Cosmological large-scale structure

Lecture 5

L. Amendola

WS2024

The structure of the large scale structure

large scales | **small scales**

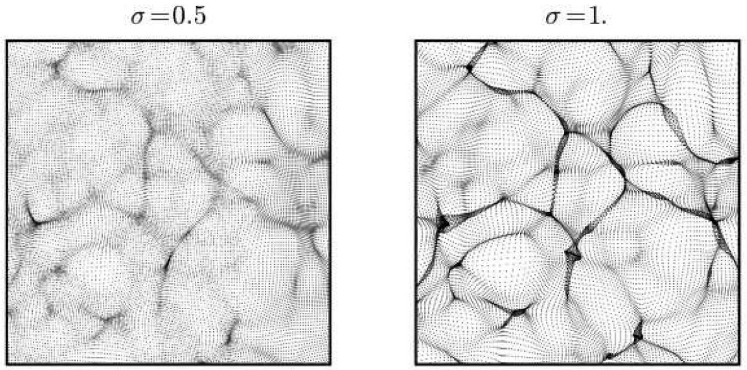
super-horizon scales		linear scales		mildly non-linear scales		strongly non-linear scales
relativistic corrections		*linear pert. theory*		*non-linear pert. theory*		*N-body simulations*
	$k \approx aH \approx$0.002 h/Mpc		$k \approx 0.1\ h/Mpc$		$k \approx 0.3\ h/Mpc$	
	$\lambda \approx 3000\ Mpc/h$		$\lambda \approx 60\ Mpc/h$		$\lambda \approx 20\ Mpc/h$	

Recap: Zel'dovich approximation

position of a particle at instant *t* $\mathbf{x}(t) = \mathbf{x}_0 + g(t)\mathbf{s}(\mathbf{x}_0)$

density field $$\boxed{\rho(x,t) = \frac{\rho_0(t)}{(1-g\lambda_1)(1-g\lambda_2)(1-g\lambda_3)}}$$

formation of pancakes,
filaments, halos



Recap: Spherical Collapse

shell equation

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} = -\frac{GM(R)}{R^2} = -\frac{4}{3}\pi G\rho R\,,$$

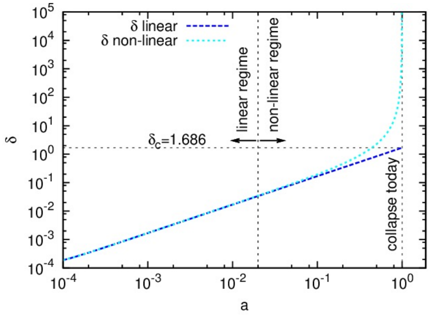
relation with densit contrast

$$\delta = \frac{\rho}{\rho_0} - 1 = \left(\frac{a(t)R_0}{R}\right)^3 - 1$$

solution

$$\delta = \frac{9}{2}\frac{(\tau - \sin\tau)^2}{(1-\cos\tau)^3} - 1\,,$$

$$\delta_L = \frac{3}{5}\left[\frac{3}{4}(\tau - \sin\tau)\right]^{2/3}\,,$$



density of virialized halos

$$\mathrm{d}n = \frac{N}{V} = \frac{\mathrm{d}p}{V_M} = \frac{\rho}{M}\left|\frac{\partial p(M,z)|_{\delta>\delta_{\mathrm{coll}}}}{\partial M}\right|\mathrm{d}M = \sqrt{\frac{2}{\pi}}\frac{\rho}{M^2}\frac{\delta_{\mathrm{coll}}}{\sigma_M}\left|\frac{\mathrm{d}\,\ln\sigma_M}{\mathrm{d}\,\ln M}\right| e^{-\delta^2_{\mathrm{coll}}/(2\sigma^2_M)}\mathrm{d}M\,.$$

Roadmap for today

- Eulerian Standard Perturbation Theory
- Convolution kernels
- Bias and redshift distortion

Back to the linear fluidodynamical equations

$$\boxed{\begin{aligned}\dot{\delta} &= -\theta \\ \dot{\theta} &= -\mathcal{H}\theta + c_s^2 k^2\delta + k^2\phi \\ k^2\phi &= -\frac{3}{2}\mathcal{H}^2\Omega_m\delta\end{aligned}}$$

$$\theta = ik^i v_i$$

$$\mathcal{H}^2 = \frac{8\pi}{3}\Omega_m a^2\rho$$

We obtain an eq. for δ

$$\delta'' + (1+\frac{\mathcal{H}'}{\mathcal{H}})\delta' - \frac{3}{2}\Omega_m\delta = 0$$

$$f = \frac{\delta'}{\delta}$$

$$f' = \frac{\delta''}{\delta} - (\frac{\delta'}{\delta})^2$$

$$\frac{\delta''}{\delta} = f' + f^2$$

$$F = 1 + \mathcal{H}'/\mathcal{H} = \tfrac{1}{2}(1 - 3w_{DE}\Omega_{DE})$$

$$S = 3\Omega_m/2.$$

General form for the growth rate

$$f' + f^2 + Ff - S = 0$$

Back to the linear fluidodynamical equations

$$\dot{\delta} = -\theta$$
$$\dot{\theta} = -\mathcal{H}\theta + c_s^2 k^2 \delta + k^2 \phi$$
$$k^2 \phi = -\frac{3}{2}\mathcal{H}^2 \Omega_m \delta$$

$$\theta = i k^i v_i$$
$$\mathcal{H}^2 = \frac{8\pi}{3}\Omega_m a^2 \rho$$

Assuming **v** Is irrotational

$$\mathbf{v} = \nabla \phi_v$$
$$\mathbf{v} = i\mathbf{k}\phi_v = A\mathbf{k}$$

$$\mathbf{v} = -i\theta \frac{\mathbf{k}}{k^2} \qquad \mathbf{v} = i\mathcal{H}\delta_k f \frac{\mathbf{k}}{k^2} \qquad \theta = -\mathcal{H} f \delta$$

At the linear level, we can use v,θ, δ interchangeably

Back to the linear fluidodynamical equations

$$\dot{\rho} + \mathbf{v}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{v} \qquad \text{conservation}$$
$$\rho(\dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p - \rho\nabla\Phi \qquad \text{Euler}$$
$$\nabla^2\Phi = 4\pi\rho \qquad \text{Poisson}$$

Same steps as in linear theory, but we keep second order terms

Define density contrast $\delta = (\rho(x,t) - \rho_0(t))/\rho_0(t)$

Define peculiar velocity $\mathbf{v} = \mathbf{v}_p + H\mathbf{x}$

Insert in the cont. eq. $\dot{\rho_0}(1+\delta) + \rho_0\dot{\delta} + \rho_0(\mathbf{v_p}+H\mathbf{x})\cdot\nabla\delta = -\rho_0(1+\delta)(\nabla\cdot\mathbf{v_p} + 3H)$

Adopt conformal time and comoving coordinates…

Full fluidodynamical equations

$$\dot{\rho} + \mathbf{v}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{v} \qquad \text{conservation}$$
$$\rho(\dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p - \rho\nabla\Phi \qquad \text{Euler}$$
$$\nabla^2\Phi = 4\pi\rho \qquad \text{Poisson}$$

Same steps as in linear theory, but we keep second order terms

Linear equations

$$\dot{\theta} + H\theta + \frac{3}{2}a^2H^2\delta = 0$$
$$\dot{\delta} = -\nabla\mathbf{v}$$

Non-linear equations for pressureless fluid and EdS

$$\left\{\begin{array}{l} \dot{\theta} + H\theta + \frac{3}{2}a^2H^2\delta = -\nabla(\mathbf{v}\cdot\nabla\mathbf{v}) \\ \dot{\delta} = -\nabla(1+\delta)\mathbf{v} \end{array}\right.$$

Back to the linear fluidodynamical equations

$$\boxed{\dot{\delta} = -\nabla(1+\delta)\mathbf{v}}$$

Fourier transforms
$$\int \dot{\delta}_k e^{i\mathbf{kx}} d^3k = -\nabla(1 + \int \delta_k e^{i\mathbf{kx}} d^3k)(\int \mathbf{v}_{k'} e^{i\mathbf{k'x}} d^3k')$$

Apply the Nabla
$$\int \dot{\delta}_k e^{i\mathbf{kx}} d^3k = -i \int \mathbf{v}_k \mathbf{k} e^{i\mathbf{kx}} d^3k - i \int \delta_k \mathbf{v}_{k'}(\mathbf{k}+\mathbf{k}') e^{i(\mathbf{k}+\mathbf{k}')\mathbf{x}} d^3k d^3k'$$

Integrate over $(2\pi)^{-3} e^{-i\mathbf{k}''\mathbf{x}} d^3x$ the LHS…
$$\int \frac{\dot{\delta}_k}{(2\pi)^3} e^{i(\mathbf{k}-\mathbf{k}'')\mathbf{x}} d^3k d^3x = \int \dot{\delta}_k d^3k \delta_D(\mathbf{k}-\mathbf{k}'') = \dot{\delta}_{k''}$$

…and the RHS
$$-i\mathbf{v}_{k''}\mathbf{k}'' - i \int \delta_k \mathbf{v}_{k'}(\mathbf{k}+\mathbf{k}')\delta_D(\mathbf{k}+\mathbf{k}'-\mathbf{k}'') d^3k d^3k'$$

Perturbation expansion: continuity

$$\boxed{\dot{\delta} = -\nabla(1+\delta)\mathbf{v}}$$

Then we get this:

$$\dot{\delta}_k + i\mathbf{v}_k\mathbf{k} = -i\int \delta_{k_1}\mathbf{v}_{k_2}(\mathbf{k}_1+\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1 d^3k_2$$

θ

Now we expand:

$$\delta = \varepsilon\delta^{(1)} + \varepsilon^2\delta^{(2)} + \ldots$$
$$\theta = \varepsilon\theta^{(1)} + \varepsilon^2\theta^{(2)} + \ldots$$

…and write the eq order by order in ε

At first order we get the linear eq.

At the second:

$$\varepsilon^2\dot{\delta}_{\mathbf{k}}^{(2)} + \varepsilon^2\theta_{\mathbf{k}}^{(2)} = \varepsilon^2\mathcal{H}f\int \delta_{k_1}^{(1)}\delta_{k_2}^{(1)}\frac{\mathbf{k}_2}{k_2^2}\cdot(\mathbf{k}_1+\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1 d^3k_2$$

Mode coupling!

Perturbation expansion: continuity

$$\boxed{\varepsilon^2\dot{\delta}_{\mathbf{k}}^{(2)} + \varepsilon^2\theta_{\mathbf{k}}^{(2)} = \varepsilon^2\mathcal{H}f\int \delta_{k_1}^{(1)}\delta_{k_2}^{(1)}\frac{\mathbf{k}_2}{k_2^2}\cdot(\mathbf{k}_1+\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1d^3k_2}$$

$$\delta = G(z)\delta_0$$

present value

Adopt *log a* as time variable

$$\mathcal{H}\delta' + \mathcal{H}\theta = \mathcal{H}G^2 f\int \delta_1\delta_2\frac{\mathbf{k}_2\mathbf{k}_1 + \mathbf{k}_2\mathbf{k}_2}{k_2^2}\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1d^3k_2$$

Final form of continuity eq.

$$\theta + \delta' = G^2 f\int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1d^3k_2 \equiv C$$

Symmetrization of the kernel

$$\begin{aligned}\alpha &= \frac{1}{2}\left[\frac{\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2)}{k_2^2} + \frac{\mathbf{k}_1\cdot(\mathbf{k}_1+\mathbf{k}_2)}{k_1^2}\right] \\ &= \frac{1}{2}\left(\frac{\mathbf{k}_2}{k_1^2} + \frac{\mathbf{k}_1}{k_2^2}\right)\cdot(\mathbf{k}_1+\mathbf{k}_2)\end{aligned}$$

Perturbation expansion: Euler

Euler equation

$$\boxed{\dot{\theta} + H\theta + \frac{3}{2}a^2H^2\delta = -\nabla(\mathbf{v}\cdot\nabla\mathbf{v})}$$

...in *log a* time

$$\theta' + F\theta + S\delta = -\nabla(\mathbf{v}\cdot\nabla\mathbf{v})$$

$$F = 1 + \mathcal{H}'/\mathcal{H} = \tfrac{1}{2}(1 - 3w_{DE}\Omega_{DE})$$

$$S = 3\Omega_m/2.$$

expanding in Fourier

$$\begin{aligned}\theta' + F\theta + S\delta &= -\nabla\left[\left(\int \mathbf{v}_k e^{i\mathbf{kx}} d^3k\right)\cdot\nabla\int \mathbf{v}_{k'} e^{i\mathbf{k'x}} d^3k'\right] \\ &= -i\nabla\left[\int(\mathbf{v}_k\cdot\mathbf{k}')\mathbf{v}_{k'} e^{i\mathbf{kx}} e^{i\mathbf{k'x}} d^3k d^3k'\right] \\ &= \int(\mathbf{v}_k\cdot\mathbf{k}')(\mathbf{v}_{k'}\cdot(\mathbf{k}+\mathbf{k}'))e^{i(\mathbf{k}'+\mathbf{k})\mathbf{x}} d^3k' d^3k\end{aligned}$$

Perturbation expansion: Euler

$$\theta' + F\theta + S\delta = -\nabla\left[\left(\int \mathbf{v}_k e^{i\mathbf{kx}} d^3k\right) \cdot \nabla \int \mathbf{v}_{k'} e^{i\mathbf{k}'\mathbf{x}} d^3k'\right]$$
$$= -i\nabla\left[\int (\mathbf{v}_k \cdot \mathbf{k}')\mathbf{v}_{k'} e^{i\mathbf{kx}} e^{i\mathbf{k}'\mathbf{x}} d^3k d^3k'\right]$$
$$= \int (\mathbf{v}_k \cdot \mathbf{k}')(\mathbf{v}_{k'} \cdot (\mathbf{k}+\mathbf{k}')) e^{i(\mathbf{k}'+\mathbf{k})\mathbf{x}} d^3k' d^3k$$

on the RH we can use linear theory: $\mathbf{v} = -i\theta \dfrac{\mathbf{k}}{k^2}$

so we get

$$\theta' + F\theta + S\delta = -G^2 \int \theta_k \theta_{k'} \left(\frac{\mathbf{k}}{k^2} \cdot \mathbf{k}'\right)\left(\frac{\mathbf{k}'}{k'^2} \cdot (\mathbf{k}+\mathbf{k}')\right) e^{i(\mathbf{k}'+\mathbf{k})\mathbf{x}} d^3k' d^3k$$
$$= -G^2 \int \theta_k \theta_{k'} \left(\frac{\mathbf{k} \cdot \mathbf{k}'}{k^2 k'^2} \mathbf{k}' \cdot (\mathbf{k}+\mathbf{k}')\right) e^{i(\mathbf{k}'+\mathbf{k})\mathbf{x}} d^3k' d^3k$$

with a symmetrized kernel $\beta(\mathbf{k}_1, \mathbf{k}_2) = \dfrac{1}{2}\left[\left(\dfrac{\mathbf{k} \cdot \mathbf{k}'}{k^2 k'^2} \mathbf{k}' \cdot (\mathbf{k}+\mathbf{k}')\right) + (\mathbf{k} \leftrightarrow \mathbf{k}')\right] = \dfrac{1}{2k^2k'^2}\left[\mathbf{k} \cdot \mathbf{k}'(\mathbf{k}'+\mathbf{k}) \cdot (\mathbf{k}+\mathbf{k}')\right] = \dfrac{\mathbf{k} \cdot \mathbf{k}'(\mathbf{k}'+\mathbf{k})^2}{2k^2k'^2}$

Perturbation expansion: Euler

real space

$$\boxed{\begin{aligned}\theta' + F\theta + S\delta &= -G^2 \int \theta_k \theta_{k'} (\frac{\mathbf{k}}{k^2} \cdot \mathbf{k}')(\frac{\mathbf{k}'}{k'^2} \cdot (\mathbf{k}+\mathbf{k}'))e^{i(\mathbf{k}'+\mathbf{k})\mathbf{x}} d^3k' d^3k \\ &= -G^2 \int \theta_k \theta_{k'} (\frac{\mathbf{k}\cdot\mathbf{k}'}{k^2 k'^2} \mathbf{k}' \cdot (\mathbf{k}+\mathbf{k}'))e^{i(\mathbf{k}'+\mathbf{k})\mathbf{x}} d^3k' d^3k\end{aligned}}$$

Integrate over $(2\pi)^{-3} e^{-i\mathbf{k}''\mathbf{x}} d^3x$
to get Fourier transf.

$$\begin{aligned}\theta' + F\theta + S\delta &= -G^2 \int \theta_1 \theta_2 \beta(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k_1} + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 \\ &= -G^2 f^2 \int \delta_1 \delta_2 \beta(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 \equiv E\end{aligned}$$

Summary

$$\boxed{\theta + \delta' = G^2 f \int \delta_1 \delta_2 \alpha(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 \equiv C}$$

$$\boxed{\begin{aligned} \theta' + F\theta + S\delta &= -G^2 \int \theta_1 \theta_2 \beta(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 \\ &= -G^2 f^2 \int \delta_1 \delta_2 \beta(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 \equiv E \end{aligned}}$$

and the time derivative of the first:

$$\theta' + \delta'' = 2G^2 (f^2 + \frac{1}{2} f') \int \delta_1 \delta_2 \alpha(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 = C'$$

Summary

$$\theta + \delta' = G^2 f \int \delta_1 \delta_2 \alpha(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 \equiv C$$

$$\theta' + F\theta + S\delta = -G^2 \int \theta_1 \theta_2 \beta(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2$$

$$= -G^2 f^2 \int \delta_1 \delta_2 \beta(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 \equiv E$$

$$\theta' + \delta'' = 2G^2 (f^2 + \frac{1}{2} f') \int \delta_1 \delta_2 \alpha(\mathbf{k}_1, \mathbf{k}_2) \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d^3k_1 d^3k_2 = C'$$

then we get

$$\delta'' + F\delta' - S\delta = \underbrace{C' - E + FC}$$

NL terms

growth of the second order density contrast

$$\delta^{(2)} = G^{(2)} \delta_k^{(2)}$$

Ansatz:
(valid only in EdS)

$$G^{(2)} = AG^2,$$

Summary

$$\boxed{\begin{aligned}&\delta'' + F\delta' - S\delta = C' - E + FC \\ &\delta^{(2)} = G^{(2)}\delta_k^{(2)} \qquad G^{(2)} = AG^2,\end{aligned}}$$

LHS:

$$\delta'' + F\delta' - S\delta = [(G^{(2)})'' + FG^{(2)\prime} - SG^{(2)}]\delta = (4(f^2 + \frac{1}{2}f') + 2Ff - S)AG^2\delta_k^{(2)} = \underbrace{(4(f^2 + \frac{1}{2}f') + 2Ff - S)}_{7/2}\delta^{(2)}$$

EdS $\quad f = 1 \quad F = 1/2, S = 3/2.$

all together

$$\delta^{(2)} = \frac{2}{7}(\delta'' + F\delta' - S\delta) = 2\frac{C' - E + FC}{7}$$

$$= \frac{2G^2}{7}\int \delta_1\delta_2\underbrace{[2\alpha + \beta + \frac{1}{2}\alpha]}_{}\delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d^3k_1 d^3k_2$$

new combined kernel

Final expression

$$\boxed{\begin{aligned}\delta^{(2)} &= \frac{2}{7}(\delta'' + F\delta' - S\delta) = 2\frac{C' - E + FC}{7} \\ &= \frac{2G^2}{7}\int \delta_1\delta_2[2\alpha + \beta + \frac{1}{2}\alpha]\delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d^3k_1 d^3k_2\end{aligned}}$$

final expression

$$\begin{aligned}\delta^{(2)} &= \frac{G^2}{7}\int \delta_1\delta_2[5\alpha + 2\beta]\delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d^3k_1 d^3k_2 \\ &= \frac{2G^2}{7}\int \delta_1\delta_2[5\frac{\mathbf{k}}{4}(\frac{\mathbf{k}_1}{k_1^2} + \frac{\mathbf{k}_2}{k_2^2}) + \frac{(\mathbf{k}_1 + \mathbf{k}_2)^2(\mathbf{k}_1\mathbf{k}_2)}{2k_1^2k_2^2}]\delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d^3k_1 d^3k_2 \\ &= G^2\int \delta_1\delta_2 F_2(\mathbf{k}_1, \mathbf{k}_2)\delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})d^3k_1 d^3k_2\end{aligned}$$

new EdS kernel

$$F_2 = \frac{5}{7} + \frac{1}{2}\mathbf{k}_1\mathbf{k}_2(\frac{1}{k_1^2} + \frac{1}{k_2^2})) + \frac{2}{7}\frac{(\mathbf{k}_1\mathbf{k}_2)^2}{k_1^2k_2^2}$$

and for θ...

$$\theta^{(2)} + \delta'^{(2)} = \theta^{(2)} + 2G^2 f\delta^{(2)} = G^2 f \int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1 d^3k_2$$

final expressions

$$\theta^{(2)} = -G^2 f \int \delta_1\delta_2 G_2(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$

$$\delta^{(2)} = G^2 \int \delta_1\delta_2 F_2(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$

$$F_2 = \frac{5}{7} + \frac{\mathbf{k}_1\mathbf{k}_2}{2k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{2}{7}\left(\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2}\right)^2$$

$$G_2 = \frac{3}{7} + \frac{\mathbf{k}_1\mathbf{k}_2}{2k_1k_2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right) + \frac{4}{7}\left(\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2}\right)^2$$

and so on to higher orders…

$$
\begin{aligned}
\theta^{(3)} &= G^3 \int \theta_1\theta_2\theta_3 G_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3} \\
&= -G^3 \int \delta_1\delta_2\delta_3 G_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3} \\
\delta^{(3)} &= G^3 \int \delta_1\delta_2\delta_3 F_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}
\end{aligned}
$$

$$
F_3 = \frac{1}{18}[(7\alpha(k_1,k_2+k_3)F_2(k_2,k_3)+2\beta(k_1,k_2+k_3)G_2(k_2,k_3))+G_2(k_1,k_2)(7\alpha(k_1+k_2,k_3)+2\beta(k_1+k_2,k_3))] \tag{5.3.53}
$$

$$
G_3 = \frac{1}{18}[3\alpha(k_1,k_2+k_3)F_2(k_2,k_3)+6\beta(k_1,k_2+k_3)G_2(k_2,k_3)+G_2(k_1,k_2)(3\alpha(k_1+k_2,k_3)+6\beta(k_1+k_2,k_3))]
$$

(to be symmetrized)

Recursion relation

$$F_n(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\sum_{m=1}^{n-1}\frac{G_m(\mathbf{q}_1,\ldots,\mathbf{q}_m)}{(2n+3)(n-1)}\Big[(2n+1)\alpha(\mathbf{k}_1,\mathbf{k}_2)F_{n-m}(\mathbf{q}_{m+1},\ldots,\mathbf{q}_n)$$
$$+2\beta(\mathbf{k}_1,\mathbf{k}_2)G_{n-m}(\mathbf{q}_{m+1},\ldots,\mathbf{q}_n)\Big], \qquad (43)$$

$$G_n(\mathbf{q}_1,\ldots,\mathbf{q}_n)=\sum_{m=1}^{n-1}\frac{G_m(\mathbf{q}_1,\ldots,\mathbf{q}_m)}{(2n+3)(n-1)}\Big[3\alpha(\mathbf{k}_1,\mathbf{k}_2)F_{n-m}(\mathbf{q}_{m+1},\ldots,\mathbf{q}_n)$$
$$+2n\beta(\mathbf{k}_1,\mathbf{k}_2)G_{n-m}(\mathbf{q}_{m+1},\ldots,\mathbf{q}_n)\Big], \qquad (44)$$

$\mathbf{k}_1\equiv\mathbf{q}_1+\ldots+\mathbf{q}_m$, $\mathbf{k}_2\equiv\mathbf{q}_{m+1}+\ldots+\mathbf{q}_n$, $\mathbf{k}\equiv\mathbf{k}_1+\mathbf{k}_2$, and $F_1=G_1\equiv1$

(Bernardeau et al. 2001)

Quiz time

1. Is the EdS assumption necessary?
2. What is missing in order to compare to observations?
3. Up to which order do we need to go?
4. Which random variable is Gaussian?

is the assumption $G^{(2)} = AG^2$ necessary?

now we don't assume anything

$$\delta'' + F\delta' - S\delta = C' - E + FC = 2G^2(f^2 + \frac{1}{2}f')\int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$- G^2 f^2 \int \delta_1\delta_2\beta(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1 d^3k_2$$
$$+ FG^2 f \int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$= G^2(2f^2 + f' + Ff)\int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$+ G^2 f^2 \int \delta_1\delta_2\beta(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$

trial solution
(no longer separable)

$$\delta^{(2)} = G_{2A}(a)A(\mathbf{k}) + G_{2B}(a)B(\mathbf{k})$$

$$A(k) = \frac{5}{7}G^2\int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$B(k) = \frac{2}{7}G^2\int \delta_1\delta_2\beta(\mathbf{k}_1,\mathbf{k}_2)(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$

is the assumption $G^{(2)} = AG^2$ necessary?

$$\delta'' + F\delta' - S\delta = C' - E + FC = 2G^2(f^2 + \frac{1}{2}f')\int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$- G^2 f^2 \int \delta_1\delta_2\beta(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})d^3k_1 d^3k_2$$
$$+ FG^2 f \int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$= G^2(2f^2 + f' + Ff)\int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$+ G^2 f^2 \int \delta_1\delta_2\beta(\mathbf{k}_1,\mathbf{k}_2)\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$

$$\delta^{(2)} = G_{2A}(a)A(\mathbf{k}) + G_{2B}(a)B(\mathbf{k})$$

$$A(k) = \frac{5}{7}G^2\int \delta_1\delta_2\alpha(\mathbf{k}_1,\mathbf{k}_2)(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$
$$B(k) = \frac{2}{7}G^2\int \delta_1\delta_2\beta(\mathbf{k}_1,\mathbf{k}_2)(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}$$

we get two equations!

$$G''_{2A} + FG'_{2A} - SG_{2A} = \frac{7}{5}G^2(2f^2 + f' + Ff) = \frac{7}{5}G^2(f^2 + S)$$
$$G''_{2B} + FG'_{2B} - SG_{2B} = \frac{7}{2}G^2 f^2$$

…to be solved numerically!

Bias and RSD

back to the
real space-redshift space mapping

$$\boxed{\mathbf{s} = \mathbf{r}\left[1 + \frac{u(r)}{r}\right]}$$

we obtained
this expression

$$\delta_s = \frac{n(s)dV_s}{n_0 dV_s} - 1 = \frac{n(r)dV_r}{n_0 dV_r \left(1 + \frac{\Delta \mathbf{u}(r)}{r}\right)^2 |J|} - 1$$

$$|J| = 1 + \frac{du}{dr}$$

velocity along LOS

$$u = \mathcal{H}^{-1}\mathbf{v} \cdot \frac{\mathbf{r}}{r}$$

but now, we don't
linearize it!

$$\delta_s = \frac{n(r)dV_r}{n_0 dV_r |J|} - 1 = \frac{1 + \delta(r)}{1 + \frac{du}{dr}} - 1 = \frac{\delta(r) - \frac{du}{dr}}{1 + \frac{du}{dr}}$$

Bias and RSD

$$\boxed{\delta_s = \frac{n(r)dV_r}{n_0 dV_r |J|} - 1 = \frac{1+\delta(r)}{1+\frac{du}{dr}} - 1 = \frac{\delta(r) - \frac{du}{dr}}{1+\frac{du}{dr}}}$$

$$\boxed{\mathbf{s} = \mathbf{r}\left[1 + \frac{u(r)}{r}\right]}$$

in Fourier space $$\delta_s(k) = \int d^3s [\frac{\delta(r) - \frac{du}{dr}}{1+\frac{du}{dr}}] e^{i\mathbf{ks}} = \int \frac{d^3s}{1+\frac{du}{dr}} [\delta(r) - \frac{du}{dr}] e^{i\mathbf{kr} + i\mathbf{k}\frac{\mathbf{r}}{r}u} = \int d^3r \underbrace{[\delta(r) - \frac{du}{dr}]} e^{i\mathbf{kr} + i\mathbf{k}\frac{\mathbf{r}}{r}u}$$

we used $$d^3s = |J| d^3r$$

Redshift Space Distortion (RSD)

Fingers-of-God (FoG)

Bias and RSD

$$\boxed{\delta_s(k) = \int d^3s[\frac{\delta(r) - \frac{du}{dr}}{1+\frac{du}{dr}}]e^{i\mathbf{ks}} = \int \frac{d^3s}{1+\frac{du}{dr}}[\delta(r) - \frac{du}{dr}]e^{i\mathbf{kr}+i\mathbf{k}\frac{\mathbf{r}}{r}u} = \int d^3r[\delta(r) - \frac{du}{dr}]e^{i\mathbf{kr}+i\mathbf{k}\frac{\mathbf{r}}{r}u}}$$

$$\mathbf{v}\cdot\frac{\mathbf{r}}{r} = v\mu_\theta \qquad \theta = ik_\theta v/\mathcal{H}$$

$$e^{i\mathbf{k}\frac{\mathbf{r}}{r}u} = e^{i\mathbf{k}\frac{\mathbf{r}}{r}\frac{v}{\mathcal{H}}\mu_\theta} = e^{ik\mu\frac{v}{\mathcal{H}}\mu_\theta} = e^{\frac{k}{k_\theta}\theta\mu\mu_\theta}$$

Bias and RSD

$$\boxed{\delta_s(k) = \int d^3 s [\frac{\delta(r) - \frac{du}{dr}}{1 + \frac{du}{dr}}] e^{i\mathbf{ks}} = \int \frac{d^3 s}{1 + \frac{du}{dr}} [\delta(r) - \frac{du}{dr}] e^{i\mathbf{kr} + i\mathbf{k}\frac{\mathbf{r}}{r}u} = \int d^3 r [\delta(r) - \frac{du}{dr}] e^{i\mathbf{kr} + i\mathbf{k}\frac{\mathbf{r}}{r}u}}$$

$$e^{i\mathbf{k}\frac{\mathbf{r}}{r}u} = e^{i\mathbf{k}\frac{\mathbf{r}}{r}\frac{v}{\mathcal{H}}\mu_\theta} = e^{ik\mu\frac{v}{\mathcal{H}}\mu_\theta} = e^{\frac{k}{k_\theta}\theta\mu\mu_\theta}$$

expand the exponential and replace every factor by its Fourier series

$$e^{k\mu\theta\frac{\mu_\theta}{k_\theta}} = \sum_{n=0} \frac{(k\mu)^n}{n!} [\frac{\mu_\theta}{k_\theta}\theta(\mathbf{r})]^n = 1 +$$

$$\sum_{n=1} \frac{(k\mu)^n}{n!} \int \frac{d^3 q_1}{(2\pi)^3} \frac{\mu_1}{q_1} \theta(\mathbf{q}_1) e^{-i\mathbf{q}_1\mathbf{r}} \int \frac{d^3 q_2}{(2\pi)^3} \frac{\mu_2}{q_2} \theta(\mathbf{q}_2) e^{-i\mathbf{q}_2\mathbf{r}} .. \int \frac{d^3 q_n}{(2\pi)^3} \frac{\mu_n}{q_n} \theta(\mathbf{q}_n) e^{-i\mathbf{q}_n\mathbf{r}}$$

$$= 1 + \sum_{n=1} \frac{(k\mu)^n}{n!} \int \frac{d^3 q_1}{(2\pi)^3} \frac{\mu_1}{q_1} \theta(\mathbf{q}_1) \int \frac{d^3 q_2}{(2\pi)^3} \frac{\mu_2}{q_2} \theta(\mathbf{q}_2) .. \int \frac{d^3 q_n}{(2\pi)^3} \frac{\mu_n}{q_n} \theta(\mathbf{q}_n) e^{-i\sum_i^n \mathbf{q}_i\mathbf{r}}$$

Bias and RSD

$$\delta_s(k) = \int d^3s[\frac{\delta(r) - \frac{du}{dr}}{1+\frac{du}{dr}}]e^{i\mathbf{ks}} = \int \frac{d^3s}{1+\frac{du}{dr}}[\delta(r) - \frac{du}{dr}]e^{i\mathbf{kr}+i\mathbf{k}\frac{\mathbf{r}}{r}u} = \int d^3r[\delta(r) - \frac{du}{dr}]e^{i\mathbf{kr}+i\mathbf{k}\frac{\mathbf{r}}{r}u}$$

$$e^{i\mathbf{k}\frac{\mathbf{r}}{r}u} = e^{i\mathbf{k}\frac{\mathbf{r}}{r}\frac{v}{\mathcal{H}}\mu_\theta} = e^{ik\mu\frac{v}{\mathcal{H}}\mu_\theta} = e^{\frac{k}{k_\theta}\theta\mu\mu_\theta}$$

$$e^{k\mu\theta\frac{\mu_\theta}{k_\theta}} = \sum_{n=0}\frac{(k\mu)^n}{n!}[\frac{\mu_\theta}{k_\theta}\theta(\mathbf{r})]^n = 1+$$

$$\sum_{n=1}\frac{(k\mu)^n}{n!}\int\frac{d^3q_1}{(2\pi)^3}\frac{\mu_1}{q_1}\theta(\mathbf{q}_1)e^{-i\mathbf{q}_1\mathbf{r}}\int\frac{d^3q_2}{(2\pi)^3}\frac{\mu_2}{q_2}\theta(\mathbf{q}_2)e^{-i\mathbf{q}_2\mathbf{r}}..\int\frac{d^3q_n}{(2\pi)^3}\frac{\mu_n}{q_n}\theta(\mathbf{q}_n)e^{-i\mathbf{q}_n\mathbf{r}}$$

$$= 1+\sum_{n=1}\frac{(k\mu)^n}{n!}\int\frac{d^3q_1}{(2\pi)^3}\frac{\mu_1}{q_1}\theta(\mathbf{q}_1)\int\frac{d^3q_2}{(2\pi)^3}\frac{\mu_2}{q_2}\theta(\mathbf{q}_2)..\int\frac{d^3q_n}{(2\pi)^3}\frac{\mu_n}{q_n}\theta(\mathbf{q}_n)e^{-i\sum_i^n\mathbf{q}_i\mathbf{r}}$$

full expansion

$$\delta_s(\mathbf{k}) = \int d^3r[\delta(\mathbf{r}) - \frac{du}{dr}]\{e^{i\mathbf{kr}} + \sum_{n=1}\frac{(k\mu)^n}{n!}\int\frac{d^3q_1}{(2\pi)^3}\frac{\mu_1}{q_1}\theta(\mathbf{q}_1)\int\frac{d^3q_2}{(2\pi)^3}\frac{\mu_2}{q_2}\theta(\mathbf{q}_2)..\int\frac{d^3q_n}{(2\pi)^3}\frac{\mu_n}{q_n}\theta(\mathbf{q}_n)e^{i(\mathbf{k}-\sum_i^n\mathbf{q}_i)\mathbf{r}}\}$$

Bias and RSD

$$\delta_s(\mathbf{k}) = \int d^3r[\delta(\mathbf{r}) - \frac{du}{dr}]\{e^{i\mathbf{kr}} + \sum_{n=1} \frac{(k\mu)^n}{n!} \int \frac{d^3q_1}{(2\pi)^3} \frac{\mu_1}{q_1}\theta(\mathbf{q}_1) \int \frac{d^3q_2}{(2\pi)^3} \frac{\mu_2}{q_2}\theta(\mathbf{q}_2).. \int \frac{d^3q_n}{(2\pi)^3} \frac{\mu_n}{q_n}\theta(\mathbf{q}_n)e^{i(\mathbf{k}-\sum_i^n \mathbf{q}_i)\mathbf{r}}\}$$

We assumed that the angle μ_θ is a constant. This is called *flat-field* approximation: the galaxies are so far, and the field of view is so small in angular extension and in depth, that the angles between the vectors $\mathbf{k}, \mathbf{v}$ and the line of sight r, are constant.

now we introduce a bias expansion…

$$\delta_g(\mathbf{r}) = b_1\delta(\mathbf{r}) + \frac{1}{2}b_2\delta(\mathbf{r})^2 + ...$$

but unbiased velocities!

$$\theta_g = \theta.$$

Bias and RSD

$$\boxed{\delta_s(\mathbf{k}) = \int d^3r[\delta(\mathbf{r}) - \frac{du}{dr}]\{e^{i\mathbf{kr}} + \sum_{n=1} \frac{(k\mu)^n}{n!} \int \frac{d^3q_1}{(2\pi)^3}\frac{\mu_1}{q_1}\theta(\mathbf{q}_1) \int \frac{d^3q_2}{(2\pi)^3}\frac{\mu_2}{q_2}\theta(\mathbf{q}_2)..\int \frac{d^3q_n}{(2\pi)^3}\frac{\mu_n}{q_n}\theta(\mathbf{q}_n) e^{i(\mathbf{k}-\sum_i^n \mathbf{q}_i)\mathbf{r}}\}}$$

real space

$$\delta_g(\mathbf{r}) = b_1\delta(\mathbf{r}) + \frac{1}{2}b_2\delta(\mathbf{r})^2 + ...$$

Fourier space

$$\delta_g(\mathbf{k}) = b_1\delta^{(1)}(\mathbf{k}) + \frac{1}{2}b_2\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\delta_D(\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2)\delta^{(1)}(\mathbf{q}_1)\delta^{(1)}(\mathbf{q}_2) + ...$$

first-order term

$$\delta_g^{(1)}(\mathbf{k}) = \int d^3r[b_1\delta(\mathbf{r}) - \frac{du}{dr}]e^{i\mathbf{kr}} = \int d^3r b_1\delta(\mathbf{r})e^{i\mathbf{kr}} - \int d^3r\frac{du}{dr}e^{i\mathbf{kr}}$$
$$= \delta^{(1)}(\mathbf{k})(b_1 + f\mu^2)$$

Bias and RSD

$$\boxed{\delta_s(\mathbf{k}) = \int d^3r[\delta(\mathbf{r}) - \frac{du}{dr}]\{e^{i\mathbf{kr}} + \sum_{n=1} \frac{(k\mu)^n}{n!} \int \frac{d^3q_1}{(2\pi)^3}\frac{\mu_1}{q_1}\theta(\mathbf{q}_1)\int \frac{d^3q_2}{(2\pi)^3}\frac{\mu_2}{q_2}\theta(\mathbf{q}_2)..\int \frac{d^3q_n}{(2\pi)^3}\frac{\mu_n}{q_n}\theta(\mathbf{q}_n)e^{i(\mathbf{k}-\sum_i^n \mathbf{q}_i)\mathbf{r}}\}}$$

real space

$$\delta_g(\mathbf{r}) = b_1\delta(\mathbf{r}) + \frac{1}{2}b_2\delta(\mathbf{r})^2 + ...$$

Fourier space

$$\delta_g(\mathbf{k}) = b_1\delta^{(1)}(\mathbf{k}) + \frac{1}{2}b_2\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}\delta_D(\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2)\delta^{(1)}(\mathbf{q}_1)\delta^{(1)}(\mathbf{q}_2) + ...$$

first-order term

$$\delta_g^{(1)}(\mathbf{k}) = \int d^3r[b_1\delta(\mathbf{r}) - \frac{du}{dr}]e^{i\mathbf{kr}} = \int d^3r b_1\delta(\mathbf{r})e^{i\mathbf{kr}} - \int d^3r\frac{du}{dr}e^{i\mathbf{kr}}$$
$$= \delta^{(1)}(\mathbf{k})(b_1 + f\mu^2)$$