Cosmological large-scale structure

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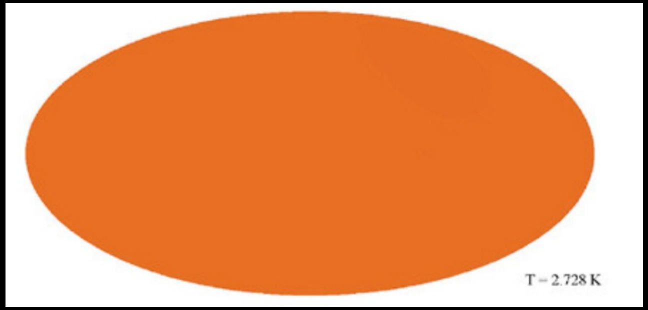
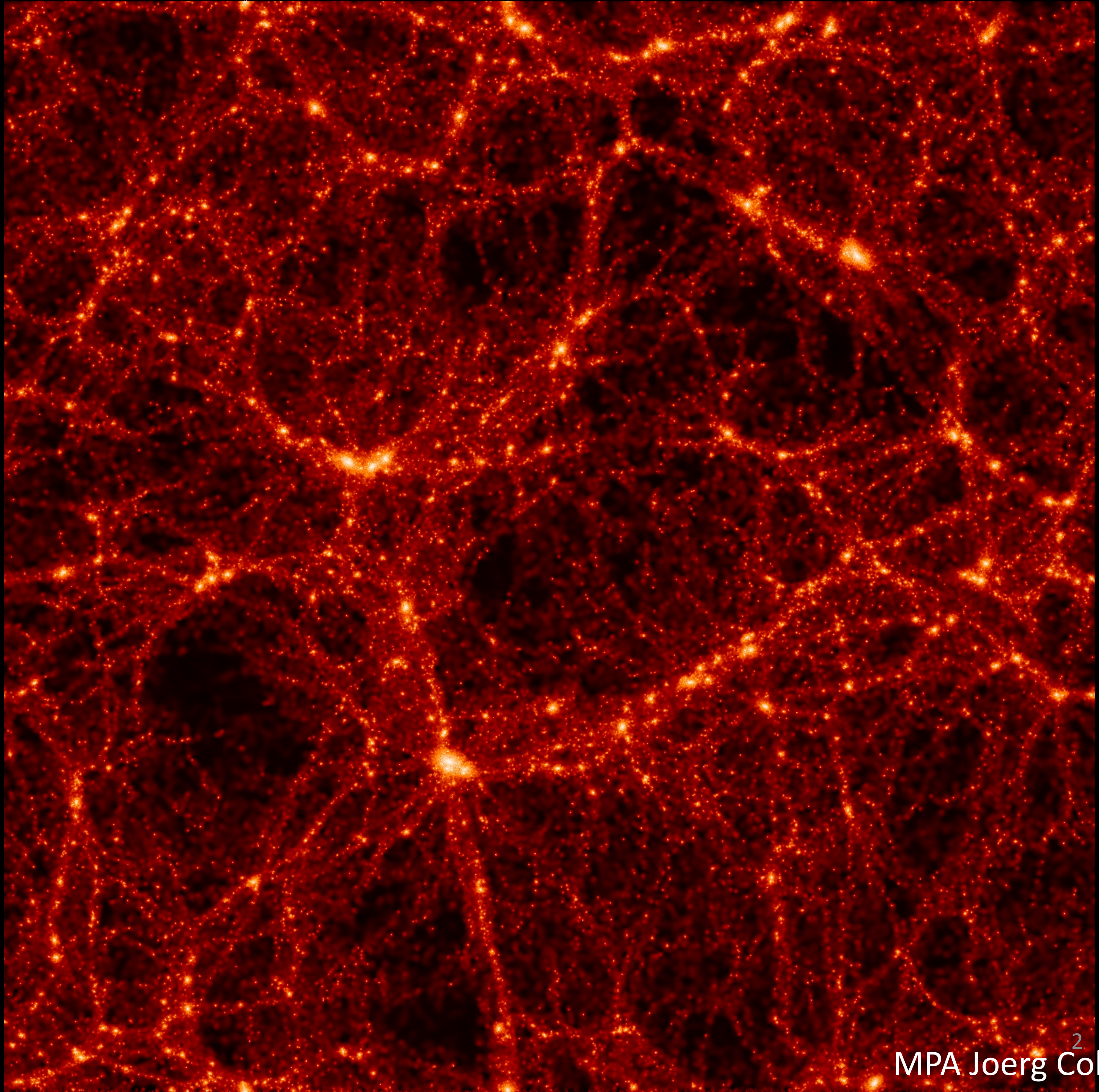
WS2024

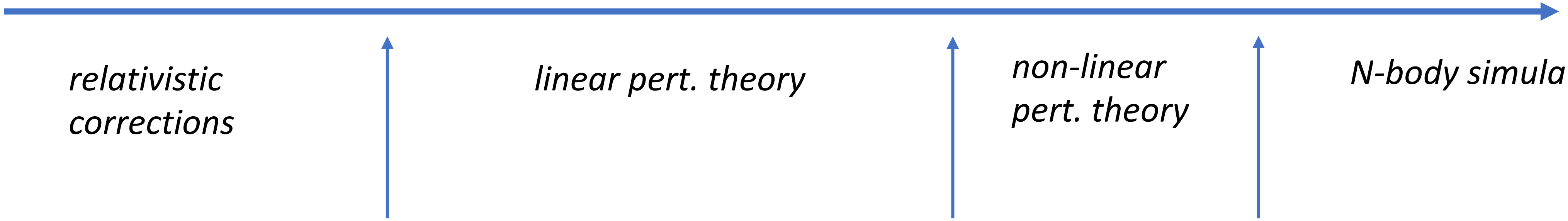


COBE CMB map



MPA Joerg Colberg

The structure of the large scale structure



large scales

small scales

super-horizon scales

linear scales

mildly
non-linear scales

strongly
non-linear scales

*relativistic
corrections*

linear pert. theory

*non-linear
pert. theory*

N-body simulations

$k \approx aH$ ≈0.002 h/Mpc

$\lambda \approx 3000\ Mpc/h$

$k \approx 0.1\ h/Mpc$

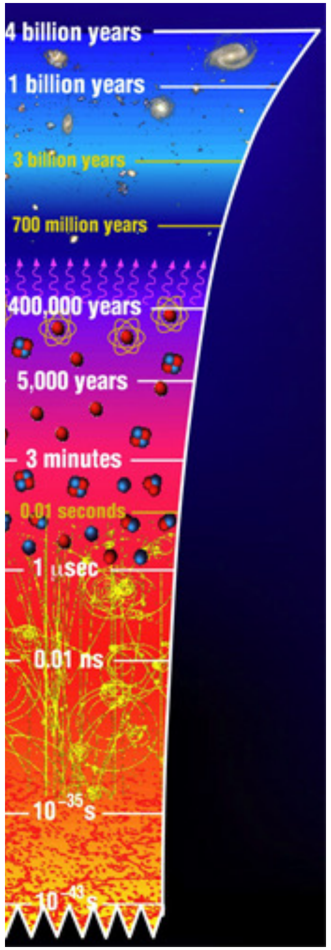
$\lambda \approx 60\ Mpc/h$

$k \approx 0.3\ h/Mpc$

$\lambda \approx 20\ Mpc/h$

The structure of the large scale structure

Late time Universe



Early Universe

large scales

small scales

super-horizon scales

relativistic corrections

$k \approx aH$ ≈0.002 h/Mpc

$\lambda \approx 3000\ Mpc/h$

linear scales

linear pert. theory

mildly non-linear scales

non-linear pert. theory

$k \approx 0.1\ h/Mpc$

$\lambda \approx 60\ Mpc/h$

strongly non-linear scales

N-body simulations

$k \approx 0.3\ h/Mpc$

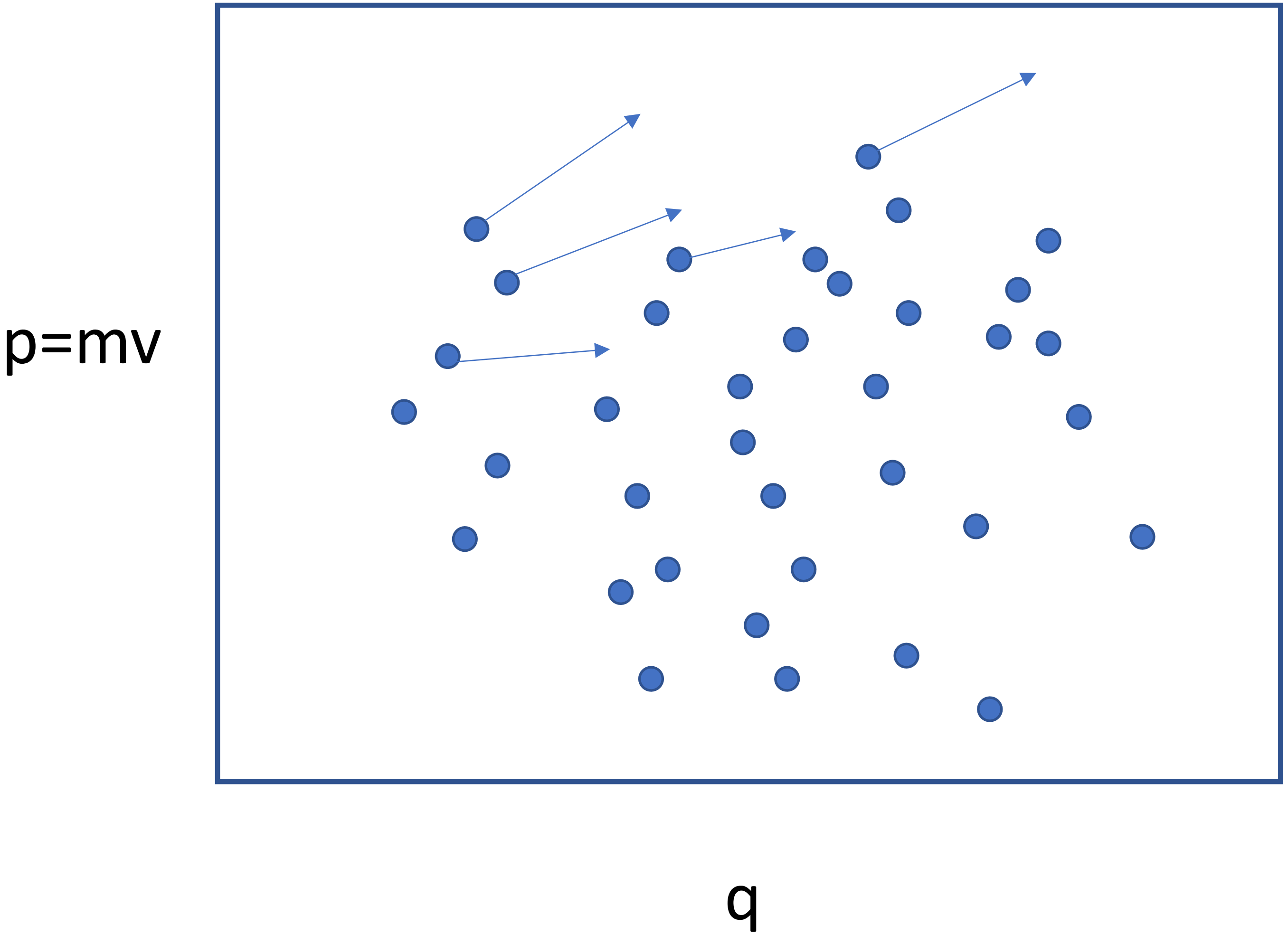
$\lambda \approx 20\ Mpc/h$

Outline

1. Newtonian equations
2. Statistical descriptors
3. Linear Galaxy power spectrum
4. Non-linearity: simplified treatment
5. Non-linearity: standard perturbation theory, I
6. Non-linearity: standard perturbation theory, II
7. Non-linearity: standard perturbation theory, III
8. Relativistic corrections

The Boltzmann equation

collisionless particles in phase space

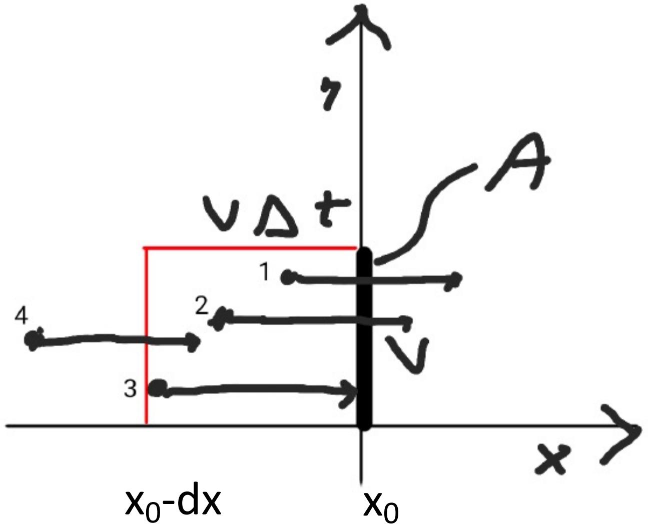


The Boltzmann equation

$$v_x dt = dx$$

$\rho(x)$ = number density

$v_x(x)$ = x-comp. of velocity field



flux of particles across A

$$N = \rho v_x dt dy dz$$

$$\Delta N = (\rho v_x)_{x_0 - dx} dt dy dz - (\rho v_x)_{x_0} dt dy dz = -d(\rho v_x) dt dy dz$$

change in particle number in the volume

$$d\rho dx dy dz = -d(\rho v_x) dt dy dz$$

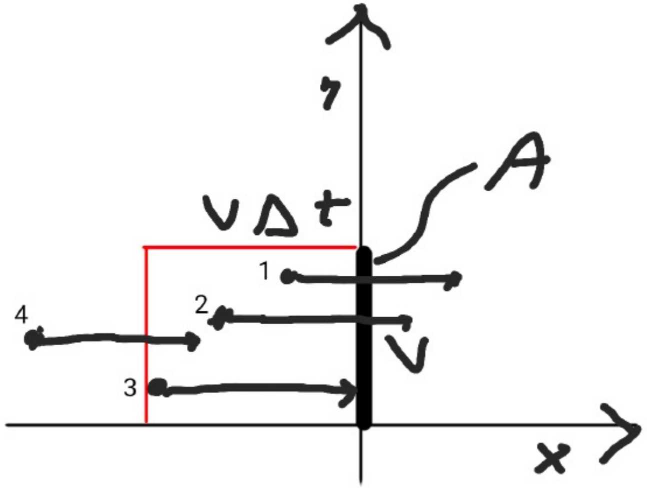
continuity along x

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho v_x)}{\partial x}$$

The Boltzmann equation: from 1D to 6D

$$d\rho dx dy dz = -d(\rho v_x) dt dy dz$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_x)}{\partial x}$$



f = phase-space density

$$dN = f(t, \mathbf{x}, \mathbf{p}) dx dy dz dp_x dp_y dp_z$$

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \frac{\partial(f \dot{w}_i)}{\partial w_i} = 0$$

continuity in 6 phase-space dimensions

$$w = \{x, y, z, p_x, p_y, p_z\} = \{\mathbf{q}, \mathbf{p}\}$$

The Boltzmann equation: Hamiltonian dynamics

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \frac{\partial (f \dot{w}_i)}{\partial w_i} = 0$$

$$\dot{\mathbf{q}} = \partial H / \partial \mathbf{p} \ \text{and} \ \dot{\mathbf{p}} = -\partial H / \partial \mathbf{q} \, ,$$

$$\sum_{i=1}^{6} \frac{\partial (f \, \dot{w}_i)}{\partial w_i} = \sum_{i=1}^{6} \left(f \frac{\partial \, \dot{w}_i}{\partial w_i} + \dot{w}_i \frac{\partial f}{\partial w_i} \right)$$

$$\square = f \left(\frac{\partial^2 H}{\partial \mathbf{q} \partial \mathbf{p}} - \frac{\partial^2 H}{\partial \mathbf{p} \partial \mathbf{q}} \right) + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f}{\partial w_i}$$

$$\square = \sum_{i=1}^{6} \dot{w}_i \frac{\partial f}{\partial w_i}$$

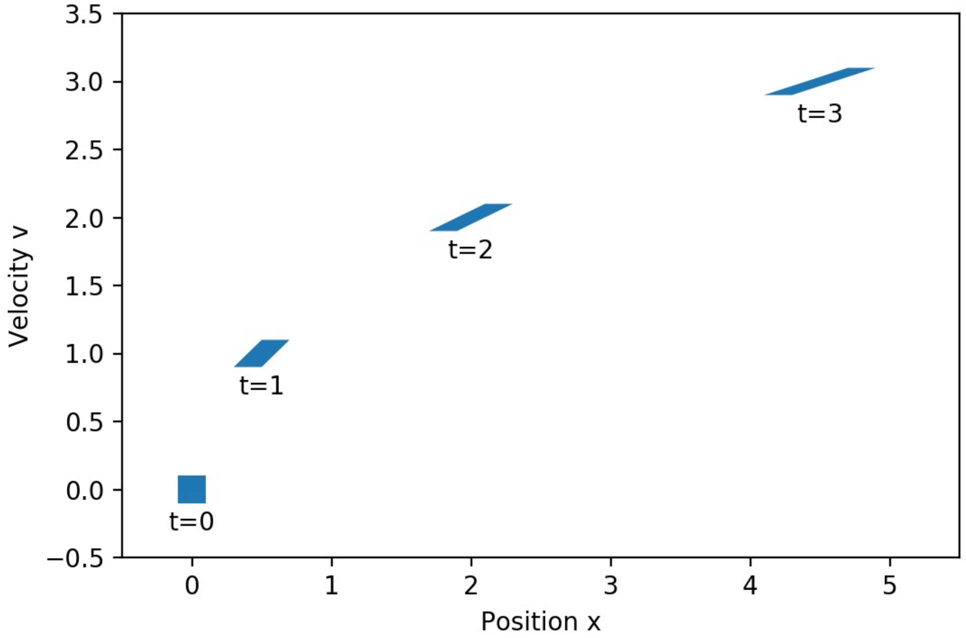
Collisionless
Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{n=1}^{6} \frac{\partial w_n}{\partial t} \frac{\partial f}{\partial w_n} = \frac{\partial f}{\partial t} + \frac{\partial q_i}{\partial t} \frac{\partial f}{\partial q_i} + \frac{\partial p_i}{\partial t} \frac{\partial f}{\partial p_i} = 0$$

Phase-space density remains constant

Collisionless
Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{n=1}^{6} \frac{\partial w_n}{\partial t}\frac{\partial f}{\partial w_n} = \frac{\partial f}{\partial t} + \frac{\partial q_i}{\partial t}\frac{\partial f}{\partial q_i} + \frac{\partial p_i}{\partial t}\frac{\partial f}{\partial p_i} = 0$$



https://physics.stackexchange.com/ user8153

From laboratory to cosmology

Potential for a distribution of matter

$$\Phi(\mathbf{x}) = -G \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|}$$

Poisson equation

$$\Delta\Phi = 4\pi\rho = 4\pi(\rho_0 + \delta\rho)$$

Solving Poisson equation
for the background

$$\Phi_0 = \frac{2\pi}{3}\rho_0 x^2 + C(t)$$

Potential due to fluctuations
above background

$$\phi = \Phi - \frac{2\pi}{3}\rho_0 x^2$$

Cosmological Poisson equation

$$\Delta\phi = 4\pi\rho_0\delta$$

The Vlasov-Poisson equation

Boltzmann

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{n=1}^{6} \frac{\partial w_n}{\partial t}\frac{\partial f}{\partial w_n} = \frac{\partial f}{\partial t} + \frac{\partial q_i}{\partial t}\frac{\partial f}{\partial q_i} + \frac{\partial p_i}{\partial t}\frac{\partial f}{\partial p_i} = 0$$

Equations of motion
in gravity

$$\frac{\partial p_i}{\partial t} = -m\nabla_i\Phi$$

$$\frac{\partial q_i}{\partial t} = \frac{p_i}{m}$$

Vlasov-Poisson equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla f - m\nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{p}} = 0$$

$$\Delta\Phi = 4\pi\rho = 4\pi(\rho_0 + \delta\rho)$$

Definition of moments

$$\int f(x)dx = 1$$

$$\langle x \rangle = \int x f(x)dx$$

$$\langle x^2 \rangle = \int x^2 f(x)dx$$

$$\bar{M}_n = \int x^n f(x)dx$$

$$M_n = \int (x - \langle x \rangle)^n f(x)dx$$

Moments of the VP equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - m \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

Moments:

normalization: total mass

$$\int f(\mathbf{q}, \mathbf{p}, t) d^3 q \, d^3 p = M$$

zero-th: density

$$\rho(x, t) \equiv \int f(\mathbf{q}, \mathbf{p}, t) d^3 p$$

first: average velocity

$$\rho v_i \equiv \int \frac{p_i}{m} f(\mathbf{q}, \mathbf{p}, t) d^3 p$$

second: velocity variance

$$\sigma_{ij} \equiv \int (\frac{p_i}{m} - v_i)(\frac{p_j}{m} - v_j) f(\mathbf{q}, \mathbf{p}, t) d^3 p$$

Moments of the VP equation

$$\boxed{\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla f - m\nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{p}} = 0}$$

zero-th moment: integrate over **p**

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho v_i - m\nabla\Phi\cdot\cancel{\frac{\partial}{\partial \mathbf{p}}\int f d^3p} = 0$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{v} = 0$$

Moments of the VP equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - m\nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

first moment: multiply by **p**/m and integrate over **p**

$$\frac{\partial}{\partial t} \int \frac{p_i}{m} f\, d^3 p + \int \frac{p_j}{m} \frac{p_i}{m} \nabla^j f\, d^3 p - m \nabla_j \Phi \int \frac{p_i}{m} \frac{\partial}{\partial p_j} f\, d^3 p =$$

$$\frac{\partial}{\partial t} \rho v_i + \nabla^j \int \frac{p_j}{m} \frac{p_i}{m} f\, d^3 p + \nabla_j \Phi \int \frac{\partial p_i}{\partial p_j} f\, d^3 p =$$

$$\frac{\partial}{\partial t} \rho v_i + \nabla^j (\sigma_{ij} + \rho v_i v_j) + \nabla_j \Phi\, \delta_i^j \int f\, d^3 p =$$

$$\frac{\partial \rho}{\partial t} v_i + \rho \frac{\partial}{\partial t} v_i + \nabla^j (\sigma_{ij} + \rho v_i v_j) + \rho \nabla_i \Phi =$$

$$- v_i \nabla^j \rho v_j + \rho \frac{\partial}{\partial t} v_i + \nabla^j (\sigma_{ij} + \rho v_i v_j) + \rho \nabla_i \Phi = 0$$

Exercise

$$\rho(x,t) \equiv \int f(\mathbf{q},\mathbf{p},t)d^3p$$

$$\rho v_i \equiv \int \frac{p_i}{m} f(\mathbf{q},\mathbf{p},t)d^3p$$

$$\sigma_{ij} \equiv \int (\frac{p_i}{m} - v_i)(\frac{p_j}{m} - v_j) f(\mathbf{q},\mathbf{p},t)d^3p$$

⬇

$$\int \frac{p_i}{m}\frac{p_j}{m} f(\mathbf{q},\mathbf{p},t)d^3p = \sigma_{ij} + \rho v_i v_j$$

Moments of the VP equation

$$\begin{aligned}
\frac{\partial}{\partial t}\int \frac{p_i}{m} f\, d^3p + \int \frac{p_j}{m}\frac{p_i}{m}\nabla^j f\, d^3p - m\nabla_j\Phi \int \frac{p_i}{m}\frac{\partial}{\partial p_j} f\, d^3p &= \\
\frac{\partial}{\partial t}\rho v_i + \nabla^j \int \frac{p_j}{m}\frac{p_i}{m} f\, d^3p + \nabla_j\Phi \int \frac{\partial p_i}{\partial p_j} f\, d^3p &= \\
\frac{\partial}{\partial t}\rho v_i + \nabla^j(\sigma_{ij} + \rho v_i v_j) + \nabla_j\Phi\, \delta_i^j \int f\, d^3p &= \\
\frac{\partial \rho}{\partial t} v_i + \rho\frac{\partial}{\partial t} v_i + \nabla^j(\sigma_{ij} + \rho v_i v_j) + \rho\nabla_i\Phi &= \\
-v_i\nabla^j\rho v_j + \rho\frac{\partial}{\partial t} v_i + \nabla^j(\sigma_{ij} + \rho v_i v_j) + \rho\nabla_i\Phi &= 0
\end{aligned}$$

…some more manipulation:

Euler equation:

$$\frac{\partial}{\partial t}v_i + v_j\nabla^j v_i = -\nabla_i\Phi - \frac{1}{\rho}\nabla^j\sigma_{ij}$$

Stress tensor

$$\sigma_{ij} \equiv \int (\frac{p_i}{m} - v_i)(\frac{p_j}{m} - v_j) f(\mathbf{q}, \mathbf{p}, t) d^3 p$$

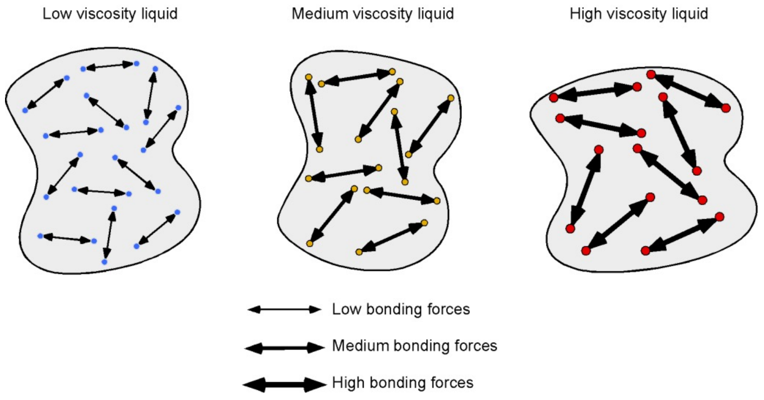
from fluidodynamics…

$$\sigma_{ij} = p\delta_{ij} - \eta(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}\nabla\cdot v) - \zeta\delta_{ij}\nabla\cdot v,$$

isotropic pressure

shear viscosity

bulk viscosity



https://eaglepubs.erau.edu/

Brutal approximation: $\sigma_{ij} = 0$

(single-stream)

Less brutal approximation: $\sigma_{ij} = p\delta_{ij}$

(perfect fluid)

Later on: full picture

Exercise

$$\frac{\partial}{\partial t} v_i + v_j \nabla^j v_i = -\nabla_i \Phi - \frac{1}{\rho} \nabla^j \sigma_{ij}$$

$$\sigma_{ij} = 0$$

$$\mathbf{v} = H\mathbf{x},\ \Phi = \phi_0 = \tfrac{2\pi}{3} \rho_0 x^2.$$

→ Second Friedmann equation (acceleration)
for a pressureless fluid

Roadmap

1. Expand to first order
2. introduce comoving coordinates
3. adopt conformal time
4. solve equations

First order in an expanding space: continuity

Approximation: $\sigma_{ij} = p\delta_{ij}$

$$\boxed{\begin{array}{c} \dot{\rho} + \mathbf{v}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{v}\underline{\qquad}\text{conservation} \\ \rho(\dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p - \rho\nabla\Phi\underline{\qquad}\text{Euler} \\ \nabla^2\Phi = 4\pi\rho\underline{\qquad}\text{Poisson} \end{array}}$$

expand

$$\left\{\begin{array}{l} \rho = \rho_0 + \delta\rho, \;\; \mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}, \qquad\qquad \mathbf{v}_0 = H\mathbf{x} \\ \dot{\rho}_0 + \dot{\delta\rho} + (\mathbf{v}_0 + \delta\mathbf{v})\cdot\nabla\delta\rho = -(\rho_0 + \delta\rho)\nabla\cdot(\mathbf{v}_0 + \delta\mathbf{v}) \end{array}\right.$$

zero-th order

$$\dot{\rho}_0 = -\rho_0 H\nabla\cdot\mathbf{x} = -3H\rho_0$$

first order

$$\dot{\delta\rho} + \mathbf{v}_0\cdot\nabla\delta\rho = -\rho_0\nabla\cdot\delta\mathbf{v} - 3H\delta\rho$$

simplify

$$\delta \equiv \frac{\delta\rho}{\rho_0} \qquad\qquad \dot{\delta} + \mathbf{v}_0\cdot\nabla\delta = -\nabla\cdot\delta\mathbf{v}$$

First order in an expanding space: Euler

$$\dot{\rho} + \mathbf{v}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{v}\underline{\qquad}\text{conservation}$$
$$\rho(\dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p - \rho\nabla\Phi\underline{\qquad}\text{Euler}$$
$$\nabla^2\Phi = 4\pi\rho\underline{\qquad}\text{Poisson}$$

Euler equation

$$\frac{d\delta\mathbf{v}}{dt} = \delta\dot{\mathbf{v}} + \mathbf{v}_0\cdot\nabla\delta\mathbf{v} = -\frac{\nabla\delta p}{\rho_0} - \nabla\delta\Phi - (\delta\mathbf{v}\cdot\nabla)\mathbf{v}_0$$
$$\nabla^2\delta\Phi = 4\pi\rho_0\delta$$

from now on $\mathbf{v}$ for $\delta\mathbf{v}$ and ϕ for $\delta\Phi$.

Comoving coordinates

comoving coord **r**

$$\left\{\begin{aligned} &\mathbf{x}(t) = a(t)\mathbf{r} \\ &\nabla = a^{-1}\,\nabla_r . \end{aligned}\right.$$

total differential

$$\dot{\delta} + \mathbf{v}_0 \cdot \nabla\delta = -\nabla \cdot \delta\mathbf{v} \qquad d\delta/dt \equiv \dot{\delta} + \mathbf{v}_0 \cdot \nabla\delta .$$

(almost)
final set of eqs

$$\left\{\begin{aligned} &a\frac{d\delta}{dt} = -\nabla_r \cdot \mathbf{v} \\ &\frac{d\mathbf{v}}{dt} = -\frac{\nabla_r \delta p}{a\rho_0} - \frac{\nabla_r \phi}{a} - H\mathbf{v} \end{aligned}\right.$$

$$\nabla_r^2 \phi = 4\pi a^2 \rho_0 \delta$$

sound speed

$$c_s^2 \equiv \delta p/\delta\rho .$$

Partial derivatives in comoving coords

comoving coord **r** $\mathbf{x}(t) = a(t)\mathbf{r}$

the partial derivative wrt t at fixed X is not the same as the partial derivative at fixed x.

start with one coordinate: $x(t, X) = a(t)X$

$$dx = [\partial(aX)/\partial t]dt + [\partial(aX)/\partial X]dX = \dot{a}Xdt + adX,$$

$$df(t,x) = \left(\frac{\partial f}{\partial t}\right)_x dt + \left(\frac{\partial f}{\partial x}\right)_t dx = \left(\frac{\partial f}{\partial t}\right)_x dt + \left(\frac{\partial f}{\partial x}\right)_t (\dot{a}Xdt + adX)$$

($\dot{a}X = Hx$)

in general:

$$\square = \underbrace{\left[\left(\frac{\partial f}{\partial t}\right)_x + \left(\frac{\partial f}{\partial x}\right)_t Hx\right]}dt + \left(\frac{\partial f}{\partial x}\right)_t adX = \underbrace{\left(\frac{\partial f}{\partial t}\right)_X} dt + \left(\frac{\partial f}{\partial X}\right)_t dX$$

⬇

relation between partial der.

$$\left(\frac{\partial}{\partial t}\right)_X = \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial}{\partial x}\right)_t Hx$$

Partial derivatives in comoving coords

1D

$$\left(\frac{\partial}{\partial t}\right)_X = \left(\frac{\partial}{\partial t}\right)_x + \left(\frac{\partial}{\partial x}\right)_t Hx$$

3D

$$\left(\frac{\partial \delta}{\partial t}\right)_{\mathbf{r}} \equiv \dot{\delta} + \mathbf{v}_0 \cdot \nabla \delta = \left(\frac{d\delta}{dt}\right)_x$$

, in comoving coordinates, the total derivative wrt time is actually a partial derivative

$$\boxed{\begin{aligned} a\frac{d\delta}{dt} &= -\nabla_r \cdot \mathbf{v} \\ \frac{d\mathbf{v}}{dt} &= -\frac{\nabla_r \delta p}{a\rho_0} - \frac{\nabla_r \phi}{a} - H\mathbf{v} \\ \nabla_r^2 \phi &= 4\pi a^2 \rho_0 \delta \end{aligned}}$$

conformal time

$$d\tau = dt/a$$

(almost)
final set of eqs

$$\begin{aligned} \dot{\delta} &= -\nabla_r \cdot \mathbf{v} \\ \dot{\mathbf{v}} &= -\nabla_r c_s^2 \delta - \nabla_r \phi - Ha\mathbf{v} \\ \nabla_r^2 \phi &= 4\pi a^2 \rho_0 \delta \end{aligned}$$

Recap

Collisionless
Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{n=1}^{6} \frac{\partial w_n}{\partial t}\frac{\partial f}{\partial w_n} = \frac{\partial f}{\partial t} + \frac{\partial q_i}{\partial t}\frac{\partial f}{\partial q_i} + \frac{\partial p_i}{\partial t}\frac{\partial f}{\partial p_i} = 0$$

Vlasov-Poisson equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - m\nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

$$\rho(x,t) \equiv \int f(\mathbf{q},\mathbf{p},t) d^3p$$

Moments

$$\rho v_i \equiv \int \frac{p_i}{m} f(\mathbf{q},\mathbf{p},t) d^3p$$

$$\sigma_{ij} \equiv \int (\frac{p_i}{m} - v_i)(\frac{p_j}{m} - v_j) f(\mathbf{q},\mathbf{p},t) d^3p$$

final set of eqs
in comoving coords
and conf time

$$\dot{\theta} = -\nabla^2 c_s^2 \delta - \nabla^2 \phi - \mathcal{H}\theta$$

$$\dot{\delta} = -\theta$$

$$\theta \equiv \nabla^i v_i$$

Conservations equations in real space

$$\begin{aligned} \dot{\delta} &= -\nabla_r \cdot \mathbf{v} \\ \dot{\mathbf{v}} &= -\nabla_r c_s^2 \delta - \nabla_r \phi - Ha\mathbf{v} \\ \nabla_r^2 \phi &= 4\pi a^2 \rho_0 \delta \end{aligned}$$

last touches $\theta \equiv \nabla^i v_i$ $\mathcal{H} = \dot{a}/a = aH.$

final set of eqs
in comoving coords
and conf time

$$\begin{aligned} \dot{\theta} &= -\nabla^2 c_s^2 \delta - \nabla^2 \phi - \mathcal{H}\theta \\ \dot{\delta} &= -\theta \end{aligned}$$

Quiz time

1. Why do we use the collisionless Boltzmann equation?
2. Why do we use the Newtonian approximation?
3. Why do we discard higher order terms?
4. Can we instead do everything in GR?
5. Why do we neglect viscosity?
6. Why is it called sound speed?
7. Why do we use conformal time?
8. We derived the second Friedmann eq.; and the first?
9. What do we do next?

Fourier space

$$\theta \equiv \nabla^i v_i \qquad \mathcal{H} = \dot{a}/a = aH\,.$$

final set of eqs
in comoving coords
and conf. time

$$\dot{\theta} = -\nabla^2 c_s^2 \delta - \nabla^2 \phi - \mathcal{H}\theta$$
$$\dot{\delta} = -\theta$$
$$\nabla_r^2 \phi = 4\pi a^2 \rho_0 \delta$$

Fourier space
(comoving wavevector **k**)

$$\phi = \int e^{ikr} \phi_k \, d^3 k, \textvisiblespace$$
$$\delta = \int e^{ikr} \delta_k \, d^3 k, \textvisiblespace \theta = \int e^{ikr} \theta_k \, d^3 k$$

associated scale

$$\lambda = 2\pi a/k$$

Fourier transformations

$$\begin{aligned} \phi(x,\tau) &\rightarrow e^{ikr}\phi_k(\tau) \\ \nabla\phi(x,\tau) &\rightarrow ie^{ikr}\mathbf{k}\phi_k(\tau) \\ \nabla^2\phi(x,\tau) \equiv \nabla_i\nabla_i\phi(x,\tau) &\rightarrow -e^{ikr}k^2\phi_k(\tau) \end{aligned}$$

Fourier space

$$\theta \equiv \nabla^i v_i \qquad \mathcal{H} = \dot{a}/a = aH\,.$$

final set of eqs
in comoving coords
and conf. time

$$\begin{aligned}\dot{\theta} &= -\nabla^2 c_s^2 \delta - \nabla^2 \phi - \mathcal{H}\theta \\ \dot{\delta} &= -\theta \\ \nabla_r^2 \phi &= 4\pi a^2 \rho_0 \delta\end{aligned}$$

$$\begin{aligned}\dot{\delta} &= -\theta \\ \dot{\theta} &= -\mathcal{H}\theta + c_s^2 k^2 \delta + k^2 \phi \\ k^2 \phi &= -\frac{3}{2}\mathcal{H}^2 \Omega_m \delta\end{aligned}$$

$$\theta = ik^i v_i$$

$$\mathcal{H}^2 = \frac{8\pi}{3}\Omega_m a^2 \rho$$

differentiate the first
and insert the other two

$$\ddot{\delta} + \mathcal{H}\dot{\delta} + \left(k^2 c_s^2 - \frac{3}{2}\mathcal{H}^2 \Omega_m\right)\delta = 0$$

Jeans equation

Compare with GR equations

Newtonian equations

$$\begin{aligned}\dot{\delta} &= -\theta\\ \dot{\theta} &= -\mathcal{H}\theta + c_s^2 k^2 \delta + k^2 \phi\\ k^2\phi &= -\frac{3}{2}\mathcal{H}^2\Omega_m\delta\end{aligned}$$

Perturbed scalar metric

$$ds^2 = a^2(\tau)[-(1+2\Psi)d\tau^2 + (1+2\Phi)dx^i dx_i]$$

Energy of the gravitational field

GR perturbation equations

$$\begin{aligned}\dot{\delta} &= -\theta - 3\dot{\Phi}\\ \dot{\theta} &= -\mathcal{H}\theta + c_s^2 k^2 \delta + k^2 \Psi\\ k^2\Psi &= -\frac{3}{2}\mathcal{H}^2\Omega_m(\delta + 3\frac{\mathcal{H}}{k^2}\theta)\\ \Psi &= -\Phi\end{aligned}$$

My preferred format

$$\begin{aligned} \dot{\delta} &= -\theta - 3\dot{\Phi} \\ \dot{\theta} &= -\mathcal{H}\theta + c_s^2 k^2 \delta + k^2 \Psi \\ k^2 \Psi &= -\frac{3}{2}\mathcal{H}^2 \Omega_m (\delta + 3\frac{\mathcal{H}}{k^2}\theta) \\ \Psi &= -\Phi \end{aligned}$$

Rescaled divergence:

$$\theta = \frac{i\mathbf{k}\mathbf{v}}{\mathcal{H}}$$

$$\begin{aligned} n &= \log a \\ \delta' &= -\theta - 3\Phi' \\ \theta' &= -(1 + \frac{\mathcal{H}'}{\mathcal{H}})\theta + c_s^2 \frac{k^2}{\mathcal{H}^2}\delta + \frac{k^2}{\mathcal{H}^2}\Psi \\ \frac{k^2}{\mathcal{H}^2}\Psi &= -\frac{3}{2}\Omega_m (\delta + 3\frac{\mathcal{H}^2}{k^2}\theta) \\ \Psi &= -\Phi \end{aligned}$$

Order parameter:

$$\lambda = \frac{\mathcal{H}}{k}$$

today $\mathcal{H} = aH \to H_0 \approx 1/(3000\,\mathrm{Mpc}/h)$ so $\lambda_{\mathrm{NL}} \approx \frac{60}{3000} = 0.02$

Growth of fluctuations

$$\boxed{\ddot{\delta} + \mathcal{H}\,\dot{\delta} + \left(k^2 c_s^2 - \frac{3}{2}\mathcal{H}^2\,\Omega_m\right)\delta = 0}$$

Wave equation in the
Minkowskian limit (H=0)

$$\ddot{\delta} - c_s^2 \nabla^2 \delta = 0,$$

Damped oscillations

$$k^2 c_s^2 - \frac{3}{2}\mathcal{H}^2 \Omega_m > 0$$

Jeans scale

$\lambda = 2\pi a/k$

$$\lambda_J \approx \frac{c_s}{H}$$

Growth of fluctuations

$$\ddot{\delta} + \mathcal{H}\,\dot{\delta} + \left(k^2 c_s^2 - \frac{3}{2}\mathcal{H}^2\,\Omega_m\right)\delta = 0$$

Perturbations only grow if they are bigger then the Jeans scale

$$\lambda_J \approx \frac{c_s}{H}$$

Radiation: Jeans scale is as big as the horizon. Hardly grows at all

Baryons: sound speed is small after decoupling. They grow above 10^6 solar masses

Dark matter: sound speed is almost zero. They grow freely.

Growth of fluctuations in Einstein-deSitter

$$\boxed{\ddot{\delta}+\mathcal{H}\,\dot{\delta}+\left(k^2 c_s^2-\frac{3}{2}\,\mathcal{H}^2\,\Omega_m\right)\delta=0}$$

Above Jeans length

$$\ddot{\delta}+\mathcal{H}\dot{\delta}-\frac{3}{2}\mathcal{H}^2\Omega_m\delta=0$$

Using $\log a$ as time variable
and $\Omega_m = 1$

$$\delta''+(\frac{\mathcal{H}'}{\mathcal{H}}+1)\delta'-\frac{3}{2}\delta=0$$

Friedmann equation for a single fluid
with $p = w\rho$

$$\frac{\mathcal{H}'}{\mathcal{H}}=-\frac{1}{2}-\frac{3}{2}w$$

For pressureless matter

$$\delta''+\frac{1}{2}\delta'-\frac{3}{2}\delta=0$$

$$\delta = Ae^{m\alpha} = Aa^m$$

$$\delta_+ = Aa^1, \quad \delta_- = Ba^{-3/2}$$

growing decaying

Growth of fluctuations in ΛCDM

$$\boxed{\ddot{\delta} + \mathcal{H}\dot{\delta} - \frac{3}{2}\mathcal{H}^2\Omega_m\delta = 0}$$

Growth rate
approx. solution

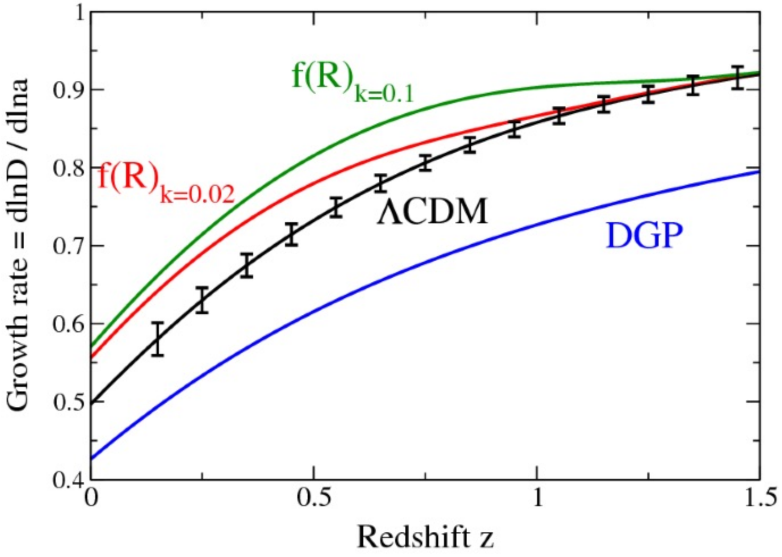
$$f \equiv \frac{d\log\delta_m}{d\log a} \approx \Omega_m^{\gamma}(z)$$

Growth index

$$\gamma \approx 0.55$$

ΛCDM

$$\Omega_m(z) = \frac{\rho_m}{\rho_{crit}} = \frac{\Omega_{m0}a^{-3}}{\Omega_{m0}a^{-3} + 1 - \Omega_m}$$



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Growth function

$$G(z) \equiv \frac{\delta_m(z)}{\delta_m(0)} = \exp\int_1^a f(z)d\log a \approx \exp\int_1^a \Omega_m^{\gamma}(z)d\log a$$