Quantum field theory in curved spacetime

Assignment 2 – May 5

Exercise 4: Conformally coupled scalar field

Motivation: In the lecture, we saw that nonminimally coupled scalar fields are not produced in an FLRW background if they are conformally coupled. But why those specific coupling values? Here, we'll work out precisely what conformal coupling means.

Consider a scalar field ϕ non-minimally coupled with gravity. It is described by the following action

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\xi}{2}R\phi^2\right). \tag{4.1}$$

(a) Consider the conformal transformation

$$g_{\mu\nu}(x,t) \to \Omega^2(x,t)g_{\mu\nu}(x,t), \tag{4.2}$$

$$\phi(x,t) \to \Omega^{-1}(x,t)\phi(x,t). \tag{4.3}$$

Calculate the value of ξ under which the action is invariant under this conformal transformation (possibly up to a total divergence).

(b) Show that the energy-momentum tensor $T^{\mu\nu}$ is expressed as

$$\begin{aligned} T^{\mu\nu} &= \nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi + \frac{1}{2}g^{\mu\nu}m^2\phi^2 - \xi\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\phi^2 \\ &+ \xi\left[g^{\mu\nu}\nabla^\alpha\nabla_\alpha(\phi^2) - \nabla^\mu\nabla^\nu(\phi^2)\right]. \end{aligned} \tag{4.4}$$

(c) Show that $T^\nu{}_\nu = 0$ when $m = 0$ and $\xi = 1/6$.

Exercise 5: Electromagnetic fields on curved backgrounds

Motivation: Non-conformally coupled scalars are copiously produced in FLRW spacetimes. But how about photons? In other words, is the universe covered in "light" of horizon wavelength?

The Maxwell action on a curved background reads

$$S = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{4.5}$$

with the field-strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \tag{4.6}$$

and the gauge field A_μ. Instead of going through the whole derivation of particle creation again, we take a shortcut.

(a) **Try to do this sub-exercise before reading the remaining ones.**

Work smarter, not harder. Before doing any calculation, think it through: Should Maxwell theory be Weyl invariant in curved spacetime? Why? What does this tell us about photon production in an FLRW background?

Let's now work through the details step by step.

(b) The gauge field transforms trivially under Weyl transformations, i.e.$A_\mu \to A_\mu$. Calculate how the Maxwell action transforms under Weyl rescalings. Is it invariant?

(c) If we rescale the FLRW metric to remove the scale factor, how do Maxwell's equations change? What does this mean for the electromagnetic vacuum in an FLRW background? Is the universe covered in "light" of horizon wave length?

Finally, let's explore how Weyl invariance shows up in the structure of the energy-momentum tensor.

(d) Compute the energy-momentum tensor $T^{\mu\nu}$ of the gauge field and show that $T^\mu_{\ \mu} = 0$.

(e) **Bonus exercise:** What does the tracelessness of the energy-momentum tensor have to do with Weyl invariance? (**Hint:** How does the matter Lagrangian change under a small Weyl rescaling? What would this imply for $T^\mu_{\ \mu}$?)

Exercise 6: Impact of general nonminimal coupling on particle production

Motivation: Last week, we found that minimally coupled scalars are generically produced in FLRW backgrounds. This does not happen for conformally coupled scalars. Here we estimate what happens for general nonminimal coupling.

Consider a nonminimally coupled massive scalar whose action is given by Eq. (4.1).

(a) Compute the equation of motion for the scalar in an FLRW background, and redefine the field $\phi \to \chi = a(\eta)\phi$ such that the friction term ($\sim \phi'$) disappears. You should obtain that the nonminimal coupling gives you an additional contribution to the effective mass.

Sanity check: What happens in the limit $\xi \to 1/6$, $m \to 0$?

(b) Assume that the background is changing slowly and consider modes with small wavelength. Find out when these two assumptions are actually equivalent.

(c) Start in the adiabatic vacuum at some conformal time $\eta = \eta_0$, namely $|0_{\mathrm{ad},\eta_0}\rangle$, and look at the resulting state at a time $\eta = \eta_0 + \Delta\eta$. Try to get at some qualitative properties of the average particle-number density $\langle 0_{\mathrm{ad},\eta_0}|n_k|0_{\mathrm{ad},\eta_0}\rangle(\eta_0 + \Delta\eta)$ without calculating it. Sketch how you expect the particle-number density to depend on ξ. (**Hint:** Keep in mind that particles are not produced if the background is static, and that in terms of some complex time-dependent Bogolyubov parameter $\beta_k(\eta)$

$$\langle 0_{\mathrm{ad},\eta_0}|n_k|0_{\mathrm{ad},\eta_0}\rangle(\eta) = |\beta_k(\eta)|^2. \tag{4.7}$$

You can find inspiration in exercise 1.)