Quantum field theory in curved spacetime

Assignment 7 – June 16

Exercise 16: Euler-Heisenberg Lagrangian

Motivation: In semi-classical gravity, one tries to also compute the backreaction of the quantum fields on the geometry. Here, we use QED as a toy model for this kind of computation, treating the electromagnetic field as a classical background, just as one does in QFT in curved spacetime. If you think Schwinger, you're exactly right (this reference may be more understandable).

Note: This sheet is a longer one, but every single sub-exercise is doable with the hints that are given. The derivation is pretty technical. If you want to skip a step, the result of the sub-exercise is usually provided, so you can continue on with the next sub-exercise. The physical interpretation of what we compute here is in sub-exercises (**l**) and (**m**). So if you're first and foremost interested in exploring the physics, concentrate on those.

The QED partition function on a flat background reads

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_{\mathrm{QED}}}, \tag{16.1}$$

with the QED action

$$S_{\mathrm{QED}} = \int_x \left[-\mathcal{F} + \bar{\psi}\left(i\not{D} - m\right)\psi\right]. \tag{16.2}$$

Here, $\mathcal{F} \equiv F_{\mu\nu}F^{\mu\nu}/4$, where $F_{\mu\nu}$ is the field-strength tensor of the gauge field A_μ. Besides, $\not{D}$ denotes the covariant Dirac operator involving the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu, \tag{16.3}$$

and e and m are the charge and the mass of the (Grassmann-valued) fermion ψ, respectively. We want to compute the one-loop effective Lagrangian for constant $F_{\mu\nu}$ by integrating out the fermion. For constant field strength, the effective Lagrangian is related to the effective action as

$$\Gamma[A] = \int \mathrm{d}^4x \mathcal{L}_{\mathrm{eff}}(F) = V\mathcal{L}_{\mathrm{eff}}(F), \tag{16.4}$$

where V denotes the spacetime volume.

(**a**) At one-loop order, we can treat the electromagnetic field as a non-dynamical background, and integrate over the fermion field. Define the effective action $\Gamma[A]$ for the background field A as

$$Z = \int \mathcal{D}Ae^{i\Gamma[A]}. \tag{16.5}$$

Show that the resulting one-loop correction to the effective action reads

$$\Gamma^{(1)}[A] = -i\log\det(i\not{D} - m). \tag{16.6}$$

This is the fermion determinant, encoding all one-loop corrections from virtual electrons in the background field.

Hints:

- Gaussian path integrals are analogous to ordinary Gaussian integrals.
- Without gravity, constants in the effective action, even if they are infinite, do not contribute to the physics, and can be neglected. You can do this in every part of this sheet.

(**b**) As it is simpler to compute determinants of scalars, let's rewrite the determinant. Show that we can express the one-loop contribution to the effective action as

$$\Gamma^{(1)}[A] = -\frac{i}{2}\log\det(\not{D}^2 + m^2). \tag{16.7}$$

Hint: Use the fact that the operator $i\not{D} - m$ is hermitian. For Hermitian $\mathcal{O}$, it is known that $\log\det\mathcal{O} = \log[\det(\mathcal{O}^\dagger\mathcal{O})]/2$.

To simplify the problem, we use the proper-time representation of the effective action by expressing the Logarithm as

$$\log\mathcal{O} = -\int_0^\infty \frac{\mathrm{d}s}{s} e^{-s\mathcal{O}} + \text{const.} \tag{16.8}$$

(**c**) Demonstrate that the proper-time representation of Eq. (16.7) reads

$$\Gamma^{(1)}[A] = \frac{i}{2}\int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2}\operatorname{tr}\left(e^{-s\not{D}^2}\right). \tag{16.9}$$

Computing the effective Lagrangian comes down to evaluating the trace of the operator $U = e^{-iHs}$, with the analogue of a Hamiltonian

$$H \equiv \not{D}^2. \tag{16.10}$$

This is why Eq. (16.8) is called proper-time representation: The operator D^2 is the generator of translations in $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. Therefore, the operator U is a translation in proper time and s, the Schwinger proper time, is the translation parameter.

(**d**) Show that

$$H = D^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} \equiv H_{\text{kin}} + H_{\text{spin}}, \tag{16.11}$$

where $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$, and H_{kin} and H_{spin} denote the kinetic term and the coupling of spin to the electromagnetic field, respectively.

The purpose of this exercise is to compute the trace for $F_{\mu\nu}$ =const. Then, the spin-interaction Hamiltonian commutes with the kinetic term such that

$$\operatorname{tr}\left(e^{-sH}\right) = \operatorname{tr}\left(e^{-sH_{\text{kin}}}\right)\operatorname{tr}\left(e^{-sH_{\text{spin}}}\right). \tag{16.12}$$

(**e**) Show that

$$\operatorname{tr}\left(e^{-\frac{es}{2}F_{\mu\nu}\sigma^{\mu\nu}}\right) = 4\cos(esa)\cosh(esb), \tag{16.13}$$

where

$$a^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}, \qquad b^2 = \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}, \tag{16.14}$$

where we defined $\mathcal{G} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu}/4$, and the dual field strength $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}/2$. Inasmuch as $a = 0$ if $\vec{E} = 0$ and $b = 0$ if $\vec{B} = 0$, we can understand a to largely measure electric contributions to the field strength, while b largely measures the magnetic ones.

Hints:

- H_{spin} is position independent, so what do we have to trace over?
- On the way, derive that $(F\sigma)^2 = 8(\mathcal{F} + i\gamma^5\mathcal{G})$. You can use the fact that

$$\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\} = 2\left(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma} + i\gamma^5\epsilon^{\mu\nu\rho\sigma}\right), \tag{16.15}$$

  and that $(\gamma^5)^2 = 1$ and $\text{tr}\gamma^5 = 0$.
- The result of the trace has to be Lorentz invariant. Are there any Lorentz invariants which contain odd powers in $F_{\mu\nu}$?

Time for the last step. The trace of H_{kin} is best computed in Euclidean signature and afterwards analytically continued back. This amounts to the transformation $t \to -ix_0$, $\partial_t \to i\partial_0$. Thus, we transform

$$D^2 = \eta^{\mu\nu}D_\mu D_\nu \to -\delta_{AB}D_A D_B \equiv -(D_A)^2, \tag{16.16}$$

where indices A, B are four-dimensional Euclidean indices. Note that raising and lowering of indices is not required in Euclidean signature

(f) Show that the operator H_{kin} can be split into two commuting operators $H_{\text{kin},a}$ and $H_{\text{kin},b}$ by an orthogonal transformation such that

$$\text{tr}\left(e^{-sH_{\text{kin}}}\right) = \text{tr}\left(e^{-sH_{\text{kin},a}}\right)\text{tr}\left(e^{-sH_{\text{kin},b}}\right). \tag{16.17}$$

From here on, we will treat these two Hamiltonians jointly by using the shorthand notation $I = a, b$, to mean either of the two.

Hints:

- Antisymmetric matrices, like F_{AB}, can be put into Darboux-form, *i. e.* into non-mixing 2-by-2 antisymmetric blocks, by an orthogonal transformation.
- Use without proof that in four dimensions and in Euclidean signature, the matrix $F_{\mu\rho}F^{\rho\nu}$ has the eigenvalues $-a^2$ and $-b^2$.
- If F_{AB} =const, we can express the gauge field in a Landau-type gauge (show that!), where

$$A = ax_0\mathrm{d}x_1 + bx_2\mathrm{d}x_3. \tag{16.18}$$

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An operator trace formally amounts to a sum over all eigenvalues of an operator, including the multiplicity if the operator has degenerate eigenstates, namely

$$\text{tr}\left(e^{-sH_{\text{kin},I}}\right) = \sum_n M_{I,n}e^{-sE_{I,n}}, \tag{16.19}$$

where the $E_{I,n}$ are the eigenvalues of $H_{\text{kin},I}$, and $M_{I,n}$ is the multiplicity of eigenstate $|E_{I,n}\rangle$ (recall that $I = a, b$) and n can collectively stand for different quantum numbers. Note, though, that operators can have continuous spectra.

(g) Compute the eigenvalues of $H_{\text{kin},I}$.

Hint: You can reduce the problem to that of a one-dimensional quantum harmonic oscillator.

(**h**) There is something fishy going on with these eigenvalues. What is the multiplicity?

Don't despair! We have seen this kind of infinity before. Recall that we want to obtain the effective Lagrangian – not the effective action. Let us, for the moment, put our theory into a box. What we found is that the multiplicity scales with the side length of that box.
The number of allowed values of k provides the multiplicity but k also shifts the centre of motion of the harmonic oscillator. Put the two two-dimensional systems into quadratic boxes of side length L positioned such that the edges are at $(x_0, x_1) = (0, 0)$ and $(x_0, x_1) = (L_a, L_a)$ as well as $(x_2, x_3) = (0, 0)$ and $(x_2, x_3) = (L_b, L_b)$, with periodic boundary conditions. As a result, the whole theory is confined to a hypercube of box length L

(**i**) Estimate the number of states at fixed n, *i. e.* $M_{I,n}$. by requiring that the centre of motion for allowed k has to be inside the box. You should obtain

$$M_{I,n} = \frac{eIL^2}{2\pi}. \tag{16.20}$$

This appears to be sleight of hand, but is actually exact in the limit $L \to \infty$ that we will take in the end. Why? **Hint:** To answer the "why"-question, consider that $\bar{\psi}$ is the eigenfunction of the one-dimensional harmonic-oscillator Hamiltonian with shifted centre of motion.

(**j**) Show that

$$\mathrm{tr}\left(e^{-sH_{\mathrm{kin}}}\right) = V_{\mathrm{E}} \frac{e^2 ab}{(4\pi)^2 \sinh(esa)\sinh(esb)}, \tag{16.21}$$

where $V_{\mathrm{E}} = L^4$ is the volume of the hypercube – the E here stands for Euclidean signature. Wick rotate back to Lorentzian signature, and provide the resulting effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{cl}} - \frac{e^2 ab}{8\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s} e^{-sm^2} \cot(esa)\coth(esb), \tag{16.22}$$

where $\mathcal{L}_{\mathrm{cl}} = -\mathcal{F}$.

Hint: Electric and magnetic fields behave differently under Wick rotation, namely $\vec{E}_{\mathrm{E}} = i\vec{E}_{\mathrm{L}}$ but $\vec{B}_{\mathrm{E}} = \vec{B}_{\mathrm{L}}$, where E stands for Euclidean and L for Lorentzian signature. How does the volume change under Wick rotation?

(**k**) The integral in Eq. (16.22) is divergent for $s \to 0$, *i. e.* in the UV. Renormalize it by subtracting solely the divergent part, *i. e.* do minimal subtraction. You should obtain

$$\mathcal{L}_{\mathrm{eff,ren}} = \mathcal{L}_{\mathrm{cl}} - \frac{1}{8\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s^3} e^{-sm^2} \left(e^2 s^2 ab \cot(esa)\coth(esb) - 1 - \frac{e^2 s^2 (b^2 - a^2)}{3} \right). \tag{16.23}$$

(**l**) **Time for physics:** Before we tackle the full integral, let's do a simplification. Consider only magnetic fields so that $\mathcal{G} = 0$ and $\mathcal{F} > 0$. Expand in $\mathcal{F}$ inside the integral and compute the lowest-order contribution, namely

$$\mathcal{L}^{(1)}_{\mathrm{eff}}|_{\mathcal{G}=0,\mathcal{F}>0} = \frac{2}{45\pi^2} \frac{e^4}{m^4} \mathcal{F}^2. \tag{16.24}$$

Why and in which regime can we expand inside the integral? Have a closer look at the resulting term: What kind of interaction did we get, *i. e.* what does backreaction do to the background fields? What would you expect at higher orders in the expansion? What does this mean for gravity?

The integral Eq. (16.23) covers the whole positive real line, where the integrand has an infinite number of poles. Thus it requires some work to be well-defined. Call the integrand

$$f(s) = \frac{e^{-sm^2}}{s^3}\left(e^2 s^2 ab \cot(esa)\coth(esb) - 1 - \frac{e^2 s^2 (b^2 - a^2)}{3}\right). \tag{16.25}$$

We can render the integral well defined by shifting the poles into the complex plane, *i. e.* by considering $s \to s + i\epsilon$ and computing

$$\int_0^\infty \mathrm{d}s f(s + i\epsilon), \tag{16.26}$$

understood as a contour integral.

(**m**) Compute the imaginary part of the effective Lagrangian. You should obtain

$$\mathrm{Im}\mathcal{L}_{\mathrm{eff}} = -\frac{e^2 ab}{8\pi}\sum_{j=1}^{\infty}\frac{e^{-\frac{j\pi m^2}{ae}}\coth\frac{j\pi b}{a}}{j\pi}. \tag{16.27}$$

Hint: The integrand is such that $\bar{f}(z) = f(\bar{z})$ for complex z, so the integral you have to evaluate is

$$\mathrm{Im}\int_0^\infty \mathrm{d}s f(s) = \frac{1}{2i}\int_0^\infty \mathrm{d}s\,(f(s+i\epsilon) - f(s-i\epsilon)). \tag{16.28}$$

This combined integral can be solved with the residue theorem.

Think before reading on: What could it mean that the imaginary part of the effective action is non-zero?

(**n**) Consider the time evolution of the vacuum. If we treat the electromagnetic field as a classical background, the amplitude describing vacuum being unchanged is given by the partition function

$$\langle 0_{\mathrm{in}} | 0_{\mathrm{out}} \rangle = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS_{\mathrm{QED}}}. \tag{16.29}$$

How does the probability of the vacuum staying the vacuum relate to the effective action? What does it mean if that probability is smaller than one? What does the quantity

$$\gamma = \frac{2\mathrm{Im}\Gamma[A]}{V} = 2\mathrm{Im}\mathcal{L}_{\mathrm{eff}} \tag{16.30}$$

measure?

Hint: Revisit the motivation at the top of the sheet.