Cosmological large-scale structure

Lecture 4

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The structure of the large scale structure

large scales

small scales

super-horizon scales

linear scales

mildly
non-linear scales

strongly
non-linear scales

relativistic corrections

linear pert. theory

non-linear pert. theory

N-body simulations

$k \approx aH \approx 0.002$ h/Mpc

$\lambda \approx 3000\ Mpc/h$

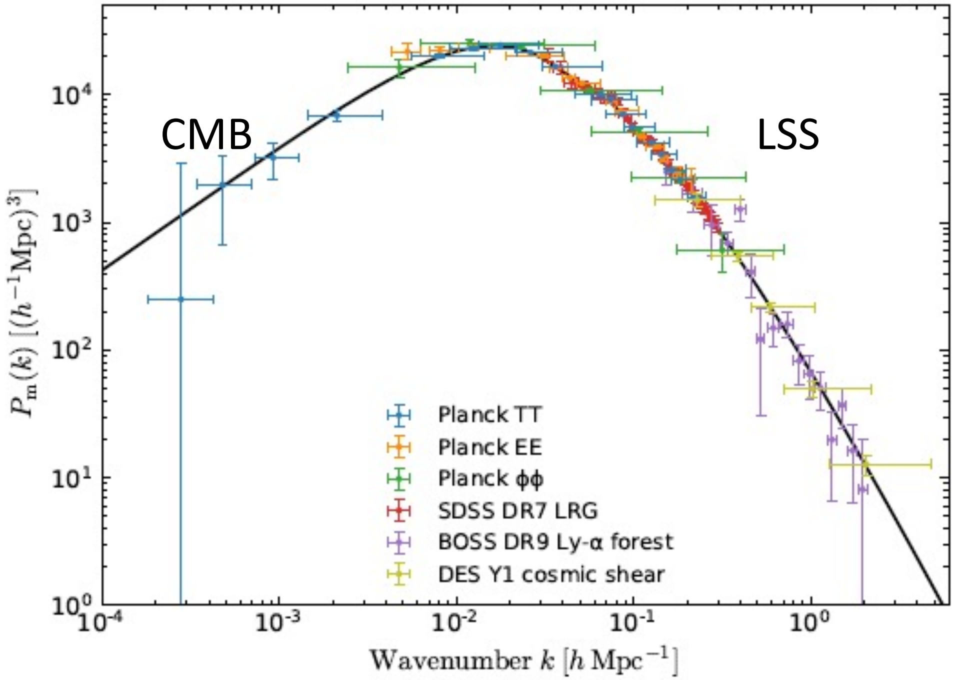
$k \approx 0.1\ h/Mpc$

$\lambda \approx 60\ Mpc/h$

$k \approx 0.3\ h/Mpc$

$\lambda \approx 20\ Mpc/h$

Recap: The observed power spectrum



ESA and the
Planck collaboration

Large scales
~1000 Mpc

Small scales
~10 Mpc

Recap

theoretical power spectrum $P(k)_{today} = Ak^{n_s}T^2(k; cosmology)$

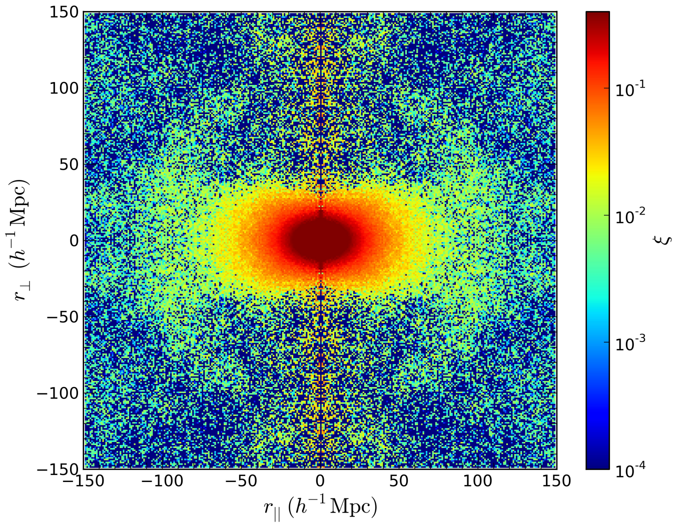
bias $P_g = b^2 P_m$

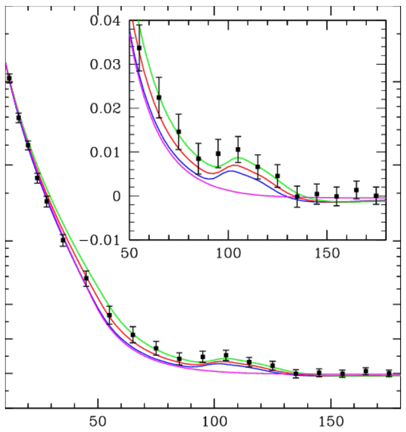
RSD $\delta_{g,s} = \delta_{g,r}(1 + \beta\mu^2)$

(almost) final form $P_g(k, \mu, z) = b^2 G^2 P_m(k, z = 0)(1 + \beta\mu^2)^2 e^{-k^2\mu^2\sigma_v^2}$

Recap

correlation function
across/along LOS





Alcock-Paczynski effect

$$k = (k_\parallel^2 + k_\perp^2)^{1/2} = \alpha k_r \,,$$

$$\mu = \frac{k_\parallel}{(k_\parallel^2 + k_\perp^2)^{1/2}} = \frac{H\mu_r}{H_r \alpha} \,,$$

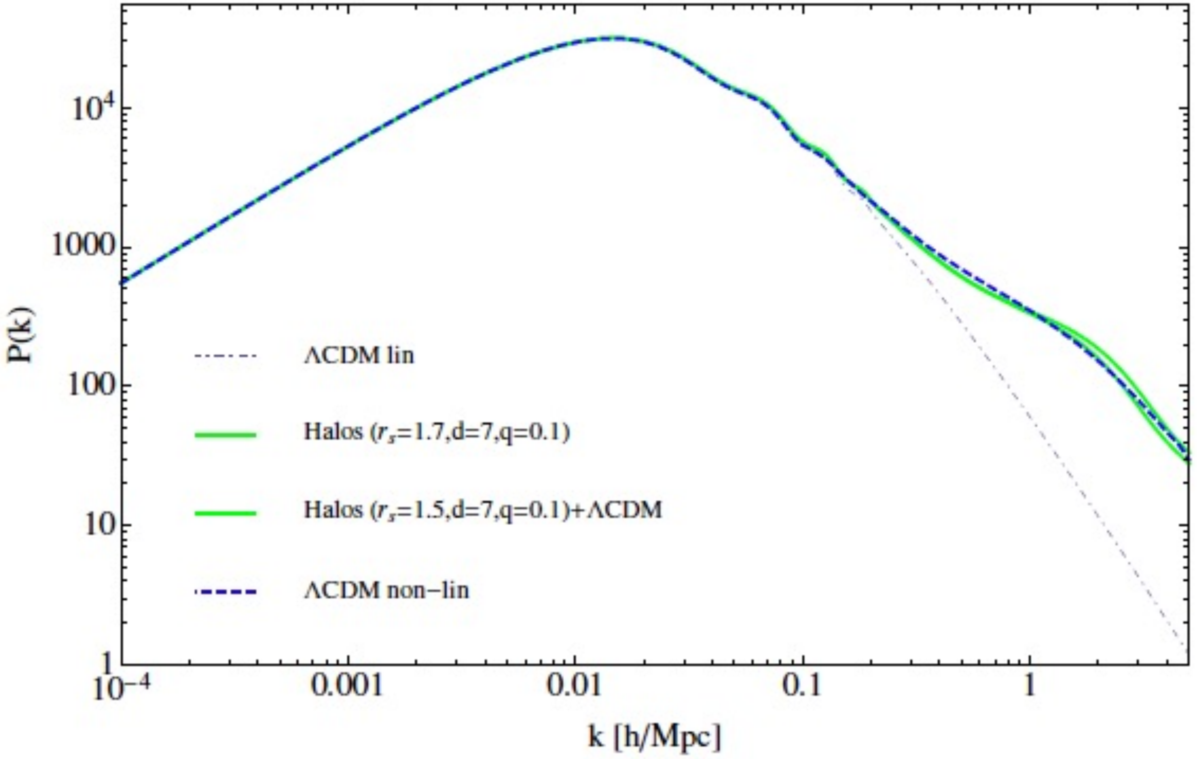
Final form linear spectrum

$$P_g(k,\mu,z) = b^2 G^2 P_m(\alpha k_r)(1 + \beta \mu_r^2 \frac{h^2}{\alpha^2})^2 e^{-k^2\mu^2\sigma_v^2}$$

Recap

$$P_{NL} = P_{LIN} + P_h$$

what we expect from non-linear corrections



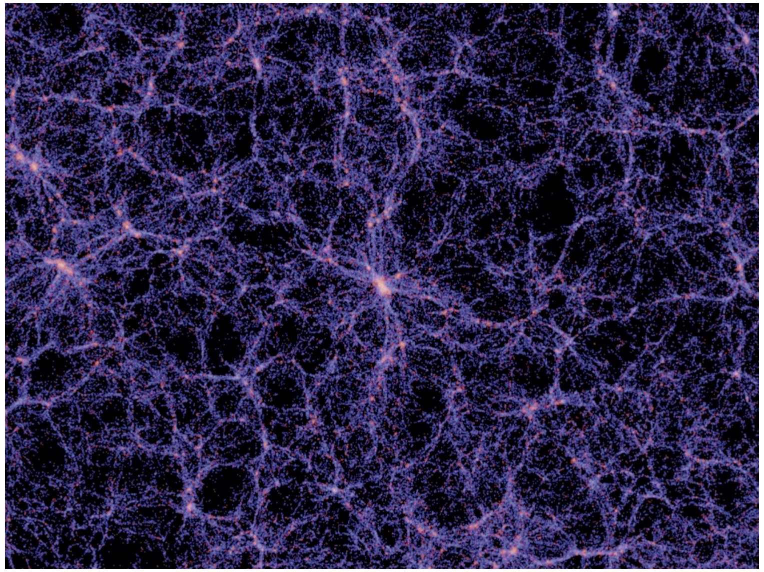
Roadmap for today

- Simplified non-linearity
- Zel'dovich approximation
- Spherical collapse
- Abundance of virialized halos

Zel'dovich approximation

Water surface

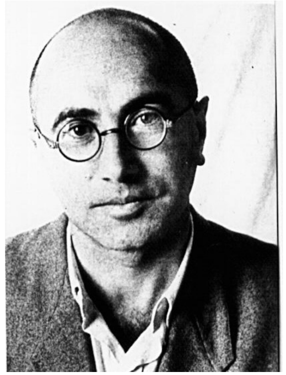
(Photo by Chris Lawton on Unsplash)



Millennium simulation

caustics form where many trajectories (of light rays or particles) converge

Zel'dovich approximation



Minsk 8 March 1914 –
Moscow 2 December 1987

Lagrangian approach

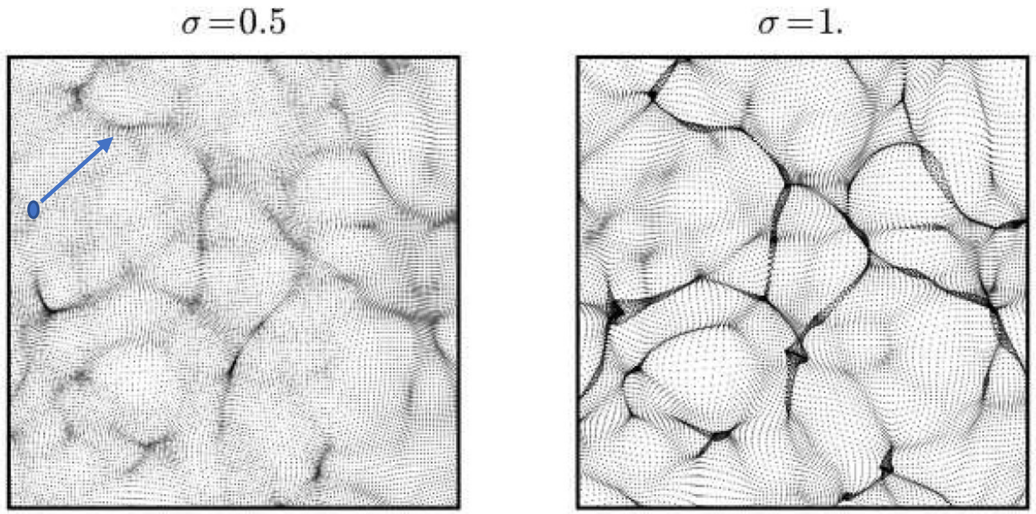
Main idea: follow the trajectories of particles until they develop a singularity (caustic)

position of a particle at instant *t*

$$\mathbf{x}(t) = \mathbf{x}_0 + g(t)\mathbf{s}(\mathbf{x}_0)$$

initial position of a particle

Ansatz: displacement is separable into time and space



this simply means particle move towards concentrations along straight lines!

Zel'dovich approximation

Main idea: follow the trajectories of particles until they develop a singularity (caustic)

$$\mathbf{x}(t) = \mathbf{x}_0 + g(t)\mathbf{s}(\mathbf{x}_0)$$

conservation of particle number $\rho(x,t)d^3x = \rho_0(t)d^3x_0$

mapping through the Jacobian $\rho(x,t) = \rho_0(t)\left|\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right|^{-1}$

diagonal deformation tensor $d_{ij} \equiv -\frac{\partial s_i}{\partial x_{0,j}}$

Eigenvalues

$$\left|\frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}\right| = \left|I + g(t)\frac{\partial \mathbf{s}(\mathbf{x}_0)}{\partial \mathbf{x}_0}\right| = |\delta_{ij} - g(t)d_{ij}| = (1 - g\lambda_1)(1 - g\lambda_2)(1 - g\lambda_3)$$

Zel'dovich approximation

$$\rho(x,t) = \frac{\rho_0(t)}{(1-g\lambda_1)(1-g\lambda_2)(1-g\lambda_3)}$$

expand for small $g\lambda$

$$\rho(x,t) \approx \rho_0(t)(1 + g(t)(\lambda_1 + \lambda_2 + \lambda_3) = \rho_0(t)(1 + g(t)\mathrm{Tr}(d_{ij}))$$

therefore…

$$\delta(t) \equiv \frac{\rho(x,t) - \rho_0(t)}{\rho_0(t)} = -g(t)\frac{\partial s_i}{\partial x_{0,i}} = -g(t)\nabla_{x_0}\mathbf{s}(x_0)$$

…this should reproduce the linear growth: $\delta(t) = G(t)\delta_0,$

Zel'dovich approximation

$$\delta(t) \equiv \frac{\rho(x,t)-\rho_0(t)}{\rho_0(t)} = -g(t)\frac{\partial s_i}{\partial x_{0,i}} = -g(t)\nabla_{x_0}\mathbf{s}(x_0)$$

$$\delta(t) = G(t)\delta_0,$$

then…

$$g(t) = G(t)$$

…and

$$-\nabla_{x_0}\mathbf{s}(x_0) = \delta_0$$

Poisson eq.

$$\nabla^2\Psi = 4\pi\rho_m\delta = \frac{3}{2}a^2H^2\Omega_m G(t)\delta_0 \quad \Rightarrow \quad \delta_0 = \frac{2}{3a^2H^2\Omega_m G}\nabla^2\Psi$$

we find that the initial displacement s is the gradient of the potential

$$\mathbf{s}(x_0) = -\frac{2}{3a^2H^2\Omega_m G}\nabla\Psi$$

Zel'dovich approximation

$$\mathbf{x}(t) = \mathbf{x}_0 + g(t)\mathbf{s}(\mathbf{x}_0)$$

$$g(t) = G(t)$$

$$\mathbf{s}(x_0) = -\frac{2}{3a^2 H^2 \Omega_m G} \nabla \Psi$$

In this way, one can run a very cheap N-body simulation: first, take the *linear* power spectrum at some early epoch for the model you want to simulate; second, convert the power spectrum for δ into a power sectrum for Ψ using Poisson equation in Fourier space; third, create a real space realization of this spectrum by overimposing sinusoidal oscillations with amplitude given by the spectrum and random phases; fourth, put particles on a regular grid; fifth, evaluate the displacement field by evaluating at every grid point (4.1.12); finally, move the particles out of their initial grid point by using (4.1.1).

until you reach a singularity!

$$\rho(x,t) = \frac{\rho_0(t)}{(1-g\lambda_1)(1-g\lambda_2)(1-g\lambda_3)}$$

growing!

Pancakes and filaments

$$\rho(x,t) = \frac{\rho_0(t)}{(1-g\lambda_1)(1-g\lambda_2)(1-g\lambda_3)}$$

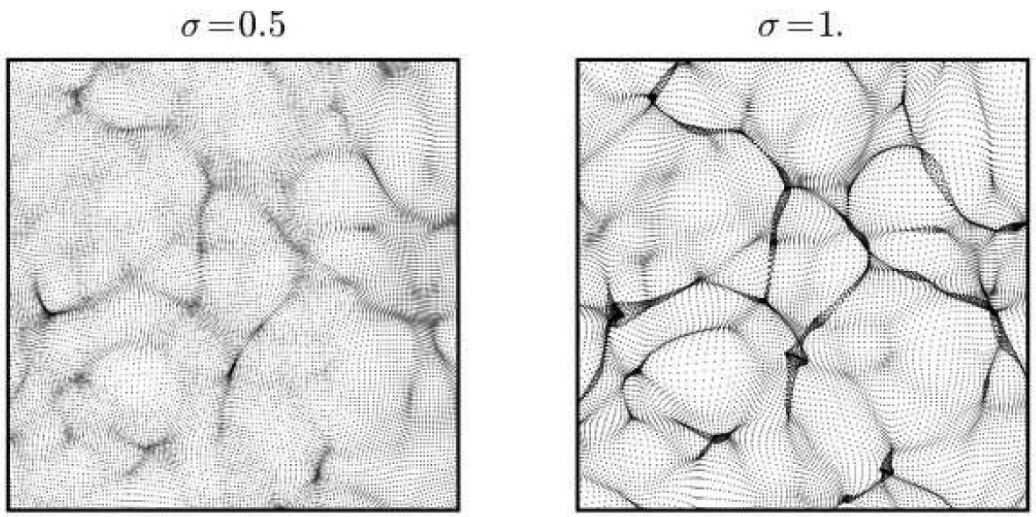
$$d_{ij} \equiv -\frac{\partial s_i}{\partial x_{0,j}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

take the largest eigenvalue:
particles will move preferentially in the direction of the associated eigenvector
the caustic will therefore form on a plane orthogonal to this direction (pancakes)

What happens next?
We can expect particle stick to this plane,
then move to the edges (filaments) and
then to the vertices (clusters)

Pancakes and filaments

particles form
pancakes, filaments and
finally clusters

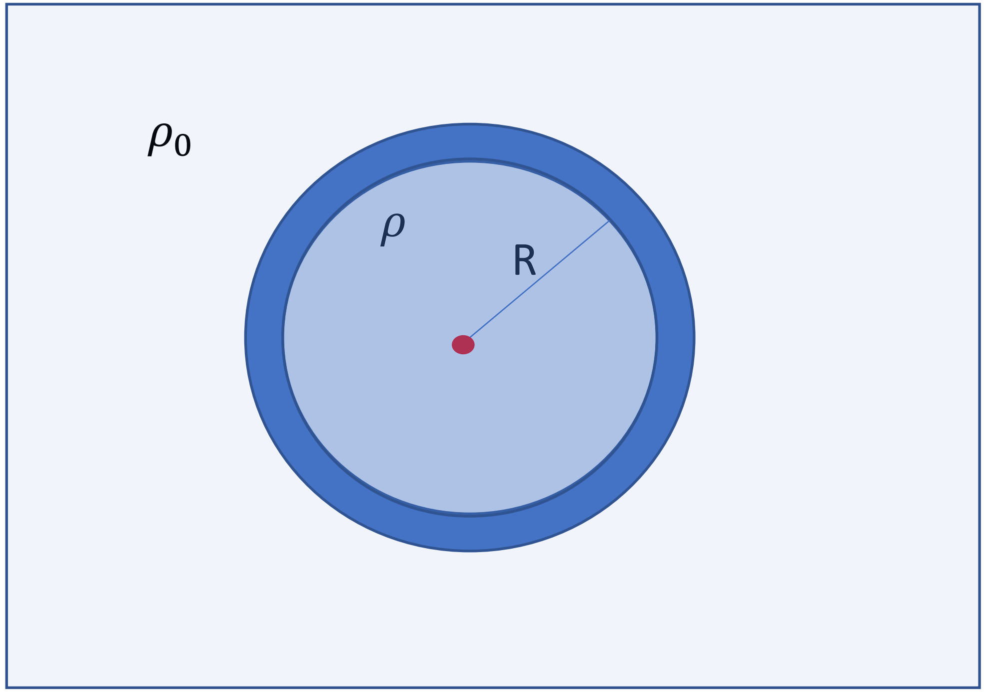


However
pancakes form around z=30 in N-body simulations: so we can use the ZA only until then

ZA describes the initial stages (pancakes)

Spherical collapse the final one (halo formation)

Spherical collapse



Top-hat distribution
$\rho > \rho_0$

Equation for the shell

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM(R)}{R^2} = -\frac{4}{3}\pi G \rho R \,,$$

Question: if initially the shell expands, what will happen?

Spherical collapse

$$\frac{d^2R}{dt^2} = -\frac{GM(R)}{R^2} = -\frac{4}{3}\pi G\rho R\,,$$

Constant mass in every shell $M(R) = 4\pi\rho R^3/3$

Density of a pressureless component $\rho_0 = (3M(R_0)/4\pi)(R_0 a(t))^{-3}$

Evolution of density contrast

$$\delta = \frac{\rho}{\rho_0} - 1 = \left(\frac{a(t)R_0}{R}\right)^3 - 1$$

Spherical collapse

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM(R)}{R^2} = -\frac{4}{3}\pi G\rho R\,,$$

$$\delta = \frac{\rho}{\rho_0} - 1 = \left(\frac{a(t)R_0}{R}\right)^3 - 1$$

Replace R with δ: non-linear equation for δ

$$\delta'' + \left(1 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\delta' - \frac{3}{2}\Omega_m\delta = \frac{4}{3}\frac{\delta'^2}{1+\delta} + \frac{3}{2}\Omega_m\delta^2\,.$$

Spherical collapse

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM(R)}{R^2} = -\frac{4}{3}\pi G \rho R \,,$$

Cycloid equation

Multiply by $2\mathrm{d}R/\mathrm{d}t$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{2GM}{R} - C \,,$$

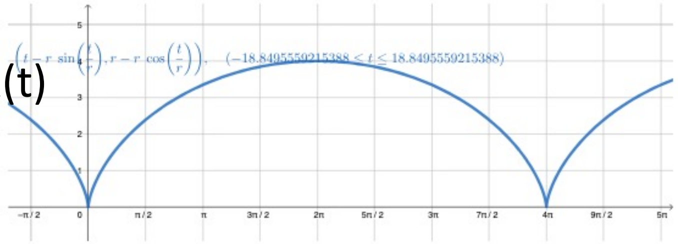
Parametric solution

$$R = GM(1 - \cos\tau)/C$$

$$t = GM(\tau - \sin\tau)/C^{3/2}$$

Expansion and collapse

R(t)



Spherical collapse

$$R = GM(1-\cos\tau)/C \qquad t = GM(\tau - \sin\tau)/C^{3/2}$$

$$\delta = \frac{\rho}{\rho_0} - 1 = \left(\frac{a(t)R_0}{R}\right)^3 - 1$$

In EdS ($\Omega_m = 1$) $\qquad a(t) = a_0(t/t_0)^{2/3}$

We obtain

$$\delta = \frac{9}{2}\frac{(\tau - \sin\tau)^2}{(1-\cos\tau)^3} - 1\,,$$

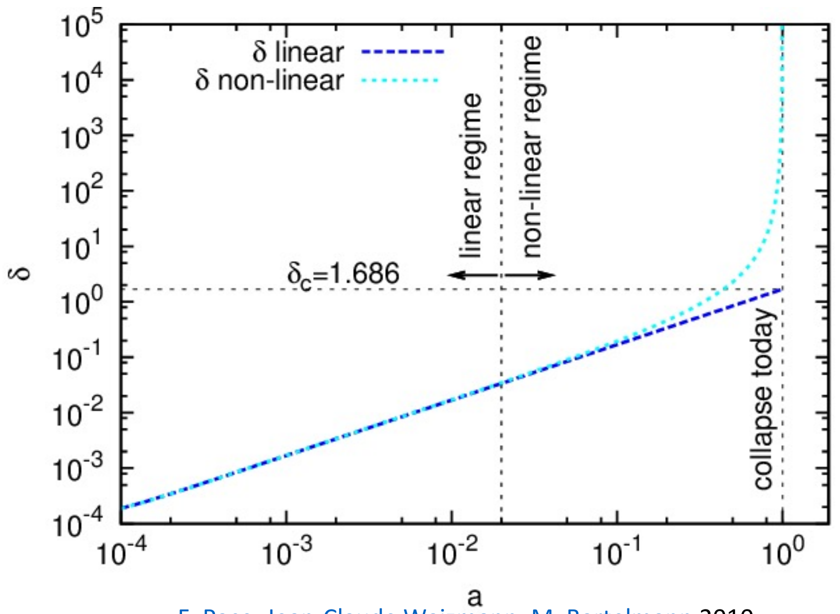
Linear part

$$\delta_L = \frac{3}{5}\left[\frac{3}{4}(\tau - \sin\tau)\right]^{2/3}\,,$$

Spherical collapse

$$\delta = \frac{9}{2}\frac{(\tau - \sin\tau)^2}{(1-\cos\tau)^3} - 1\,,$$

$$\delta_L = \frac{3}{5}\left[\frac{3}{4}(\tau - \sin\tau)\right]^{2/3}\,,$$

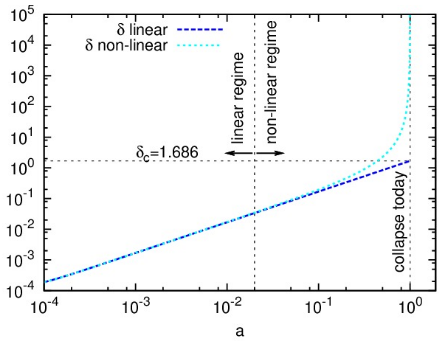


F. Pace, Jean-Claude Waizmann, M. Bartelmann 2010

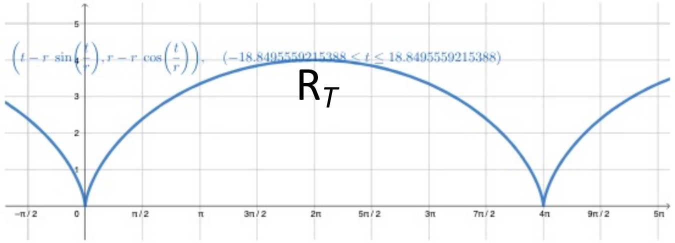
$$\delta_L = \delta_{\rm coll} = (3/5)(3\pi/2)^{2/3} \approx 1.686\,,$$

Virialization

$$\delta_L = \delta_{\rm coll} \approx 1.686\,,$$



R(t)



However, we assumed exact spherical symmetry and *M(R)*=const in the shell, valid up to shell crossing

In practice, the collapse will start deviating from spherical symmetry

We can find that at

$$R_V = R_T/2.$$

The system virializes

$$U_T = U_V + K_V = U_V/2\,.$$

which means all components of velocity should be equal: no longer radial symmetry!

Quiz time

1. Do we really observe pancakes and filaments?
2. What happens if the initial distribution is non-Gaussian?
3. What is the density contrast at virialization?
4. We assumed that the shell mass is constant. Is this realistic?
5. How can the spherical collapse model be generalized/improved?

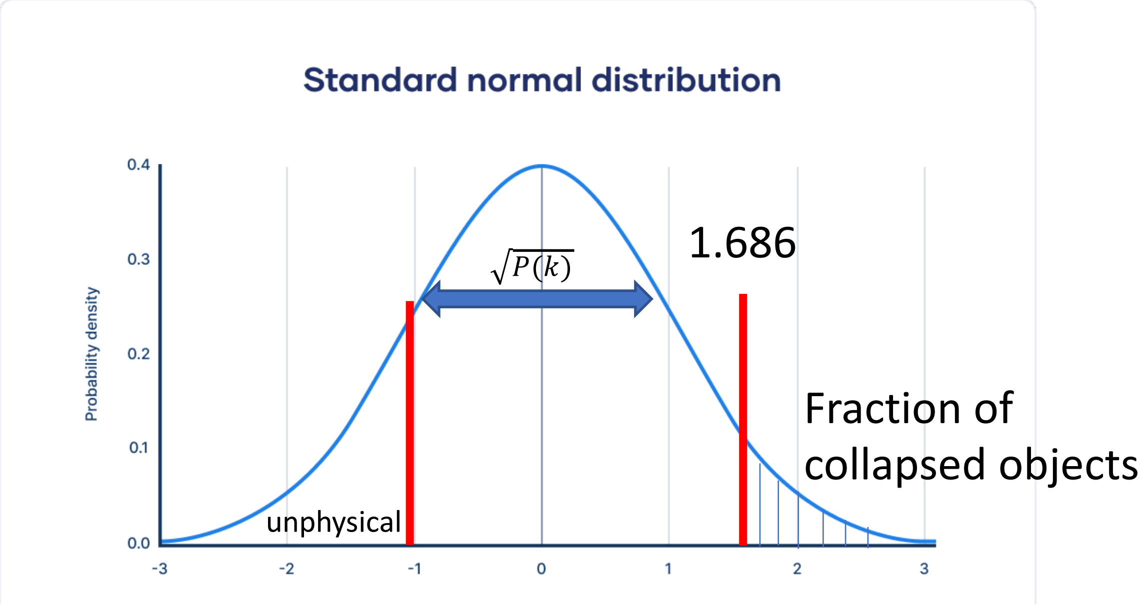
Spherical collapse: why is it useful?

Because it tells us at which value of the linear evolution objects becomes strongly non-linear, i.e. collapse and form structures

$$\delta_L = \delta_{\rm col} \approx 1.686\,,$$

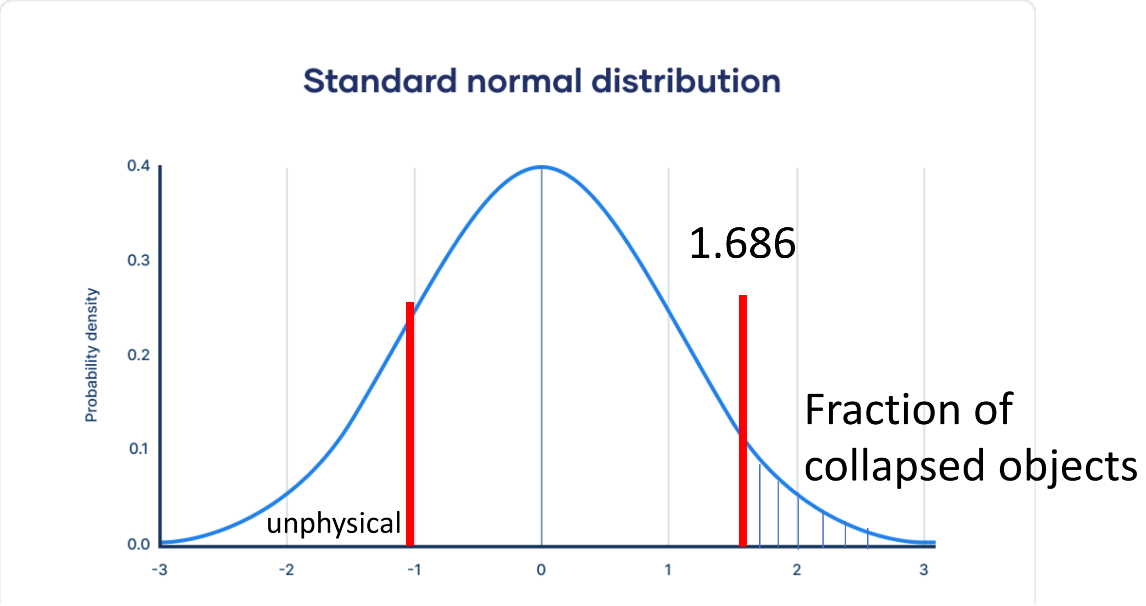
That is, we can use linear theory to predict a non-linear event!

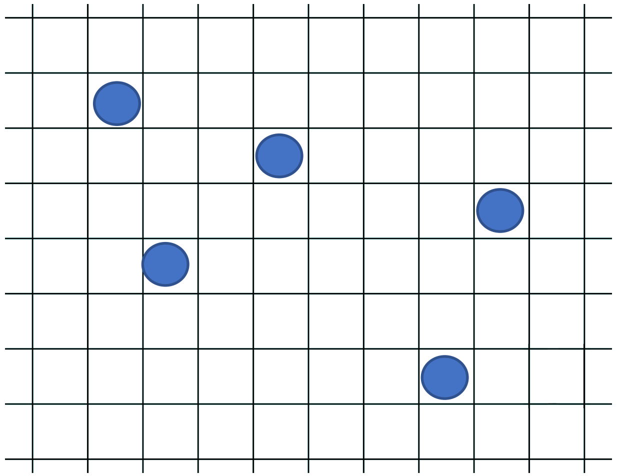
We discussed how δ evolves in time and what is its variance $\xi(r)$ or *P(k)*…
But what is its full distribution?



(broadening with time)

Spherical collapse: why is it useful?



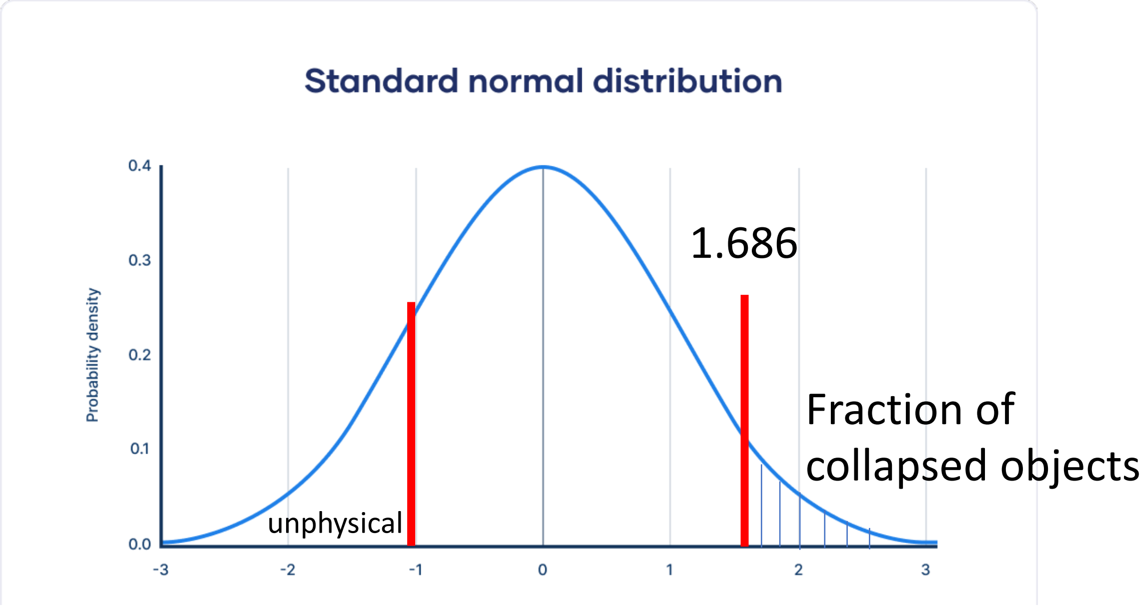


We can therefore estimate at any given time what fraction of cells of size R (containing mass M) contains a virialized object!

Variance of the counts in cells of mass M

$$\sigma_M^2(z).$$

Spherical collapse: why is it useful?



Fraction of collapsed object of mass M (inside R) at time z

$$p(M,z)|_{\delta>\delta_{\rm coll}} = \frac{1}{\sigma_M(z)\sqrt{2\pi}} \int_{\delta_{\rm coll}}^{\infty} \exp\left(-\frac{\delta_M^2}{2\sigma_M^2(z)}\right) {\rm d}\delta_M = \frac{1}{2}\,{\rm erfc}\left(\frac{\delta_{\rm coll}}{\sqrt{2}\sigma_M(z)}\right),$$

Finally (see script) we get
the number density of virialized halos in the mass range dM

$${\rm d}n = \frac{N}{V} = \frac{{\rm d}p}{V_M} = \frac{\rho}{M}\left|\frac{\partial p(M,z)|_{\delta>\delta_{\rm coll}}}{\partial M}\right| {\rm d}M = \sqrt{\frac{2}{\pi}}\frac{\rho}{M^2}\frac{\delta_{\rm coll}}{\sigma_M}\left|\frac{{\rm d}\,\ln\sigma_M}{{\rm d}\,\ln M}\right| e^{-\delta_{\rm coll}^2/(2\sigma_M^2)}{\rm d}M\,.$$

Spherical collapse: why is it not SO useful?

number density of virialized halos in the mass range dM

$$dn = \frac{N}{V} = \frac{dp}{V_M} = \frac{\rho}{M}\left|\frac{\partial p(M,z)|_{\delta>\delta_{\rm coll}}}{\partial M}\right| dM = \sqrt{\frac{2}{\pi}}\frac{\rho}{M^2}\frac{\delta_{\rm coll}}{\sigma_M}\left|\frac{d\ \ln\sigma_M}{d\ \ln M}\right| e^{-\delta^2_{\rm coll}/(2\sigma^2_M)} dM\,.$$

Because:

- It assumes isolated radial collapse
- It is analytical only for EdS
- It does not predict the correlation of the halos

Next: higher-order perturbation theory