Quantum Field Theory I

Assignment Week 10

Classroom Exercise 1: Physics of spectral functions

Motivation: The spectral function and the Källen-Lehmann representation are important ingredients to connect theory with experiments. Their physical content is rich and central in the renormalization of the fields and the mass parameters.

Consider a real scalar field $\phi(x)$ whose two-point function admits the Källén–Lehmann representation

$$\tilde{G}(p^2) = \int_0^\infty ds \, \frac{\rho(s)}{p^2 - s + i\epsilon}, \tag{1.1}$$

where $\rho(s)$ is the spectral function.

a) Explain why the spectral function $\rho(s)$ is *real and non-negative*. What would it mean for the probabilistic interpretation of particle creation from the vacuum if $\rho(s)$ were negative?

In an interacting theory the spectral function contains a one-particle contribution as an isolated peak, and also a multiparticle continuum or other isolated δ-peaks corresponding to bound states collectively denoted as $\sigma(s)$,

$$\rho(s) = Z \, \delta(s - m^2) + \sigma(s). \tag{1.2}$$

b) What is the interpretation of $Z \neq 1$, in terms of probabilities of creating one-particles states from the vacuum? Hint: Think of the difference between the vacua of an interacting and a corresponding free theory.

c) What is the physical meaning of the limits $Z \to 1$ and $Z = 0$? Hint: Recall the sum rule of the spectral function (see p. 110 Eq. (505) in the lecture notes).

d) The mass parameter appearing in the Lagrangian of a theory is generally different than the physical mass m^2, because of quantum corrections. How can we identify/define the physical mass in terms of the propagator?

Exercise 1: Spinor solutions of the Dirac equation

Motivation: The goal of this exercise is to get you comfortable with the solutions of the Dirac equation and the spin structure encoded in its solutions. We will show some important relations and properties and we will also see that these solutions describe particles with spin $\frac{1}{2}$, as was promised in the beginning of the course.

In the lecture, you saw that the solutions of the Dirac equation, are 4-component spinors that can have either positive or negative energy. For the positive energy solutions with spin s, you found that:

$$\psi_s^p(x) = u_s(p)e^{-ipx}, \;\; \text{with } p^2 = m^2, \; p^0 > 0, \; \text{and } u_s(p) = \sqrt{m}\begin{pmatrix}\xi_s \\ \xi_s\end{pmatrix}, \; s = 1,2 \; , \tag{2.3}$$

while for the negative energy solutions of spin s you found that

$$\psi_s^n(x) = v_s(p)e^{ipx}, \;\; \text{with } p^2 = m^2, \; p^0 > 0, \; \text{and } v_s(p) = \sqrt{m}\begin{pmatrix}\eta_s \\ -\eta_s\end{pmatrix}, \; s = 1,2 \; , \tag{2.4}$$

in the rest frame where $p = (m, \vec{0})$, for any two component spinors ξ_s,η_s.

a) Show that the spinor part of the above solutions satisfy the following orthonormality relations:

$$i) \;\; \bar{u}_r(p)u_s(p) = 2m\delta_{rs} \; , \tag{2.5}$$

$$ii) \;\; \bar{v}_r(p)v_s(p) = -2m\delta_{rs} \; , \tag{2.6}$$

$$iii) \;\; \bar{u}_r(p)v_s(p) = 0 \; , \tag{2.7}$$

$$iv) \;\; \bar{v}_r(p)u_s(p) = 0 \tag{2.8}$$

b) Perform a boost along the z-direction to calculate the Weyl spinors in a general frame. Show that they take the form:

$$u_s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma} \;\; \xi_s \\ \sqrt{p\cdot\bar{\sigma}} \;\; \xi_s\end{pmatrix} \;\; , \;\; v_s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma} \;\; \eta_s \\ -\sqrt{p\cdot\bar{\sigma}} \;\; \eta_s\end{pmatrix} \; , \tag{2.9}$$

where $\sigma^\mu = (1, \vec{\sigma})$,$\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and $p^\mu = (E, 0, 0, p^3)$. Hint: Recall that a Lorentz transformation along one direction can be parametrized with the quantity often called rapidity η such that:

$$\begin{pmatrix}E \\ p^3\end{pmatrix} = \begin{pmatrix}\cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta\end{pmatrix}\begin{pmatrix}m \\ 0\end{pmatrix} \; , \tag{2.10}$$

and that spinors transform in the spinor representation ($M^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$) via:

$$\psi' = S(\Lambda)\psi = \exp\left(-i\eta M^{03}\right)\psi \tag{2.11}$$

c) Now use Eqs. (2.9) and show the following completeness relations:

$$\sum_s (u_s(p))_\alpha (\bar{u}_s(p))^\beta = \left(\not{p} + m\right)_\alpha^{\;\beta} \tag{2.12}$$

$$\sum_s (v_s(p))_{\dot{\alpha}} (\bar{v}_s(p))^{\dot{\beta}} = \left(\not{p} - m\right)_{\dot{\alpha}}^{\;\dot{\beta}} \; , \tag{2.13}$$

where we have made explicit the internal spinor indices in both expressions. Note that these spinor indices are different since they correspond to different irreducible representations of the Lorentz group,

$$\alpha, \beta \to \left(\frac{1}{2}, 0\right) \;\; \& \;\; \dot{\alpha}, \dot{\beta} \to \left(0, \frac{1}{2}\right) \tag{2.14}$$

Now we would like to show that the Dirac spinors, i.e. the solutions of the Dirac equation, describe particles of spin $\frac{1}{2}$. Remember that the classification of particles with their spin is facilitated by the eigenstates Pauli-Lubanski Casimir, which in the rest frame takes the form[a],

$$W^i = J^i P^0 = mJ^i \implies W^2 = -m^2 \vec{J} \cdot \vec{J}, \tag{2.15}$$

with $J^i = \frac{1}{2}\epsilon^i_{jk} M^{jk}$.

b) Use the spinor representation of the Lorentz algebra with the Weyl representation of the γ matrices and show that

$$W^i = \frac{m}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} . \tag{2.16}$$

Finally, act on the rest frame spinors with W^2 and calculate the spin of $u_1(p)$, $u_2(p)$, $v_1(p)$ and $v_2(p)$. We know that the operator $\vec{J}^2$ has eigenvalues $s(s+1)$ so what is the total spin s of a Dirac spinor?

[a] See lecture notes p.19-20

Exercise 2: Another spinor Lagrangian

Motivation: The goal of this exercise is to explain why a quantization of fermions described by a relativistic Lagrangian depending on $\Box = \partial_\mu \partial^\mu$ fails to describe realistic fermions.

As you mentioned in the lecture, the basic requirements for a Lagrangian are Lorentz invariance and being real. One such Lagrangian could be:

$$\mathcal{L} = \bar{\Psi} \Box \Psi \tag{2.17}$$

a) Express this Lagrangian in terms of the left and right-handed spinors and calculate the equations of motion for Ψ_R and Ψ_L. How many (real) degrees of freedom does this system describe? How many degrees of freedom would you expect from realistic fermions (i.e. the electrons)?

b) Compute the equations of motion for the left and right handed spinors from the Dirac equation. What is crucially different in this case? How many degrees of freedom are described by the solutions of the Dirac equation?

c) What if we knew nothing about the degrees of freedom of electrons? Could we have a good theory starting from (2.17)? To answer this expand Eq. (2.17) in spinor components in the Dirac basis[a] of the γ-matrices where

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \tag{2.18}$$

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[a] In this basis γ^0 is diagonal while in the Weyl basis γ^5 is diagonal. Both are equivalent since they are connected by unitary transformations of the spinors.