Quantum Field Theory I

Assignment Week 12

Classroom Exercise 1: Scatterind and trace technology

Motivation: Here we will compute the square of the scattering amplitude for the Bhabha scattering and practice in trace technology.

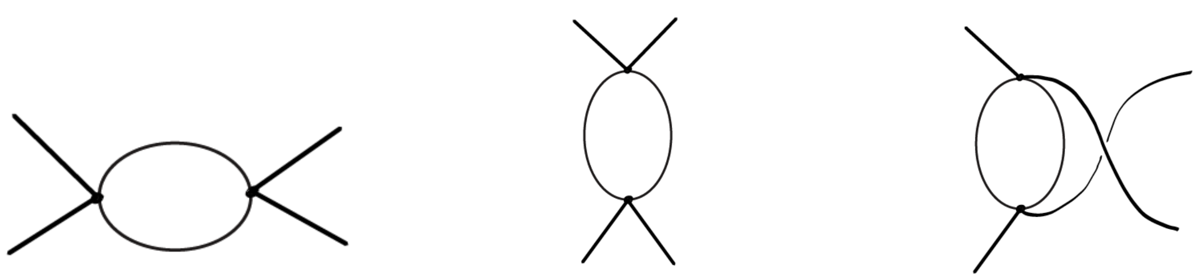
Recall that in Sheet 11 you derived the formula of the scattering amplitude for the Bhabha ($e^+e^- \to e^+e^-$) using the Feynman rules and Wick's theorem. You derived:

$$\begin{aligned} i\mathcal{M} &= (-ie)^2\Bigg[\bar{u}^s(\vec{p'})\gamma^\mu v^{s'}(\vec{k'})\frac{-i\eta_{\mu\nu}}{(p+k)^2+i\epsilon}\bar{v}^r(\vec{k})\gamma^\nu u^{r'}(\vec{p}) \\ &\qquad -\bar{u}^s(\vec{p'})\gamma^\mu u^{r'}(\vec{p})\frac{-i\eta_{\mu\nu}}{(p-p')^2+i\epsilon}\bar{v}^r(\vec{k})\gamma^\nu v^{s'}(\vec{k'})\Bigg] \end{aligned} \tag{1.1}$$

a) Show that the unpolarized polarized cross section proportional to $\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$, receives contributions from the **cross-terms** in Eq. (1.1) of the form:

$$\begin{aligned} \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= -\frac{e^4}{4st}\Big\{\mathrm{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{k}'-m)\gamma^\nu(\not{k}-m)\gamma_\mu(\not{p}+m)\gamma_\nu\right] \\ &\qquad +\mathrm{Tr}\left[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}-m)\gamma_\mu(\not{k}'-m)\gamma_\nu\right]\Big\} \end{aligned} \tag{1.2}$$

b) Compute the traces in Eq.(1.2), using the identities in p. 143-145 in the script. *Hint: Note that the two traces are mapped through each other via $p \to -k'$.*



(a) (b) (c)

Figure 1: One-particle irreducible diagrams (1PI) at 1-loop for ϕ^4-theory.

Exercise 1: Calculation of a 1-loop diagram

Motivation: In lecture you saw that loop corrections in scattering amplitudes are represented by Feynamn diagrams with loops. Here we are going to compute the first order quantum correction to 2-to-2 scattering for the ϕ^4-theory, using dimensional regularization. This is a very standard calculation and it exists in most textbooks.

Let our theory be the usual ϕ^4-theory with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \frac{\lambda}{4!}\phi^4$. The first order correction to the coupling λ is given by the amputated "candy" diagrams in Fig. (1)[a]

a) Write down explicitly the momenta along the lines of the diagrams and write their analytic expression using Feynman rules in momentum space. Show that the s-channel diagram in Fig. (1a) takes the form:

$$\mathcal{I}_s = +\frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - m^2 + i\epsilon}\frac{1}{(p_1+p_2-k)^2 - m^2 + i\epsilon}\ , \tag{1.3}$$

where p_1, p_2 are the external incoming momenta, and k is the internal momentum running inside the loop. How does this integral diverge[b] in the $k \to \infty$ limit?

b) Focus now on the s-channel diagram in Fig. (1a). As a first step re-write the above integral by applying the **Feynman trick**:

$$\frac{1}{AB} = \int_0^1 \frac{dz}{\left[Az + B(1-z)\right]^2}\ , \tag{1.4}$$

and then perform the change of variable $l = k + z(p_1+p_2)$. You should find:

$$\mathcal{I}_s = \frac{\lambda^2}{2}\int_0^1 dz \int \frac{d^4l}{(2\pi)^4}\frac{1}{\left[l^2 + z(1-z)(p_1+p_2) - m^2\right]^2} \tag{1.5}$$

c) Now perform a Wick rotation $l_E^0 = -il^0$ and perform the dimensional continuation by writing the integral as a d-dimensional integral with $d = 4 - \epsilon$[c]. *Note that you have to introduce a new energy scale μ to some appropriate power so that the integral remains dimensionless.*

d) Follow the steps in the lecture to calculate the d-dimensional integral in terms of Γ-functions, and expand the result in the limit $d \to 4$. You should find in the limit $\epsilon \to 0$,

$$\mathcal{I}_s = \frac{i\lambda^2}{32\pi^2}\left\{\frac{2}{\epsilon} - \gamma\right\} - \frac{i\lambda^2}{32\pi^2}\int_0^1 dz \log\left[\frac{sz(1-z) - m^2}{4\pi\mu^2}\right]\ . \tag{1.6}$$

This quantity is divergent for $\epsilon \to 0 (d \to 4)$, however we have managed to separate the purely divergent from the finite terms. The idea of renormalization, which you will discuss in more detail next semester, is to add **counter-terms** in the Lagrangian that would introduce divergent terms in Eq. (1.6) in such a way that the divergences in $\mathcal{I}_s$ would cancel.

[a] It suffices to focus only on these diagrams since these are involved in the calculation of the effective action (or potential). A diagram is one-particle irreducible (1PI) if it does not reduce to product of other 1-loop diagrams upon cutting one edge. In QFT2 you will see that these diagrams contain all information about the quantum corrections.

[b] We mean what is the functional dependence of $V(k^2)$ for large k

[c] You can ignore the $i\epsilon$ term in the denominator from here on

Exercise 2: Coulomb potential from QFT

Motivation: In this exercise we show how to retrieve classical electromagnetism from first principles using quantum field theory. We will also show that the massless nature of meadiating photons is what leads to long-range electromagnetic interactions.

In non-relativistic quantum mechanics, the scattering of an electron off a stationary potential is given at first order in perturbation theory by the formula[a]

$$\langle p'|iT|p\rangle = -i(2\pi)\delta(E_{\vec{p}} - E_{\vec{p}'}) \int d^3x V(\vec{x}) e^{i(\vec{p}-\vec{p}')\vec{x}} \, . \tag{1.7}$$

Now we would like to compute the potential between two negatively charged particles, which should be repulsive. For simplicity, we will focus on the process $e^-\mu^- \to e^-\mu^-$.

a) Draw the corresponding Feynman diagram and write down its analytic expression. Then show that in the non-relativistic limit it takes the form:

$$i\mathcal{M} = -i\frac{e^2}{|\vec{p}-\vec{p}'|^2}(2m_e)(2m_\mu)(2\pi)^4\delta^{(4)}(p-p') \, . \tag{1.8}$$

Hint: You can follow Peskin & Schröder Ch. 4.7-4.8 and/or T. Weigand's notes p. 122-124.

b) Integrate the amplitude with respect to the momentum of the incoming muon and by matching with Eq. (1.7)[b], show that the potential takes the following form,

$$V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\frac{e^2}{|\vec{q}|^2}e^{i\vec{q}\vec{x}} = \frac{e^2}{4\pi r} \, . \tag{1.9}$$

What is the range of this potential? Is it attractive or repulsive? Hint: Introduce an infinitesimal imaginary part in the denominator ($-i\epsilon$) and calculate the integral using Cauchy's theorem for complex integrals.

c) If we consider that the mediator particle (here the photon) is massive, then the above formula for the amplitude becomes,

$$i\mathcal{M} = -i\frac{e^2}{|\vec{p}-\vec{p}'|^2 + m_\gamma^2}(2m_e)(2m_\mu)(2\pi)^4\delta^{(4)}(p-p') \, . \tag{1.10}$$

Calculate the new potential takes the following form,

$$V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\frac{e^2}{|\vec{q}|^2 + m_\gamma^2}e^{i\,\vec{q}\,\vec{x}} = \frac{e^2}{4\pi r}e^{-m_\gamma r} \, . \tag{1.11}$$

What is the range of this interaction?

d) Draw the corresponding tree-level Feynman diagram and show that it takes the following form,

$$i\mathcal{M} = e^2\left[\bar{u}_e^s(\vec{p}')\gamma^\mu u_e^{s'}(\vec{p})\right]\frac{-i\eta_{\mu\nu}}{(p-p')^2 + i\epsilon}\left[v_\mu^r(\vec{k})\gamma^\nu v_\mu^{r'}(\vec{k}')\right](2\pi)^4\delta^{(4)}(p-p') \, , \tag{1.12}$$

This is identical to the previous scattering amplitude up to a minus sign, and so it recovers an attractive Coulomb potential in the non-relativistic limit.

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[a] This is usually called the **Born approximation** about which you can read in any introductory textbook on quantum mechanics.

[b] Ignore the factors $(2m)$ since they come from our different choice of normalization for the in/out states.