilon)$$

$$\rho(\varepsilon_4)\dot{f}(\varepsilon_4) = \frac{m\sigma}{\pi^2\hbar^2} \int d\varepsilon_1 \, d\varepsilon_2 \, d\varepsilon_3 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \rho(\min[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]) \{f(\varepsilon_1) f(\varepsilon_2) - f(\varepsilon_3) f(\varepsilon_4)\}$$

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Numerical solution of the kinetic equation

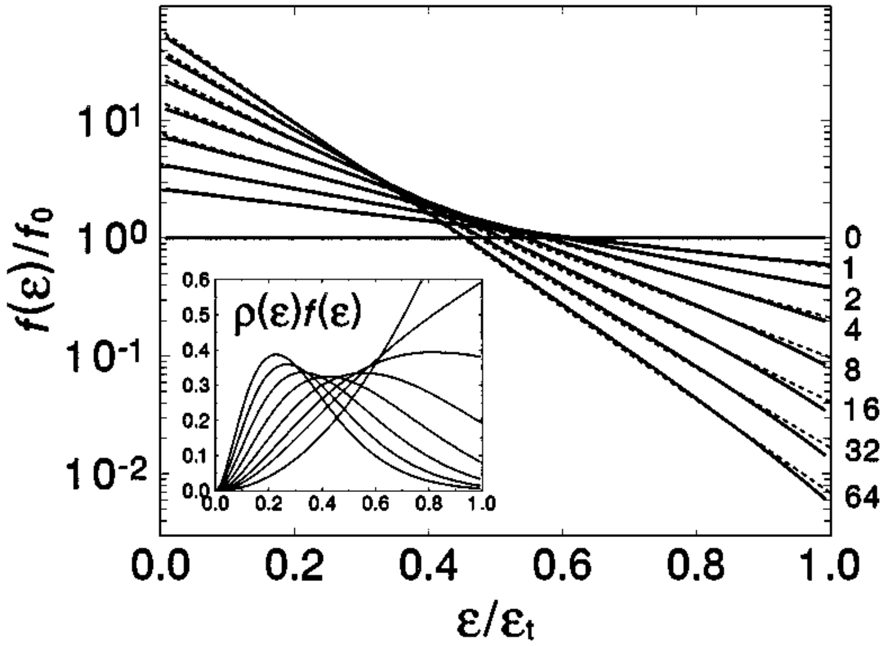

Figure: Evolution of the distribution function $f(\epsilon)$, [4]

$$\rho(\varepsilon_4)\dot{f}(\varepsilon_4) = \frac{m\sigma}{\pi^2\hbar^2}\int d\varepsilon_1\, d\varepsilon_2\, d\varepsilon_3 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\rho(\min[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4])\{f(\varepsilon_1)f(\varepsilon_2) - f(\varepsilon_3)f(\varepsilon_4)\}$$

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Evaporation Rate

$$\dot{N}_{ev} = -\int_{\varepsilon_t}^{\infty} d\varepsilon_4\, \rho(\varepsilon_4)\dot{f}(\varepsilon_4)$$

$$= -\frac{m\sigma}{\pi^2\hbar^2}\int_0^{\varepsilon_t} d\varepsilon_3 \int_{\varepsilon_3}^{\varepsilon_t} d\varepsilon_2 \int_{\varepsilon_3+\varepsilon_t-\varepsilon_2}^{\varepsilon_t} d\varepsilon_1\, \rho(\varepsilon_3)f(\varepsilon_1)f(\varepsilon_2)$$

$$= -n_0^2\bar{v}\sigma \mathrm{e}^{-\eta}V_{ev}$$

$$\varepsilon_4 > \varepsilon_t > \varepsilon_1, \varepsilon_2$$

$$\varepsilon_3 = \varepsilon_1 + \varepsilon_2 - \varepsilon_4$$

$$\bar{v} \equiv (8kT/\pi m)^{1/2}$$

$$V_{ev} = \frac{\Lambda^3}{kT}\int_0^{\varepsilon_t} d\varepsilon\, \rho(\varepsilon)\left((\varepsilon_t - \varepsilon - kT)\mathrm{e}^{-\varepsilon/kT} + kT\mathrm{e}^{-\varepsilon_t/kT}\right)$$

Kinetic equation: $\rho(\varepsilon_4)\dot{f}(\varepsilon_4) = \frac{m\sigma}{\pi^2\hbar^2}\int d\varepsilon_1\, d\varepsilon_2\, d\varepsilon_3 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\rho(\min[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4])\{f(\varepsilon_1)f(\varepsilon_2) - f(\varepsilon_3)f(\varepsilon_4)\}$

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Loss of internal energy

$$\dot{E} = \dot{E}_{ev} + \dot{E}_t$$

$$\dot{E} = \dot{E}_{ev} = -\int_{\varepsilon_t}^{\infty} d\varepsilon_4\, \varepsilon_4\, \rho(\varepsilon_4)\dot{f}(\varepsilon_4)$$

$$= \dot{N}_{ev}(\varepsilon_t + (1 - X_{ev}/V_{ev})kT)$$

with $$X_{\mathrm{ev}} = \frac{\Lambda^3}{kT}\int_0^{\varepsilon_t} d\varepsilon\, \rho(\varepsilon)[kT\mathrm{e}^{-\varepsilon/kT} - (\varepsilon_t - \varepsilon + kT)\mathrm{e}^{-\varepsilon_t/kT}]$$

forced evaporation

$$\dot{E}_t = \varepsilon_t \rho(\varepsilon_t) f(\varepsilon_t)\dot{\varepsilon}_t = \varepsilon_t \dot{N}_t$$

Kinetic equation: $$\rho(\varepsilon_4)\dot{f}(\varepsilon_4) = \frac{m\sigma}{\pi^2 h^2}\int d\varepsilon_1\, d\varepsilon_2 d\varepsilon_3 \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\rho(\min[\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4])\{f(\varepsilon_1)f(\varepsilon_2) - f(\varepsilon_3)f(\varepsilon_4)\}$$

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Comparison of results

$$\dot{E} = C\dot{T} + \mu\dot{N}$$

differential equations

$$\dot{T} = \frac{\dot{E}_{ev} - \mu\dot{N}_{ev}}{C}$$

$$\dot{N} = \dot{N}_{ev}$$

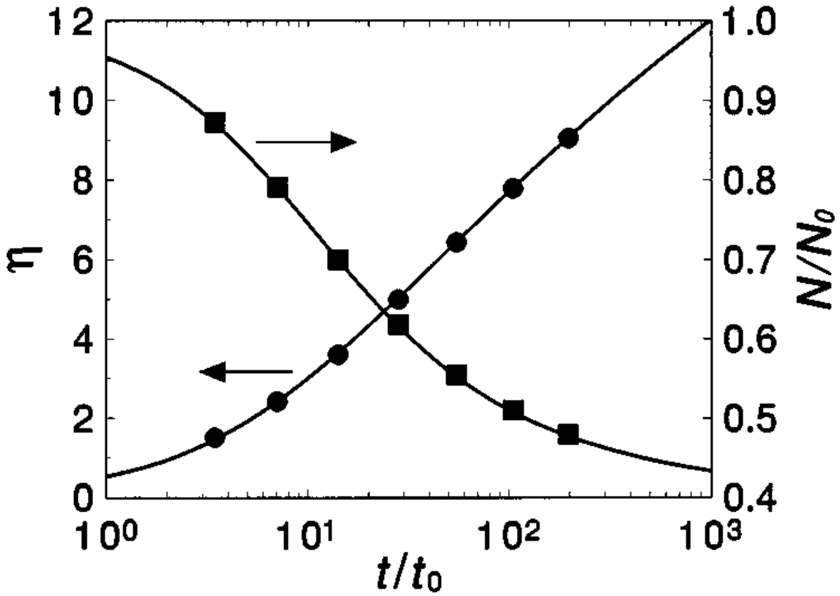

Figure: Truncation parameter η and fraction of atoms remaining in the trap N/N_0 as a function of reduced time t/t_0 , [4]

$$\dot{E} = \dot{E}_{ev} + E_t$$

$$\dot{E}_t = \varepsilon_t \rho(\varepsilon_t) f(\varepsilon_t) \dot{\varepsilon}_t = \varepsilon_t \dot{N}_t$$

$$\dot{N} = \dot{N}_{ev} + \dot{N}_t$$

Relaxation & Heating

$$\dot{N}_{rel} = -\int d^3r\, G(\boldsymbol{r}) n^2(\boldsymbol{r})$$

$$\Rightarrow \quad \tau_{rel}^{-1} = n_0\; G\, V_{2e}/V_e$$

$$\dot{E}_{rel} = \dot{N}_{rel}\left(\frac{3}{2} + \tilde{\gamma}_2\right) kT$$

$$\tau_{rel}^{-1} = -\left(\dot{N}_{rel}/N\right)$$

$$V_{2\mathrm{e}} = \int d^3r\, (n\,(\boldsymbol{r})/n_0)^2$$

$$\tilde{\gamma}_2 = (T/(2V_{2e}))\partial\, V_{2e}/\partial\, T$$

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Thermalization ratio

$$\dot{E}_{ev} + \dot{E}_{rel} = \dot{N}_{ev}\left(\eta + (1 - X_{ev}/V_{ev}) + \left(\frac{3}{2} + \tilde{\gamma}_2\right)/R\right)kT$$

thermalisation ratio: $R = \frac{\bar{v}\sigma}{G}\frac{V_{ev}}{V_{2e}}\mathrm{e}^{-\eta}$

$$\dot{N}_{ev} + \dot{N}_{rel} = \dot{N}_{ev}(1 + 1/R)$$

$$\dot{E}_{ev} = \dot{N}_{ev}(\varepsilon_t + (1 - X_{ev}/V_{ev})kT)$$

$$\dot{E}_{rel} = \dot{N}_{rel}\left(\frac{3}{2} + \tilde{\gamma}_2\right)kT$$

$$R = \dot{N}_{ev}/\dot{N}_{rel}$$

$$\tau_{ev}^{-1} = -(\dot{N}_{ev}/N)$$
$$= n_0\,\bar{v}\sigma\mathrm{e}^{-\eta}V_{ev}$$

Cooling atomic hydrogen

$$kT^* = \frac{\pi m G^2}{16\sigma^2}$$

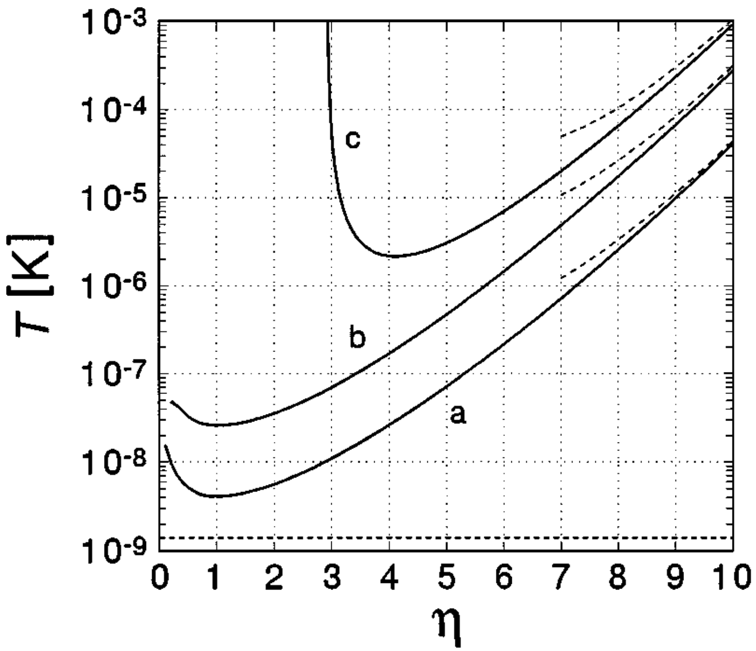

Figure: Characteristic temperatures of H in a harmonic trap as a function of η, [4]

$G \cong 10^{-15} \mathrm{cm}^3 \mathrm{s}^{-1}$

$a \cong 0.072 \mathrm{nm}$

$T^* \cong 1.4 \mathrm{nK}$

Scaling Laws

$$\alpha = \frac{d(\ln T)}{d(\ln N)} = \frac{\dot{T}/T}{\dot{N}/N}$$

$$\alpha = \frac{\eta + \kappa}{\delta + 3/2} - 1$$

SCALING LAWS FOR EVAPORATIVE COOLING IN A d-DIMENSIONAL POTENTIAL $U(r) \propto r^{d/\delta}$

Quantity	Exponent , x
Number of atoms, N	1
Temperature, T	α
Volume, V	$\delta\alpha$
Density, n	$1 - \delta\alpha$
Phase-space density, D	$1 - \alpha(\delta + 3/2)$
Elastic collision rate, $n\sigma v$	$1 - \alpha(\delta - 1/2)$

[5]

$$\kappa \equiv 1 - X_{ev}/V_{ev}$$

Runaway Evaporation

$$\frac{\mathrm{d}(n\sigma v)}{\mathrm{d}t}/\mathrm{n}\sigma v = \frac{1}{\tau_{el}}\left(\frac{\alpha(\delta-1/2)-1}{\lambda} - \frac{1}{R}\right)$$

$$R \geq R_{min} = \frac{\lambda}{\alpha(\delta-1/2)-1}$$

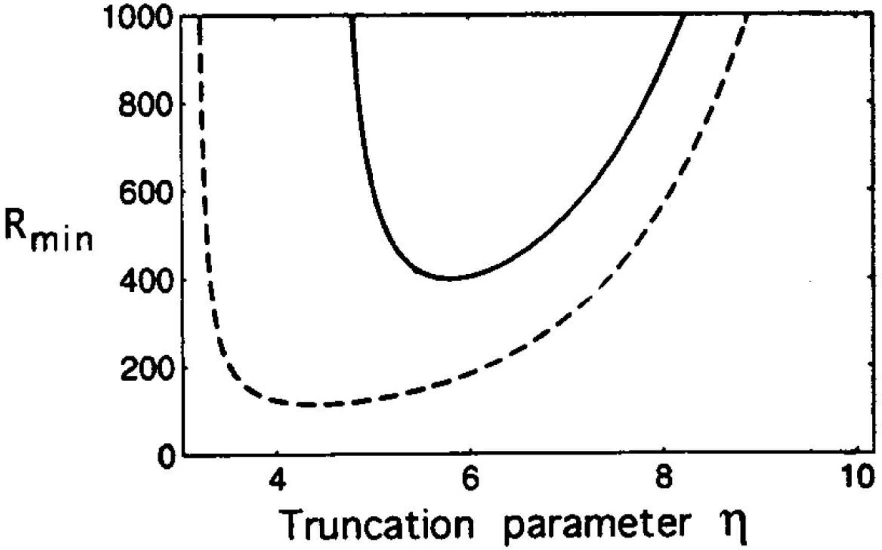

Figure: The minimum ratio R_{min} versus the truncation parameter η, [5]

$$\dot{N} = -Nn_0\sigma\bar{v}\eta e^{-\eta} = -\frac{N}{\tau_{ev}}$$

$$\lambda = \tau_{ev}/\tau_{el}$$

$$1/\tau_{el} = n_0\sigma\bar{v}\sqrt{2}$$

Quantity	Exponent, x
$n\sigma v$	$1-\alpha(\delta-1/2)$

Runaway Evaporation

$$\beta = 100\tau_{el}\frac{\mathrm{d}}{\mathrm{d}t}(\log_{10} D)$$

$$= \frac{100}{\ln 10}\left(\frac{\alpha(\delta + 3/2) - 1}{\lambda} - \frac{1}{R}\right)$$

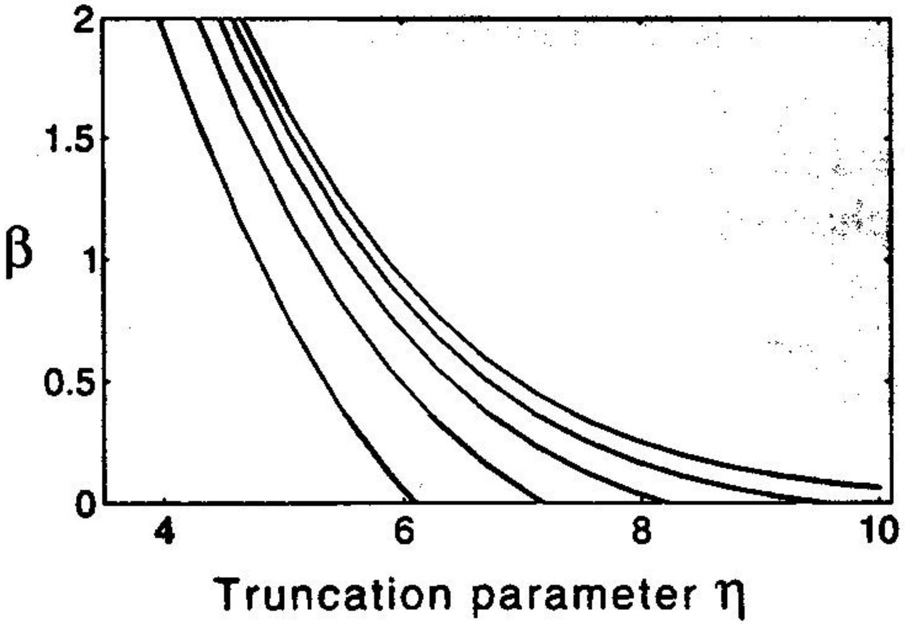

Figure: Logarithmic phase-space density increase β per 100 elastic collision times, [5]

Quantity	Exponent, x
D	$1 - \alpha(\delta + 3/2)$

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Maximizing Phase-Space Density

$$\gamma = -\frac{d(\ln D)}{d(\ln N)} = \frac{\alpha(\delta + 3/2)}{1 + \lambda/R} - 1$$

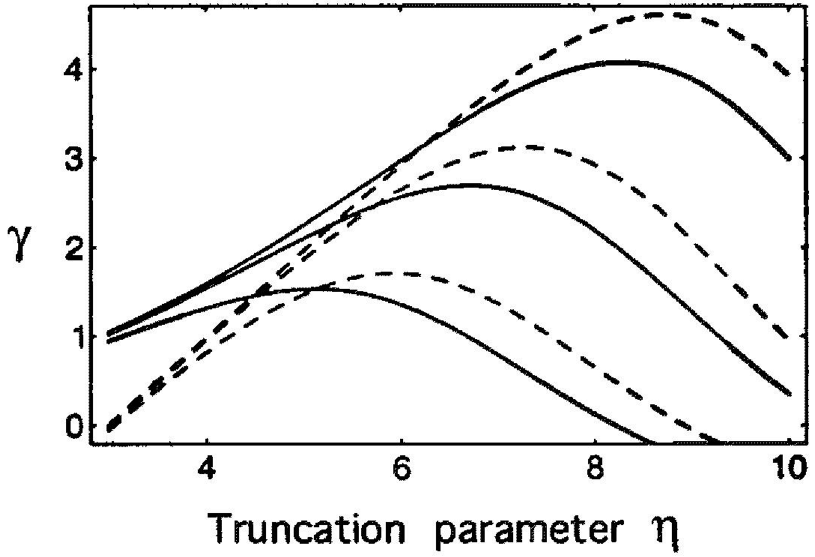

Figure: Efficiency parameter γ versus the truncation parameter η, [5]

$$-\frac{d(\ln R)}{d(\ln N)} = \frac{\alpha(\delta - 1/2)}{1 + \lambda/R} - 1$$

Alkali atoms
Background gas collisions

increased R during evaporation

Three-body recombination/ dipolar relaxation

Atomic hydrogen
Inelastic binary collision

increased R during evaporation

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Nonlinear boson diffusion equation (NBDE)

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \varepsilon}\left(vn(1+n) + n\frac{\partial D}{\partial \varepsilon}\right) + \frac{\partial^2}{\partial \varepsilon^2}(Dn)$$

in the limit of constant transport coefficients v, D: $\frac{\partial n}{\partial t} = -v\frac{\partial}{\partial \varepsilon}(\mathrm{n}(1+n)) + D\frac{\partial^2 n}{\partial \varepsilon^2}$

$$n(\varepsilon,t) = -\frac{D}{v}\frac{\partial}{\partial \varepsilon}\ln Z\,(\varepsilon,t) - \frac{1}{2} = -\frac{D}{v}\frac{1}{Z}\frac{\partial Z}{\partial \varepsilon} - \frac{1}{2}$$

$$n_{eq}(\varepsilon) = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T} - 1}$$

$$T = -D/v$$

linear diffusion equation:

$$\frac{\partial}{\partial t}Z(\varepsilon,t) = D\frac{\partial^2}{\partial \varepsilon^2}Z(\varepsilon,t)$$

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Free solutions

$$Z_{free}(\varepsilon,t) = a(t) \int_{-\infty}^{+\infty} G_{free}(\varepsilon,x,t)\, F(x)\, \mathrm{d}x$$

with $G_{free}(\varepsilon,x,t) = \exp\left(-\frac{(\varepsilon - x)^2}{4Dt}\right)$

$$\mathrm{F(x)} = \exp\left(-\frac{1}{2D}\left(vx + 2v\underbrace{\int_0^x n_i(y)\,\mathrm{d}y}_{= A_i(x)}\right)\right)$$

Inclusion of a singularity at x = μ . . .

$$\mu' = \frac{D}{v}\ln(z^{-1} - \exp(-\varepsilon_i/T))$$

$$z = \exp(\mu/T)$$

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Exact solutions with boundary conditions

$$Z(\varepsilon,t) = \int_0^\infty G(\varepsilon,x,t)F(x+\mu)\,\mathrm{d}x$$

with $G(\varepsilon,x,t) = G_{free}(\varepsilon-\mu,x,t) - G_{free}(\varepsilon-\mu,-x,t)$

truncated thermal eqilibrium distribution

$$n_i(\varepsilon) = \frac{1}{\mathrm{e}^{(\varepsilon-\mu)/T}-1}\theta(1-\varepsilon/\varepsilon_i)$$

$$n(\epsilon,t) = \frac{1}{\exp\left(\frac{\varepsilon-\mu}{T}\right)L(\varepsilon,t)-1}$$

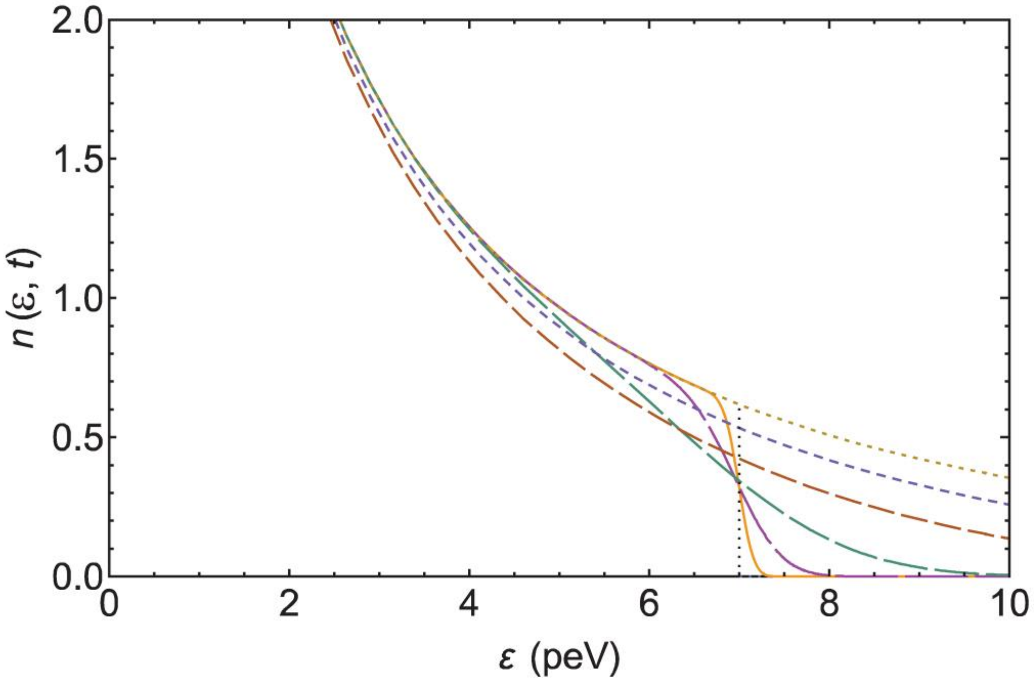

Figure: Equilibration of a finite Bose system, transport coefficients: $D = 8\times 10^3 \mathrm{peV}^2 s^{-1}, \mathrm{v} = -1\times 10^3 \mathrm{peV} s^{-1}$, $\mathrm{T} = -\mathrm{D}/\mathrm{v} = 8\,\mathrm{peV} \cong 93\mathrm{nK}\,(\mathrm{const.})$
time evolution: t = 0.001, 0.01, 0.1, 1, 4 and 40ms (decreasing dash lengths), [13]

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Relaxation-time approximation

$$\partial n_{rel}/\partial t = \left(n_{eq} - n_{rel}\right)/\tau_{eq}$$

$$n_{rel}(\varepsilon, t) = n_i(\varepsilon)\ \mathrm{e}^{-t/\tau_{eq}} + n_{eq}(\varepsilon)\left(1 - \mathrm{e}^{-t/\tau_{eq}}\right)$$

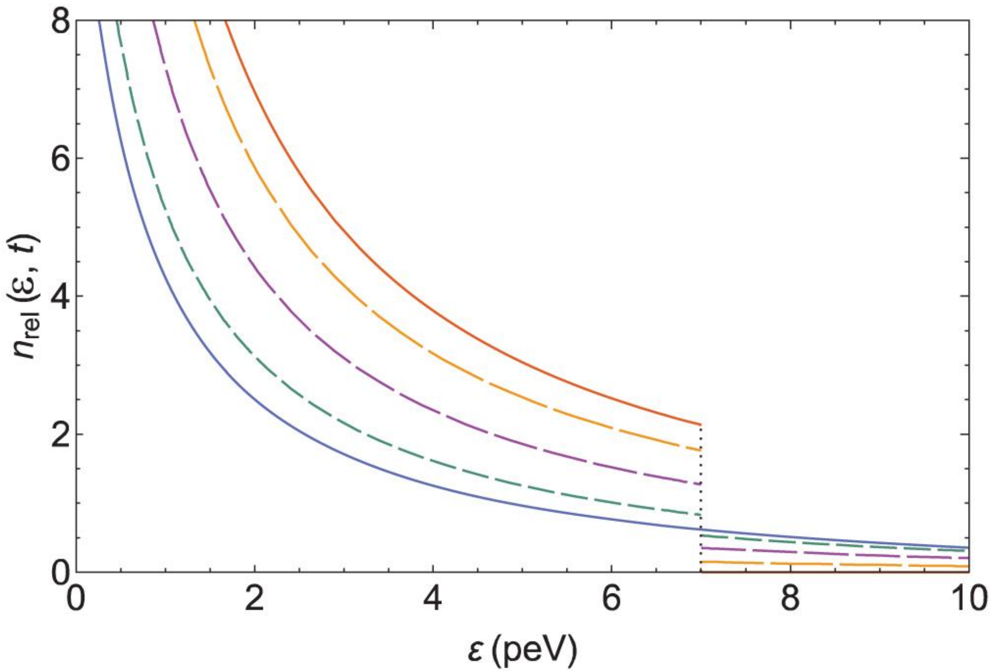

Figure: Evaporative cooling in a bosonic system, $T_i = 20\ \mathrm{peV} \cong 232\mathrm{nK}$ (upper solid curve), $T_f = 8\mathrm{peV} \cong 93\mathrm{nK}$ (lower solid curve), time evolution: $t = 1, 3$ and $7\mathrm{ms}$ (decreasing dash lengths), [13]

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Nonlinear boson diffusion equation

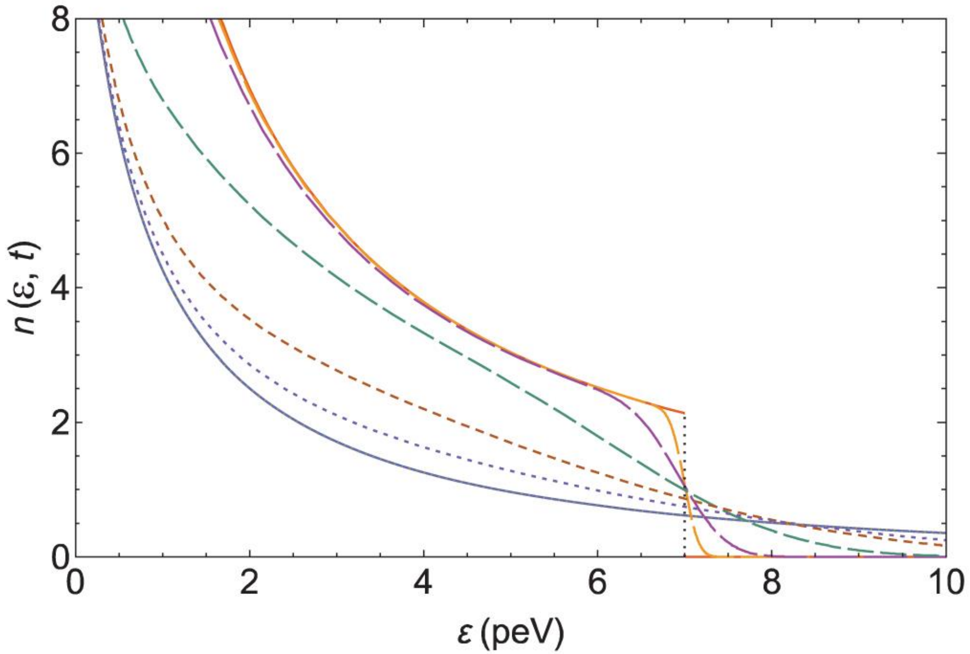

Figure: Evaporative cooling in a bosonic system, $T_i = 20\text{ peV} \cong 232\text{nK}$ (upper solid curve), $T_f = 8\text{peV} \cong 93\text{nK}$ (lower solid curve), time evolution: t = 0.001, 0.01, 0.1, 0,4 and 0.8ms (decreasing dash lengths), [13]

Relaxation–time approximation

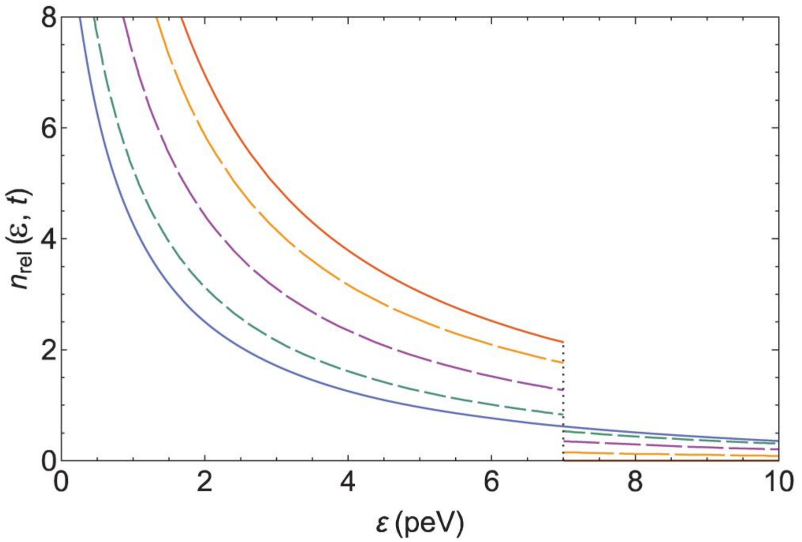

Figure: Evaporative cooling in a bosonic system, $T_i = 20\text{ peV} \cong 232\text{nK}$ (upper solid curve), $T_f = 8\text{peV} \cong 93\text{nK}$ (lower solid curve), time evolution: $t = 1, 3$ and 7ms (decreasing dash lengths), [13]

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Time-dependent entropy

$$S(t) = \int_0^\infty g(\varepsilon)\left[\ln\left(1 + n(\varepsilon,t)\right) + n(\varepsilon,t)\ln\left(1 + 1/n(\varepsilon,t)\right)\right] \mathrm{d}\varepsilon$$

Power-law: $g(\varepsilon) = g_0\sqrt{\varepsilon}$

$$g_0 = (2m)^{3/2}V/(4\pi^2)$$

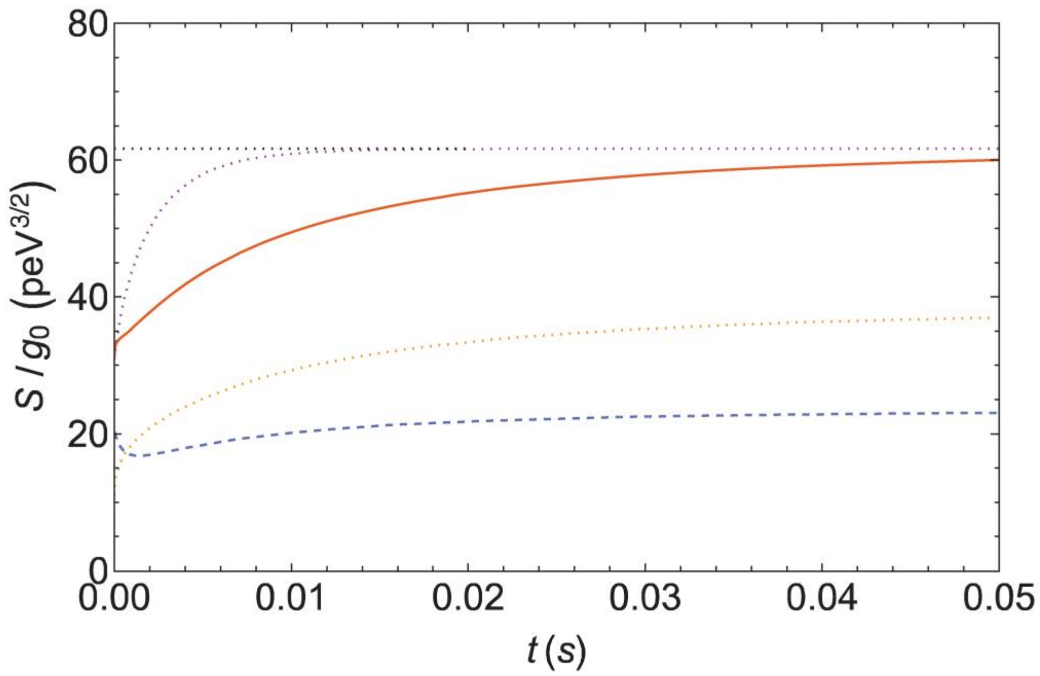
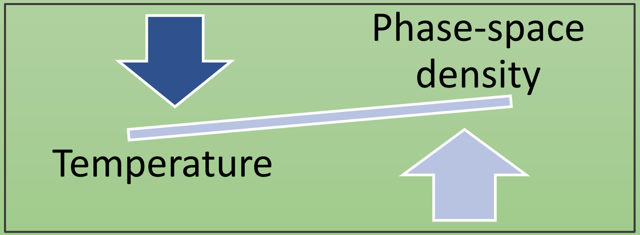


Figure: Time evolution of the entropy $S(t)/g_0$ in an equilibrating Bose system, [13]

Summary

Evaporative cooling



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boundary conditions at the singularity for physically meaningful solutions

Calculation of the time-dependent entropy

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energy distribution can be approximated by a truncated Boltzmann distribution

BEC can be obtained in magnetically trapped atomic hydrogen

Outlook

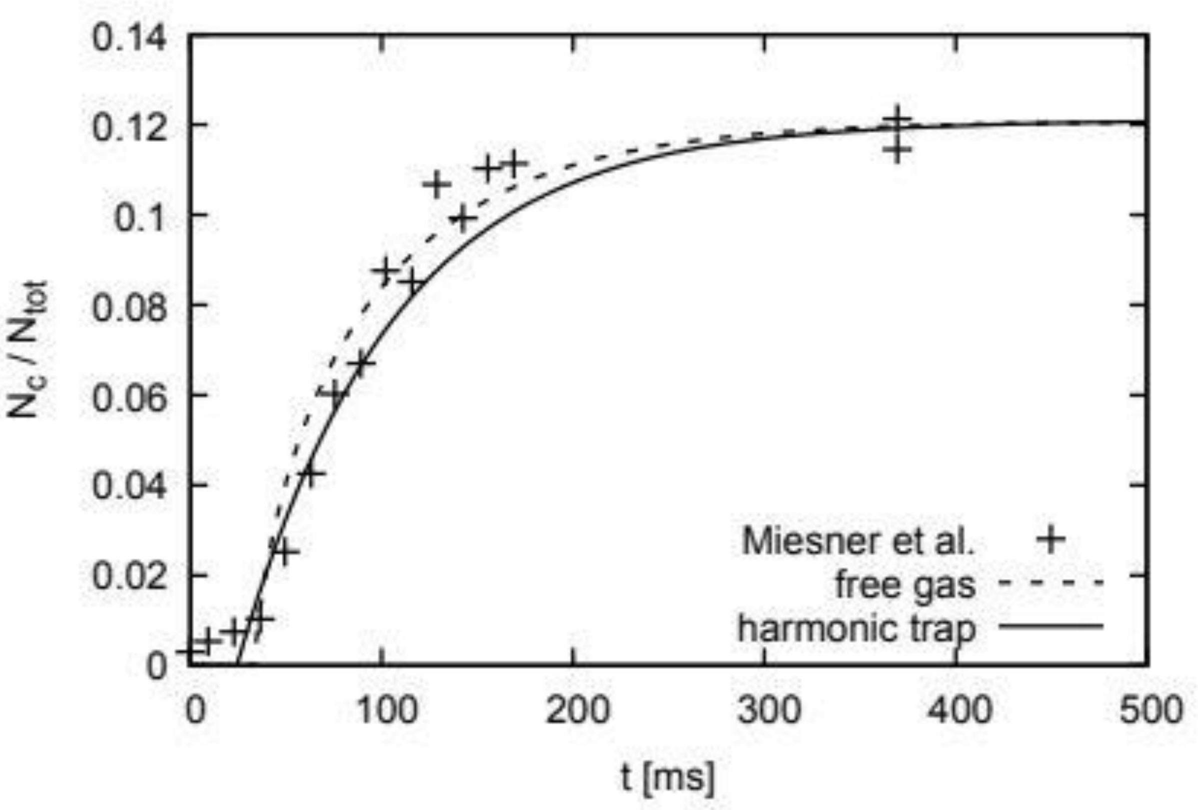

Figure: Comparison with data using different density of states, data from Miesner et al., Science, 279, 1005 (1998)

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