Heidelberg University

Department of Physics and Astronomy

Continuum mechanics

Prof. Ulrich Schwarz

Winter term 2023/24

Last update: April 11, 2025

Contents

CHAPTER 1

Introduction

Continuum Mechanics describes the movement of matter on a sufficiently large length scale (typically $\geq 100\,nm$) such that it can be considered to be continuously distributed, rather than consisting of discrete atoms or molecules. The appropriate mathematical concept is that of continuous fields, in particular:

$\rho(\mathbf{r}, t)$	mass density field
$\mathbf{u}(\mathbf{r}, t)$	deformation field
$\mathbf{v}(\mathbf{r}, t)$	velocity field

Fig. 1.1 shows examples: in (A) we see the deformation field of an elastic solids, and in (B) we see the flow field around a sphere.

These two examples in fact represent the two major areas into which continuum mechanics is usually divided: solid mechanics and fluid dynamics. Fig. 1.2 shows how one can further subdivide these fields. The core of solid mechanics is elasticity theory, but also defects and flow are important. Fluid dynamics is usually subdivided into flow of incompressible fluids like water (hydrodynamics) and compressible fluids like air (aerodynamics). In this course we will focus on solid mechanics, in particular on elasticity theory, but also cover the grey zone to hydrodynamics, namely viscoelasticity. We will begin with a one-dimensional version and only later turn to the full three-dimensional theory. Despite our focus on elasticity theory,

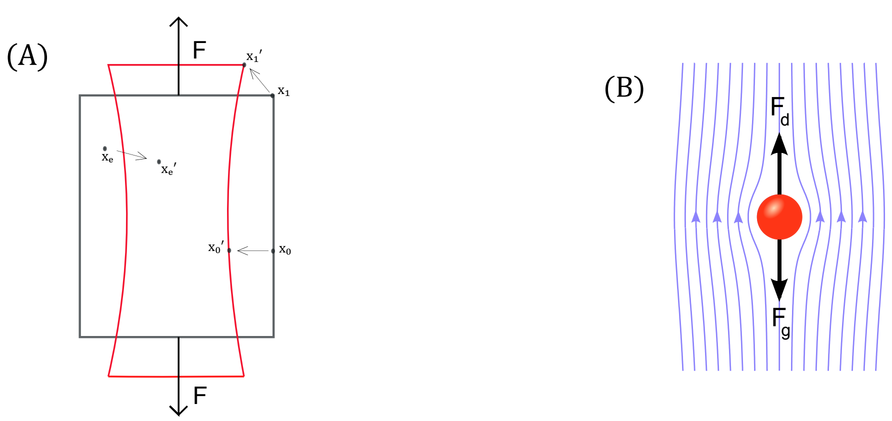


Figure 1.1.: (A) By deforming an elastic body, in general all points will change their position. By mapping the original onto the deformed shape, one obtains the *deformation field* $\mathbf{u}(\mathbf{r}, t)$. (B) The flow of a fluid around a sphere is described by a *velocity field* $\mathbf{v}(\mathbf{r}, t)$. Both vector fields are depicted by arrows.

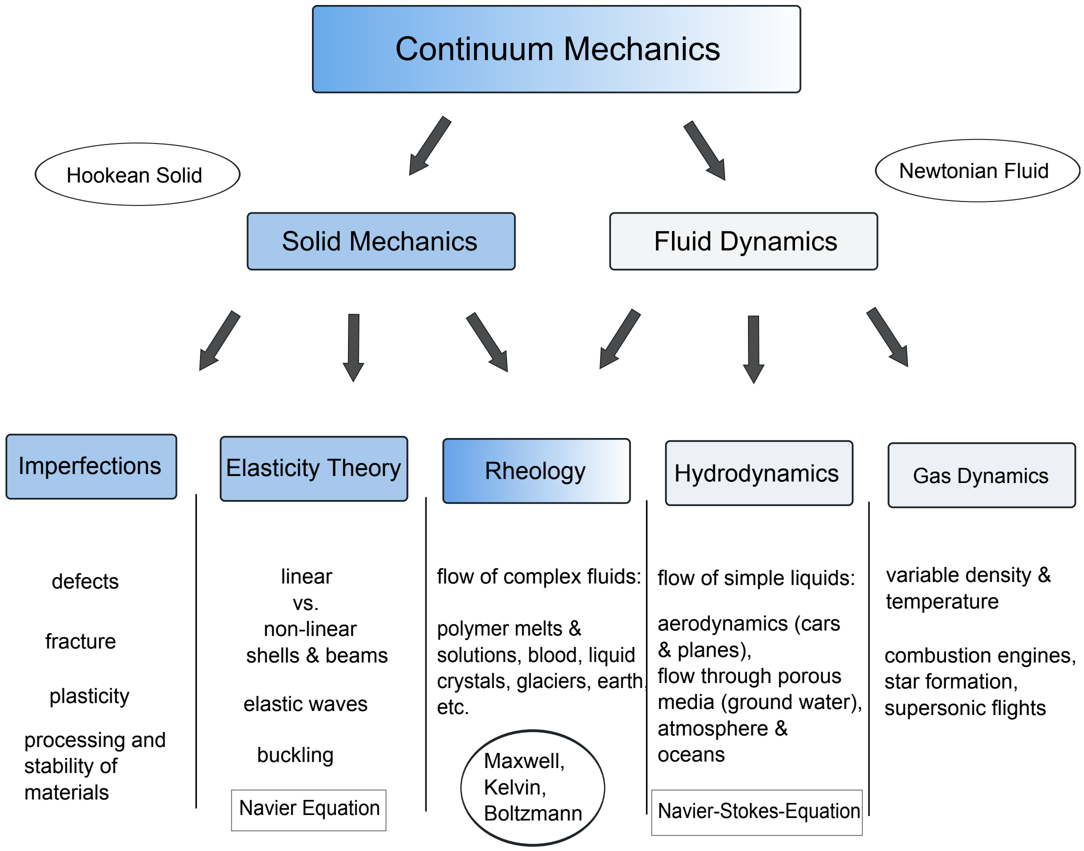


Figure 1.2.: Continuum mechanics can be subdivided into many different subdisciplines. Here we focus on elasticity theory, but also include viscoelasticity and rheology, which is the intersection with hydrodynamics.

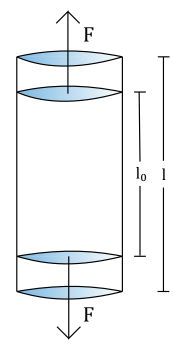
many of the methods we will encounter are also common to fluid dynamics, because in all cases, one deals with partial differential equations (PDEs) that can be solved e.g. with the finite element method.

Continuum mechanics is a classical field theory like electrodynamics and thus we will use similar concepts, e.g. Green's functions. It can be combined with other fundamental parts of physics, e.g. thermodynamics (if temperature, heat, work and entropy are important) or electrodynamics (if electric and/or magnetic fields exist). Such combined theories are often called *multiphysics*. Important applications for continuum mechanics are in engineering, biophysics, materials science (including biomaterials), environmental sciences and astrophysics. A very modern branch is *active systems*, where movement results from local energy sources.

CHAPTER 2

Linear viscoelasticity in 1d

2.1. Motivation



Continuum mechanics is a field theory for vectors and tensors of rank two. However, before we develop the three-dimensional theory, we first consider its 1d (scalar) version. In particular we introduce the concepts of viscoelasticity and the complex modulus. A typical experiment in this context would be the mechanical test of a fiber (see the figure on the right). There exist three typical setups for this test:

- Relaxation experiment: one prescribes the deformation and records the force.
- Creep experiment: one prescribes the force and records the deformation.
- Cyclic loading: force is varied in a period manner and one measure phase shift and amplitude of the deformation.

Fig. 2.1 depicts a relaxation experiment. In a perfect elastic system the force will follow exactly the stretch profile, whereas in a viscoelastic system, the asymmetry between loading and unloading leads to a *hysteresis cycle*. The energy $W = \int F dl \neq 0$, corresponding to the area of the rectangle, is dissipated as heat and thus presents a possibility to distinguish between elastic and viscoelastic systems.

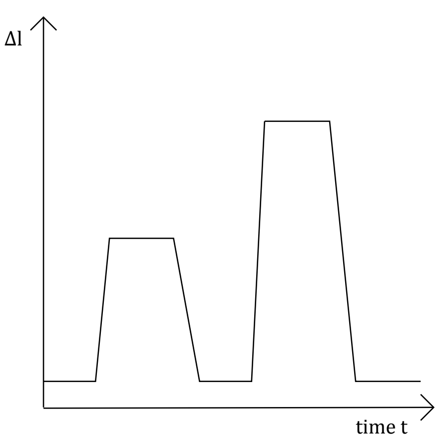


Figure 2.1.: Input

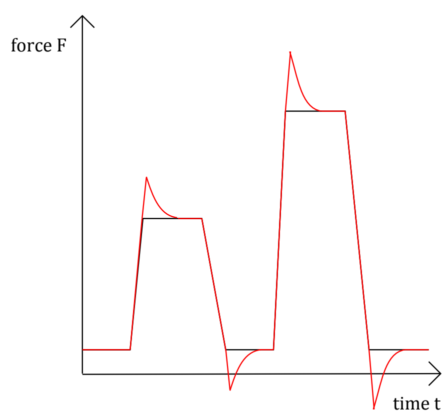


Figure 2.2.: Output

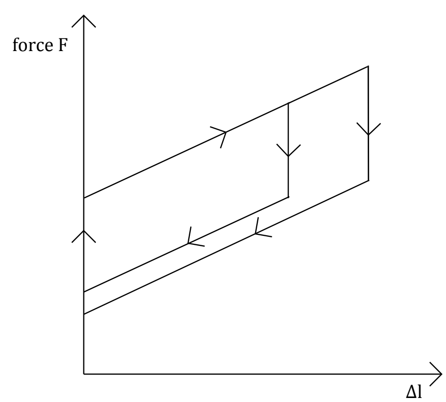


Figure 2.3.: Hysteresis cycle

2.2. Elastic response

As the simplest model consider a spring with spring constant k (fig.2.4).
The simplest possible mechanical response to a force is an elastic one: $F = k \cdot (l - l_o)$.

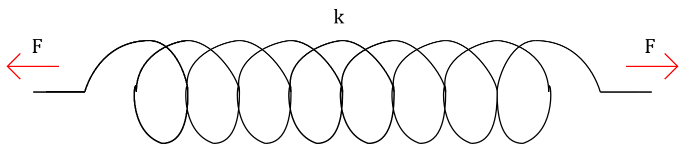


Figure 2.4.: The response of an elastic spring, characterized by its spring constant k, to an applied force **F** is in linear approximation given by *Hooke's Law.*

We define:

$$\textit{stretch} \qquad \lambda = \frac{l}{l_0} = \frac{l_0 + \Delta l}{l_0} \tag{2.1}$$

$$\textit{strain}^{1} \qquad \epsilon = \frac{\Delta l}{l_0} \tag{2.2}$$

Equation 2.2 can be used to define the *1d modulus* C which characterizes material properties of the fiber:

$$F = k \cdot \Delta l = (k \cdot l_0) \cdot \epsilon \equiv C \cdot \epsilon \tag{2.3}$$

In a 3d stretch experiment, a force F applied over an area A stretches the material from length l to Δl.
A linear elastic response implies:

$$\frac{F}{A} = E \cdot \frac{\Delta l}{l} \tag{2.4}$$

where *stress* $\sigma \equiv \frac{F}{A}$ and strain ϵ act as cause and effect

$$\sigma = E \cdot \epsilon \tag{2.5}$$

where E is the so called *Young's modulus* or *rigidity* of the material. From here we find $C = E \cdot A$.
Equation 2.5 might be recognized as Hooke's law conjuring up the image of macroscopic deformation as the result of the stretching of a large set of microscopic springs corresponding to the elastic elements within the material.
A dimensional analysis of the quantities in question reveals the following:

$$[\epsilon] = 1 \tag{2.6}$$

$$[C] = [F] = N, \tag{2.7}$$

$$[\sigma] = \frac{N}{m^2} = Pa \tag{2.8}$$

$$[E] = [\sigma] = Pa \tag{2.9}$$

For solid materials the Young's modulus is typically in the range of GPa, much larger than for soft matter, e.g. cells with rigidity in the order of $10\,kPa$.
Stretching the relationship between stress and strain a bit further, we can rewrite Equation 2.5 as

$$F = (E \cdot A) \cdot \epsilon = C \cdot \epsilon = (k \cdot l_0) \cdot \epsilon \tag{2.10}$$

$$\Rightarrow \qquad k = \frac{E \cdot A}{l_0} \tag{2.11}$$

where k is considered to be the 'spring constant' of the material.

[1]Linear elasticity theory (LET) is an expansion in small ϵ

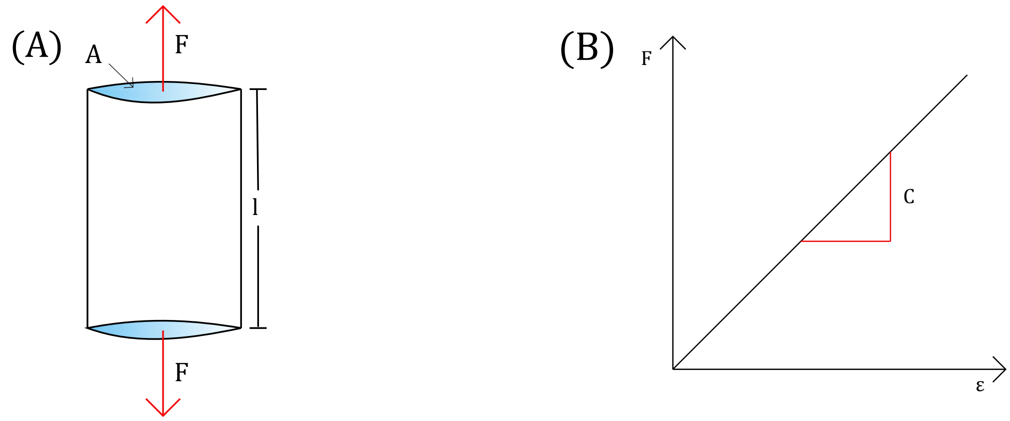


Figure 2.5.: (A) Elastic material of length l and cross-sectional area A is stretched by a force F which will result in a deformation of the material.
(B) Unlike viscoelastic material (compare fig. 2.3) an elastic material shows no hysteresis and does not dissipate energy.

2.3. Viscous response

Most biological materials show *viscoelastic* behaviour. In this section we will cast this behaviour in a one-dimensional mathematical format.

For a viscous element, force results from movement:

$$F = C_\eta \frac{1}{l}\frac{dl}{dt} \tag{2.12}$$

where C_η is the *damping coefficient* and

$$D \equiv \frac{1}{l}\frac{dl}{dt} \tag{2.13}$$

is called the *rate of deformation*. Recalling the stretch parameter $\lambda = \frac{l}{l_0}$ (Equation 2.1) and the relation $\lambda = 1 + \epsilon$ we can write

$$D = \frac{1}{l}\frac{dl}{dt} = \frac{1}{\lambda}\frac{d\lambda}{dt} = \frac{1}{1+\epsilon}\dot{\epsilon} \approx (1-\epsilon)\dot{\epsilon} \approx \dot{\epsilon} \tag{2.14}$$

for small strain $\epsilon \ll 1$ such that

$$D \approx \dot{\epsilon} = \frac{1}{l_0}\frac{dl}{dt} \tag{2.15}$$

and

$$\boxed{F = C_\eta \cdot D \approx C_\eta \cdot \dot{\epsilon}} \tag{2.16}$$

To perform the experiment correctly for all strain values one has to implement a constant deformation rate $D = const$:

$$\Rightarrow \frac{1}{l}\frac{dl}{dt} = \frac{d\ln l}{dt} = D = const \tag{2.17}$$

with solution

$$l = l_0 \cdot e^{Dt} \tag{2.18}$$

subject to the initial conditon $l(t=0) = l_0$, meaning that the endpoint has to be displaced exponentially in time in order to maintain a constant deformation rate.

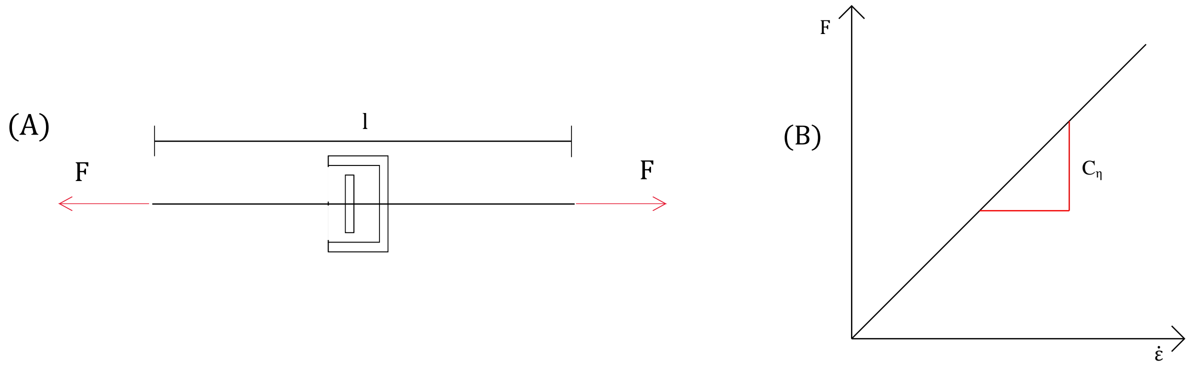


Figure 2.6.: (A) A dashpot is a damping device which resists motion via friction and serves as the mechanical equivalent of a viscous fibre.
(B) Response curve

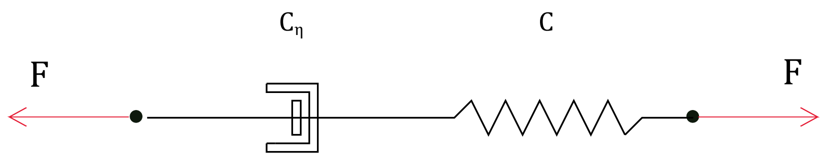


Figure 2.7.: Maxwell model: Dashpot and spring in series

2.4. Maxwell model

The Maxwell model is the simplest spring-and-dashpot model for a *viscoelastic fluid* (it flows on long time scales). The single elements of the model are given by

$$F_s = C \cdot \epsilon_s, \qquad F_d = C_\eta \cdot \dot{\epsilon}_d \tag{2.19}$$

for spring and dashpot respectively. The strains add up to ϵ

$$\epsilon = \epsilon_s + \epsilon_d \tag{2.20}$$

implying

$$\dot{\epsilon} = \dot{\epsilon}_s + \dot{\epsilon}_d. \tag{2.21}$$

The forces in the spring and the dashpot are the same, hence the overall strain rate can be written as

$$\dot{\epsilon} = \dot{\epsilon}_s + \dot{\epsilon}_d = \frac{1}{C}\dot{F} + \frac{1}{C_\eta}F \tag{2.22}$$

We consider a relaxation experiment, that is the strain ϵ is given and the force F has to be calculated: Multiplying Equation 2.22 with C and rearranging we get an ordinary differential equation (ODE) in F:

$$\dot{F} + \frac{C}{C_\eta}F = C\dot{\epsilon} \tag{2.23}$$

Introducing the *relaxation time* $\frac{C_\eta}{C} \equiv \tau$ we obtain

$$\dot{F} + \frac{1}{\tau}F = C\dot{\epsilon} \tag{2.24}$$

The general solution of the ODE is given by the homogeneous solution and one particular solution $F = F_h + F_p$. The homogeneous solution to

$$\dot{F_h} + \frac{1}{\tau}F_h = 0 \tag{2.25}$$

is given by

$$F_h = A_1 \cdot e^{-\frac{t}{\tau}}, \qquad A_1 = const \tag{2.26}$$

and one particular solution to the inhomogeneous equation 2.24 by

$$F_p = A(t) \cdot e^{-\frac{t}{\tau}} \tag{2.27}$$

Substitution into Equation 2.24 leads to an expression for A:

$$\frac{dA}{dt} = C \cdot e^{\frac{t}{\tau}} \cdot \dot{\epsilon} \tag{2.28}$$

hence

$$A = C \cdot \int_0^t dt' \cdot e^{\frac{t'}{\tau}} \cdot \dot{\epsilon}(t') \tag{2.29}$$

subject to the condition that for $t < 0$ the strain rate vanishes $\dot{\epsilon} \equiv 0$.
Likewise imposing $F = 0$ for $t < 0$ leads to $A_1 = 0$ and collecting all the pieces we arrive at the integral solution

$$\boxed{F(t) = C \cdot \int_0^t dt' \cdot e^{-\frac{(t-t')}{\tau}} \cdot \dot{\epsilon}(t')} \tag{2.30}$$

Example: 1) Strain ramp

For a strain depending linearly on time (compare fig. 2.9) the strain rate is constant $\dot{\epsilon} = const \equiv r$. Equation 2.30 then reduces to

$$F(t) = C \cdot r \cdot e^{-\frac{t}{\tau}} \int_0^t dt' \cdot e^{\frac{t'}{\tau}} \tag{2.31}$$

$$= C \cdot r \cdot e^{-\frac{t}{\tau}} \cdot \tau \cdot [e^{\frac{t}{\tau}} - 1] \tag{2.32}$$

$$= C_\eta \cdot r \cdot (1 - e^{-\frac{t}{\tau}}) \tag{2.33}$$

In the case of *short times* only the spring is extended and the response is linear and elastic (compare fig. 2.9)

$$F = C_\eta r \frac{t}{\tau} = Crt \tag{2.34}$$

For *long times* we have a constant and viscous response

$$F = C_\eta r, \tag{2.35}$$

the spring has a constant extension and the force is dominated by the dashpot.

Example: 2) Relaxation experiment

We keep ϵ constant starting at time t^*. Then the strain rate is zero for $t > t^*$

$$\dot{\epsilon} = 0 \tag{2.36}$$

and the force is given by

$$F = F^* \cdot e^{-\frac{(t-t^*)}{\tau}} \tag{2.37}$$

The spring relaxes back to zero, no energy is stored but dissipated as heat.

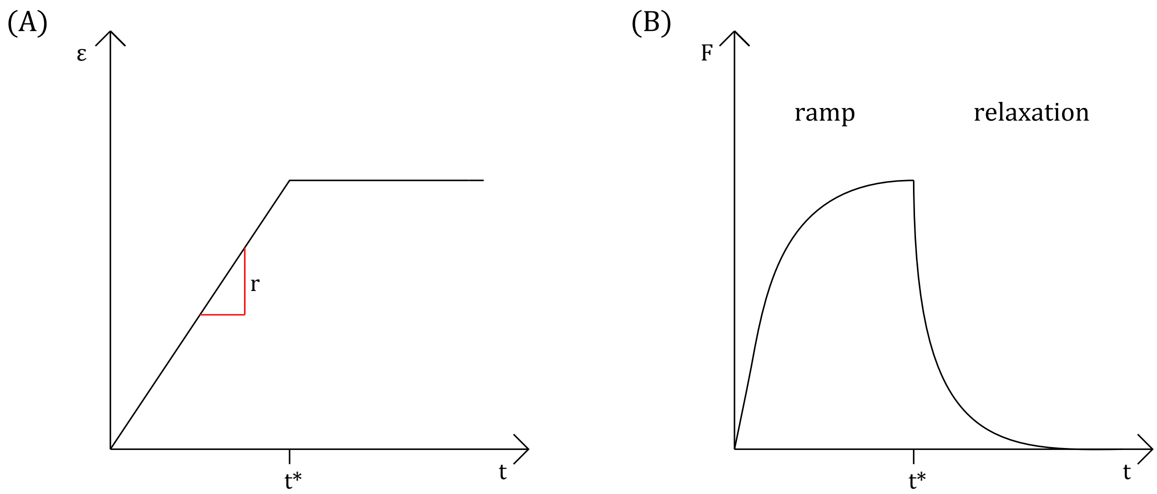


Figure 2.8.: (A) Strain as a function of time and (B) the force response for the spring-dashpot model.

2.5. Laplace transformation

For the Maxwell model we had to solve the ODE

$$\dot{F} + \frac{1}{\tau}F = C\dot{\epsilon} = f(t) \tag{2.38}$$

for $t \geq 0$ and with known initial condition $F(0)$. This can be nicely done with *Laplace transforms*.

Definition 1. Let f(t) be a function defined for $t \geq 0$:

$$\hat{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty dt\, f(t) \cdot e^{-st}, \qquad s \in \mathbb{C}$$

The *back transforms*

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \hat{f}(s) \cdot e^{st} \tag{2.39}$$

follow from complex analysis and are tabulated in many books.
We now show how to solve the ODE (Equation 2.38):

$$\dot{F} + BF = f(t) \tag{2.40}$$

$$\Rightarrow \qquad s\hat{F}(s) - F_0 + B\hat{F}(s) = \mathcal{L}[f(t)] = \hat{f}(s) \tag{2.41}$$

$$\Rightarrow \qquad \hat{F}(s) = \frac{\hat{f}(s) + F_0}{s + B} \tag{2.42}$$

Taking the strain ramp as an example from above:

$$f(t) = C \cdot \dot{\epsilon} = C \cdot r = const, \qquad B = \frac{1}{\tau}, \qquad F_0 = 0 \tag{2.43}$$

$$\Rightarrow \qquad \hat{f}(s) = \frac{C \cdot r}{s} \tag{2.44}$$

$$\Rightarrow \qquad \hat{F}(s) = \frac{C \cdot r}{s \cdot (s + B)} \tag{2.45}$$

$$\Rightarrow \qquad F(t) = C \cdot r \cdot \tau(1 - e^{-\frac{t}{\tau}}) \tag{2.46}$$

Table 2.1.: Some examples of forward Laplace transformations

$f(t)$	$\hat{f}(s)$
1	$\int_0^\infty dt\, 1 e^{-st} = \frac{1}{s}$
e^{at}	$\int_0^\infty dt\, e^{-(s-a)t} = \frac{1}{s-a}$
$e^{at} - e^{bt}$	$\frac{1}{s-a} - \frac{1}{s-b} = \frac{a-b}{(s-a)(s-b)}$
t	$\int_0^\infty dt\, t e^{-st} = \left[-\frac{t}{s}e^{-st}\right]_0^\infty + \int_0^\infty dt\, \frac{1}{s}e^{-st} = \frac{1}{s^2}$
$t \cdot f(t)$	$-\frac{d}{ds}\int_0^\infty dt\, f(t)e^{-st} = -\frac{d}{ds}\hat{f}(s)$
t^n	$\int_0^\infty dt\, e^{-st} t \cdot t^{n-1} = -\frac{d}{ds}\mathcal{L}[t^{n-1}] = (-1)^{n-1}\frac{d^{n-1}}{ds^{n-1}}\mathcal{L}[t] = \frac{n!}{s^{n+1}}$
$f'(t)$	$\int_0^\infty dt\, e^{-st}\frac{df}{dt} = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty dt\, e^{-st}f(t) = -f(0) + s\hat{f}(s)$
convolution integral: $\int y(t-t')x(t')dt' \equiv f(t)$	$\int_0^\infty dt\, e^{-st}\int dt' y(t-t')x(t') = \mathcal{L}[x(t)] \cdot \mathcal{L}[y(t)], \quad t = t'' + t'$

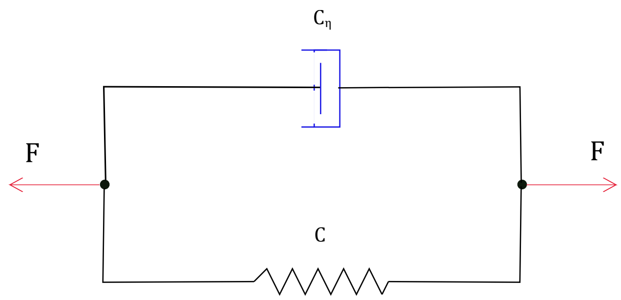


Figure 2.9.: Spring-dashpot in parallel is a Kelvin-Voigt arrangement

2.6. Kelvin-Voigt model

The *Kelvin-Voigt model* as another example of combined viscoelastic behaviour is the simplest model for a *viscoelastic solid* (it does not flow on long time scales).
In analogy to an electric circuit, the forces add up so that the total force equals the sum of the forces due to the elastic spring F_s and the viscous damper F_d:

$$F = F_s + F_d = C \cdot \epsilon + C_\eta \cdot \dot{\epsilon} \tag{2.47}$$

The natural way to analyse this situation is a *creep experiment* where the force is prescribed. Following the mathematical treatment of the Maxwell modell, the constitutive ODE for the Kelvin-Voigt model is given by

$$\boxed{\dot{\epsilon} + \frac{1}{\tau}\epsilon = \frac{F}{C_\eta}} \qquad \text{linear ODE for } \epsilon \tag{2.48}$$

Like equation 2.30, but with ϵ and F exchanged, the solution is

$$\epsilon(t) = \frac{1}{C_\eta} \int_0^t dt'\, e^{-(t-t')/\tau}\, F(t') \tag{2.49}$$

where we assume that at $t = 0$ we start to pull on the setup. In the case of a *force jump* to a constant value F_0 at $t = 0$, the *creep function* is given by

$$J(t) = \frac{\epsilon(t)}{F_0} = \frac{1}{C}(1 - e^{-\frac{t}{\tau}}) \tag{2.50}$$

The strain response is such that it initially increases and then plateaus which is called *creep* (see fig. 2.11). Looking at the limits we have for $t \ll \tau$:

$$\epsilon = \frac{F_0 \cdot t}{C_\eta} \qquad \text{a linear viscous response} \tag{2.51}$$

and for $t \gg \tau$:

$$\epsilon = \frac{F_0}{C} \qquad \text{a constant elastic response} \tag{2.52}$$

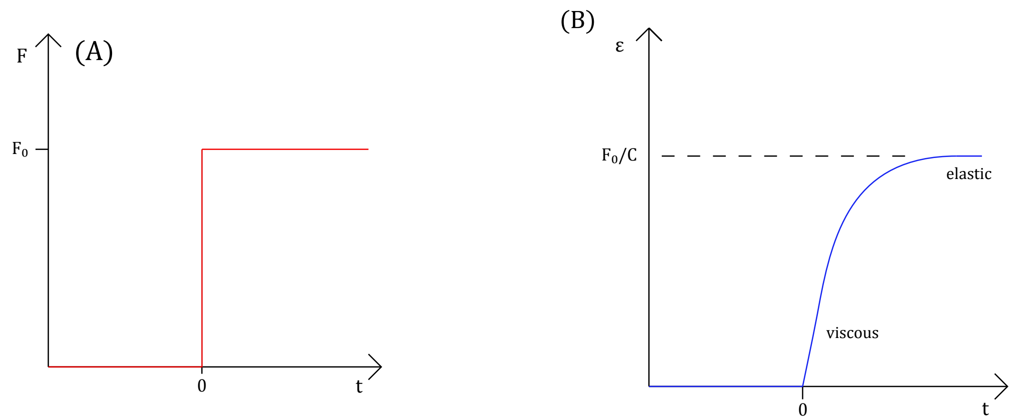


Figure 2.10.: (A) The simple Heaviside step function is often used as a loading protocol for viscoelastic models.
(B) The behaviour of the Maxwell model is reversed, showing that the response not only depends on scaling but also on the arrangements of elements in the setup.

2.7. Standard linear model

For the standard linear model (see cartoon in fig. 2.12) we now combine the Maxwell and Kelvin-Voigt models. For the linear elements the force has to be the same, whereas in parallel the forces add up. Hence we get the following relations:

$$F = F_1 + F_2 \tag{2.53}$$
$$= C_1 \cdot (\epsilon - \epsilon_d) + C_2 \cdot \epsilon \tag{2.54}$$
$$\dot{\epsilon}_d = \frac{F_1}{C_\eta} = \frac{F - F_2}{C_\eta} = \frac{F - C_2\epsilon}{C_\eta} \tag{2.55}$$

Inserting (2.54) into (2.55) and thus eliminating F_1, F_2 and $\dot{\epsilon}_d$ and again introducing the relaxation time τ (equation 2.4)

$$\Rightarrow \qquad \dot{F} = C_1 \cdot (\dot{\epsilon} - \dot{\epsilon}_d) + C_2 \cdot \dot{\epsilon} \tag{2.56}$$
$$= (C_1 + C_2) \cdot \dot{\epsilon} - C_1 \cdot (\frac{F - C_2 \cdot \epsilon}{C_\eta}) \tag{2.57}$$

we finally obtain the constitutive equation for the standard linear model:

$$\Rightarrow \qquad \boxed{\tau\dot{F} + F = (C_1 + C_2)\tau\dot{\epsilon} + C_2\epsilon} \tag{2.58}$$

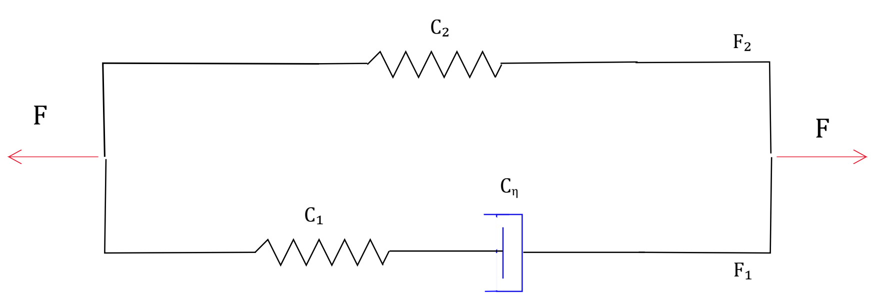


Figure 2.11.: The standard linear model with one dashpot and two springs.

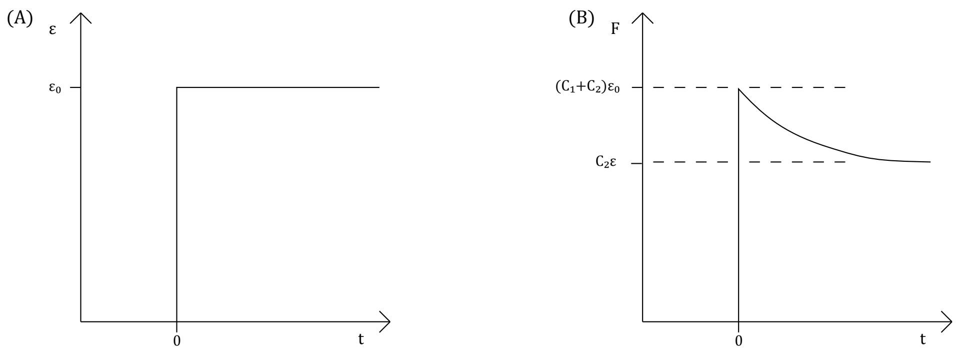


Figure 2.12.: (A) Strain jump.(B) Force response

By taking away the upper branch, i.e. $C_2 \to 0$ we regain the Maxwell model and in the limit $C_1 \to \infty$ the elastic elements become infinitely rigid, leading back to the Kelvin-Voigt model:

$$C_2 \to 0 \qquad \Rightarrow \qquad \tau \dot{F} + F = C_1 \tau \dot{\epsilon} \qquad \text{Maxwell} \tag{2.59}$$

$$C_1 \to \infty \qquad \Rightarrow \qquad F = C_\eta \dot{\epsilon} + C_2 \epsilon \qquad \text{Kelvin-Voigt} \tag{2.60}$$

Equation 2.58 allows for both a relaxation and a creep experiment by specification of the source term.

Example: Strain jump (relaxation experiment, see fig. 2.13)

The relaxation function is given by

$$F(t) = \epsilon_0 \cdot (C_2 + C_1 e^{-\frac{t}{\tau}}) \tag{2.61}$$

Getting the jump at $t = 0$ is not trivial. We introduce two times $t = 0^-$ and $t = 0^+$ at the left and right of $t = 0$ and rewrite the ODE as:

$$\frac{F(0^-) + F(0^+)}{2} + \tau \cdot \frac{F(0^+) - F(0^-)}{\Delta t} = C_2 \cdot \frac{\epsilon(0^-) + \epsilon(0^+)}{2} + (C_1 + C_2)\tau \cdot \frac{\epsilon(0^+) - \epsilon(0^-)}{\Delta t} \tag{2.62}$$

Multiplying with Δt and using the one-sided nature of the function (hence $F(0^-) = 0 = \epsilon(0^-)$) we find in the limit $\Delta t \to 0$

$$F(0^+) = (C_1 + C_2) \cdot \epsilon(0^+) \tag{2.63}$$

a finite force jump with magnitude $(C_1 + C_2)\epsilon_0$.

2.8. Boltzmanns Theory of Linear Viscoelasticity

The spring-and-dashpot models discussed above can be generalized to a class of materials called *linear viscoelastic bodies* (Boltzmann 1876). The basic assumption here is superposition: Individual loading histories add up linearly to the combined loading history. Therefore, all we need to know is the response to a unit-step perturbation.
Consider a creep experiment where we prescribe the force $F = H(t)$ with Heaviside function $H(t)$.
The strain in response will follow the force denoted by $\epsilon(t) = J(t)$ which is called the *creep compliance* or simply the *creep function*. The superposition principle is graphically shown below:

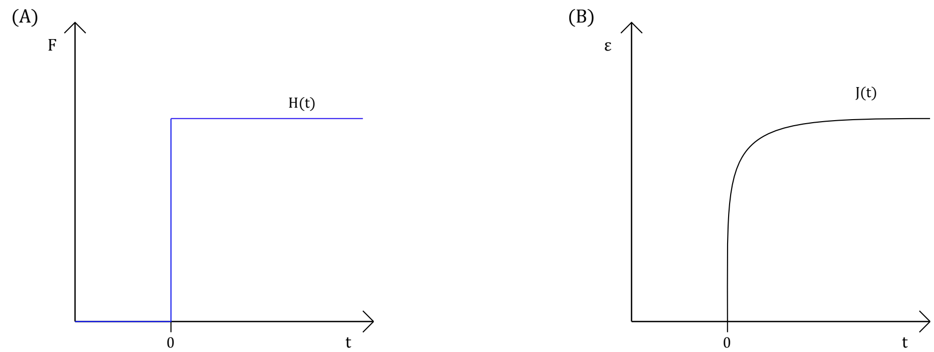


Figure 2.13.: Example of a strain response to a prescribed unit-step in force.

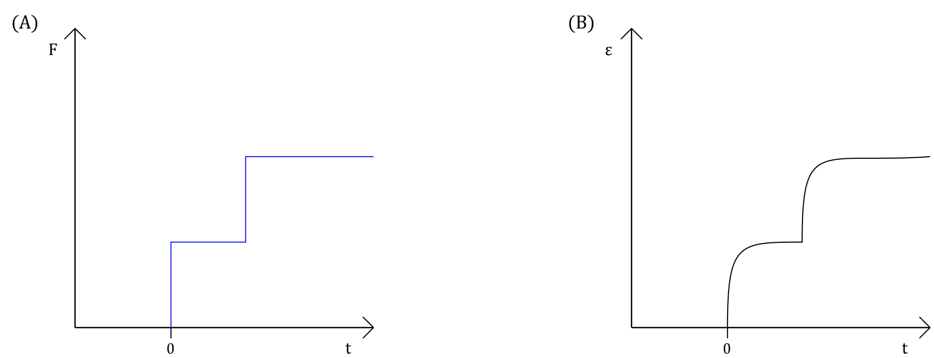


Figure 2.14.: Linear superposition of responses.

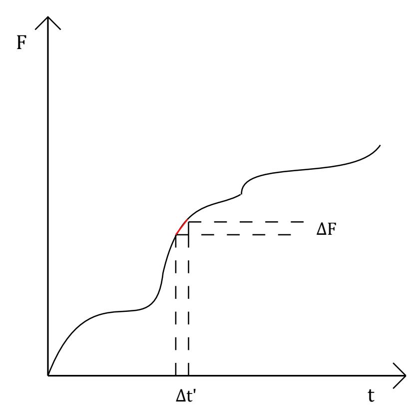


Figure 2.15.: An arbitrary force linearized.

An arbitrary perturbation can be considered as an infinite number of small steps in the force: The increase in the force is then given by

$$\Delta F(t') = \frac{dF(t')}{dt'}dt' \tag{2.64}$$

and it follows for the strain response that

$$\Delta\epsilon(t) = \dot{F}(t')dt' J(t-t') \tag{2.65}$$

Taking the time intervals $\Delta t'$ infinitesimally small and using the superposition principle, we can add up all the responses to steps in the force to get an integral expression first derived by Boltzmann in 1876:

$$\boxed{\epsilon(t) = \int_{-\infty}^{t} dt'\, J(t-t')\dot{F}(t')} \qquad \text{Boltzmann 1876} \tag{2.66}$$

In the same way one can write for a relaxation experiment:

$$\boxed{F(t) = \int_{-\infty}^{t} dt'\, G(t-t')\dot{\epsilon}(t')} \tag{2.67}$$

where $G(t)$ is the relaxation function. Obviously G and J must be related to each other[2].
Again the natural framework for this are *Laplace transforms*. The two integral equations become

$$\hat{\epsilon}(s) = \hat{J}(s) \cdot s \cdot \hat{F}(s) \tag{2.68}$$

$$\hat{F}(s) = \hat{G}(s) \cdot s \cdot \hat{\epsilon}(s) \tag{2.69}$$

$$\Rightarrow \qquad \hat{G}(s) \cdot \hat{F}(s) = \frac{1}{s^2} \tag{2.70}$$

$$\Rightarrow \qquad \boxed{\int_{0}^{t} dt'\, J(t-t')G(t') = t} \tag{2.71}$$

Creep function J and relaxation function G are thus related by an integral equation. In principal it is sufficient to know one of them.

2.9. Complex modulus

So far we have introduced the 1d elastic modulus C and the viscous modulus C_η (damping coefficient) by

$$F_s = C \cdot \epsilon_s \qquad [C] = N \tag{2.72}$$

$$F_d = C_\eta \cdot \dot{\epsilon}_d \qquad [C_\eta] = Ns \tag{2.73}$$

For composite systems, we have seen that the overall response depends on time scales. It therefore makes sense to consider harmonic excitations, when loading has the form of a sine or cosine (Fourier analysis):

$$\epsilon(t) = \epsilon_0 \cdot \cos(\omega t) \tag{2.74}$$

[2] G and J act as propagators. In the creep experiment, think of this as creating a perturbation at t' and then propagate it in a linear manner to the present time and integrate over the past history. G and J are both Green's functions to the ODE of the viscoelastic model of interest and can be calculated respectively by specifying the source term.

$$\Rightarrow \qquad F(t) = \int_{-\infty}^{t} dt'\, G(t-t')\dot{\epsilon}(t) \qquad \text{Boltzmann linear viscoelasticity} \tag{2.75}$$

$$= -\int_{\infty}^{0} ds\, G(s)\dot{\epsilon}(t-s), \qquad s = t - t' \tag{2.76}$$

$$= \int_{0}^{\infty} ds\, G(s)\dot{\epsilon}(t-s) \tag{2.77}$$

Inserting

$$\dot{\epsilon}(t-s) = -\epsilon_0\omega\sin(\omega(t-s)) \tag{2.78}$$

$$= -\epsilon_0\omega\sin(\omega t)\cos(\omega s) + \epsilon_0\omega\cos(\omega t)\sin(\omega s) \tag{2.79}$$

gives

$$F(t) = \underbrace{\epsilon_0\cos(\omega t)}_{\text{in phase, elastic}} \cdot \underbrace{\left[\omega\int_0^\infty ds\, G(s)\sin(\omega t)\right]}_{\equiv E_1(\omega),\ \text{"storage modulus"}} - \underbrace{\epsilon_0\sin(\omega t)}_{\text{out of phase, viscous}} \cdot \underbrace{\left[\omega\int_0^\infty ds\, G(s)\cos(\omega s)\right]}_{\equiv E_2(\omega)\ \text{"loss modulus"}} \tag{2.80}$$

$$= \epsilon_0 E_1\cos(\omega t) - \epsilon_0 E_2\sin(\omega t) \tag{2.81}$$

$$= F_0\cdot\cos(\omega t + \delta) \tag{2.82}$$

with

$$\delta = \arctan(\frac{E_2}{E_1}) \qquad \text{phase shift} \tag{2.83}$$

$$F_0 = \epsilon_0\sqrt{E_1^2 + E_2^2} \qquad \text{amplitude} \tag{2.84}$$

We can again look at the elastic and viscous limit

$$\text{elastic limit:} \qquad E_2 = 0, \quad \delta = 0, \quad F_0 = \epsilon_0\cdot E_1 \tag{2.85}$$

$$\text{viscous limit:} \qquad E_1 = 0, \quad \delta = \frac{\pi}{2}, \quad F_0 = \epsilon_0\cdot E_2 \tag{2.86}$$

where in the elastic limit no energy is dissipated and the force in the viscous limit is shifted by half a cycle. The amount of work for one loading cycle can be calculated

$$W = \int_0^{\frac{2\pi}{\omega}} dt\, F\cdot\dot{\epsilon} \tag{2.87}$$

$$= -\int_0^{\frac{2\pi}{\omega}} dt\, \left[\epsilon_0 E_1\cos(\omega t) - \epsilon_0 E_2\sin(\omega t)\right]\epsilon_0\omega\sin(\omega t) \tag{2.88}$$

$$= \pi\epsilon_0^2 E_2 \qquad \text{This work is disspated as heat} \tag{2.89}$$

The work represented by E_1 is released again during unloading ($\rightarrow$ *storage modulus*), but not the work represented by E_2 ($\rightarrow$ *loss modulus*).

In harmonic calculations it is often convenient to use comlex numbers. Instead of equation 2.74 write

$$\epsilon(t) = \mathrm{Re}\left(\epsilon_0 e^{i\omega}t\right) \tag{2.90}$$

Then the convolution integral becomes

$$F(t) = \mathrm{Re}\left(\int dt'\, G(t-t')\dot{\epsilon}(t')\right) \tag{2.91}$$

$$= \mathrm{Re}\left(\int ds\, G(s)\dot{\epsilon}(t-s)\right) \tag{2.92}$$

$$= \mathrm{Re}\left(i\omega\epsilon_0 e^{i\omega t}\underbrace{\int ds\, G(s)e^{-i\omega s}}_{G^*(\omega)}\right) \tag{2.93}$$

where we recognize $G^*(\omega)$ as the Fourier transform of $G(t)$.
We then define the *complex modulus* as

Definition 2. $E^*(\omega) = i\omega G^*(\omega) = E_1(\omega) + iE_2(\omega)$

$$\Rightarrow \qquad F(t) = \mathrm{Re}\left(\epsilon_0 e^{i\omega t}E^*(\omega)\right) \tag{2.94}$$

$$= \mathrm{Re}\left(\epsilon_0(\cos(\omega t) + i\sin(\omega t))(E_1(\omega) + iE_2(\omega))\right) \tag{2.95}$$

$$= \epsilon_0 E_1\cos(\omega t) - \epsilon_0 E_2 \sin(\omega t) \tag{2.96}$$

thus E_1 and E_2 are the storage and loss modulus as defined above.
With these definitions, the fundamental relation between force and strain

$$F(t) = \int_{-\infty}^{\infty} ds\, G(s)\dot{\epsilon}(t-s) \tag{2.97}$$

simply becomes

$$\boxed{\begin{aligned} F^*(\omega) &= G^*(\omega)i\omega\epsilon^*(\omega) \\ &= E^*(\omega)\epsilon^*(\omega) \end{aligned}} \tag{2.98, 2.99}$$

In the physics literature the complex modulus $E^*(\omega)$ is usually denoted as

$$G(\omega) = G'(\omega) + i\cdot G''(\omega) \tag{2.100}$$

where $G'(\omega)$ is the storage and $G''(\omega)$ the loss modulus.
$G(\omega)$ carries the complete viscoelastic information of a system. It is measured in a *rheometer*.

A typical result for a polymer melt (liquid) is shown in fig. 2.18:
The polymer is elastic at high frequencies ($G' > G''$) and viscous at low frequencies ($G'' > G'$) (this corresponds to the longtime limit).
Here

$$G'' = C_\eta \omega \tag{2.101}$$

and

$$G' \approx \frac{C_\eta^2\omega^2}{C} \tag{2.102}$$

. Equating at the intersection gives an expression for ω and thus a simple way to extract the relaxation time τ:

$$\omega = \frac{C}{C_\eta} = \frac{1}{\tau} \tag{2.103}$$

Perfect elastic networks like PDMS (Polydimethylsiloxan) have a large and constant storage modulus and small loss modulus.
The Newtonian liquid water has a constant loss modulus for $10^{-5}\,\mathrm{Hz} < \dot{\epsilon} < 10^{5}\,\mathrm{Hz}$ (viscoelasticity is expected for $10^{10}\,\mathrm{Hz}$!)

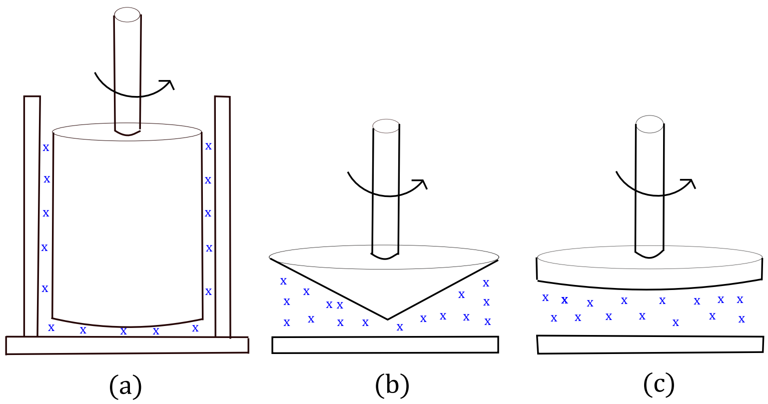


Figure 2.16.: Sketches of rotational rheometers: (a) cylindrical, (b) cone and plate, (c) parallel plate

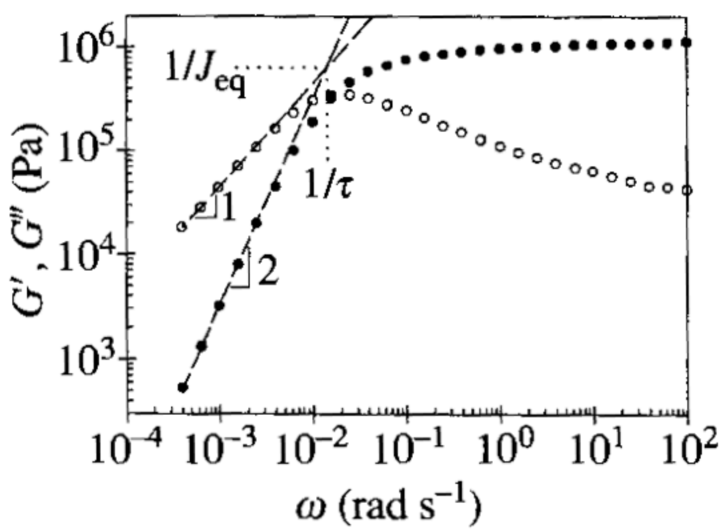


Figure 2.17.: Response of a linear polybutadiene at 25 °C. Filled symbols are the storage modulus G' and open symbols are the loss modulus G'' (Rubinstein and Colby, Polymer Physics, 2003, p. 293).

CHAPTER 3

Distributed forces in 1d

In the preceding chapter, deformations of 1d structures were represented only by one variable. In this chapter we address the deformation of 1d structures in more detail, in particular the effect of spatially distributed deformations and forces.

3.1. Continuum equation for an elastic bar

Consider a bar loaded by a force F at $x = L$ and clamped in space at $x = 0$ (fig. 3.1) . Different from Chapter 2 we now allow the cross sectional area A and the elastic modulus E to be functions of x. We also allow for some force per volume Q to act in x-direction (e.g. gravity) and the coupling to an elastic foundation, which introduces a traction force.
Consider a slice of thickness Δx. Here we have

$$N(x) = N(x + \Delta x) + Q(x)A(x) \cdot \Delta x - Ku(x) \tag{3.1}$$

where $N(x)$ is the normal force on the slice surface. With $q(x) = Q(x)A(x)$, ρ the spring constant K per length and in the continuum limit ($\Delta x \to 0$) we obtain

$$\frac{dN}{dx} + q(x) - \rho u(x) = 0 \tag{3.2}$$

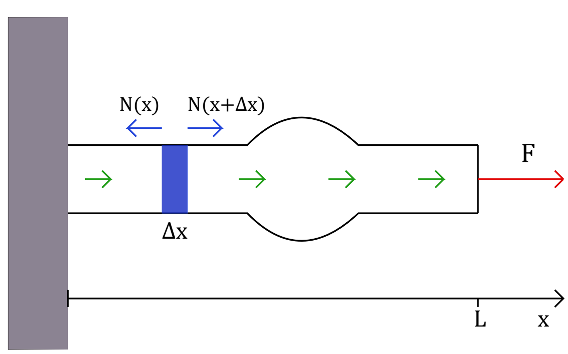


Figure 3.1.: Representation of an elastic bar with magnification of the forces acting on a slice of thickness Δx.

If the force per volume Q and the traction are zero, the normal force N is constant throughout the bar. We now introduce a *displacement field* $u(x)$. Then the strain at position x reads

$$\epsilon = \frac{u(x+\Delta x) - u(x)}{\Delta x} \overset{\Delta x \to 0}{=} \frac{du(x)}{dx} \tag{3.3}$$

$$[\epsilon] = \frac{m}{m} = 1 \tag{3.4}$$

Strain is related to stress by Hooke's law:

$$\sigma = E \cdot \epsilon \tag{3.5}$$

where the stress σ is defined as

$$\sigma = \frac{N}{A} \tag{3.6}$$

Substitution of $N = A\sigma$ into equation 3.2 yields a second order differential equation for the displacement $u(x)$

$$-q + \rho u = \frac{dN}{dx} = \frac{d}{dx}(\sigma A) \tag{3.7}$$

$$= \frac{d}{dx}(EA\epsilon) \tag{3.8}$$

$$= \frac{d}{dx}\left(C(x) \cdot \frac{du}{dx}\right) \tag{3.9}$$

where $C(x) = E(x) \cdot A(x)$ is the 1d modulus.

$$\Rightarrow \qquad \boxed{\frac{d}{dx}\left(C(x) \cdot \frac{du}{dx}\right) + q(x) - \rho u(x) = 0} \qquad \text{central ODE} \tag{3.10}$$

Example: 1) Homogeneous bar without gravity (see fig. 3.2)

$A = const$, $E = const$, $Q = 0$, $\rho = 0$ and the ODE reduces to

$$C \cdot u'' = 0 \tag{3.11}$$

$$\Rightarrow \qquad u' = const = a \tag{3.12}$$

$$\Rightarrow \qquad u = a \cdot x + b \tag{3.13}$$

subject to the boundary conditions of suppressed displacement at the origin (3.15) and a force F applied at the free end of the bar (3.16):

$$u(x = 0) = 0 \tag{3.14}$$

$$N(x = L) = F = C \cdot u'(L) \tag{3.15}$$

hence the solution reads

$$b = 0, \qquad a = \frac{F}{C} \qquad \Rightarrow \qquad \boxed{u = \frac{F}{C}x} \tag{3.16}$$

The stress is then given by

$$\sigma = E \cdot u' = \frac{F}{A} = const \tag{3.17}$$

Figure 3.2.: A bar of length L, with uniform cross section A and constant Young's modulus E. No volume force or traction is present.

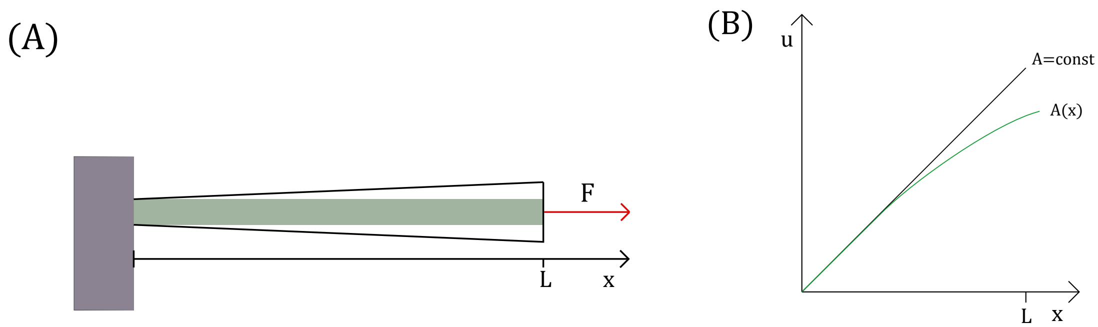


Figure 3.3.: (A) The Young's modulus E is constant throughout the bar of length L, but the cross section $A(x)$ varies along the x-axis.
(B) The displacement along the bar is linear for a homogeneous cross section A whereas for an increasing cross section the displacement becomes weaker for larger A.

Example: 2) Increasing cross section (see fig. 3.3)

Let

$$A(x) = A_0 \cdot \left(1 + \frac{x}{3L}\right) \tag{3.18}$$

Integration of

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0 \tag{3.19}$$

yields

$$\frac{du}{dx} = \frac{a}{EA_0 \cdot (1 + \frac{x}{3L})} \tag{3.20}$$

Integration again gives

$$u = \frac{3aL}{EA_0} \ln\left(1 + \frac{x}{3L}\right) + b \tag{3.21}$$

with integration constants a and b to be specified by the boundary conditions:

$$u(0) = 0 \qquad \Rightarrow b = 0 \tag{3.22}$$

$$EA \cdot u'(L) = F \qquad \Rightarrow u'(L) = \frac{3aL}{EA_0}\frac{1/3L}{4/3} = \frac{3a}{4EA_0} = \frac{F}{EA_0 4/3} \qquad \Rightarrow a = F \tag{3.23}$$

and all in all the displacement reads

$$\boxed{u = \frac{3FL}{EA_0}\ln\left(1 + \frac{x}{3L}\right)} \quad \approx \frac{F}{EA_0}x \quad \text{for } x \ll L,\ A = const \tag{3.24}$$

Example: 3) Homogeneous bar with gravity (see fig. 3.4)

$A = const$, $E = const$, $q = const = \rho \cdot g \cdot A$ and the central equation becomes

$$\frac{d}{dx}\left(C \cdot \frac{du}{dx}\right) = -q \tag{3.25}$$

$$\Rightarrow \qquad u = -\frac{q}{2C}x^2 + \frac{a}{C}x + b \tag{3.26}$$

subject to the boundary conditions

$$u(x = 0) = 0 \Rightarrow\ b = 0 \tag{3.27}$$

$$C \cdot u'(L) = F = -q \cdot L + a \Rightarrow\ a = F + qL \tag{3.28}$$

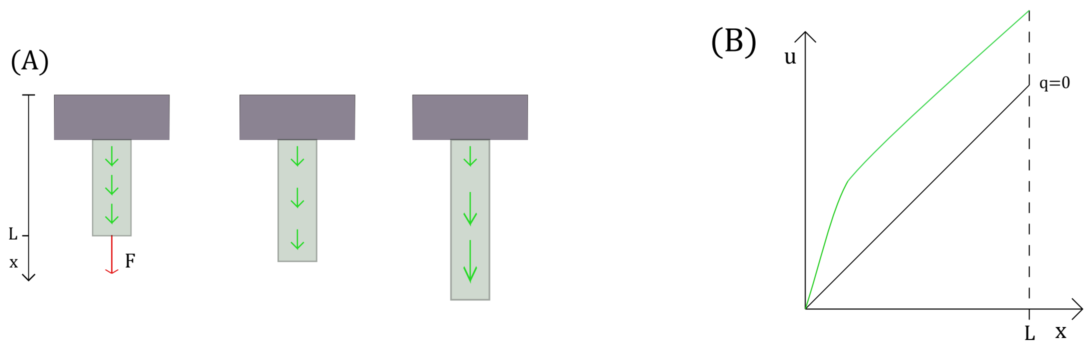


Figure 3.4.: (A) For a bar exposed to gravity, the largest elongation occurs at the end, whereas the largest force acts close to the bounding surface and defines a possible breaking point.
(B) A comparison to the reference case with linear displacement (example 1)) shows an increased slope at small x. The boundary condition at $x = L$ forces a crossover to the same slope.

Thus

$$\boxed{u = -\frac{q}{2C}x^2 + \frac{F + qL}{C}x} \tag{3.29}$$

For $q = 0$ this reduces to the linear behaviour of the elastic bar in example 1):

$$u = \frac{F}{C} \cdot x \tag{3.30}$$

The full solution is depicted in fig. 3.4.

3.2. Elastic chain

We now start with a discrete approach and derive the 1d continuum equation for an elastic chain (”coarse-graining” or ”homogenization”). We also include elastic coupling to the ground (fig. 3.5).
We have $n = 0, 1, ..., N$ beads at positions $x_n = a \cdot n$ with $x_0 = 0$ and $x_N = L$.
For a given n in the chain the force at equilibrium is

$$0 = F_N = k_{n+1} \cdot (u_{n+1} - u_n) - k_n \cdot (u_n - u_{n-1}) - K_n \cdot u_n - qa \tag{3.31}$$

where qa denotes the amount of volume force on bead n.
A Taylor expansion of k_{n+1}, u_{n+1} and u_{n-1} for $a \ll L$, i.e. in $\frac{a}{L} = \frac{1}{N}$, and taking u_n as a reference term gives

$$0 = F_N \tag{3.32}$$

$$= (k_n + a \cdot \partial_x k_n + \frac{a^2}{2}\partial_x^2 k_n) \cdot (u_n + a \cdot \partial_x u_n + \frac{a^2}{2}\partial_x^2 u_n - u_n) \tag{3.33}$$

$$+k_n \cdot (u_n - (u_n - a \cdot \partial_x u_n + \frac{a^2}{2}\partial_x^2 u_n)) + qa - K_n u_n \tag{3.34}$$

$$= k_n a^2 \partial_x^2 u_n + a^2(\partial_x k_n) \cdot (\partial_x u_n) - K_n u_n + qa + \mathcal{O}(a^3) \tag{3.35}$$

$$= k(x)a^2 u''(x) + a^2 k'(x)u'(x) - Ku(x) + qa \tag{3.36}$$

$$\Rightarrow \qquad \boxed{\partial_x(C(x) \cdot u'(x)) - \rho \cdot u(x) + q = 0} \tag{3.37}$$

with $C = E \cdot A = k \cdot a$ and $\rho = {}^{K}\!/_{a}$.
This is the same equation as above where we started with the continuum description in the first place.

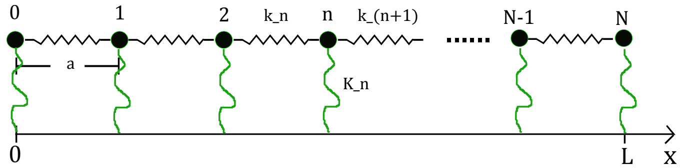


Figure 3.5.: The beads, separated at a distance a, are laterally connected by springs with spatially varying spring constants k_n and attached to the surface with springs with constants K_n.

Example: 1) Stretching a bar with elastic foundation, no gravity (see fig. 3.6)

Without gravity, $q = 0$, the constitutive equation becomes, in the case $\rho = 0$:

$$Cu'' = 0 \quad \Rightarrow \quad N = Cu' = F \quad \Rightarrow \quad u = \frac{F}{C} \cdot x \tag{3.38}$$

like before.
And for $\rho \neq 0$:

$$Cu'' - \rho u = 0 \quad \Rightarrow \quad l^2 u'' - u = 0 \tag{3.39}$$

where we define the *localization length* as $l = \sqrt{\frac{C}{\rho}} = \sqrt{\frac{k}{K}}a$.

$$\Rightarrow \quad u = A \cdot e^{x/l} + B \cdot e^{-x/l} \tag{3.40}$$

$$u(0) = 0 \Rightarrow \quad B = -A \tag{3.41}$$

$$\Rightarrow \quad u = 2A \cdot \sinh(x/l) \tag{3.42}$$

$$C \cdot u'(L) = F = \frac{C2A}{l} \cdot \cosh(L/l) \tag{3.43}$$

resulting in

$$\boxed{u = \frac{Fl}{C} \cdot \frac{\sinh(x/l)}{\cosh(L/l)}} \tag{3.44}$$

The force on the foundation $F_T = k \cdot u$ is known as the *traction force*.

Example: 2) Contracting bar with elastic foundation, no gravity (see fig. 3.7)

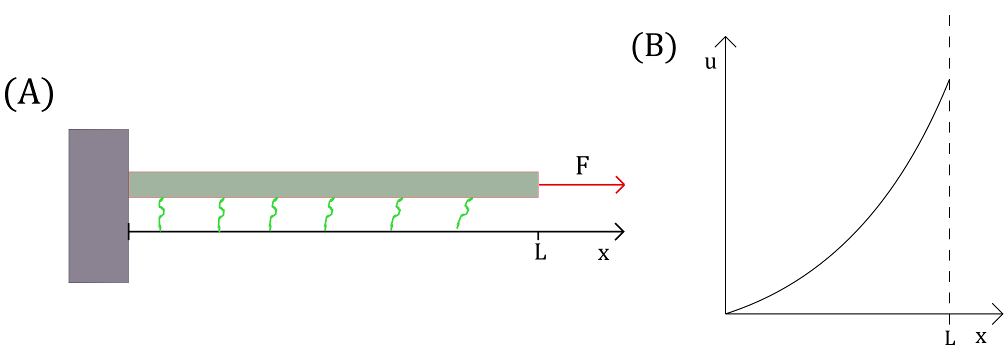


Figure 3.6.: (A) For a bar with elastic foundation, the displacement of string attachments increases with x.
(B) The displacement is an exponentially increasing function of x.

We now consider that deformation arises not from an external force, but by internal contraction (e.g. thermal or active contraction). We add *force dipoles* along the bar:

$$N = Cu' + P, \qquad P = const \tag{3.45}$$
$$\Rightarrow \qquad l^2 u'' - u = -P'/\rho = 0 \tag{3.46}$$

Thus the displacement has the same solution as before

$$u = 2A \cdot \sinh\left({}^{x}/_{l}\right) \tag{3.47}$$

but the boundary condition is different at $x = L$:

$$N(L) = 0 \tag{3.48}$$
$$\Rightarrow \qquad C \cdot u'(L) + P = 0 \tag{3.49}$$
$$\Rightarrow \qquad -\frac{P}{C} = \frac{2 \cdot A}{l} \cdot \cosh\left({}^{L}/_{l}\right) \tag{3.50}$$
$$\Rightarrow \qquad \boxed{u(x) = -\frac{Pl}{C} \cdot \frac{\sinh({}^{x}/_{l})}{\cosh({}^{L}/_{l})}} \tag{3.51}$$

Thus the deformation vanishes if the contraction is zero $P = 0$. Note also that the displacement now is to the left.

We can now easily calculate the internal force in the bar:

$$N(x) = Cu' + P = P\left(1 - \frac{\cosh({}^{x}/_{l})}{\cosh({}^{L}/_{l})}\right) \tag{3.52}$$

This force is maximal at $x = 0$ and falls to cero at $x = L$.

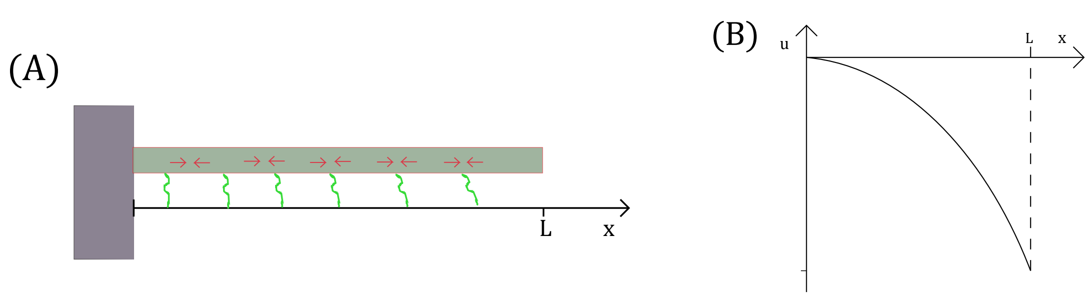


Figure 3.7.: (A) A bar with a distribution of contractile forces. (B) Displacement for to a contracting bar.

CHAPTER 4

Elasticity theory in 3d

Up to this point we dealt with one-dimensional equations describing the relationship between stress and strain for fibres and bars. We now turn to the derivation of the fundamental concepts and equations for a $3d$ theory of elastic deformations.

4.1. Material and spatial temporal derivatives

In order to describe the deformation and movement of a three-dimensional object in time, we consider how it deforms under surface or volume forces (see fig.4.1). We consider a *material point* P at $\mathbf{x}_0$ in a continuum body with initial volume V_0. This point is moved to $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ at time t and deformed by surface or volume forces. The vector $\mathbf{x_0}$ is bound to the material point for all configurations parameterized by t and thus when $\mathbf{x}_0$ is constant we can trace the material point in time where $\mathbf{x}(\mathbf{x}_0, t)$ is the kinetic path taken by the material point P.
Partial differentiation with respect to time t results in the velocity and acceleration fields of the material point under consideration:

$$\mathbf{v} = \dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}_0, t) \qquad \text{velocity field} \tag{4.1}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{x}} = \mathbf{a}(\mathbf{x}_0, t) \qquad \text{acceleration field} \tag{4.2}$$

For each vector $\mathbf{x}, \mathbf{v}, \mathbf{a}$, its three components are scalar fields depending on $\mathbf{x}_0$ and t. The same would be true for any other scalar field, take for example a temperature field $T(\mathbf{x}_0, t)$ where the temperature is defined on every point of the body and the temporal evolution monitored.

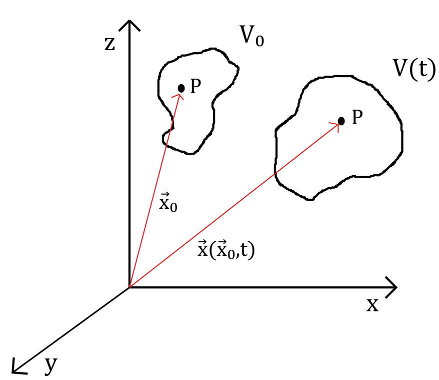


Figure 4.1.: The material point P is labeled on the reference configuration V_0 and then traced in time.

Lagrangian and Eulerian frame

We can define two different types of time derivatives:

(1) **Material time derivative** $\dot{T} = \frac{\partial T}{\partial t}\Big|_{\mathbf{x}_0 = const}$

$\dot{T}$ is the change in unit time of the temperature at material point P identified by $\mathbf{x}_0$ and moving through space. When the the temperature field in the configuration $V(t)$ is mapped onto the reference configuration V_0 the description is referred to as *Lagrangian.* The Lagrangian description is the natural choice for solids where it is always possible to return to the initial configuration by removing surface or volume forces.

(2) **Spatial time derivative** $\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t}\Big|_{\mathbf{x} = const}$

In the *Eulerian description*, $\mathbf{x}$ is a fixed point in $V(t)$. At each time t, a different material particle might be present at this location. Thus $\frac{\delta T}{\delta t}$ is the change in unit time of the temperature at a fixed point $\mathbf{x}$ in space. The Eulerian frame is the natural choice for fluids, because upon deformation there is no way to trace back the reference configuration of a fluid.

In order to derive a **relation between the material and the spatial time derivative** we start with the Eulerian description of the scalar temperature field $T = T(\mathbf{x}, t)$. The total derivative can be rewritten with the gradient operator $\boldsymbol{\nabla}$:

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz + \frac{\partial T}{\partial t}dt \tag{4.3}$$

$$= (d\mathbf{x} \cdot \boldsymbol{\nabla})\, T + \frac{\delta T}{\delta t}\Big|_{\mathbf{x}=const} dt \tag{4.4}$$

$$= (\mathbf{v}dt \cdot \boldsymbol{\nabla})\, T + \frac{\delta T}{\delta t} dt \qquad \equiv \dot{T} dt \tag{4.5}$$

where $d\mathbf{x} = \mathbf{v}dt$ implies movement with the flow and accordingly a change in temperature $dT = \dot{T}dt$. Dividing equation 4.5 by dt gives

$$\boxed{\underbrace{\dot{T}}_{\text{material derivative}} = (\mathbf{v} \cdot \boldsymbol{\nabla})\, T + \underbrace{\frac{\delta T}{\delta t}}_{\text{spatial derivative}}} \tag{4.6}$$

Special care has to be taken with the two different types of derivatives involved. If $\mathbf{v} = 0$, i.e. there is no movement/deformation, the derivatives are equal. The difference between material and spatial derivatives is called the *convective contribution* given by $(\mathbf{v} \cdot \boldsymbol{\nabla})$.
The temperature field was an introductory example, but the same can now also be applied to vectors:

$$\dot{\mathbf{x}} = (\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{x} + \underbrace{\frac{\delta \mathbf{x}}{\delta t}}_{\text{fixed position in space}} = \mathbf{v} \cdot \mathbb{1} + \mathbf{0} = \mathbf{v}$$

or with the convective part written in components

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{x} = (v_j \partial_j)\, x_i = v_j \delta_{ij} = v_i \tag{4.7}$$

The velocity is of course the correct result when following the motion in the Eulerian frame.
Applying the same operation to the velocity $\mathbf{v}$ and taking the i^{th} component of the convective term results

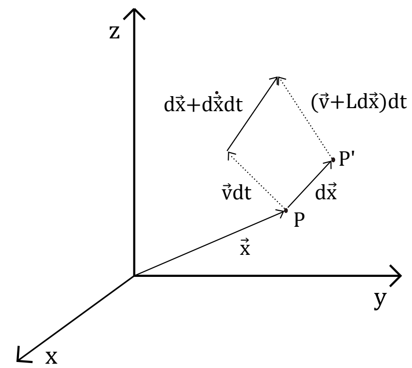


Figure 4.2.: Change of a material line element $d\mathbf{x}$ expressed by the the velocity gradient tensor L.

in

$$\begin{aligned}\dot{\mathbf{v}} &= \underbrace{(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}}_{(v_j\partial_j)v_i} + \frac{\delta\mathbf{v}}{\delta t} = \underbrace{\mathbf{v}\cdot(\boldsymbol{\nabla}\otimes\mathbf{v})}_{v_j\partial_j v_i} + \frac{\delta\mathbf{v}}{\delta t} \\ &= \underbrace{(\boldsymbol{\nabla}\otimes\mathbf{v})^T\cdot\mathbf{v}}_{\partial_j v_i v_j} + \frac{\partial\mathbf{v}}{\partial t} = \underbrace{\mathbf{L}}_{\text{velocity gradient tensor}} \cdot\mathbf{v} + \frac{\delta\mathbf{v}}{\delta t}\end{aligned}$$

This is the first time we encounter *tensors* and the *dyadic product* $\otimes$. We also have used $b_i = M_{ij}a_j = a_j(M^T)_{ji}$ for $\mathbf{b} = \mathbf{M}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{M}^T$. The difference between the material and the spatial derivative is in this case simply a linear operation on the velocity.

Tensors

Elasticity theory in 3d makes use of 2nd rank tensors, which are linear maps of one vector onto another. The simplest way to generate such a tensor is the *dyadic* or *tensor product*:

$$(\mathbf{a}\otimes\mathbf{b})_{ij} \equiv a_i b_j$$

We take the dyadic product of gradient and velocity

$$\boldsymbol{\nabla}\otimes\mathbf{v} = \begin{pmatrix} \partial_x v_x & \partial_x v_y & \partial_x v_z \\ \partial_y v_x & \partial_y v_y & \partial_y v_z \\ \partial_z v_x & \partial_z v_y & \partial_z v_z \end{pmatrix} \tag{4.8}$$

therefore the velocity gradient tensor reads

$$\mathbf{L} = (\boldsymbol{\nabla}\otimes\mathbf{v})^T = \begin{pmatrix} \partial_x v_x & \partial_y v_x & \partial_z v_x \\ \partial_x v_y & \partial_y v_y & \partial_z v_y \\ \partial_x v_z & \partial_y v_z & \partial_z v_z \end{pmatrix} \tag{4.9}$$

The tensor $\mathbf{L}$ measures the change in unit time of a material line element $d\mathbf{x}$ (see fig.4.2):

- The tail of $d\mathbf{x}$ (P) moves to $\mathbf{x} + \mathbf{v}dt$
- The head of $d\mathbf{x}$ (P') moves to $\mathbf{x} + d\mathbf{x} + (\mathbf{v} + \underbrace{d\mathbf{x}(\boldsymbol{\nabla}\times\mathbf{v})}_{\mathbf{L}d\mathbf{x}})dt$

After an infinitesimal increase in time dt the line element in the volume V will change into $d\mathbf{x} + d\dot{\mathbf{x}}$ and we can read of fig.4.2

$$\begin{aligned} &\Rightarrow \qquad d\mathbf{x} + d\dot{\mathbf{x}}dt = \mathbf{x} + d\mathbf{x} + (\mathbf{v} + \mathbf{L}d\mathbf{x})dt - (\mathbf{x} + \mathbf{v}dt) \\ &\Rightarrow \qquad \boxed{d\dot{\mathbf{x}} = \mathbf{L}d\mathbf{x}} \end{aligned}$$

The line element thus changes by the linear element $\mathbf{L}$ which measures changes both in length and in orientation, gives complete information as the 3d metric of the system.

Since in the following, extensive use of 2nd rank tensors will me made, this section summarizes some general statements:
Typically one uses Cartesian coordinates with a set of basis vectors $\{\mathbf{e}_i\}$. Then an arbitrary 2nd rank tensor can be written as

$$\mathbf{M} = \sum_{ij} M_{ij} \left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \tag{4.10}$$

and we note that the dyadic product is compatible with the inner product (by definition and the linearity of vectors)

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{f} = \mathbf{a}(\mathbf{b} \cdot \mathbf{f}) \quad \Rightarrow \quad (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbf{e}_k = \mathbf{a}_i \cdot \delta_{jk} \tag{4.11}$$

and thus

$$\begin{aligned} \mathbf{b} = \mathbf{M} \cdot \mathbf{a} &= \left(\sum_{ij} M_{ij} \left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \right) \cdot \left(\sum_k a_k \mathbf{e}_k \right) \\ &= \sum_{ij} M_{ij} a_j \mathbf{e}_i \quad \Rightarrow \quad b_i = M_{ij} a_j \end{aligned}$$

The *trace* of $\mathbf{M}$ is defined as

$$\mathrm{tr}(\mathbf{M}) = \sum_i M_{ii} \tag{4.12}$$

i.e. as the sum of diagonal elements.
The *deviatoric part* of $\mathbf{M}$ is defined as

$$\mathbf{M}^d = \mathbf{M} - \frac{1}{3} \mathrm{tr}(\mathbf{M}) \mathbb{1} \quad \Rightarrow \quad \mathrm{tr}(\mathbf{M}^d) = 0 \tag{4.13}$$

Most of the tensors we encounter will be symmetric and can therefore always be diagonalized. The *eigenvalues* of $\mathbf{M}$ are defined by the *characteristic equation*

$$0 = \det\left(\mathbf{M} - \lambda \mathbb{1}\right) = \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 \tag{4.14}$$

in 3d with the three invariants of $\mathbf{M}$:

$$I_1 = \mathrm{tr}(\mathbf{M}) = M_{xx} + M_{yy} + M_{zz} \tag{4.15}$$

$$I_2 = \frac{1}{2} \left[(\mathrm{tr}(\mathbf{M}))^2 - \mathrm{tr}(\mathbf{M} \cdot \mathbf{M}) \right] = M_{xx} M_{yy} + M_{xx} M_{zz} + M_{yy} M_{zz} - M_{xy}^2 - M_{yz}^2 - M_{xz}^2 \tag{4.16}$$

$$I_3 = \det(\mathbf{M}) = M_{xx} \left(M_{yy} M_{zz} - M_{zy} M_{yz} \right) - M_{xy} \left(M_{yx} M_{zz} - M_{zx} M_{yz} \right) + M_{xz} \left(M_{yx} M_{zy} - M_{zx} M_{yy} \right) \tag{4.17}$$

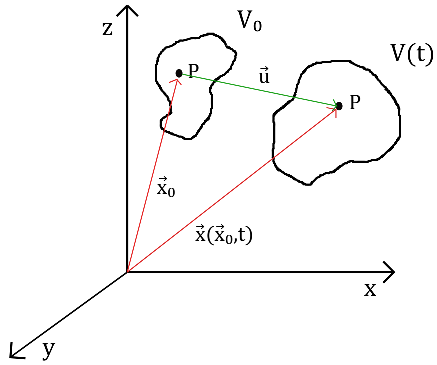


Figure 4.3.: The displacement vector for a material point P.

4.2. The displacement vector field

We now introduce the most central quantity of elasticity theory, the *displacement vector field*. Again considering the displacement of a material point P from a reference configuration V_0 to the current configuration $V(t)$ (see fig.4.3), the displacement vector $\mathbf{u}$ satisfies

$$\mathbf{u} = \mathbf{x} - \mathbf{x}_0 = \begin{cases} \mathbf{x}(\mathbf{x}_0, t) - \mathbf{x}_0 = \mathbf{u}(\mathbf{x}_0, t) & \text{Lagrangian frame} \\ \mathbf{x} - \mathbf{x}_0(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) & \text{Eulerian frame} \end{cases} \tag{4.18}$$

Many problems in solid mechanics amount to determining the displacement field $\mathbf{u}$ corresponding to a given set of applied forces.
For a scalar field like temperature T, we can define two gradients depending on the frame that should become identical for $\mathbf{u} = 0$:

- spatial gradient $\boldsymbol{\nabla} T = \mathbf{e}_x \frac{\partial T}{\partial x} + \mathbf{e}_y \frac{\partial T}{\partial y} + \mathbf{e}_z \frac{\partial T}{\partial z}$
- material gradient $\boldsymbol{\nabla}_0 T = \mathbf{e}_x \frac{\partial T}{\partial x_0} + \mathbf{e}_y \frac{\partial T}{\partial y_0} + \mathbf{e}_z \frac{\partial T}{\partial z_0}$

These two can be related by the chain rule as exemplified for the x-component at a fixed time t

$$\frac{\partial T}{\partial x_0} = \frac{\partial T}{\partial x_0} \frac{\partial x}{\partial x_0} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial x_0} + \frac{\partial T}{\partial z} \frac{\partial z}{\partial x_0} \tag{4.19}$$

and likewise for the other two components. Using the procedure from section 4.1 we can summarize this as

$$\boldsymbol{\nabla}_0 T = \mathbf{F}^T \cdot \boldsymbol{\nabla} T \tag{4.20}$$

with the transpose of the *deformation gradient tensor* $\mathbf{F}$ from the current configuration with respect to the reference configuration, or mathematically with the Jacobian matrix of the coordinate transformation.

$$\mathbf{F}^T = \begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} & \frac{\partial z}{\partial x_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} & \frac{\partial z}{\partial y_0} \\ \frac{\partial x}{\partial z_0} & \frac{\partial y}{\partial z_0} & \frac{\partial z}{\partial z_0} \end{pmatrix} \quad \Rightarrow \quad \mathbf{F} = \begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{pmatrix} \tag{4.21}$$

We can also write, including the deformation vector $\mathbf{u} = \mathbf{x} - \mathbf{x}_0$

$$\mathbf{F}^T = \boldsymbol{\nabla}_0 \otimes \mathbf{x} = \boldsymbol{\nabla}_0 \otimes (\mathbf{x}_0 + \mathbf{u}) = \mathbb{1} + \boldsymbol{\nabla}_0 \otimes \mathbf{u} \tag{4.22}$$

$$\mathbf{F} = (\boldsymbol{\nabla}_0 \otimes \mathbf{x})^T = \mathbb{1} + (\boldsymbol{\nabla}_0 \otimes \mathbf{u})^T \tag{4.23}$$

If there is no deformation ($\mathbf{u} = 0$):

$$\mathbf{F}^T = \mathbf{F} \quad \Rightarrow \quad \boldsymbol{\nabla} = \boldsymbol{\nabla}_0 \tag{4.24}$$

Again, if the reference and current state are equal, then the deformation is zero and $\mathbf{F} = \mathbb{1}$ and the gradient operators are also identical as demanded.

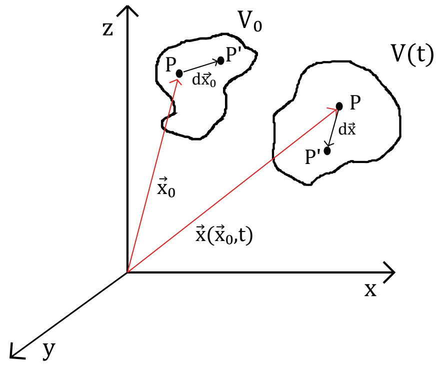


Figure 4.4.: The change in distance and direction between the two material points can be described with the strain tensor.

4.3. The strain tensor

We now consider a material line segment $d\mathbf{x}_0$ that changes both length and orientation during deformation:

- $dl_0 \to dl$ length
- $\mathbf{e}_0 \to \mathbf{e}$ orientation

The orientation and length of the respective line segments are defined by:

$$d\mathbf{x}_0 = \mathbf{e}_0 dl_0 \qquad dl_0 = \sqrt{d\mathbf{x}_0 \cdot d\mathbf{x}_0} \tag{4.25}$$

$$d\mathbf{x} = \mathbf{e} dl \qquad dl = \sqrt{d\mathbf{x} \cdot d\mathbf{x}} \tag{4.26}$$

For the relation between $d\mathbf{x}_0$ and $d\mathbf{x}$ we look at the total differential again (e.g. for the x-component)

$$dx = \frac{\partial x}{\partial x_0} dx_0 + \frac{\partial x}{\partial y_0} dy_0 + \frac{\partial x}{\partial z_0} dz_0 \tag{4.27}$$

and the same for the other components such that in a more compact form the deformation gradient tensor can be used

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{x}_0 \tag{4.28}$$

This can be rewritten with the help of equations 4.25 and 4.26

$$\mathbf{e} dl = \mathbf{F} \cdot \mathbf{e}_0 dl_0 \qquad |()^2 \tag{4.29}$$

$$\Rightarrow \quad \mathbf{e} \cdot \mathbf{e} dl^2 = \mathbf{e}_0 \cdot \mathbf{F}^T \mathbf{F} \cdot \mathbf{e}_0 dl_0^2 \tag{4.30}$$

$$\Rightarrow \quad \text{stretch ratio} \quad \boxed{\lambda = \frac{dl}{dl_0} = \sqrt{\mathbf{e}_0 \cdot \underbrace{\mathbf{F}^T \cdot \mathbf{F}}_{\equiv \mathbf{C}} \cdot \mathbf{e}_0}} \tag{4.31}$$

The stretch ratio is defined as the ratio between the lengths of the line segments in the corresponding configurations. The stretch ratio λ for the material line segment is thus determined by the *right Cauchy-Green deformation tensor C* (Lagrangian description).
The new orientation of the material line segment can be calculated as

$$\mathbf{e} = \mathbf{F} \cdot \mathbf{e}_0 \frac{dl_0}{dl} = \mathbf{F} \cdot \mathbf{e}_0 \frac{1}{\lambda} = \frac{\mathbf{F} \cdot \mathbf{e}_0}{\sqrt{\mathbf{e}_0 \cdot \mathbf{C} \cdot \mathbf{e}_0}} \tag{4.32}$$

The procedure can also be inverted, starting from $(dl, \mathbf{e})$ we can get to $(dl_0, \mathbf{e}_0)$:

$$\mathbf{F}^{-1} \cdot d\mathbf{x} = d\mathbf{x}_0 \quad \Leftrightarrow \quad \mathbf{F}^{-1} \cdot \mathbf{e} dl = \mathbf{e}_0 dl_0 \tag{4.33}$$

$$\Rightarrow \quad \mathbf{e} \cdot \mathbf{F}^{-T} \mathbf{F}^{-1} \cdot \mathbf{e} dl^2 = \mathbf{e}_0 \cdot \mathbf{e}_0 dl_0^2 \tag{4.34}$$

$$\Rightarrow \quad \boxed{\lambda = \frac{dl}{dl_0} = \frac{1}{\sqrt{\mathbf{e} \cdot \underbrace{\mathbf{F}^{-T}\mathbf{F}^{-1}}_{\equiv \mathbf{B}^{-1}} \cdot \mathbf{e}}}} \tag{4.35}$$

The tensor $B \equiv \mathbf{F} \cdot \mathbf{F}^T$ is called the *left Cauchy-Green deformation tensor* in the Eulerian frame. The old direction of the material line segment is then given by

$$\mathbf{e}_0 = \mathbf{F}^{-1} \cdot \mathbf{e} \cdot \frac{dl}{dl_0} = \frac{\mathbf{F}^{-1} \cdot \mathbf{e}}{\sqrt{\mathbf{e} \cdot \mathbf{B}^{-1} \cdot \mathbf{e}}} \tag{4.36}$$

As explained earlier, often it is more convenient to introduce a variable which vanishes for vanishing deformation (*strain*). We now define various strain tensors. One can show that each of them is invariant under a rigid body transformation (translation and rotation).
In the Lagrangian frame we found

$$\lambda^2 = \mathbf{e}_0 \cdot \mathbf{C} \cdot \mathbf{e}_0 \tag{4.37}$$

coupled to this one can now introduce the *Green-Lagrange strain* ϵ_{GL}

First definition:

$$\boxed{\epsilon_{GL} = \frac{\lambda^2 - 1}{2} = \mathbf{e}_0 \underbrace{\frac{1}{2}(\mathbf{C} - \mathbb{1})}_{\equiv \mathbf{E}} \cdot \mathbf{e}_0} \tag{4.38}$$

with the *Green-Lagrange strain tensor* $\mathbf{E}$.
Above we showed

$$\mathbf{F}^T = \mathbb{1} + (\boldsymbol{\nabla}_0 \otimes \mathbf{u}) \qquad \mathbf{F} = \mathbb{1} + (\boldsymbol{\nabla}_0 \otimes \mathbf{u})^T \tag{4.39}$$

Substitution gives an expression for $\mathbf{E}$

$$\boxed{\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbb{1}\right) = \frac{1}{2}\left[\underbrace{(\boldsymbol{\nabla}_0 \otimes \mathbf{u}) + (\boldsymbol{\nabla}_0 \otimes \mathbf{u})^T}_{\text{linear in u}} + \underbrace{(\boldsymbol{\nabla}_0 \otimes \mathbf{u})(\boldsymbol{\nabla}_0 \otimes \mathbf{u})^T}_{\text{quadratic}}\right]} \tag{4.40}$$

The linear strain ϵ_{lin} is defined as

Second definition:

$$\boxed{\epsilon_{lin} = \lambda - 1 = \sqrt{\mathbf{e}_0 \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{e}_0} - 1} \tag{4.41}$$

Due to the square root this expression is not easy to use. For small deformations $\mathbf{F} \approx \mathbb{1}$ we can write

$$\epsilon_{lin} = \sqrt{1 + \mathbf{e}_0 \cdot (\mathbf{F}^T \cdot \mathbf{F} - \mathbb{1})\, \mathbf{e}_0} - 1 \tag{4.42}$$

$$\approx \frac{1}{2}\mathbf{e}_0 \left(\mathbf{F}^T \cdot \mathbf{F} - \mathbb{1}\right) \cdot \mathbf{e}_0 = \mathbf{e}_0 \cdot \mathbf{E} \cdot \mathbf{e}_0 = \epsilon_{GL} \tag{4.43}$$

We further linearise to define the *linear strain tensor* ϵ expressed as

$$\boxed{\epsilon = \frac{1}{2}\left(\mathbf{F}^T + \mathbf{F} - 2\mathbb{1}\right) = \frac{1}{2}\left((\boldsymbol{\nabla}_0 \otimes \mathbf{u}) + (\boldsymbol{\nabla}_0 \otimes \mathbf{u})^T\right)} \tag{4.44}$$

This is the most often used formulation of a strain tensor in linear elasticity theory (LET).
In the Eulerian frame we found

$$\frac{1}{\lambda^2} = \mathbf{e} \cdot \mathbf{B}^{-1} \cdot \mathbf{e} \tag{4.45}$$

Coupled to this we define the *Almansi Euler strain* ϵ_{AE} as

Third definition:

$$\boxed{\epsilon_{AE} = \frac{1 - \frac{1}{\lambda^2}}{2} = \mathbf{e} \cdot \underbrace{\frac{1}{2} \cdot \left(\mathbb{1} - \mathbf{B}^{-1}\right)}_{\equiv \mathbf{A}} \cdot \mathbf{e}} \tag{4.46}$$

with the *Almansi-Euler strain tensor* **A**.

In summary we have found that for large deformation, geometrical non-linearities appear in the various strain tensors. In linear elasticity theory (LET), one assumes small deformations. Then it is sufficient to consider the linear strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = \begin{pmatrix} \frac{\partial u_x}{\partial x_0} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y_0} + \frac{\partial u_y}{\partial x_0}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z_0} + \frac{\partial u_z}{\partial x_0}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x_0} + \frac{\partial u_x}{\partial y_0}\right) & \frac{\partial u_y}{\partial y_0} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z_0} + \frac{\partial u_z}{\partial y_0}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x_0} + \frac{\partial u_x}{\partial z_0}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y_0} + \frac{\partial u_y}{\partial z_0}\right) & \frac{\partial u_z}{\partial z_0} \end{pmatrix} \tag{4.47}$$

This is a symmetrical 3x3 matrix which can be interpreted as follows:

- The diagonal terms ϵ_{ii} are the linear strains of material line segments of the reference configuration in the i-directions.
- The off-diagonal terms represent the shear in the material (compare fig. 4.5)

Volume change

We consider a small parallelepiped spanned by three linearly independent vectors $\{d\mathbf{x}_0^a, d\mathbf{x}_0^b, d\mathbf{x}_0^c\}$. The Volume is then given by

$$dV_0 = \left(d\mathbf{x}_0^a \times d\mathbf{x}_0^b\right) \cdot d\mathbf{x}_0^c \tag{4.48}$$

In the deformed configuration we have

$$d\mathbf{x}^i = \mathbf{F} \cdot d\mathbf{x}_0^i \tag{4.49}$$

One can then calculate for the relative volume

$$J = \frac{dV}{dV_0} = \det(\mathbf{F}) \tag{4.50}$$

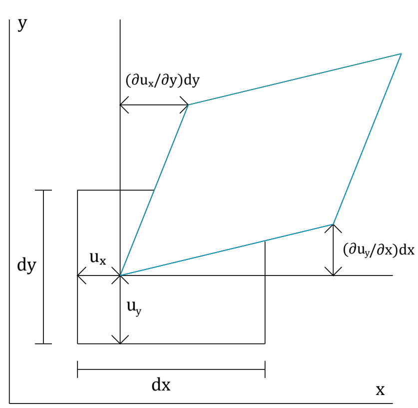


Figure 4.5.: Graphical interpretation of the linear strain tensor

Note that this is simply the Jacobian for the transformation $\mathbf{x}_0 \to \mathbf{x} : d\mathbf{x} = \mathbf{F} \cdot d\mathbf{x}_0$.
In linear approximation one finds

$$\frac{dV}{dV_0} = 1 + \mathrm{tr}(\epsilon) \quad \Rightarrow \quad \frac{dV - V_0}{dV_0} = \mathrm{tr}(\epsilon) \tag{4.51}$$

Thus the trace of the linear strain tensor is simply the relative volume change.
This result also motivates to decompose the strain tensor into a pure *shear* and a pure *compression/dilation* part:

$$\epsilon_{ij} = \underbrace{\left(\epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{ll}\right)}_{\text{pure shear}} + \underbrace{\left(\frac{1}{3}\delta_{ij}\epsilon_{ll}\right)}_{\text{compression/dilation}} \tag{4.52}$$

The trace of the strain tensor ϵ_{ll} gives the volume change in terms of compression and dilation, whereas the off-diagonal entries give the pure shear without a change of the volume (deviatoric part).

Relation to deformation in time:
Above we have considered a material line segment $d\mathbf{x}$ which evolves into a line segment $d\mathbf{x} + d\dot{\mathbf{x}}dt$ from time t to time $t + \Delta t$. We have shown that

$$d\dot{\mathbf{x}} = \mathbf{L} \cdot d\mathbf{x} \qquad \text{with} \qquad \mathbf{L} = (\boldsymbol{\nabla}_0 \otimes \mathbf{v})^T \tag{4.53}$$

where $\mathbf{L}$ is yet another tensor, the *velocity gradient tensor*, a purely kinematic variable not related to the reference configuration. On the other hand we now have

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{x}_0 \tag{4.54}$$

with the deformation gradient tensor $\mathbf{F} = (\boldsymbol{\nabla}_0 \otimes \mathbf{x})^T$. We can thus relate the two and find

$$d\dot{\mathbf{x}} = \dot{\mathbf{F}} \cdot d\mathbf{x}_0 = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \cdot d\mathbf{x} = \mathbf{L} \cdot d\mathbf{x} \tag{4.55}$$

$$\Rightarrow \qquad \boxed{\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}} \tag{4.56}$$

4.4. The stress tensor

We have seen before that for a continuum body, stress $\sigma = F/A$ rather than force F is the correct concept for the cause of a deformation.
For a 1d bar with constant cross-section A and no volume forces, we have derived as condition of mechanical equilibrium: $\frac{d\sigma}{dx} = 0$. We now generalize these results to 3d.
We first define a stress vector $\mathbf{s} = \sum_{i=1}^{3} s_i \mathbf{e}_i$ by decomposing the force ΔF onto an infinitesimally small surface element of area ΔA in a 3d continuum body:

$$\Delta F = \sum_{i=1}^{3} \Delta F_i e_i \quad \Rightarrow \quad s_i = \frac{\Delta F_i}{\Delta A} \tag{4.57}$$

Like before, stress is defined in the limit $\Delta A \to 0$, but now we deal with a 3d vector.

We next investigate the equilibrium conditions in 2d. Therefore we decompose the stress vector into different directions (see fig. 4.7(B)):

- Following the *notation* for σ_{ij} in fig.4.7, the subscript i denotes the direction in which the stress is acting, whereas the subscript j gives the direction of the normal of the surface element.

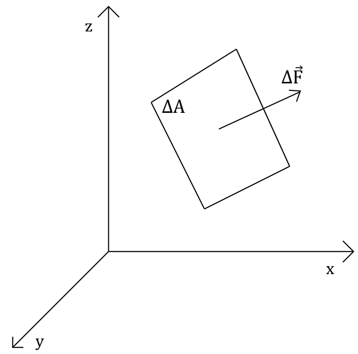


Figure 4.6.: Force ΔF acting on a small surface element ΔA in a contiuum body.

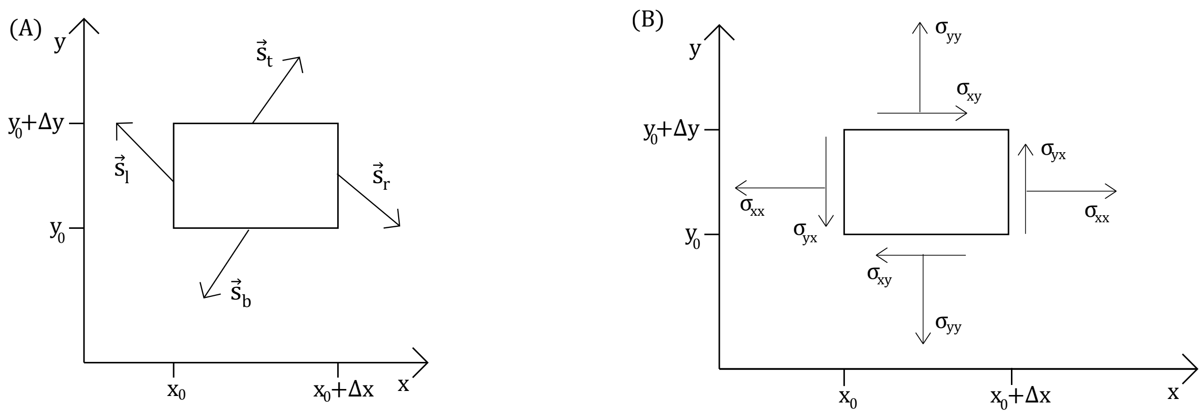


Figure 4.7.: (A) Free body diagram of the volume element with the stress vector **s** decomposed into directions (t) top, (r) right, (b) bottom and (l) left.
(B) All stress components are a function of the x- and y-position in space, but assumed constant in the z-direction.

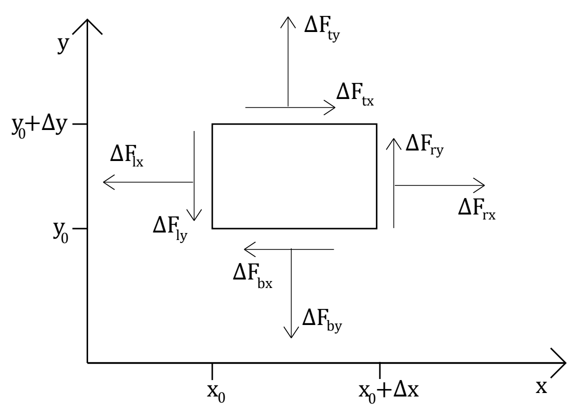


Figure 4.8.: Illustration of the forces acting on the surface element.

- As for the *sign convention*, $\sigma_{ij} > 0$ if i, j have the **same** orientation in regard to their respective coordinate direction.

Note that all $\sigma_{ij} = \sigma_{ij}(x, y)$ are functions of position!
For equilibrium the forces in each direction have to add up to zero:

- x-direction: $\Delta F_{lx} + \Delta F_{bx} = \Delta F_{rx} + \Delta F_{tx}$
- y-direction: $\Delta F_{ly} + \Delta F_{by} = \Delta F_{ty} + \Delta F_{ry}$

We now transform the equations for forces into one equation for stresses and first take the x-direction with reference point (x_0, y_0):

$$\Delta F_{lx} = h \cdot \int\limits_{y_0}^{y_0+\Delta y} dy\, \sigma_{xx}(x_0, y) \tag{4.58}$$

$$= h \cdot \int\limits_{y_0}^{y_0+\Delta y} dy \left[\sigma_{xx}(x_0, y_0) + \left.\frac{\partial \sigma_{xx}}{\partial y}\right|_{x=x_0, y=y_0} (y - y_0) + ...\right] \tag{4.59}$$

$$= \sigma_{xx} h \Delta y + \frac{\partial \sigma_{xx}}{\partial y} h \frac{\Delta y^2}{2} \tag{4.60}$$

Here h is a constant thickness in z-direction and we suppress dependances on (x_0, y_0). In a similar way we find

$$\Delta F_{rx} = h \cdot \int\limits_{y_0}^{y_0+\Delta y} dy \left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x + \frac{\partial \sigma_{xx}}{\partial y}(y - y_0) + ...\right] \tag{4.61}$$

$$= \sigma_{xx} h \Delta y + \frac{\partial \sigma_{xx}}{\partial x} h \Delta x \Delta y + \frac{\partial \sigma_{xx}}{\partial y} h \frac{\Delta y^2}{2} \tag{4.62}$$

$$\Delta F_{tx} = h \int\limits_{x_0}^{x_0+\Delta x} dx \left[\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}(x - x_0) + \frac{\partial \sigma_{xy}}{\partial y}\Delta y + ...\right] \tag{4.63}$$

$$= \sigma_{xy} h \Delta x + \frac{\partial \sigma_{xy}}{\partial x} h \frac{\Delta x^2}{2} + \frac{\partial \sigma_{xy}}{\partial y} h \Delta x \Delta y \tag{4.64}$$

$$\Delta F_{bx} = h \int\limits_{x_0}^{x_0+\Delta x} dx \left[\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}(x - x_0) + ...\right] \tag{4.65}$$

$$= \sigma_{xy} h \Delta x + \frac{\partial \sigma_{xy}}{\partial x} h \frac{\Delta x^2}{2} \tag{4.66}$$

The equilibrium condition in the x-direction now yields

$$\sigma_{xx}\Delta y + \frac{\partial \sigma_{xx}}{\partial y}\frac{\Delta y^2}{2} + \sigma_{xy}\Delta x + \frac{\partial \sigma_{xy}}{\partial x}\frac{\Delta x^2}{2} \tag{4.67}$$

$$= \sigma_{xx}\Delta y + \frac{\partial \sigma_{xy}}{\partial y}\Delta x \Delta y + \frac{\partial \sigma_{xx}}{\partial y}\frac{\Delta y^2}{2} + \sigma_{xy}\Delta x + \frac{\partial \sigma_{xy}}{\partial x}\frac{\Delta x^2}{2} + \frac{\partial \sigma_{xy}}{\partial y}\Delta x \Delta y \tag{4.68}$$

$$\Rightarrow \quad \boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0} \tag{4.69}$$

For the y-direction one finds in a similar manner

$$\boxed{\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0} \tag{4.70}$$

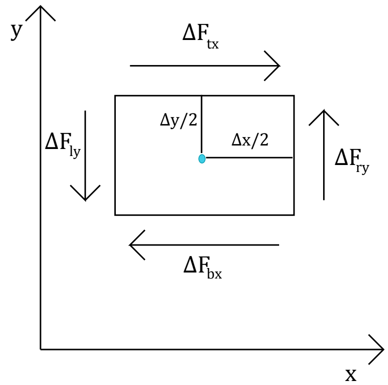


Figure 4.9.: Shear forces contribute to rotation of the prism.

Thus the equilibrium conditions amount to PDEs for σ_{ij}. This is the generalization of $\frac{\partial \sigma}{\partial x} = 0$ for 1d.

We next show that $\sigma_{xy} = \sigma_{yx}$ by balancing the moments around the midpoint (see fig.4.9). We note that only the shear forces create a moment:

$$-\frac{\Delta x}{2}\Delta F_{ly} - \frac{\Delta x}{2}\Delta F_{ry} + \frac{\Delta y}{2}\Delta F_{tx} + \frac{\Delta y}{2}\Delta F_{bx} = 0 \tag{4.71}$$

Inserting the expressions from above gives the desired result

$$\boxed{\sigma_{xy} = \sigma_{yx}} \tag{4.72}$$

The stress tensor σ_{ij} has to be **symmetric** in order to avoid rotation.

In 2d, we have three independent stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$. How do we calculate the stress vector $\mathbf{s}$ acting on an arbitrary area element from these stresses? We consider the following triangular prism (see fig. 4.10): The normal to the inclined plane has components $n_x = \sin\alpha$ and $n_y = \cos\alpha$ and the stress vector can be decomposed as $\mathbf{s} = s_x\mathbf{e}_x + s_y\mathbf{e}_y$. The forces over the whole prism have to sum up to zero again:

- x-direction: $s_x \cdot \Delta l \cdot h = \sigma_{xx} \cdot \sin\alpha \cdot \Delta l \cdot h + \sigma_{xy} \cdot \cos\alpha \cdot \Delta l \cdot h \quad \Rightarrow s_x = \sigma_{xx} \cdot n_x + \sigma_{xy} \cdot n_y$
- y-direction: $s_y \cdot \Delta l \cdot h = \sigma_{yx} \cdot \sin\alpha \cdot \Delta l \cdot h + \sigma_{yy} \cdot \cos\alpha \cdot \Delta l \cdot h \quad \Rightarrow s_y = \sigma_{yx} \cdot n_x + \sigma_{yy} \cdot n_y$

$$\Rightarrow \qquad \boxed{\mathbf{s} = \sigma \cdot \mathbf{n}} \tag{4.73}$$

The stress vector is simply the product of the *stress tensor* σ with the normal vector $\mathbf{n}$.

Generalization to 3d

In 3d there are 6 independent stress components building up the symmetric stress tensor:

$$\sigma = \sum_{i,j=1}^{3} \sigma_{ij}\, e_i \otimes e_j, \qquad \sigma_{ij} = \sigma_{ji} \tag{4.74}$$

The stress vector on an arbitrary surface element with normal $\mathbf{n}$ is simply

$$\boxed{\mathbf{s} = \sigma \cdot \mathbf{n}} \tag{4.75}$$

The condition of mechanical equilibrium is

$$\boxed{\partial_j \sigma_{ij} = 0} \tag{4.76}$$

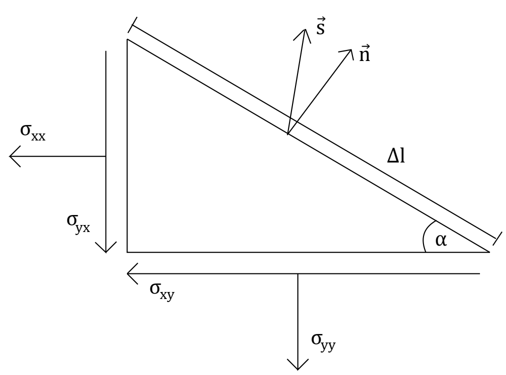


Figure 4.10.: A prism with two faces along cartesian coordinates and one face that is inclined by an angle α.

the divergence of the stress tensor has to vanish.
This central result follows in a more elegant way from the divergence theorem (DT). Since all elastic forces act over surfaces, then there must exist a tensor σ_{ij}, called *Cauchy's stress tensor*, such that

$$0 = \int_V f_i \, dV = \int_{\partial V} \sigma_{ij} \, dA_j \overset{\text{DT}}{=} \int_V \frac{\partial \sigma_{ij}}{\partial x_j} \, dV \tag{4.77}$$

This holds for any volume V and thus

$$\Rightarrow \qquad \boxed{\partial_j \sigma_{ij} = 0} \tag{4.78}$$

The detailed derivation above gave the same result and showed how to calculate and interpret σ in detail. A more general derivation starts from momentum conservation and considers Newton's second law for a small and arbitrary material volume:

$$\frac{d}{dt} \int_V \frac{\partial u_i}{\partial t} \underbrace{\rho \, dV}_{\rho J dV_0 = const} = \int_V g_i \rho \, dV + \int_{\partial V} \sigma_{ij} \, dA_j \tag{4.79}$$

$$\Rightarrow \qquad \boxed{\rho \frac{\partial^2 u_i}{\partial t^2} = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}} \qquad \text{Cauchy's momentum equation (valid both for fluids and solids)} \tag{4.80}$$

The condition $\boldsymbol{\nabla} \cdot \sigma = \mathbf{0}$ thus arises as a steady state solution for Cauchy's equation without volume forces. With the constitutive equation for LET between σ and ϵ, Cauchy's equation becomes the *Navier equation*. We now formulate an energy equation from equation 4.80 by multiplying with $\frac{\partial \mathbf{u}}{\partial t}$ and integrating over a large volume (without volume forces, $g_i = 0$):

$$\int_V \frac{\partial^2 u_i}{\partial t^2} \frac{\partial u_i}{\partial t} \rho \, dV = \frac{d}{dt} \underbrace{\int_V \frac{1}{2} \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2 \rho \, dV}_{\text{kinetic energy T}} = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} \frac{\partial u_i}{\partial t} \, dV \tag{4.81}$$

$$\overset{\text{DT, PI}}{=} \underbrace{\int_{\partial V} \frac{\partial u_i}{\partial t} \sigma_{ij} \, dA_j}_{\text{surface traction term, can usually be neglected}} - \underbrace{\int_V \sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial t} \, dV}_{\text{bulk term describing the rate at which energy is stored in the material as it deforms}} \tag{4.82}$$

where we again used the divergence theorem (DT), partial integration (PI), the symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$ and the linear strain tensor $\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ to get to the last line.
We introduce a scalar function w such that $\frac{\partial w}{\partial \epsilon_{ij}} = \sigma_{ij}$

$$\Rightarrow \qquad \dot{T} + \underbrace{\int_V \frac{\partial w}{\partial \epsilon_{ij}} \frac{\partial \epsilon_{ij}}{\partial t} \, dV}_{\dot{U}} = 0 \qquad \text{energy conservation: } T + U = const \tag{4.83}$$

$$dw = \sigma_{ij} d\epsilon_{ij} \qquad \text{strain energy density} \tag{4.84}$$

analogous to the energy stored in a stretched spring. U is the potential energy.

Principal stresses

For each point in the continuum body, the stress tensor σ describes its local stress state. Because σ is symmetric, it can be diagonalized, giving three *principal stresses* σ_i and the corresponding *principal stress directions* $\mathbf{n}_i$. Then

$$\mathbf{s}_i = \sigma \cdot \mathbf{n}_i = \sigma_i \cdot \mathbf{n}_i \qquad 1 \leq i \leq 3 \tag{4.85}$$

Thus for the directions, only normal and no shear forces are acting. We arrange the principal stresses such that $\sigma_1 \leq \sigma_2 \leq \sigma_3$ (ordered in rising magnitude).
We consider an arbitrary surface element with normal $\mathbf{n}$. Then the stress vector $\mathbf{s} = \sigma \cdot \mathbf{n}$ has normal and tangential components

$$\mathbf{s}_n = (\mathbf{s} \cdot \mathbf{n})\mathbf{n} \qquad s_n = \mathbf{s} \cdot \mathbf{n} \tag{4.86}$$

$$\mathbf{s}_t = \mathbf{s} - \mathbf{s}_n \qquad s_t = |\mathbf{s}_t| \tag{4.87}$$

One can prove that all possible combinations of (s_n, s_t) are located in the marked area between the three *Mohr's circles* in fig. 4.11:

$$(s_n)_{max} = \sigma_3 \quad (s_n)_{min} = \sigma_1 \quad (s_t)_{max} = \frac{\sigma_3 - \sigma_1}{2} \tag{4.88}$$

The eigenvalues of σ_{ij} give upper bounds for maximal stresses which are the starting point for *failure mechanics*.
Next we observe that if all three shear components are zero ($\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$) and all normal stresses are equal ($\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$), then

$$\sigma = -p\mathbb{1} \tag{4.89}$$

In this case, p can be identified with the pressure. This motivates to identify

$$p = -\frac{1}{3}\,\mathrm{tr}(\sigma) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{4.90}$$

$$\sigma^h = -p\mathbb{1} \qquad \text{hydrostatic stress tensor} \tag{4.91}$$

$$\Rightarrow \qquad \sigma = \sigma^h + \sigma^d \tag{4.92}$$

with σ^d the *deviatoric* stress tensor.
Depending on the nature of the material under consideration, it might fail (break) if different stresses are exceeded. For example for metals, the maximum shear is relevant, whereas ceramics have a threshold in extension.
In this context, often one considers the *von Mises stress*:

$$\sigma_M = \sqrt{\frac{3}{2}\,\mathrm{tr}(\sigma^d \cdot \sigma^d)} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \tag{4.93}$$

The von Mises stress is often used to color-circle the stress field in a loaded piece of material.

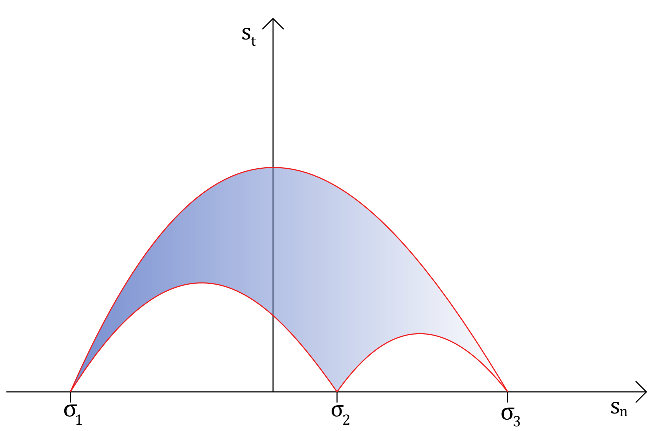


Figure 4.11.: The shaded area between the three Mohr's circles gives all possible combinations for (s_n, s_t).

4.5. Linear elasticity theory

For a purely elastic system, the deformation history is not relevant and therefore there must exist a *constitutive relation* between stress and deformation gradient tensors

$$\boxed{\sigma = \sigma(\mathbf{F})} \tag{4.94}$$

We also assume that the reference state is stress-free,

$$\boxed{\sigma(\mathbf{F} = \mathbb{1}) = \mathbf{0}} \tag{4.95}$$

thus excluding pre-stressed material (typical for biomaterials, for example wood, carots or skin, which spring open when being cut).
We first consider linear elasticity theory (LET), where one assumes that $\partial_i u_j$ is small. Then σ_{ij} depends only on the linear strain tensor $\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ as

$$\boxed{\sigma_{ij} = C_{ijkl} \cdot \epsilon_{kl}} \tag{4.96}$$

where C_{ijkl} is the tensor of elastic moduli (of rank 4).
The symmetry of σ_{ij} and ϵ_{kl} allows one to reduce the number of unknowns from 81 to 36. The minimal number of elastic moduli depends on the symmetry group of the material:

triclinic	21
hexagonal	5
cubic	3
isotropic	2

Isotropic LET

The **isotropic** case can be introduced as follows. We decompose both stress and strain tensors into isotropic and deviatoric parts:

$$\sigma = -p\mathbb{1} + \sigma^d \qquad \epsilon = \frac{1}{3}\operatorname{tr}(\epsilon)\mathbb{1} + \epsilon^d \tag{4.97}$$

where $p = -\frac{1}{3}\operatorname{tr}(\sigma)$ is the hydrostatic pressure and $\operatorname{tr}(\epsilon) = \frac{dV - dV_0}{dV_0}$ is the relative volume change. Linear elastic isotropic behaviour then assumes linear relations between the corresponding parts:

$$p = -K\operatorname{tr}(\epsilon), \qquad \sigma^d = 2G \cdot \epsilon^d \tag{4.98}$$

with K the *compression* or *bulk modulus* and G the *shear modulus*. Both are positive for thermodynamic stability.
Rewriting equation 4.97 with the help of equation 4.98 gives the $3d$ version of Hooke's law:

$$\sigma = K\operatorname{tr}(\epsilon)\mathbb{1} + \underbrace{2G \cdot \underbrace{(\epsilon - \frac{1}{3}\operatorname{tr}(\epsilon)\mathbb{1})}_{\epsilon^d}}_{\sigma^d} \tag{4.99}$$

$$= (K - \frac{2}{3}G)\operatorname{tr}(\epsilon)\mathbb{1} + 2G \cdot \epsilon \tag{4.100}$$

$$\Rightarrow \quad \boxed{\sigma = \lambda\operatorname{tr}(\epsilon)\mathbb{1} + 2\mu \cdot \epsilon} \tag{4.101}$$

3d Hooke's law, generalization of $\sigma = E \cdot \epsilon$

with the *Lamé constants* $\lambda = K - \frac{2}{3}G$ and $\mu = G$ being an alternative choice to (K, G). This choice corresponds to

$$C_{ijkl} = \lambda \cdot \delta_{ij}\delta_{kl} + 2\mu \cdot \delta_{ik}\delta_{jl} \tag{4.102}$$

The two terms represent the two possibilities to construct an isotropic tensor of rank 4 and explain why one has at least two elastic constants.
The relation between stress and strain tensors can be easily inverted:

$$\sigma_{ll} = 3\lambda \cdot \epsilon_{ll} + 2\mu \cdot \epsilon_{ll} = (3\lambda + 2\mu)\epsilon_{ll} \tag{4.103}$$

$$\Rightarrow \quad \boxed{\epsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu}\frac{1}{(3\lambda + 2\mu)}\sigma_{ll}} \tag{4.104}$$

Our equilibrium condition for stress was $\partial_j \sigma_{ij} + \rho g_i = 0$. We now can replace σ_{ij} by ϵ_{ij} and then ϵ_{ij} by u_i and thus obtain an equation for the displacement field u_i:

$$\partial_j \sigma_{ij} = \partial_j(\lambda \cdot \epsilon_{ll}\delta_{ij} + 2\mu \cdot \epsilon_{ij}) = \lambda \cdot \partial_i \epsilon_{ll} + 2\mu \cdot \partial_j \epsilon_{ij} \tag{4.105}$$

$$= \lambda \cdot \partial_i \partial_l u_l + \mu \cdot (\partial_j \partial_i u_j + \partial_j \partial_j u_i) \tag{4.106}$$

$$= (\lambda + \mu) \cdot \partial_i(\partial_l u_l) + \mu \cdot \partial_j \partial_j u_i = -\rho \cdot g_i \tag{4.107}$$

In vector notation:

$$\boxed{\mu \Delta \mathbf{u} + (\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) + \rho \mathbf{g} = \mathbf{0}} \tag{4.108}$$

where Δ is the Laplace-operator and $\boldsymbol{\nabla} \cdot \mathbf{u}$ couples different components of $\mathbf{u}$. There are two elastic constants because there are two ways to write a 2nd order derivative in 3d.
Due to the linearity of this ODE for $\mathbf{u}$, it can be solved by a *Green's function*:

$$u_i(\mathbf{r}) = \int d\mathbf{r}' \, G_{ij}(\mathbf{r} - \mathbf{r}') F_j(\mathbf{r}') \tag{4.109}$$

where the Green tensor $\mathbf{G}$ follows from solving

$$\mu \Delta \mathbf{u} + (\mu + \lambda)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) = -\mathbf{F}\delta(\mathbf{r}) \qquad \text{(no volume force)} \tag{4.110}$$

For the infinite isotropic elastic space, the solution was given in 1848 by *Lord Kelvin*:

$$G_{ij} = \frac{1}{8\pi\mu(2\mu + \lambda)} \left[\underbrace{(3\mu + \lambda)\delta_{ij}}_{\text{compression}} + \underbrace{(\mu + \lambda)\frac{x_i x_j}{r^2}}_{\text{shear}} \right] \frac{1}{r} \tag{4.111}$$

Note the $\frac{1}{r}$-scaling factor from the solution of the Laplace equation in 3d, implying a long-ranged kernel with strong dependence on boundary conditions.
We finally give an expression for the strain energy density w:

$$dw = \sigma_{ij} d\epsilon_{ij} = (\lambda \epsilon_{ll}\delta_{ij} + 2\mu\epsilon_{ij}) d\epsilon_{ij} \tag{4.112}$$

$$\Rightarrow \quad \boxed{w = \frac{1}{2}\lambda(\epsilon_{ll})^2 + \mu \epsilon_{ij}\epsilon_{ij}} \tag{4.113}$$

There are two elastic constants because there are two ways to contract ϵ_{ij} to a scalar (in quadratic order).

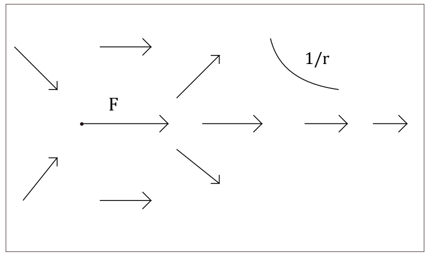


Figure 4.12.: The displacement decays with $^1/_r$ and points move according to the change in angle.

4.6. Non-linear elasticity theory

For linear elasticity theory, the difference between Lagrangian and Eulerian variables could be neglected. For non-linear elasticity theory, when deformations can be large, we now have to confront the difficulty that the balance of stresses is performed in the deformed state, while the constitutive relation refers to the reference configuration.

In the deformed state, we have for the stress vector

$$d\mathbf{s} = \sigma \cdot d\mathbf{A} \tag{4.114}$$

We have to relate $d\mathbf{A}$ to the surface element $d\mathbf{A}_0$ in the undeformed state. We consider the deformation of a small cylinder defined by $d\mathbf{A}_0$ and $d\mathbf{x}_0$:

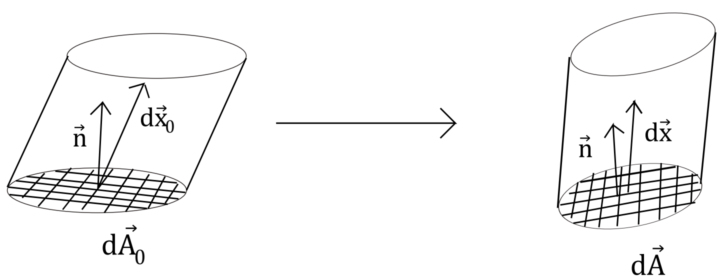


Figure 4.13.: For infinitesimal deformations of the cylinder, both normal vector $\mathbf{n}$ and axis vector $d\mathbf{x}_0$ change directions and lengths.

The volumes are related by

$$dV = d\mathbf{x} \cdot d\mathbf{A} = J \cdot dV_0 = J \cdot d\mathbf{x}_0 \cdot d\mathbf{A}_0 = (\mathbf{F} \cdot d\mathbf{x}_0) \cdot d\mathbf{A} = d\mathbf{x}_0 \cdot (\mathbf{F}^T \cdot d\mathbf{A}) \tag{4.115}$$

with $J = det(\mathbf{F})$. This is valid for all $d\mathbf{x}_0$ and hence

$$\boxed{d\mathbf{A} = J \cdot (\mathbf{F}^T)^{-1} \cdot d\mathbf{A}_0} \tag{4.116}$$

Via equation 4.114 we define

$$\boxed{d\mathbf{s} = \underbrace{J \cdot \sigma(\mathbf{F}^T)^{-1}}_{\mathbf{P}} \cdot d\mathbf{A}_0} \tag{4.117}$$

the *first Piola-Kichhoff stress tensor* $\mathbf{P}$. $\mathbf{P}$ describes the stress in the reference configuration. In contrast to the Cauchy stress tensor σ, it is **not** symmetric.
The *second Piola-Kirchhoff stress tensor*

$$\mathbf{S} \equiv \mathbf{F}^{-1}\mathbf{P} = J \cdot \mathbf{F}^{-1}\sigma(\mathbf{F}^T)^{-1} \tag{4.118}$$

is symmetric and can be shown to be energy-conjugate to the Green-Lagrange strain tensor $\mathbf{E}$. It is therefore the standard choice for a constitutive relation.
We now can write

$$d\mathbf{s}_0 = S \cdot d\mathbf{A}_0 \tag{4.119}$$

in the undeformed state, where s_0 is the back-transformed stress vector. Note however that this is a mathematical definition, because in physical reality, the stress tensor can only be measured in the deformed state.

We now revisit Cauchy's momentum equation and transfer it to the reference (Langrangian) frame. We again start with Newton's second law:

$$\frac{d}{dt}\int_V \frac{\partial x_i}{\partial t}\rho dV = \int_V g_i \rho dV + \int_{\partial V} \sigma_{ij} dA_j \tag{4.120}$$

$$\Rightarrow \quad \frac{d}{dt}\int_{V_0} \frac{\partial x_i}{\partial t}\underbrace{\rho J}_{\rho_0} dV_0 = \int_{V_0} g_i \underbrace{\rho J}_{\rho_0} dV_0 + \int_{\partial V_0} \underbrace{J\sigma(\mathbf{F}^T)^{-1}}_{\mathbf{P}} d\mathbf{A}_0 \tag{4.121}$$

$$\overset{\text{DT}}{\Rightarrow} \quad \int_{V_0} \frac{\partial^2 x_i}{\partial t^2}\rho_0 dV_0 = \int_{V_0} g_i \rho_0 dV_0 + \int_{V_0} \frac{\partial P_{ij}}{\partial x_{0j}} dV_0 \tag{4.122}$$

$$\overset{\forall \text{V}}{\Rightarrow} \quad \boxed{\rho_0 \frac{\partial^2 x_i}{\partial t^2} = \rho_0 g_i + \frac{\partial P_{ij}}{\partial x_{0j}}} \qquad \text{Lagrangian form of Cauchy's equation} \tag{4.123}$$

In steady state and without volume forces the equilibrium condition $\boldsymbol{\nabla} \cdot \sigma = \mathbf{0}$ is thus replaced by $\boldsymbol{\nabla}_0 \cdot \mathbf{P} = \boldsymbol{\nabla}_0 \cdot (\mathbf{F} \cdot \mathbf{S}) = \mathbf{0}$.

NLET in 1D

We consider a *unidirectional* displacement $u(x,t)$

$$\mathbf{F} = \begin{pmatrix} 1+\frac{\partial u}{\partial x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbb{1}) \tag{4.124}$$

$$E_{xx} = \frac{\partial u}{\partial x} + \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2}_{\text{geometrical NL}} \tag{4.125}$$

The equilibrium condition is

$$\frac{\partial P_{xx}}{\partial x} = 0 \quad \Rightarrow \quad P_{xx} = const. \tag{4.126}$$

The most general constitutive law is

$$S_{xx} = \phi(E_{xx}) \tag{4.127}$$

with some function ϕ (possibly non-linear, a *material* NL).

$$\Rightarrow \quad \boxed{P_{xx} = const = (\mathbf{F}\cdot\mathbf{S})_{xx} = \left(1+\frac{\partial u}{\partial x}\right)\phi\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)} \tag{4.128}$$

This equation is both geometrically **and** mechanically non-linear. For mechanical linearity, one has

$$\phi\left(E_{xx}\right) = (\lambda + 2\mu)E_{xx} \tag{4.129}$$

$$\Rightarrow \quad P_{xx} = \left(1+\frac{\partial u}{\partial x}\right)(\lambda+2\mu)\left(1+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)\right)\frac{\partial u}{\partial x} \tag{4.130}$$

a geometrically non-linear, cubic equation for $\frac{\partial u}{\partial x}$. In LET this reduces to

$$P_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} \tag{4.131}$$

$$P_{xx} = \det J \sigma_{xx} \frac{1}{1 + \frac{\partial u}{\partial x}} \approx \sigma_{xx} \qquad \Rightarrow \qquad \sigma_{xx} = (\lambda + 2\mu)\epsilon_{xx} \quad \text{1d LET} \tag{4.132}$$

NLET in 3D

For LET in 3D, we have assumed

$$\sigma = \sigma(\mathbf{F}), \qquad \sigma(\mathbf{F} = \mathbb{1}) = \mathbf{0} \tag{4.133}$$

For NLET in 3D, one can argue that the corresponding assumption should be

$$\mathbf{S} = \mathbf{S}(\mathbf{C}), \qquad \mathbf{S}(\mathbf{C} = \mathbb{1}) = \mathbf{0} \tag{4.134}$$

Assuming further that the material is locally isotropic, one can show that $\mathbf{S}$ has to have the form

$$\mathbf{S}(\mathbf{C}) = \phi_0(I_1, I_2, I_3)\mathbb{1} + \phi_1(I_1, I_2, I_3)\mathbf{C} + \phi_2(I_1, I_2, I_3)\mathbf{C}^2 \tag{4.135}$$

where ϕ_0, ϕ_1, ϕ_2 are functions of the three invariants of $\mathbf{C}$:

$$I_1 = \mathrm{tr}(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{4.136}$$

$$I_2 = \frac{1}{2}\left((\mathrm{tr}(\mathbf{C}))^2 - \mathrm{tr}(\mathbf{C}^2)\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \tag{4.137}$$

$$I_3 = \det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{4.138}$$

There are no higher order terms of $\mathbf{C}$ due to the *Cayley-Hamilton theorem* from Linear Algebra. The λ_i^2 are the eigenvalues of $\mathbf{C}$. In LET, the λ_i become the stretch ratios. Therefore an isotropic constitutive relation amounts to specifying the three scalar functions ϕ_1, ϕ_2, ϕ_3.
$\mathbf{S}(\mathbb{1}) = \mathbf{0}$ leads to the condition

$$\phi_0(3,3,1) + \phi_1(3,3,1) + \phi_2(3,3,1) = 0 \tag{4.139}$$

Choosing arbitrary ϕ_i usually leads to models with unphysical behaviour (e.g. material which can be used as a limitless energy source during cyclic deformation). The best solution to this problem is the use of appropriate strain energy density functions (*hyperelastic materials*).

Energy equation for NLET

We again multiply the momentum equation by the velocity and integrate over a large material volume V_0:

$$\int\limits_{V_0} \rho_0 \frac{\partial^2 x_i}{\partial t^2}\frac{\partial x_i}{\partial t} dV_0 = \int\limits_{V_0} \rho_0 g_i \frac{\partial x_i}{\partial t} dV_0 + \int\limits_{V_0} \frac{\partial P_{ij}}{\partial x_{0j}}\frac{\partial x_i}{\partial t} dV_0 \tag{4.140}$$

$$\overset{\text{DT, PI}}{\Rightarrow} \underbrace{\frac{d}{dt}\int\limits_{V_0} \frac{\rho_0}{2}\left(\frac{\partial x_i}{\partial t}\right)^2 dV_0}_{\text{rate of change in kinetic energy} = \dot{T}} + \underbrace{\int\limits_{V_0} P_{ij}\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial x_i}{\partial x_{0j}}\right)}_{F_{ij}} dV_0}_{\text{rate at which elastic energy is stored in the material} = \dot{U}} \tag{4.141}$$

$$= \underbrace{\int\limits_{V_0} \rho_0 g_i \frac{\partial x_i}{\partial t} dV_0}_{\text{rate of work by body forces}} + \underbrace{\int\limits_{\partial V_0} \frac{\partial x_i}{\partial t} P_{ij} dA_j}_{\text{rate of work on the surface}} \tag{4.142}$$

We postulate the existence of a strain energy density $w(F_{ij})$ such that

$$P_{ij} = \frac{\partial w}{\partial F_{ij}} \qquad \Rightarrow \qquad \dot{U} = \frac{d}{dt} \int\limits_{V_0} w \cdot dV_0 \tag{4.143}$$

$\dot{U}$ is the rate of change in elastic energy stored int the material. We also need to require w to have a minimum for $\mathbf{F} = \mathbb{1}$. If w is a function only of $\mathbf{C}$, then one can show that

$$S_{ij} = 2 \cdot \frac{\partial w}{\partial C_{ij}} \tag{4.144}$$

For isotropic material, $w = w(I_1, I_2, I_3)$, it holds that

$$S_{ij} = 2 \cdot \frac{\partial w}{\partial I_k} \frac{\partial I_k}{\partial C_{ij}} \tag{4.145}$$

This leads to an explicit procedure to calculate ϕ_0, ϕ_1, ϕ_2 from a given w.
Using minimization of an energy functional is computationally much easier than solving the non-linear PDEs following from $\boldsymbol{\nabla}_0(\mathbf{F} \cdot \mathbf{S}) = \mathbf{0}$. This procedure naturally leads to the *finite element method* (FEM).
For hyperelastic material (a typical example is *rubber*, which can have very large deformations), one usually assumes incompressibility, that is $I_3 = 1$.
The commonly used constitutive relations are:

(a) **Neo-Hookean:** $w = \frac{\mu}{2}(I_1 - 3)$

The Neo-Hookean description is good for plastic and rubber up to 20% strain. μ is the classical shear modulus known from LET.

(b) **Mooney-Rivelin:** $w = C_1(I_1 - 3) + C_2(I_2 - 3)$

The Mooney-Rivelin formulation becomes Neo-Hookean with $C_2 = 0$ and is good for rubber up to 100% strain.

(c) **Ogden:** $w = \sum\limits_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3)$

The Ogden relation is a generalization of 1 and 2 in the sense that it contains both the Neo-Hookean and the Mooney-Rivelin description. Usually $N = 3$ with 6 independent parameters gives a very good fit to experiments. The relation to LET is such that $\sum\limits_{p=1}^{N} \mu_p \alpha_p = 2\mu$.

The results of a typical stress test with uni-axial loading are visualized in fig. 4.14.

Example: Blowing up a balloon

Balloons as rubber-like material with very large displacements are ideal examples for NLET.
We consider a thin, spherical, incompressible rubber membrane of initial radius R and thickness $H \ll R$.
The two angular stretches have to be the same (biaxial loading in the sheet):

$$\lambda_\theta = \lambda_\phi = \frac{2\pi r}{2\pi R} = \frac{r}{R} \tag{4.146}$$

The normal stretch is determined by incompressibility:

$$I_3 = \lambda_\theta^2 \lambda_\phi^2 \lambda_r^2 = 1 \qquad \Rightarrow \qquad \lambda_r = \frac{R^2}{r^2} \tag{4.147}$$

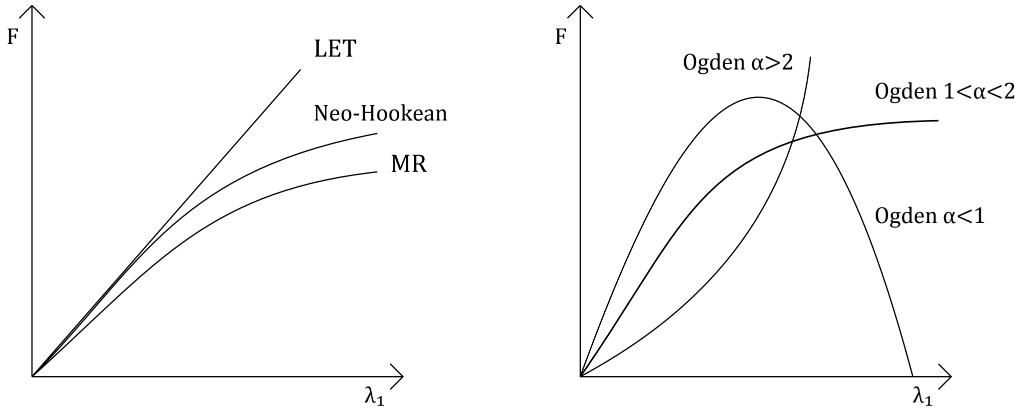


Figure 4.14.: In each case the slope of the curve corresponds to the elastic constant of the material probed. Neo-Hookean behaviour with strain-softening is typical for synthetic polymer networks, whereas strain-stiffening (Ogden $\alpha > 2$) is typical for biopolymer networks. Material instability or failure behavior occurs in the Ogden model for $\alpha < 1$.

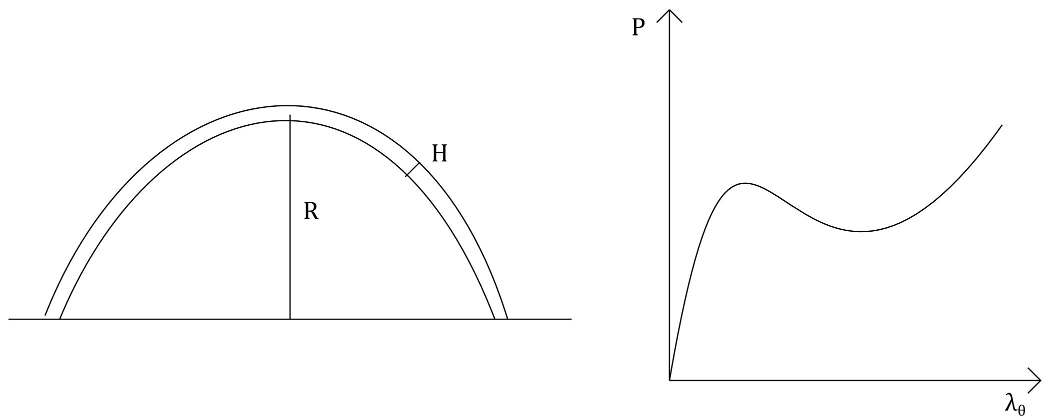


Figure 4.15.: The pressure required to inflate a balloon with radius R much larger than thickness H initially increases, but then decreases. This corresponds to the familiar experience that blowing up a balloon becomes easier after an initial barrier.

A typical value for the classical shear modulus is $\mu = 0.4\,MPa$. This defines the Neo-Hookean model. For Mooney-Rivelin one can use $C_1 = 0.44\mu$, $C' = \frac{C_2}{C_1} = \frac{1}{7}$.

For the Ogden relation one can use $N = 3$ and

$$\alpha_1 = 1.3 \qquad \mu_1 = 0.6\,MPa \tag{4.148}$$

$$\alpha_2 = 5.0 \qquad \mu = 0.01\,MPa \tag{4.149}$$

$$\alpha_3 = -2.0 \qquad \mu = -0.01\,MPa \tag{4.150}$$

All three material laws give similar results for this example.

We now consider that the balloon is inflated by some internal pressure p. This gives rise to a Laplace relation

$$d(E_p) = d(E_T) \quad \Rightarrow \quad d(\frac{4\pi}{3}r^3 p) = d(4\pi r^2 T) \tag{4.151}$$

$$\Rightarrow \quad pr^2 dr = T2rdr \quad \Leftrightarrow \quad \boxed{p = \frac{2T}{r}} \tag{4.152}$$

For Mooney-Rivelin, one can combine these elements to show:

$$p = \frac{4C_1 H}{R} \cdot \frac{(1 + C'\lambda_\theta^2)(\lambda_\theta^6 - 1)}{\lambda_\theta^7} \tag{4.153}$$

which is only defined for $\lambda_\theta \geq 1$.
Equation 4.153 is plotted in fig. 4.15.

CHAPTER 5

Applications of LET

This chapter will deal with elastic problems that can be analytically solved in LET. Exact analytical solutions to realistic problems are often not possible and require numerical methods like the finite element method (FEM), but the systems discussed here are of large importance and help to understand more complex situations. Before addressing the actual problems, we will quickly recapitulate the most important concepts of LET.

5.1. Reminder on isotropic LET

In LET we have two elastic constants and we have already encountered two different possible choices:

- (K, G) the bulk and shear moduli
- (λ, μ) the Lamé coefficients

which are related by $\lambda = K - \frac{2}{3}G$, $\mu = G$.

The three main concepts are the displacement vector field $\mathbf{u}(\mathbf{x})$ and the tensors for strain and stress, ϵ_{ij} and σ_{ij}. The equilibrium condition for stress and displacement respectively is

$$\mathbf{0} = \boldsymbol{\nabla}\sigma + \rho\mathbf{g} = \mu\Delta\mathbf{u} + (\lambda + \mu)\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\cdot\mathbf{u}) + \rho\mathbf{g} \tag{5.1}$$

and the strain energy density w is given by

$$w = \sigma_{ij}\epsilon_{ij} = \frac{1}{2}\lambda\,(\epsilon_{ll})^2 + \mu\,\epsilon_{ij}\,\epsilon_{ij} \tag{5.2}$$

5.2. Pure Compression

As a first example, we look at a case where the material is compressed and there are no shear forces, as illustrated in fig. 5.2. The stress tensor is then given by

$$\sigma_{ij} = -p\,\delta_{ij} \tag{5.3}$$

where the pressure p is related to the relative volume change by

$$dw = \sigma_{ij}\,d(\epsilon_{ij}) = -p\,d(\epsilon_{ll}) = -p\frac{dV}{V_0} \tag{5.4}$$

$$\sigma_{ll} = -3p = 3K\epsilon_{ll} \qquad \Rightarrow \qquad \frac{1}{K} = -\frac{1}{V_0}\frac{\partial V}{\partial p} \tag{5.5}$$

That the volume change should be negative as pressure increases is a familiar result from thermodynamics. The bulk modulus K is the *isothermal compressibility* and has to be positive, $K > 0$, as a stability criterium.

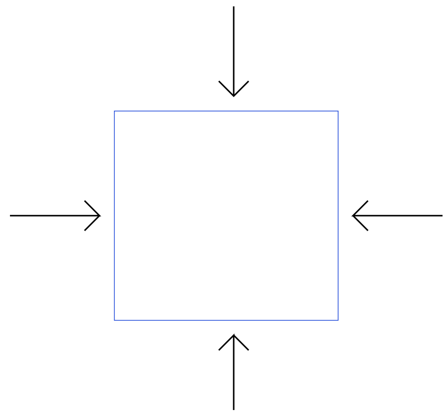


Figure 5.1.: We take a piece of material and compress it equally from all sides. There are only normal forces acting.

5.3. Simple shear

As a second example we look at a plate under force F and with area A leading to pure, one-dimensional shear in the x-direction (fig. 5.3). The stress is the force per area $s = \frac{F}{A}$ and the stress tensor is now given by

$$\sigma = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.6}$$

With vanishing trace, $\sigma_{ll} = 0$, the inverted relation is obtained easily and the strain tensor has the same symmetrical structure:

$$\epsilon = \begin{pmatrix} 0 & \frac{s}{2\mu} & 0 \\ \frac{s}{2\mu} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.7}$$

The displacement field corresponding to these stress and strain tensors is then

$$\mathbf{u} = \left(\frac{sy}{\mu}, 0, 0\right) \qquad \alpha = \frac{u_1}{y} = \frac{s}{\mu} \tag{5.8}$$

The displacement has a linear profile in the x-direction and is dependent only on the shear modulus μ and not on the bulk modulus. The larger the shear modulus, the smaller the displacement and the shear angle α.

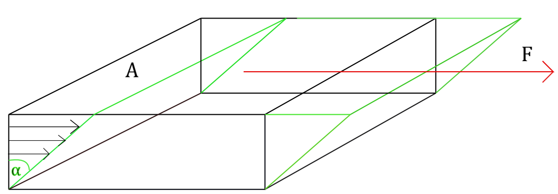


Figure 5.2.: Shear forces acting on a plate of area A. The deformed state is characterized by the shear angle α.

5.4. Uni-axial stretch

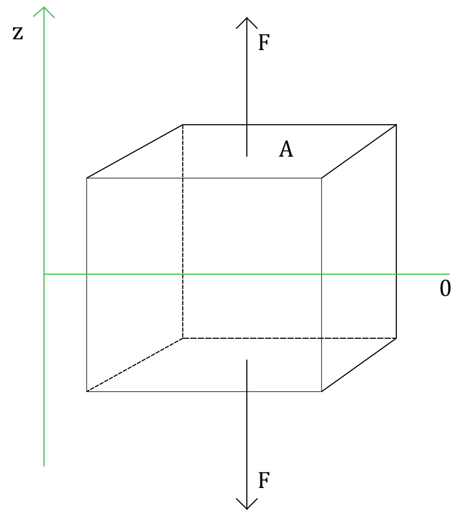


Figure 5.3.: Illustration of uni-axial stretching. The material is stretched in the z-direction.

In the case of uni-axial stretching (fig. 5.4), we again know the stress $p = \frac{F}{A}$ and can immediately write for the stress tensor

$$\sigma_{zz} = p, \qquad \sigma_{ij} = 0 \quad \text{for all other components} \tag{5.9}$$

In order to construct the strain tensor, we procede component-by-component:

$$\epsilon_{zz} = \frac{1}{2\mu}\sigma_{zz} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\underbrace{\sigma_{ll}}_{=\sigma_{zz}} = \frac{(\lambda+\mu)}{\mu(3\lambda+2\mu)}p \equiv \frac{p}{E} \tag{5.10}$$

$$\epsilon_{xx} = \epsilon_{yy} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}p \equiv -\nu\epsilon_{zz} = -\frac{\nu}{E}p \tag{5.11}$$

Here we have defined a new set of elastic constants:

$$E = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)} \qquad \text{Young's modulus} \tag{5.12}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \qquad \text{Poisson's ratio} \tag{5.13}$$

The strain tensor and matching displacement are then given by

$$\epsilon = \frac{p}{E}\cdot\begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{u} = \frac{p}{E}\begin{pmatrix} -\nu x \\ -\nu y \\ z \end{pmatrix} \tag{5.14}$$

$$\epsilon_{ll} = \frac{p}{E}\cdot(1-2\nu) \tag{5.15}$$

The *Young's modulus* is commonly stated in the literature to characterize the *stiffness* or *rigidity* of a given material. Some typical values are given in table 5.1.

The Poisson ratio ν has no physical unit and describes the coupling between different directions in the material. It is the ratio between longitudinal expansion and lateral contraction. In the example of uni-axial stretch, the displacement in x- and y- direction is negative and the material moves in from the sides when being stretched (Poisson effect).

Since bulk and shear moduli must be positive for thermodynamic reasons, you can also convince yourself that the values for the dimensionless Poisson's ratio ν lie in a narrow range:

$$G, K > 0 \quad \Rightarrow \quad \boxed{-1 < \nu < \frac{1}{2}} \tag{5.16}$$

E [Pa]	material
TPa	graphene
GPa	crystals
MPa	rubber
kPa	cells

Table 5.1.: Typical values for the Young's modulus for different types of materials

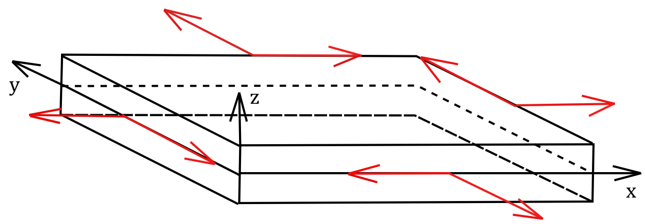


Figure 5.4.: The plate is strained in the (x,y)-plane.

The upper bound $\nu = \frac{1}{2}$ corresponds to the limit $\lambda \to \infty$ ($K \to \infty$) and the material becomes *incompressible* ($\epsilon_{ll} = \frac{p}{E}(1-2\nu) = 0$). This is the case for most biomaterials which are incompressible due to the large amount of water in the material (volume conservation in biological systems).

For negative Poisson's ratio, $\nu < 0$, so-called *auxetic* materials, when expanded in one direction, they also expand in the other directions (take for example a crumpled piece of paper and expand it uni-axially).

(E, ν) is an alternative choice to (λ, μ) or (K, G)

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{5.17}$$

With this choice, the constitutive relation is given by

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{ll}\delta_{ij} \tag{5.18}$$

and the strain energy density reads

$$w = \frac{p^2}{2E} \qquad \text{compare spring:} \;\; U = \frac{F^2}{2k} \tag{5.19}$$

5.5. Biaxial strain

Compression and shearing forces act as shown in fig. 5.5. We assume linear in-plane deformations and shrinking by a factor γ in the z-direction:

$$\mathbf{u} = \begin{pmatrix} ax+by \\ cx+dy \\ -\gamma z \end{pmatrix} \qquad \Rightarrow \qquad \epsilon = \begin{pmatrix} a & \frac{1}{2}(b+c) & 0 \\ \frac{1}{2}(b+c) & d & 0 \\ 0 & 0 & -\gamma \end{pmatrix} \tag{5.20}$$

ϵ_{ij} and σ_{ij} are both constant.

By a linear mapping we get for the stress tensor (assuming free surfaces on the top and bottom):

$$\sigma_{zz} = \lambda\,\epsilon_{ll} + 2\mu\,\epsilon_{zz} = \lambda(a+d-\gamma) + 2\mu(-\gamma) = 0 \tag{5.21}$$

$$\Rightarrow \qquad \gamma = \frac{\lambda(a+d)}{(\lambda+2\mu)} = (\frac{\nu}{1-\nu})(a+d) \tag{5.22}$$

$$\Rightarrow \qquad \sigma = \begin{pmatrix} \frac{E(a+\nu d)}{1-\nu^2} & \frac{E(b+c)}{2(1+\nu)} & 0 \\ \frac{E(b+c)}{2(1+\nu)} & \frac{E(\nu a+d)}{1-\nu^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.23}$$

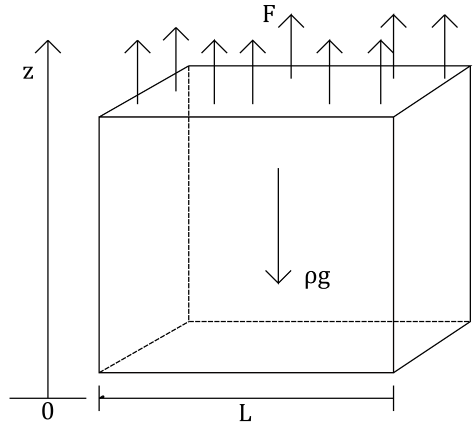


Figure 5.5.: Elastic cube with dimenson L subject to gravity. Homogeneous stress at the top surface is holding the cube.

If **no force** is applied in the y-direction: $\sigma_{yx} = \sigma_{xy} = \sigma_{yy} = 0$

$$\Rightarrow \quad d = -\nu a, \quad c = -b \tag{5.24}$$

$$\Rightarrow \quad \sigma = \begin{pmatrix} Ea & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} a & 0 & 0 \\ 0 & -\nu a & 0 \\ 0 & 0 & -\nu a \end{pmatrix} \tag{5.25}$$

We recover the solution for uni-axial stretching for $E \cdot a = p$.

Alternatively, one can also obtain displacement only in the (x,z)-plane, **no displacement** in the y-direction:

$$b = c = d = 0 \quad \Rightarrow \quad \sigma = \begin{pmatrix} \frac{Ea}{1-\nu^2} & 0 & 0 \\ 0 & \frac{E\nu a}{1-\nu^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.26}$$

Thus, a transverse stress σ_{yy} must be applied to prevent the plate from contracting in the y-direction as we stretch in x-direction. We obtain an effective elastic modulus $\frac{E}{1-\nu^2} > E$, so that 2d stretching is more strenuous than uni-axial stretching.

5.6. Elastic cube under its own weight

The cube and corresponding stresses are illustrated in fig. 5.6. The cube under its own weight is subject to the body force $f_i = -\rho g \delta_{iz}$. We ask for stress free boundaries at the sides and bottom, $\sigma \cdot \mathbf{n} = \mathbf{0}$, but hold the cube from above by a homogeneous surface stress.

The equilibrium condition is given by the steady Navier equation with constant gravity

$$\partial_i \sigma_{ij} + f_i = 0 \tag{5.27}$$

$$\Rightarrow \quad \frac{\partial \sigma_{zz}}{\partial z} = \rho g \quad \Rightarrow \quad \boxed{\sigma_{zz} = \rho g z} \tag{5.28}$$

Gravitation is balanced everywhere in the material and the total force on the upper surface, $\rho g L \cdot L^2$, exactly balances the overall gravitation.

Strain tensor and displacement field are given by

$$\epsilon = \begin{pmatrix} -\frac{\nu}{E}\rho g z & 0 & 0 \\ 0 & -\frac{\nu}{E}\rho g z & 0 \\ 0 & 0 & \frac{\rho g}{E} z \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} -\nu \frac{\rho g}{E} xz \\ -\nu \frac{\rho g}{E} yz \\ \frac{\rho g}{2E}(z^2 + \nu(x^2 + y^2)) \end{pmatrix} \tag{5.29}$$

Upper and lower surfaces become parabolic and the cube, with the point $(0, 0, 0)$ fixed in space, broadens from top to bottom (see fig. 5.7).

5.7. Torsion of a bar

We twist a bar of *arbitrary* cross-section D by applying moments to its ends and make the following ansatz for the displacement vector:

$$\mathbf{u} = \begin{pmatrix} -\Omega\, yz \\ \Omega\, xz \\ \Omega\, \Psi(x,y) \end{pmatrix} \tag{5.30}$$

Hence we assume a rotation, with Ω representing the *twist* of the bar, in the (x,y)-plane, and translational invariance, with the *torsion function* Ψ yet to be determined, in the z-direction.

The strain and stress tensors are then given by

$$\epsilon = \begin{pmatrix} 0 & 0 & \frac{\Omega}{2}(\partial_x\Psi - y) \\ 0 & 0 & \frac{\Omega}{2}(\partial_y\Psi + x) \\ \frac{\Omega}{2}(\partial_x\Psi - y) & \frac{\Omega}{2}(\partial_y\Psi + x) & 0 \end{pmatrix} \qquad \sigma = 2\mu\epsilon \tag{5.31}$$

The trace of the strain tensor vanishes, $\epsilon_{ll} = 0$, so that this is a pure shear experiment. The stress tensor has the same structure as the strain tensor.

The condition for equilibrium is again given by the Navier equation:

$$\partial_j\sigma_{ij} = \Omega\mu(\partial_x^2\,\Psi + \partial_y^2\,\Psi) = 0 \tag{5.32}$$

$$\Rightarrow \quad \boxed{\boldsymbol{\nabla}^2\Psi = 0} \qquad \text{Ψ has to satisfy the Laplace equation on D} \tag{5.33}$$

We parameterize the boundary ∂D as $\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix}$. The normal in the (x,y)-plane is given by $\mathbf{n} = \begin{pmatrix} Y' \\ -X' \end{pmatrix}$.

Assuming stress-free boundary conditions, we get

$$\sigma_{zj}n_j = \Omega\mu\left[(\partial_x\Psi - Y)Y' - (\partial_y\Psi + X)X'\right] = 0 \tag{5.34}$$

$$\Rightarrow \quad \partial_x\Psi Y' - \partial_y\Psi X' = \boldsymbol{\nabla}\Psi\cdot\mathbf{n} \tag{5.35}$$

$$= \boxed{\partial_n\Psi = \frac{1}{2}\frac{d}{ds}(X^2+Y^2)} \qquad \text{on } \partial D \tag{5.36}$$

The solution $\Psi(x,y)$ for this Neumann problem is unique up to an arbitrary constant corresponding to an arbitrary uniform translation.

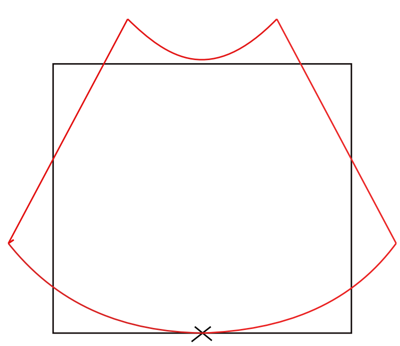


Figure 5.6.: Image of the deformed cube from one side. The cube is fixed at the point $(0,0,0)$. Again the Poisson effect brings the sides of the cube in, but lower and upper side are parabola shaped. Gravity lets the system sag down.

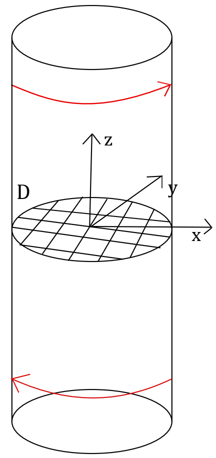


Figure 5.7.: A moment couple applied to a general elastic rod with domain D. Due to symmetry there will always be one plane without movement, here at $z = 0$.

Once the solution is known, the moment applied at each end of the bar is given by

$$\mathbf{M} = \int_D dxdy\, \mathbf{r} \times \mathbf{F} = \int_D dxdy \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \\ 0 \end{pmatrix} \tag{5.37}$$

$$= \int_D dxdy \begin{pmatrix} 0 \\ 0 \\ (x\sigma_{yz} - y\sigma_{xz}) \end{pmatrix} \tag{5.38}$$

$$\Rightarrow \quad M = M_z = \underbrace{\mu \int_D dxdy \left[x\partial_y \Psi - y\partial_x \Psi + (x^2 + y^2)\right]}_{\equiv R \text{ torsional rigidity}} \Omega \tag{5.39}$$

The *torsional rigidity* R is the factor of proportionality between moment M and twist Ω, analogue to the spring constant.

For simple cross-sectional shapes, R can be calculated analytically. The simplest case is the circular bar. Then, D is a disc of radius a and our Neumann problem is

$$\Delta\Psi = \frac{1}{r}\partial_r(r\partial_r\Psi) = 0, \qquad r < a \tag{5.40}$$

$$\partial_n\Psi = 0, \qquad r = a \tag{5.41}$$

We then find

$$\Psi = const \tag{5.42}$$

$$R = 2\pi\mu \int_0^a rdr\, r^2 = \boxed{\frac{\pi\mu a^4}{2} = R} \tag{5.43}$$

The torsional rigidity of a circular bar is linear in the shear modulus μ, as expected in LET, but increases with the 4^{th} power of its radius, showing a large geometrical dependence! This is formally the same problem as the Hagen-Poiseuille law for viscous fluid flow due to a pressure gradient through a pipe of radius a in hydrodynamics.

Only for the circular bar we get $\partial_n\Psi = 0$ and $\Psi = const$. For all other cases, one would get more complicated boundary conditions and a non-trivial component $u_z = \Omega\Psi(x, y)$.

We now introduce an alternative and **more general way** to solve this problem. Due to the special form of the stress tensor, the steady Navier equation reads

$$\partial_x \sigma_{zx} + \partial_y \sigma_{zy} = 0 \tag{5.44}$$

This can be satisfied by postulating the existence of a *stress function* $\phi(x,y)$ such that

$$\sigma_{zx} = \mu\Omega\partial_y\phi, \qquad \sigma_{zy} = -\mu\Omega\partial_x\phi \tag{5.45}$$

The factors of $\mu\Omega$ are introduced for later convenience. ϕ has the role of a scalar potential and is defined up to addition of an arbitrary constant.

Comparing this with

$$\sigma_{zx} = \mu\Omega(\partial_x\Psi - y), \qquad \sigma_{zy} = \mu\Omega(\partial_y\Psi + x) \tag{5.46}$$

we can relate ϕ to Ψ by

$$\partial_x\Psi = \partial_y\phi + y, \qquad \partial_y\Psi = -\partial_x\phi - x \tag{5.47}$$

$$\Rightarrow \qquad \Delta\phi = (-\partial_x\partial_y\Psi - 1) + (\partial_y\partial_x\Psi - 1) = -2 \tag{5.48}$$

Thus ϕ satisfies *Poisson's equation*

$$\boxed{\Delta\phi = -2} \tag{5.49}$$

in the domain D.

The zero-stress boundary condition now reads

$$0 = \sigma_{zx}Y' - \sigma_{zy}X' = \mu\Omega(\partial_y\phi Y' + \partial_x\phi X') = \mu\Omega\phi' \quad \text{on } \partial D \tag{5.50}$$

$$\Rightarrow \qquad \phi = const = 0 \quad \text{on } \partial D \text{ without loss of generality} \tag{5.51}$$

Thus the Neumann problem for Ψ has now been converted into a Dirichlet problem for ϕ, which is easier to solve. For the torsional rigidity R we find:

$$R = \int_D dxdy\,(x\sigma_{zy} - y\sigma_{zx}) \tag{5.52}$$

$$= \mu\int_D dxdy\,\underbrace{(-x\partial_x\phi - y\partial_y\phi)}_{=-\boldsymbol{\nabla}(\phi\mathbf{r})+2\phi} = 2\mu\int_D dxdy\,\phi \tag{5.53}$$

In application to a circular bar this becomes:

$$\Delta\phi = \frac{1}{r}\partial_r(r\partial_r\phi) = -2, \qquad \phi = 0 \text{ for } r = a \tag{5.54}$$

$$\Rightarrow \qquad \phi = \frac{a^2 - r^2}{2} \qquad \Rightarrow \qquad R = 2\pi\mu\int_0^a (a^2 - r^2)rdr = \frac{\pi\mu a^4}{2} \tag{5.55}$$

We recover the same result as in the ansatz with the torsion function Ψ above.

5.8. Contact of two elastic spheres (Hertz solution 1881)

We consider two elastic spheres of radii R and R', which are pressed onto each other by a force F (see fig. 5.9). Their elastic constants are (E,ν) and (E',ν'), respectively. The undeformed spheres around the point of contact have the shapes

$$z = \kappa r^2, \qquad z' = \kappa' r^2 \tag{5.56}$$

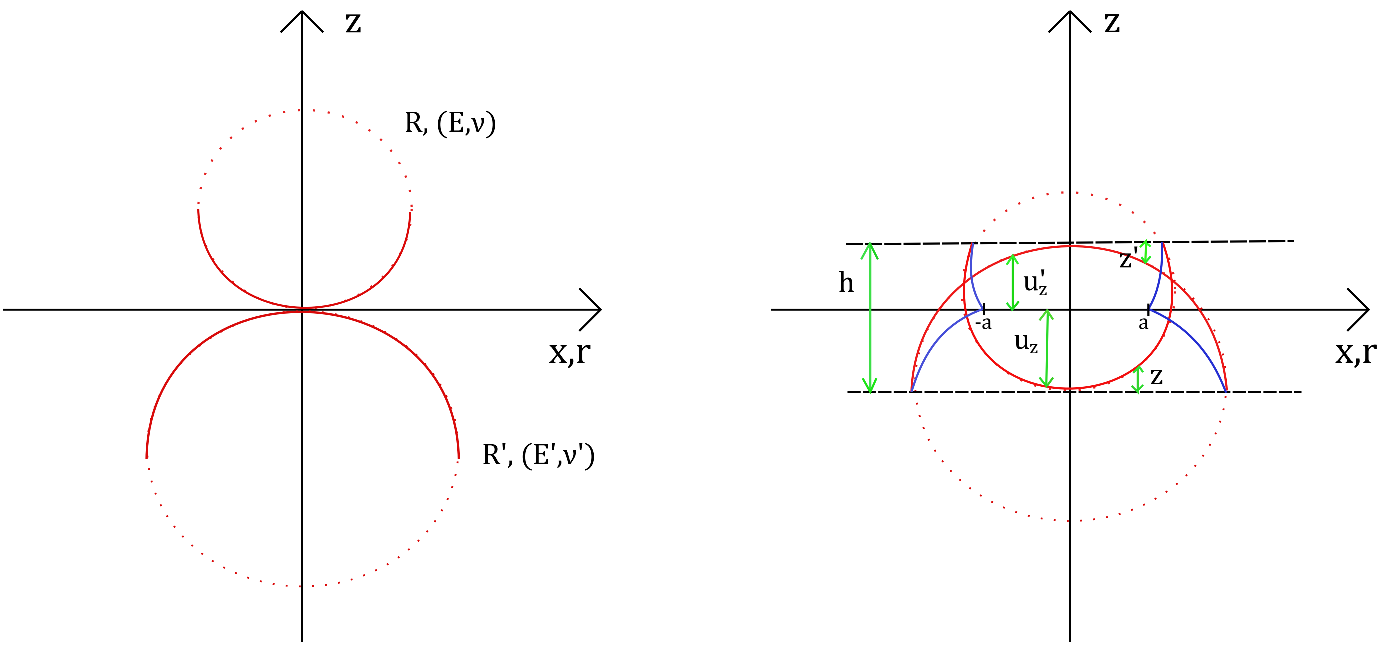


Figure 5.8.: The Hertz problem is a mixed boundary contact problem with restrictions on u and σ. Two elastic spheres are pressed onto each other by a force F. If they would not feel each other, the spheres would move into each other with indentation depth h. In reality, the spheres develop a contact area A, a circular disc of radius a. In general, the contact area does **not** need to be flat as ahown here. For $(R', E') \to \infty$, the problem becomes one of pressing an elastic ball onto a flat rigid substrate.

with $\kappa = \frac{1}{2R}$ the *mean curvature* and the radial distance $r^2 = x^2 + y^2$.

The indentation length is denoted by h. We then have (see Fig. 5.9)

$$(z + u_z) + (z' + u'_z) = h \tag{5.57}$$

$$\Rightarrow \quad \underbrace{(\kappa + \kappa')}_{\equiv A} r^2 + u_z + u'_z = h \tag{5.58}$$

within the contact area. For symmetry reasons, this has to be a circular disc of radius a.

We assume that only a normal stress $p_z(x, y)$ acts inside the contact area. The resulting displacement fields are obtained from the Green's function for a surface force acting on an elastic halfspace (*Boussinesq solution*):

$$u_z(x, y) = \frac{1 - \nu^2}{\pi E} \int dx' dy' \, \frac{p_z(x', y')}{s} \tag{5.59}$$

$$u'_z(x, y) = \frac{1 - \nu'^2}{\pi E'} \int dx' dy' \, \frac{p_z(x', y')}{s} \tag{5.60}$$

$$s = \left| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x' \\ y' \end{pmatrix} \right| = \sqrt{(x - x')^2 + (y - y')^2} \tag{5.61}$$

Note the $1/r$-relation typical for 3d LET. From (5.58) we now get

$$\underbrace{\frac{1}{\pi} \left(\frac{1 - \nu^2}{E} + \frac{1 - \nu'^2}{E'} \right)}_{\equiv \frac{4}{3} D} \int dx' dy' \, \frac{p_z(x', y')}{s} = h - A r^2 \tag{5.62}$$

This integral equation determines the stress distribution $p_z(x, y)$ in the contact area. We solve this by noting that it corresponds to the potential of a uniformly charged disc known from electrostatics:

$$\boxed{p_z(r) = \frac{3F}{2\pi a^2}\sqrt{1-\left(\frac{r}{a}\right)^2}} \qquad \text{Hertz-stress} \tag{5.63}$$

The stress is normalized such that

$$\int dx'dy'\, p_z(x',y') = F \tag{5.64}$$

The maximal stress (at $r = 0$) is

$$\frac{3F}{2\pi a^2} = \frac{3}{2} \underbrace{(\frac{F}{\pi a^2})}_{\text{average stress}} \tag{5.65}$$

Combining the expression for the Hertz-stress with the integral equation (equation 5.63) allows us to express indentation length h and contact area with radius a as a function of the force F (this requires more integrals from potential theory). One then finds:

$$\boxed{a^3 = FD\frac{RR'}{R+R'}, \qquad h^3 = F^2D^2(\frac{1}{R}+\frac{1}{R'})} \tag{5.66}$$

The *Hertz law* $a \sim F^{1/3}$ and the relation $h \sim F^{2/3}$ for a spherical indenter are the most famous results of contact mechanics.

From $-F = -\frac{\partial U}{\partial h}$ we obtain for the potential energy:

$$\boxed{U = \frac{2}{5}h^{5/2}\frac{1}{D}\left(\frac{RR'}{R+R'}\right)^{1/2}} \tag{5.67}$$

In the limit $R' \to \infty$: $a^3 = FDR$, $h^3 = F^2\frac{D^2}{R}$, $U = \frac{2}{5}h^{5/2}R^{1/2}/D$.

These relations are often used to measure the stiffness of a material, for example by placing a steel ball of radius R on a material with modulus E' ($E \to \infty$, $F = G = \frac{4\pi R^3}{3}\rho g$) or by indenting with an AFM. The scaling laws $a \sim F^{1/3}$ and $h \sim F^{2/3}$ hold true also for non-spherical indenters with finite curvature (but not for conical or flat indenters), compare Fig. 5.10. Here is a simple way to predict the scaling law for the spherical indenter (see Fig. 5.11):

$$U \sim VE\epsilon^2 \tag{5.68}$$

where V is the deformed volume and ϵ is the strain. The indentation length is usually the quantity monitored. From Fig. 5.11 we obtain:

$$R^2 = l^2 + (R-h)^2 \quad \Rightarrow \quad l \sim \sqrt{Rh} \tag{5.69}$$

$$\Rightarrow \quad V \sim l^3, \qquad \epsilon \sim \frac{h}{l} \tag{5.70}$$

Then, force and potential energy scale like

$$U \sim (Rh)^{3/2}E\left(\frac{h}{(Rh)^{1/2}}\right)^2 = E\cdot R^{1/2}\cdot h^{5/2} \tag{5.71}$$

$$F \sim \frac{\partial U}{\partial h} \sim E\cdot R^{1/2}\cdot h^{3/2} \tag{5.72}$$

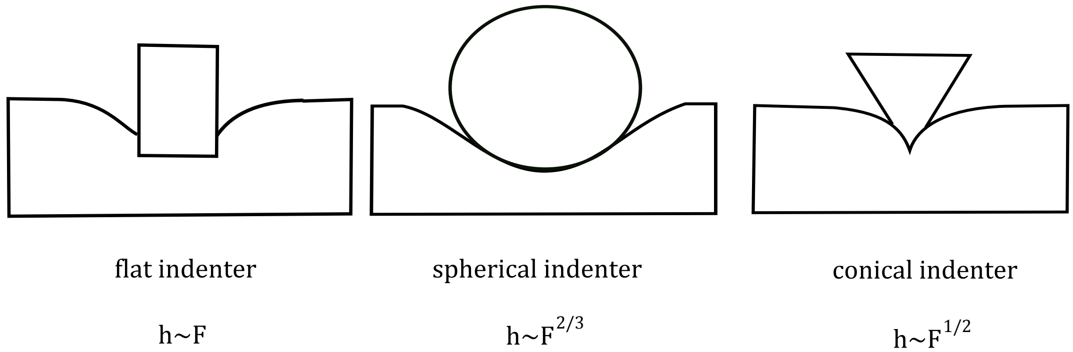


Figure 5.9.: A standard way to measure the rigidity of a material is by means of measuring the indentation length h as a function of force F.

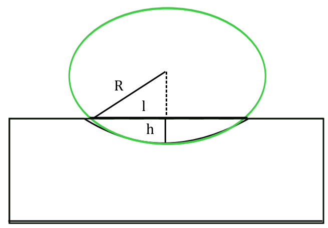


Figure 5.10.: Illustration of a spherical indenter.

5.9. Compatibility conditions

The steady Navier equation $\partial_j \sigma_{ij} + \rho g_i = 0$ can be regarded as 3 equations for the 3 displacment components u_i. However, the strain tensor ϵ_{ij} has 6 components, correspondong to 6 equations for $\mathbf{u}$ if σ_{ij} and the constitutive equations are known. Therefore ϵ_{ij} is actually overdetermined and has to satisfy 3 additional requirements , the 3 *compatibility conditions*. Only if these conditions are satisified, then ϵ_{ij} corresponds to a single-valued, and thus physically acceptable, displacement field.

We first consider a plain strain problem, that is $\mathbf{u} = (u_x(x,y), u_y(x,y), 0)$. If u_x and u_y are twice continuously differentiable single-valued functions:

$$\partial_x \partial_y u_x = \partial_y \underbrace{\partial_x u_x}_{\epsilon_{xx}}, \qquad \partial_x \underbrace{\partial_y u_y}_{\epsilon_{yy}} = \partial_y \partial_x u_y \tag{5.73}$$

$$\Rightarrow \quad \partial_y^2 \epsilon_{xx} + \partial_x^2 \epsilon_{yy} = \partial_y \partial_x (\partial_y u_x) + \partial_x \partial_y (\partial_x u_y) = \partial_x \partial_y (\partial_y u_x + \partial_x u_y) = 2 \partial_x \partial_y \epsilon_{xy} \tag{5.74}$$

$$\Rightarrow \quad \boxed{\partial_y^2 \epsilon_{xx} + \partial_x^2 \epsilon_{yy} - 2 \partial_x \partial_y \epsilon_{xy} = 0} \tag{5.75}$$

The same considerations for the (x,z)- and (y,z)-planes give

$$\partial_z^2 \epsilon_{yy} + \partial_y^2 \epsilon_{zz} - 2 \partial_y \partial_z \epsilon_{yz} = 0 \tag{5.76}$$

$$\partial_x^2 \epsilon_{zz} + \partial_z^2 \epsilon_{xx} - 2 \partial_z \partial_x \epsilon_{zx} = 0 \tag{5.77}$$

One can derive 3 more compatibility conditions, but only 3 out of the 6 are independent. The first equation from above (equation 5.76) in linear elasticity can be written in terms of stresses as

$$\boxed{\partial_y^2 \sigma_{xx} + \partial_x^2 \sigma_{yy} - 2 \partial_x \partial_y \sigma_{xy} = \frac{\nu}{1+\nu} (\partial_x^2 + \partial_y^2)\, \mathrm{tr}(\sigma)} \tag{5.78}$$

and similar expressions result from the other 5 compatibility conditions in strain.

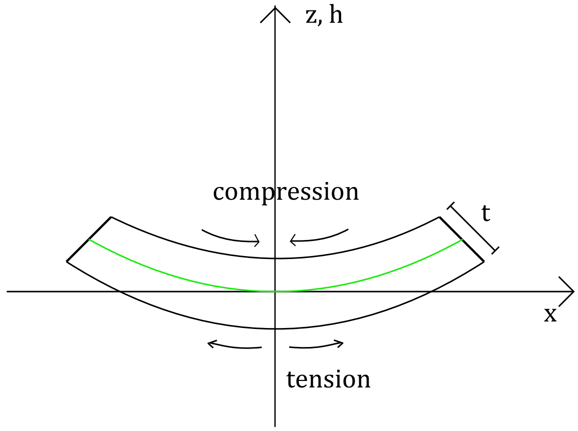


Figure 5.11.: Bending of a thin plate in z-direction. There must exist a plane with neither compression nor tension which defines a neutral surface.

5.10. Bending of a plate

We consider a weakly bent plate with the surface normal in z-direction and a thickness t which is much smaller than the lateral extension (see Fig. 5.12). The upper side is in compression and the lower side is in tension. No stresses thus exist in the *neutral* middle surface, which we describe by the height function h(x,y). To first order in h, the displacement vector for the middle surface is

$$\mathbf{u}^0 = \begin{pmatrix} 0 \\ 0 \\ h(x,y) \end{pmatrix} \qquad \text{Monge parametrization} \tag{5.79}$$

At the bottom and top surfaces we have the boundary conditions:

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \tag{5.80}$$

Because the plate is very thin, these equations must be valid also in its interior

$$\Rightarrow \qquad \sigma_{zx} = \frac{E}{1+\nu}\epsilon_{zx} = \frac{E}{2(1+\nu)}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \tag{5.81}$$

$$\sigma_{zy} = \frac{E}{1+\nu}\epsilon_{zy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0 \tag{5.82}$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{zz} + \nu(\epsilon_{xx} + \epsilon_{yy})\right] = 0 \tag{5.83}$$

Since

$$\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{ll}\delta_{ij}\right] \tag{5.84}$$

the inverted expression gives the strain tensor

$$\sigma_{ij} = \frac{E}{(1+\nu)}\epsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\delta_{ij}\epsilon_{ll} \tag{5.85}$$

and with

$$\frac{\partial u_x}{\partial z} = -\frac{\partial u_z}{\partial x} = -\partial_x h \qquad \Rightarrow \qquad u_x = -z\partial_x h \tag{5.86}$$

$$\frac{\partial u_y}{\partial z} = -\frac{\partial u_z}{\partial y} = -\partial_y h \qquad \Rightarrow \qquad u_y = -z\partial_y h \tag{5.87}$$

$$\epsilon_{zz} = -\frac{\nu}{(1-\nu)}(\epsilon_{xx} + \epsilon_{yy}) \tag{5.88}$$

we now know the strain tensor:

$$\epsilon_{ij} = \begin{pmatrix} -z\partial_x^2 h & -z\partial_x\partial_y h & 0 \\ -z\partial_x\partial_y h & -z\partial_y^2 h & 0 \\ 0 & 0 & \frac{z\nu}{(1-\nu)}(\partial_x^2 h + \partial_y^2 h) \end{pmatrix} \tag{5.89}$$

Note that for ϵ_{31} we used, to first order in h, that

$$\epsilon_{31} = \frac{1}{2}\left(\partial_x u_z + \partial_z u_x\right) = \frac{1}{2}\left(\partial_x h - \partial_x h\right) = 0 \tag{5.90}$$

The strain energy density w leads to the bending energy of the plate:

$$w = \frac{E}{2(1+\nu)}(\epsilon_{ij}^2 + \frac{\nu}{1-2\nu}\epsilon_{ll}^2) \tag{5.91}$$

$$= z^2 \frac{E}{(1+\nu)}\left[\frac{2}{(1-\nu)}(\underbrace{\frac{1}{2}(\partial_x^2 h + \partial_y^2 h)}_{\equiv H \text{ mean curvature}})^2 - \underbrace{(\partial_x^2 h \partial_y^2 h - (\partial_x\partial_y h)^2)}_{\equiv K \text{ Gaussian curvature}}\right] \tag{5.92}$$

Mean curvature and Gaussian curvature are definitions from differential geometry

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \qquad \text{mean curvature} \tag{5.93}$$

$$K = \kappa_1 \cdot \kappa_2 \qquad \text{Gaussian curvature} \tag{5.94}$$

where κ_1 and κ_2 are the principal curvatures of the surface. These geometric quantities are fundamental in all theories of plates and shells.

For a thin plate, we intend to integrate out the z-component and obtain for the net strain energy:

$$U = \int_{-t/2}^{t/2} dz \int dxdy\, w \tag{5.95}$$

$$= \underbrace{2\kappa \int dxdy\, H^2}_{\text{bending energy}} + \underbrace{\tilde{\kappa} \int dxdy\, K}_{= \, const, \text{ due to Gauss-Bonnet theorem}} \tag{5.96}$$

The *bending stiffness* κ is then given by

$$\kappa = \frac{E}{(1-\nu^2)} \int_{-t/2}^{t/2} dz\, z^2 = \boxed{\frac{Et^3}{12(1-\nu^2)} = \kappa} \tag{5.97}$$

Again, the bending rigidity (in units of energy) shows a strong dependence on geometry, in terms of the thickness, of the material. $\tilde{\kappa} = \kappa(1-\nu)$ is another curvature elastic constant of interest, called the *splay modulus*.

Minimizing U a a functional of h using the calculus of variation is non-trivial but gives a clear result:

$$\kappa \boldsymbol{\nabla}^4 h = -\rho t g \qquad \text{biharmonic equation} \tag{5.98}$$

In addition to derivatives of 4^{th} order, the boundary conditions for the plate problem with body forces are non-trivial and simplest for the clamped case ($h = \partial_n h = 0$ at the rim).

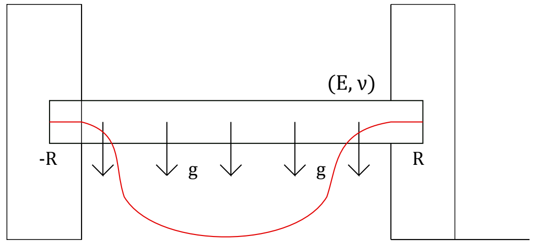


Figure 5.12.: The plate is clamped at both sides and gravity pulling everywhere lets it sag down.

Example: Circular plate with gravity

For example, in order to describe the sagging of an elastic disc of radius R under gravity with clamped boundary conditions, (5.98) reads

$$\nabla^4 h = \left(\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})\right)^2 h = 64\beta \text{ with } \beta = \frac{3\rho g(1-\nu^2)}{16t^2 E} \tag{5.99}$$

$$\Rightarrow \quad h = \beta r^4 + ar^2 + b + cr^2\ln(\frac{r}{R}) + d\ln(\frac{r}{R}) \tag{5.100}$$

We make a polynomial ansatz for the height function $h(r)$ in cylindrical polar coordinates and demand $d = c = 0$ to avoid singularities. The constants a and b are calculated from the boundary conditions $h = \partial_r h = 0$ at $r = R$ and we find that

$$h = \beta(R^2 - r^2)^2 \tag{5.101}$$

The height of the clamped plate varies parabolically in r (compare Fig. 5.13). A **technical application** of the mechanics of bending a plate is, for example, the manufacture of curved wind screens. A glass plate is heated with an inhomogeneous temperature field to achieve the required sag.

5.11. Bending of a rod

Like for the plate, we have tension on one side, compression on the other, and a neutral surface inbetween. For weak bending, torsion is a higher order effect and can be neglected. We choose the z-axis for the long axis of the rod. In this case

$$n_z = 0 \quad \Rightarrow \quad \sigma_{ix}n_x + \sigma_{iy}n_y = 0 \tag{5.102}$$

for example, for $i = x$. At point P we have

$$n_y = 0 \quad \Rightarrow \quad \sigma_{xx} = 0 \tag{5.103}$$

everywhere since the rod is thin. σ_{ij} vanishes except for σ_{zz} and we only have tension or compression along the z-axis. Basically this is a stretch experiment.

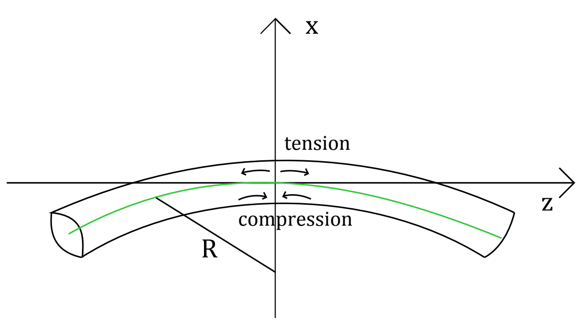


Figure 5.13.: Bending of a rod. Basically this is a stretch experiment with tension for $x > 0$ and compression for $x < 0$.

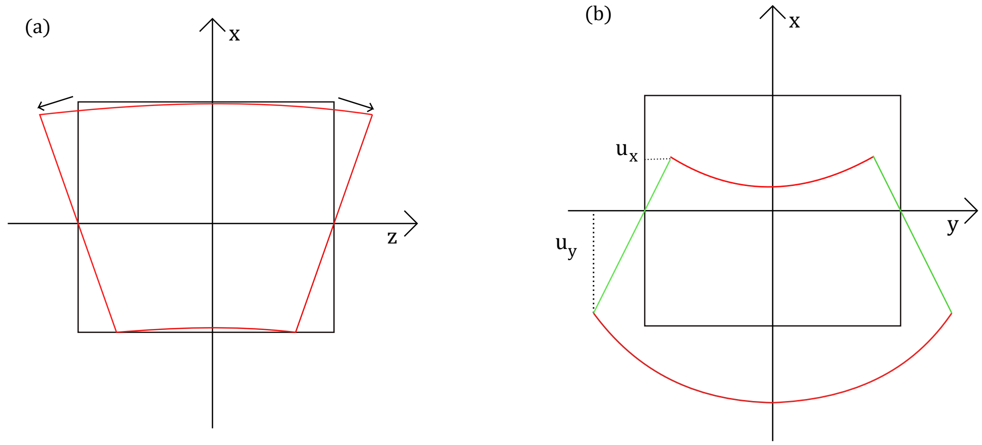


Figure 5.14.: (a) View from the side. (b) Cut through the rod. The sides of the rectangular cross-section are tilted with linear dependence in x-direction, but remain planar, whereas the upper and lower planes deform parabolically.

For the stretch in x-direction we get

$$\frac{dz'}{dz} = \frac{2\pi(R+x)}{2\pi R} = (1+\frac{x}{R}) \quad \Rightarrow \quad \frac{dz'-dz}{dz} = \epsilon_{zz} = \frac{x}{R} \tag{5.104}$$

We then find for the displacement field and stress tensor:

$$\epsilon = \begin{pmatrix} -\nu\frac{x}{R} & \frac{(-\nu y+\nu y)}{R} = 0 & \frac{(z-z)}{R} = 0 \\ 0 & -\nu\frac{x}{R} & 0 \\ 0 & 0 & \frac{x}{R} \end{pmatrix} \qquad \mathbf{u} = \frac{1}{R}\begin{pmatrix} -\frac{1}{2}(z^2+\nu(x^2-y^2)) \\ -\nu xy \\ xz \end{pmatrix} \tag{5.105}$$

where we used the same intermediate steps as in section 5.10 for the bending of a plate, respectively

$$\sigma_{zz} = E\frac{x}{R} \tag{5.106}$$

$$\epsilon_{xx} = \epsilon_{yy} = -\nu\epsilon_{zz} = -\nu\frac{x}{R} \quad \Rightarrow \quad \partial_x u_x = \partial_y u_y = -\nu\frac{x}{R} \tag{5.107}$$

A cross-section at constant $z = z_0$ has $u_z = z_0\frac{x}{R}$. It stays planar but is rotated (except at the origin), as shown in Fig. 5.15a. However, the shape of the cross-section is changed as shown in Fig. 5.15b. For example for a rectangular cross-section at $y = \pm y_0$ the two sides stay planar, but are rotated:

$$u_y = \frac{-\nu y_0}{R}x \tag{5.108}$$

At $x = \pm x_0$, the top and bottom sides become parabolic:

$$u_x = \frac{-1}{2R}(z_0^2 + \nu(x_0^2 - y^2)) \tag{5.109}$$

From the strain energy density w we now obtain a strain energy per length $\frac{U}{L}$:

$$w = \frac{\sigma_{ik}\epsilon_{ik}}{2} = \frac{\sigma_{zz}\epsilon_{zz}}{2} = \frac{Ex^2}{2R^2} \tag{5.110}$$

$$\Rightarrow \quad \frac{U}{L} = \frac{E}{2R^2}\underbrace{\int x^2 dA}_{\equiv I_y} \tag{5.111}$$

where I_y is the moment of inertia with respect to the y-axis. For a rectangular cross-section with dimensions a and b in x- and y-directions, respectively, we have $I_y = \frac{a^3 b}{12}$. For a circular cross-section we have $I_y = \frac{\pi R^4}{4}$. For the whole rod we now have

$$U = \frac{EI}{2}\int ds \frac{1}{R^2} = \frac{EI}{2}\int ds \left(\frac{d^2\mathbf{r}}{dz^2}\right) \tag{5.112}$$

where I is the moment of inertia with regard to the axis around which we bend. This is the basis of the *worm-like chain* model for polymers. $\kappa = E \cdot I$ is called the bending stiffness which is related to the *persistence length* l_p, i.e. the length on which the rod stays bend, by $\kappa = l_p \cdot k_B T$.

For bending in a plane, $\mathbf{r} = (x, y, 0)$ and the corresponding Euler-Lagrange equation gives

$$\frac{EI}{2}\left(\frac{d^2}{ds^2}\frac{\partial \mathcal{L}}{\partial X''}\right) = \boxed{EIX'''' = K_x} \tag{5.113}$$

$$\frac{EI}{2}\left(\frac{d^2}{ds^2}\frac{\partial \mathcal{L}}{\partial Y''}\right) = \boxed{EIY'''' = K_y} \tag{5.114}$$

where K_x, K_y are external forces per length. Thus for weakly bent rods, we have to solve a differential equation of 4^{th} order, like for weakly bent plates.

Example 1:

An initially horizontal rod is clamped at $s = 0$ and free at $s = L$. How does it deform under its own weight?

$$Y'''' = \frac{g}{EI} \quad \Rightarrow \quad Y = \frac{g}{24EI}\cdot s^2(s^2 - 4Ls + 6L^2) \quad \Rightarrow \quad Y(L) = \frac{gL^4}{8EI} \tag{5.115}$$

Again we encounter a strong dependence on geometry. Cross-checking the solution depicted in Fig. 5.16 gives the correct results:

$$Y'''' = \frac{g}{EI} \qquad Y(0) = Y'(0) = 0 \tag{5.116}$$

$$\text{moment} \qquad M_x(L) = -EIY''(L) = 0 \tag{5.117}$$

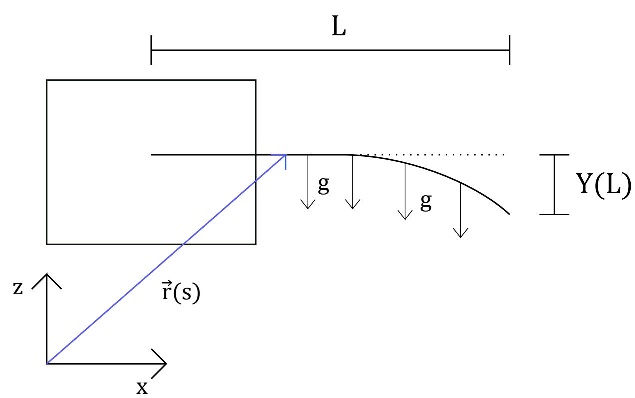


Figure 5.15.: Clamped rod subject to gravity.

Example 2:

Now the rod is deformed by a point force F at its free end:

$$Y = \frac{F}{6EI}\cdot s^2(3L - s) \quad \Rightarrow \quad Y(L) = \frac{FL^3}{3EI} = \frac{4FL^3}{3\pi ER^4} \tag{5.118}$$

where the last formula is valid for a spherical cross-section with radius R. The proportionality factor between force and deformation is the spring constant of the rod, given by

$$k = \frac{3\pi E R^4}{4L^3} \tag{5.119}$$

where the rod radius R enters to 4^{th} power. Cross-checking the solution gives

$$Y'''' = 0, \qquad Y(0) = Y'(0) = 0 \tag{5.120}$$

$$M_x(L) = 0, \qquad F_y = -EIY''' = -F \tag{5.121}$$

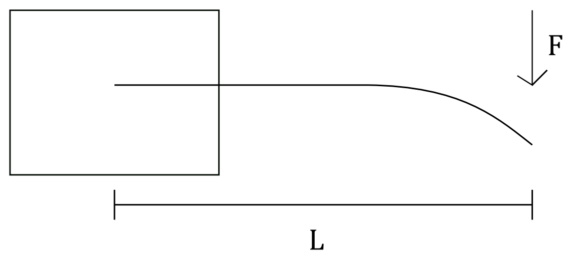


Figure 5.16.: Clamped rod subject to a point force at its end.

CHAPTER 6

The finite element method

Continuum mechanics leads to partial differential equations (PDEs), and in particular elasticity theory leadds to the Navier-Cauchy equations (like fluid dynamics leads to the Navier-Stokes equations). Many different techniques have been developed to solve PDEs numerically, including

- finite differences
- finite volumes
- finite element method (FEM)
- boundary element method
- spectral methods

In contrast to ordinary differential equations (ODEs), there is no general mathematical theory for the solvability of PDEs. Rather different numerical schemes have been developed for different classes of PDEs. In each case, one has to check for which assumption a solution exists, if it is unique, and how it depends on the parameters (existence, uniqueness, robustness). Here we will focus on the FEM, which is the standard tool in many applications. FEM combines different aspects, but the two most important ones are usage of the weak formulation, which converts the differential equation into an integral equation, and transformation of this integral equation into an algebraic equation by use of elements and shape functions.

6.1. Classification of PDEs

A linear PDE of second order has the form ($a_{ij} = a_{ji}$ since $\partial_j \partial_i u = \partial_i \partial_j u$):

$$Lu = -a_{ij}\partial_j\partial_i u + b_i \partial_i u + cu = f \tag{6.1}$$

It is called

(a) elliptic if all eigenvalues of $\mathbf{a}$ are non-zero and have the same sign. For example: Laplace equation

$$\boxed{\Delta u = 0} \qquad \mathbf{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.2}$$

(b) hyperbolic if all eigenvalues are non-zero, $n-1$ have the same sign and the remaining one the opposite sign. For example: Wave equation

$$\boxed{\partial_t^2 u = \Delta u} \qquad \mathbf{a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.3}$$

(c) parabolic if one eigenvalue is zero and the remaining ones have the same sign. For example: Heat equation

$$\boxed{\partial_t u = \Delta u} \qquad \mathbf{a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.4}$$

These names are taken from the case $n = 2$ when the quadratic form $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = 0$ describes an ellipse, a hyperbola or a parabola. For each of these types and depending on boundary conditions, theorems on existence and uniqueness and robustness can be derived. However, for practical purposes it is important to note that also functions that do not have the derivatives required for a rigorous solution of a given PDE qualify as reasonable solutions. It is these *weak solutions* which are obtained with the FEM. Below we will choose the elliptical 1d equation (actually an ODE)

$$\frac{d}{dx}(c\frac{du}{dx}) + f = 0 \tag{6.5}$$

to introduce the FEM. Here c could have a space-dependance, $c = c(x)$. Note that this equation occurs in many different physical situations. Later we generalise to 3d and PDEs.

diffusion equation	u concentration c diffusion constant f particle production
heat conduction (steady state)	u temperature c thermal conductivity f heat source term
mechanics of a bar	u displacement $c = E \cdot A$ f body force

6.2. The weak form

Rather than solving the PDE directly (*strong form*), we transform it into an integral equation by multiplying with a weighting function $w(x)$ and integrating over the domain $[a, b]$:

$$I = \int_a^b w \underbrace{\left[\frac{d}{dx}(c\frac{du}{dx}) + f\right]}_{\text{residual } R(x)} dx = 0 \tag{6.6}$$

This has to hold for **all** weighting functions $w(x)$. Then it is also true for $w(x) = R(x)$ and thus from the requirement $I = 0 \ \forall w$ it follows that $R = 0$. The reverse direction is obviously treu as well.

Integration by parts gives:

$$I = w \underbrace{c\frac{du}{dx}\bigg|_a^b}_{\substack{= J_b - J_a \equiv B, \\ \text{with flux } J = c\frac{du}{dx}}} - \int_a^b \frac{dw}{dx} c \frac{du}{dx} dx + \int_a^b wf\, dx \tag{6.7}$$

$$\Rightarrow \quad \boxed{\int_a^b \frac{dw}{dx} c \frac{du}{dx} dx = \int_a^b wf\, dx + B} \tag{6.8}$$

The *weak form* only involves first order derivatives. This formulation forces the residual to vanish in a spatially averaged sense. Mathematically the weak formulation corresponds to the introduction of a scalar product in a *Sobolev space*. The main idea of FEM is to solve the weak problem in a finite dimensional subspace. The *Lax-Milgram theorem* then ensures solvability in the subspace. For a grid size h going to zero, this solution converges to the full solution. As we will see below, the discretized version can be solved algebraically (by matrix inversion).

6.3. Shape functions

We discretize the problem by introducing n *nodes* x_i ($1 \leq i \leq n$) in the domain. The polynomial approximation to u then is

$$u_h(x) = \sum_{i=0}^{n-1} a_i x^i \tag{6.9}$$

where h represents the grid spacing. The coefficients a_i follow from solving

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ & & & \cdot & \\ & & & \cdot & \\ & & & \cdot & \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} \tag{6.10}$$

The a_i depend linearly on the u_i

$$\Rightarrow \quad u_h(x) = \sum_{i=1}^{n} N_i(x) u_i \tag{6.11}$$

The *shape functions* $N_i(x)$ are polynomial expressions of order $n-1$ in x.

Example: n=2

$$\Rightarrow \quad u_h(x) = N_1(x)u_1 + N_2(x)u_2 \tag{6.12}$$

with

$$N_1(x) = 1 - \frac{x - x_1}{x_2 - x_1} \qquad N_2(x) = \frac{x - x_1}{x_2 - x_1} \tag{6.13}$$

6.4. Galerkin approximation

We now transform the weak form into a linear set of equations. We divide the domain Ω into N_{el} subdomains Ω_e (*elements*). Within each element both the weighting function $w(x)$ and the unknown functin $u(x)$ are

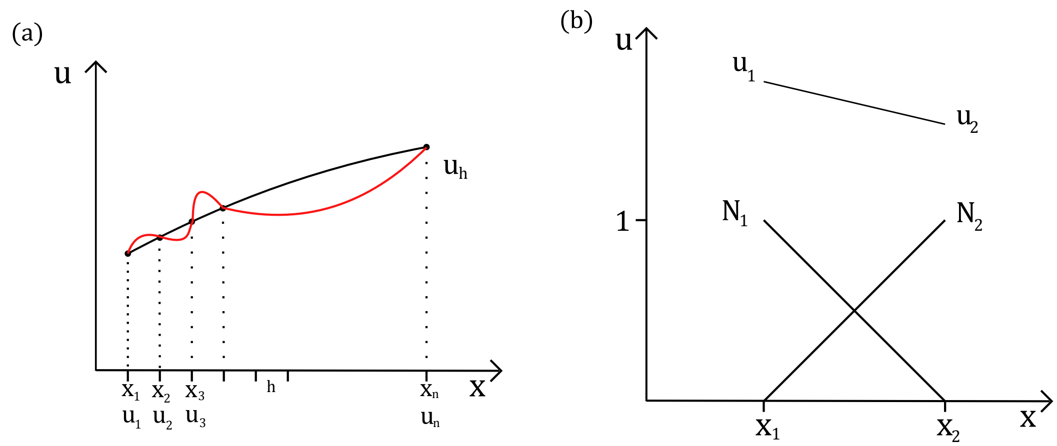


Figure 6.1.: (a) We introduce n nodes x_i with typcial grid spacing h, not necessarily equidistant. The polynomial approximation for $u_h(x)$ is exact at the grid points, but can deviate inbetween. (b) Shape functions in the simple case of two nodes.

approximated by a polynomial in Ω_e:

$$u_h^e(x) = \sum_{i=1}^{n} N_i(x) u_i^e = \mathbf{N}(x) \cdot \mathbf{u}_e, \qquad w_h^e(x) = \sum_{i=1}^{n} N_i(x) w_i^e = \mathbf{N}(x) \cdot \mathbf{w}_e \tag{6.14}$$

$$\Rightarrow \quad \frac{du_h^e}{dx} = \frac{d\mathbf{N}}{dx} \cdot \mathbf{u}_e, \qquad \frac{dw_h^e}{dx} = \frac{d\mathbf{N}}{dx} \cdot \mathbf{w}_e \tag{6.15}$$

$$\Rightarrow \quad \int_{\Omega_e} \frac{dw_h^e}{dx} c \frac{du_h^e}{dx} dx = \int_{\Omega_e} \frac{d\mathbf{N}}{dx} \cdot \mathbf{w}_e \, c \frac{d\mathbf{N}}{dx} \cdot \mathbf{u}_e \, dx \tag{6.16}$$

$$= \mathbf{w}_e \cdot \underbrace{\left(\int_{\Omega_e} \frac{d\mathbf{N}}{dx} \otimes \frac{d\mathbf{N}}{dx} c \, dx \right)}_{\equiv \mathbf{K}_e} \cdot \mathbf{u}_e \tag{6.17}$$

where $\mathbf{K}_e$ is the *stiffness matrix* of element e (can be calculated by numerical integration with trapezoidal or Gaussian rules). For the last term in (6.7) we define the *loading vector* of element e:

$$\mathbf{f}_e = \int_{\Omega_e} \mathbf{N} \, f \, dx \qquad \Rightarrow \qquad \int_{\Omega_e} w_h^e \, f \, dx = \mathbf{w}_e \cdot \mathbf{f}_e \tag{6.18}$$

We now *assemble* the local elements into a global stiffness matrix $\mathbf{K}$ and a global loading vector $\mathbf{f}$: For example for 4 nodes we have 3 elements:

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 \tag{6.19}$$

$$\mathbf{K}_1 = \begin{pmatrix} x & x & 0 & 0 \\ x & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{pmatrix} \tag{6.20}$$

$$\mathbf{f} = \mathbf{f}_{int} + f_{ext} \tag{6.21}$$

with

$$\mathbf{f}_{ext} \equiv \begin{pmatrix} -J_a \\ 0 \\ 0 \\ J_b \end{pmatrix} \tag{6.22}$$

the *external loading vector* specifying the flux condition such that B in the integral equation (6.8) can be written as

$$B = \mathbf{w} \cdot \mathbf{f}_{ext} \tag{6.23}$$

and

$$\mathbf{f}_{int} \equiv \sum_{e=1}^{N_{el}} f_e \tag{6.24}$$

the *internal loading vector*.
All in all,

$$\mathbf{w} \cdot \mathbf{K} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{f} \tag{6.25}$$

and because this must hold for all $\mathbf{w}$, we finally have

$$\boxed{\mathbf{K} \cdot \mathbf{u} = \mathbf{f}} \tag{6.26}$$

The final step depends on the boundary conditions. As an example, let us prescribe u for $x = a$ and J for $x = b$. We label the known part of $\mathbf{u}$ as $\mathbf{u}_p$. The remaining part $\mathbf{u}_u$ is the unknown one.

$$\Rightarrow \quad \begin{pmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{pmatrix} \begin{pmatrix} \mathbf{u}_u \\ \mathbf{u}_p \end{pmatrix} = \begin{pmatrix} \mathbf{f}_u \\ \mathbf{f}_p \end{pmatrix} \tag{6.27}$$

$\mathbf{f}_u$ will be known and thus we can solve

$$\mathbf{K}_{uu} \cdot \mathbf{u}_u = \mathbf{f}_u - \mathbf{K}_{up} \cdot \mathbf{u}_p \tag{6.28}$$

for $\mathbf{u}_u$. Then we can calculate

$$\mathbf{f}_p = \mathbf{K}_{pu} \cdot \mathbf{u}_u + \mathbf{K}_{pp} \mathbf{u}_p \tag{6.29}$$

because $\mathbf{u}_p$ is known. With this, the problem is completely solved! In summary, the objective of a FEM-program is to compute the coefficient or stiffness matrix $\mathbf{K}$ and the loading vector $\mathbf{f}$, and to solve the resulting system of algebraic equations taking the boundary conditions into account.

6.5. FEM for 3d LET

We now deal with a PDE in 3d:

$$\boldsymbol{\nabla} \cdot \sigma + \mathbf{f} = \mathbf{0} \qquad \text{steady Cauchy equation} \tag{6.30}$$

For the weak formulation, the weight function now has to be a vector field $\mathbf{w}(\mathbf{x})$:

$$\int_\Omega \mathbf{w} \cdot (\boldsymbol{\nabla} \cdot \sigma + \mathbf{f})\, dV = 0 \tag{6.31}$$

We integrate by parts using

$$\partial_i(\sigma_{ij} w_j) = (\partial_i \sigma_{ij}) w_j + \sigma_{ij}(\partial_i w_j) \tag{6.32}$$

$$\Rightarrow \quad \int_\Omega \boldsymbol{\nabla} \cdot (\sigma \cdot \mathbf{w}) dV - \int_\Omega \sigma : (\boldsymbol{\nabla} \otimes \mathbf{w})^T dV + \int_\Omega \mathbf{w} \cdot \mathbf{f} dV = 0 \tag{6.33}$$

with

$$\mathbf{A} : \mathbf{B} \equiv \mathrm{tr}(\mathbf{A} \cdot \mathbf{B}) = A_{ij} B_{ji} \qquad \text{double dot product} \tag{6.34}$$

By the divergence theorem we get

$$\int_\Omega (\boldsymbol{\nabla} \otimes \mathbf{w})^T : \sigma\, dV = \int_{\partial\Omega} \mathbf{w} \cdot \sigma \cdot dA + \int_\Omega \mathbf{w} \cdot \mathbf{f}\, dV \tag{6.35}$$

The LHS can be simplyfied: We split the dyadic product $(\boldsymbol{\nabla}\otimes\mathbf{w})^T$ into a symmetric and a skew-symmetric part:

$$(\boldsymbol{\nabla}\otimes\mathbf{w})^T = \underbrace{\frac{1}{2}\left[(\boldsymbol{\nabla}\otimes\mathbf{w}) + (\boldsymbol{\nabla}\otimes\mathbf{w})^T\right]}_{\equiv\epsilon^w} - \frac{1}{2}\left[(\boldsymbol{\nabla}\otimes\mathbf{w}) - (\boldsymbol{\nabla}\otimes\mathbf{w})^T\right] \tag{6.36}$$

Because σ is symmetric, only the symmetric part ϵ^w contributes to the double dot product:

$$(\boldsymbol{\nabla}\otimes\mathbf{w})^T : \sigma = \epsilon^w : \sigma \tag{6.37}$$

Because it is linear in the tensor components, it also can be written as the dot product of two appropriately defined vectors:

$$\epsilon^w : \sigma = \epsilon^w \cdot \sigma \tag{6.38}$$

The same applies for the constitutive equation, which we also write with vectors:

$$\sigma = \mathbf{H}\,\epsilon \tag{6.39}$$

where ϵ is an appropriate definition following from the strain tensor ϵ.

$$\Rightarrow \qquad \epsilon^w \cdot \sigma = \epsilon^w\,\mathbf{H}\,\epsilon \tag{6.40}$$

We also have

$$\epsilon = \mathbf{B}\,\mathbf{u}, \qquad \epsilon^w = \mathbf{B}\,\mathbf{w} \tag{6.41}$$

where $\mathbf{B}$ is a differential operator and $\mathbf{u}$ the displacement field. Expressing $\mathbf{u}$ and $\mathbf{w}$ by shape functions, $\mathbf{B}$ becomes the *strain displacement matrix* with entries of the type $\frac{\partial N_i}{\partial x_j}$.

$$\Rightarrow \qquad \int_\Omega (\boldsymbol{\nabla}\otimes\mathbf{w})^T : \sigma\,dV = \mathbf{w}\cdot \underbrace{\int_\Omega \mathbf{B}^T\mathbf{H}\,\mathbf{B}\,dV}_{\equiv\,\mathbf{K}\text{ stiffness matrix}} \cdot\mathbf{u} \tag{6.42}$$

With an appropriately defined loading vector $\mathbf{f}$, we finally have the same result as before:

$$\mathbf{w}\,\mathbf{K}\,\mathbf{u} = \mathbf{w}\cdot\mathbf{f} \quad \overset{\forall\mathbf{w}}{\Rightarrow} \quad \boxed{\mathbf{K}\,\mathbf{u} = \mathbf{f}} \tag{6.43}$$

which can be solved as explained earlier for given boundary conditions.

6.6. Software for FEM

The standard GUI-based commercial software for solving FEM-problems is *Comsol Multiphysics*. 2d problems (plain strain, plain stress) can also be solved with the PDE toolbox in *Matlab*. Basic PDE-tools are also provided by *Mathematica*. Very powerful but complicated commercial FEM-software used for industrial applications (for example in the automobile or airplane industries) are for example *Abaqus*, *Ansys* and *Adina*. There exist several non-commercial (sometimes open source) FEM-packages, mainly from academic groups in applied math. At Heidelberg, this includes *deal.II* and *Dune*. A very good open source choice is *FEniCS* (https://fenicsproject.org), which can be run from python scripts that start directly from the weak form. The FEniCS-tutorial is a good introduction into FEM.

APPENDIX A

Overview of tensors in elasticity theory

$F_{ij} = \frac{\partial x_i}{\partial x_{0j}} = \delta_{ij} + \frac{\partial u_i}{\partial x_{0j}}$	deformation gradient tensor
$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$	right Cauchy-Green deformation tensor
$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$	left Cauchy-Green deformation tensor (aka Finger tensor)
$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbb{1})$	Green-Lagrange strain tensor
$\mathbf{A} = \frac{1}{2}(\mathbb{1} - \mathbf{B}^{-1})$	Almansi-Euler strain tensor
$\epsilon = \frac{1}{2}(\mathbf{F}^T + \mathbf{F} - 2\mathbb{1})$	linear strain tensor
σ	Cauchy stress tensor
C_{ijkl}	tensor of elastic moduli
$\mathbf{P} = det(\mathbf{F})\sigma(\mathbf{F}^T)^{-1}$	1st Piola-Kirchhoff stress tensor
$\mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$	2nd Piola-Kirchhoff stress tensor