Quantum Field Theory I

Assignment Week 11

Classroom Exercise 1: Wick's theorem

Motivation: In the lecture you mentioned Wick's theorem which allows to express time-ordered correlation functions in terms and is a central result in QFT. Here you will complete the proof of this theorem.

In the lecture, you showed the following relation for the product of $n = 2$ fields[a],

$$\mathcal{T}\left\{\phi(x), \phi(y)\right\} =: \phi(x)\phi(y) : + \overbrace{\phi(x)\phi(y)} \, , \tag{1.1}$$

with $\overbrace{\phi(x)\phi(y)} = D_F(x-y)$.

We would like to generalize the theorem to an arbitrary number of fields n as:

$$\mathcal{T}\left\{\phi(x_1)\dots\phi(x_n)\right\} =: \phi(x_1)\dots\phi(x_n) + \left(\sum \text{all possible contractions}\right) : \, , \tag{1.2}$$

which we prove by induction, by assuming the relation holds for $(n-1)$ fields and showing that it holds for n-fields. We assume without the loss of generality that our fields are already time-ordered (i.e. $x_1^0 > x_2^0 > \dots > x_n^0$).

a) Use the decomposition $\phi(x) = \phi^+(x) + \phi^-(x)$ to show that:

$$\left[\phi^+(x_1)\, , : \phi(x_2)\dots\phi(x_n) :\right] \;=\; : \left(\sum \text{single contractions with } \phi(x_1)\right) : \, .$$

b) Using the above result prove the general Wick's theorem by showing the following

$$\begin{aligned}\mathcal{T}\left\{\phi(x_1)\dots\phi(x_n)\right\} &= (\phi^+(x_1) + \phi^-(x_1))\Big[: \phi(x_2)\dots\phi(x_n) : \\ &\qquad + \sum \text{all contractions not involving } \phi(x_1) : \Big] \qquad (1.3)\\ &= : \phi(x_1)\dots\phi(x_n) : + : \sum \text{all possible contractions} : \qquad (1.4)\end{aligned}$$

[a] All field operators here are meant to be in the interaction picture.

Exercise 1: Scattering amplitudes at tree level

Motivation: This exercise showcases the use of the Feynman rules in computing elementary processes in QED at tree level. Particularly regarding fermions, signs are very important and require careful analysis in interaction picture, using Wick's theorem for fermions. You are advised to consult Peskin & Schröder Ch. 4.7-4.8 and/or Tong CH. 5.6-5.7

Draw the tree-level diagrams of the following scattering processes, and write down their analytic expressions using Feynman rules:

a) Bhabha scattering ($e^+e^- \to e^+e^-$)

b) Compton Scattering ($e^-\gamma \to e^-\gamma$)

Hint: To find the correct signs for all diagrams you need to start from the general formulas in the interaction picture and take into account the anti-commuting character of fermions.

Exercise 2: Time-ordering in the LSZ reduction formula

Motivation: The LSZ reduction formula allows us to compute on-shell amplitudes describing scattering experiments, from time-ordered correlations functions. We will see explicitly how the time-ordering and the disconnected parts come up in the formula.

Central in the derivation of the LSZ reduction formula, and for deriving the time-ordering,is the **weak asymptotic condition**[a],

$$\lim_{t\to\pm\infty} \langle a|\phi(x)|b\rangle = \sqrt{Z}\,\langle\alpha|\phi_{in}^{out}(x)|\beta\rangle\ . \tag{1.5}$$

We also going to assume that all external momenta are different. Then the case where some momenta are equal ("forward scattering") can be obtained by analytic continuation.

a) Write the S-matrix element in terms of creation and annihilation operators of in/out states. Explain why the result:

$$\langle p_1,\dots,p_n;\text{out}|q_1,\dots,q_m;\text{out}\rangle\Big|_{\text{on-shell}} = \langle 0|a_{\vec{p}_1,\text{out}}\cdots a_{\vec{p}_n,out}\,a^\dagger_{\vec{q}_1,\text{in}}\cdots a^\dagger_{\vec{q}_m,\text{in}}|0\rangle \tag{1.6}$$

is already time-ordered.

Now we continue with the reduction, using the relation you derived in the lecture (essentially Eq. (539) on p. 114 in the script).[b]:

$$a^\dagger_{\vec{p},\text{out}} - a^\dagger_{\vec{p},\text{in}} = -iZ^{-1/2}\int d^4x\, e^{-ipx}\left(\Box + m^2\right)\phi(x) \tag{1.7}$$

b) Use Eq. (1.7) and the time-ordering symbol and the following identities, to show that (suppressing here the on-shell condition):

$$\langle p_1,\dots,p_n;\text{out}|q_1,\dots,q_m;\text{in}\rangle = \left(\frac{i}{\sqrt{Z}}\right)^{n+m}\int\left(\prod_k^n d^4y_k\,e^{-ip_ky_k}\right)\left(\prod_k^n d^4x_k\,e^{-iq_kx_k}\right)$$
$$\cdot\left(\Box_{y_1}+m^2\right)\cdots\left(\Box_{x_m}+m^2\right)\langle 0|\mathcal{T}\left[\phi(y_1)\cdots\phi(x_m)\right]|0\rangle + \sum(\text{disconnected terms}) \tag{1.8}$$

Hint: Do the reduction step by step substituting the asymptotic creation/annihilation operators via Eq.(1.7), until you are convinced of the general formula (1.8). You can also consult Sec. (2.4) in T. Weigand's notes, which are available online.

[a]The t-limit means that the interacting fields ϕ as configurations match the free-fields $\phi_{in/out}$. $\phi_{\text{in/out}}$ are still time-dependent in the Heisenberg picture.

[b]The weak asymptotic condition is crucial to derive this formula and it also requires that we take localized wave-packets and not plane waves as the in/out states. You can pull out the Klein-Gordon operator from the time-ordering at the expense of contact terms that nevertheless do not contribute to scattering when we take the on-shell limit and thus, you can ignore them.