Quantum Field Theory I

Assignment Week 3

Classroom Exercise 1: Dirac δ-function Part 1: Basic properties

Motivation: The goal here is to remind you of some basic properties of the Dirac delta function which is actually a generalised function or a distribution. It might be obvious already that understanding the properties of this distribution is very important for Quantum Field Theory since it appears in the commutation relations. Additionally it is very important for modelling point-like sources, e.g. in electromagnetism, or in solving differential equations with the method of Green's functions.

In this lecture (and in others), you have seen the Dirac delta function $\delta(x)$. It is intuitively understood as the zero function with an infinitely high spike at $x = 0$,

$$\delta(x) = \begin{cases} 0 & , \ x \neq 0 \\ +\infty & , \ x = 0 \end{cases} \tag{1.1}$$

with the normalization condition $\int_{-\infty}^{\infty} \delta(x) dx = 1$. The above intuition helps us understand the fundamental property of this "function"

$$f(0) = \int_{-\infty}^{+\infty} \delta(x) f(x) dx \ , \tag{1.2}$$

where $f(x)$ is "well-behaved" function often called test-function[a]. In reality however, such a function does not exist as an ordinary function but only as a **generalized function** or a **distribution**. In other words, $\delta(x)$ is defined as a mathematical object that makes sense only inside an integral with the property (1.2).

a) Using the defining relation (1.2), change of variables and partial integration, show the following identities, all meant to hold inside an integral:

$$\text{i)} \quad \delta(x) = \delta(-x) \ , \quad \text{ii)} \ f(a) = \int_{-\infty}^{+\infty} \delta(x-a) f(x) dx$$

$$\text{iii)} \quad \delta(kx) = \frac{\delta(x)}{|k|} \quad , \quad iv) \ \ x\delta(x) = 0 \ .$$

b) Assuming that the smooth test functions $f(x)$ and its derivatives fall off sufficiently fast in the limit $x \to \pm\infty$ such that they are integrable, show that

$$\frac{d}{dx}\Theta(x) = \delta(x) \ , \text{ with } \Theta(x) = \begin{cases} 1 \ , & x > 0 \\ 0 \ , & x < 0 \end{cases} \ . \tag{1.3}$$

The function Θ is often called the "step" or "Heaviside" function.

Finally, another way to define distributions is by limits of sequences of regular functions $\delta_n(x)$. For a sequence of functions $\delta_n(x)$ to be a δ-function, they need to be normalized to 1 and additionally satisfy the following:

$$\int_{-\infty}^{+\infty} \delta(x) f(x) dx \equiv \lim_{n\to\infty} \int_{-\infty}^{+\infty} \delta_n(x) f(x) dx = f(0) \ . \tag{1.4}$$

In practice, such a sequence converges to the δ-function if

$$\lim_{n\to\infty} \delta_n(x) = \begin{cases} \infty & x=0 \\ 0 & ,x\neq 0 \end{cases} \tag{1.5}$$

and if also integrates to one, i.e. $\int_{-\infty}^{+\infty} \delta_n(x)dx = 1, \forall n$.

c) Calculate the sequence

$$\delta_n(x) = \frac{1}{2\pi}\int_{-n}^{n} e^{ixt}dt \,, \tag{1.6}$$

and show that it converges to the δ-function as in (1.5). Do you recognize the limit $n\to\infty$ of the integral (1.6)? Hint: for the normalization you will need that $\int_0^{+\infty} \frac{\sin(x)}{x}dx = \frac{\pi}{2}$.

[a] Here the term "well-behaved" corresponds to a class of **smooth** functions whose derivatives are rapidly decreasing. These are usually called **Schwartz functions**. We will not be interested in the details of these functions that are important for a rigorous definition of distributions but instead consider all formal manipulations here as permissible for all test functions.

Exercise 1: The Dirac delta Part 2: Lorentz invariance

Motivation: In this exercise we expand on the classroom exercise by deriving an important property of the δ-function. The property of composition with a function $g(x)$ is important to understand the Lorentz invariance of the momentum integrals that appear again and again in quantum field theory. It is also important to understand the normalization of the one particle states. Feel free to read the classroom exercise and consult its results that you are going to derive in class.

Let $g(x)$ be a smooth function with a simple zero at $x = x_0$, i.e.

$$g(x_0) = 0 \quad , \quad g'(x_0) \neq 0 \,, \tag{2.7}$$

that is also monotonic, i.e. $g'(x) \neq 0$, $\forall x$.

a) Using change of variables and the above identities show that

$$\delta(g(x)) = \frac{\delta(x-x_0)}{|g'(x_0)|} \,. \tag{2.8}$$

Such an identity generalizes to functions that have multiple zeros $x_1,\ldots,x_n$ and that are not necessarily monotonic. Let $g(x)$ be a smooth function with finite number of roots $(x_1,\ldots,x_n)$, then we generally have

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|} \,. \tag{2.9}$$

A rigorous proof is not difficult but requires the use of a **partition of unity** and the **inverse function theorem**. We will discuss such a proof in the solutions. For now do the following:

b) Think of the δ-function as "function" that is zero everywhere apart from where its argument is zero and convince yourself that for a general $g(x)$ with a finite number of roots, the identity (2.9) holds.

c) Use the generalized identity to show the following,

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}f(p) = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\Theta(p^0)f(p)\ , \tag{2.10}$$

where p denotes the 4-momentum $p^\mu = (p^0, \vec{p})$ and $f(p)$ a (Lorentz) scalar function. Is the integral in the left-hand side invariant under the proper orthochronous Lorentz group?

d) Show that for the one dimensional case the quantity $E_p\delta(p-q)$ is Lorentz invariant,with $E_p = \sqrt{p^2+m^2}$, i.e. show that

$$E_p\delta(p-q) = E_{p'}\delta(p'-q')\ , \tag{2.11}$$

where $E_{p'}$ and p, q' are the Lorentz transformed energy and momenta. Hint: Calculate $p'-q'$ in terms of p and q, as well as $E_{p'}$ in terms of E_p and p. Apply the δ-function identities we have already discussed. This relation straightforwardly generalizes to the 3-dimensional case as

$$E_p\delta^{(3)}(p-q) = E_{p'}\delta^{(3)}(p'-q') \tag{2.12}$$

Exercise 2: Quantization of the scalar field

Motivation: This exercise serves as a revision of the quantization of the free scalar field. It is good practice to repeat the important calculations involving commutators and creation/annihilation operators.

We work in the Schrödinger picture where all field operators are time-independent. For the free scalar field you assumed the canonical commutation relations

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^{(3)}(x-y)\ , \tag{3.13}$$

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 = [\Pi(\vec{x}), \Pi(\vec{y})]\ . \tag{3.14}$$

a) Show that the commutation relation of the Fourier transforms $\tilde{\phi}(\vec{p})$, $\tilde{\Pi}(\vec{p})$ is the following:

$$\left[\tilde{\phi}(\vec{p}), \tilde{\Pi}(q)\right] = i(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\ , \tag{3.15}$$

b) Calculate the Hamiltonian in Fourier space and write it in terms of creation/annihilation operators, $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$, following the quantization of the harmonic oscillator in quantum mechanics. What are the forms of $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(q)$?

c) Use the equations for $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(q)$ in terms of the creation/annihilation operators and Eq. (3.15), to show the commutation relations

$$\left[\,a_{\vec{p}}\,, a^\dagger_{\vec{q}}\,\right] = (2\pi)^3\delta^{(3)}(p-q)\ , \tag{3.16}$$

$$[\,a_{\vec{p}}\,, a_{\vec{q}}\,] = 0 = \left[\,a^\dagger_{\vec{p}}\,, a^\dagger_{\vec{q}}\,\right] \tag{3.17}$$

Exercise 3: Noether's theorem Part 2: the $O(N)$ model

Motivation: In a previous exercise you derived the Noether charges associated to Lorentz symmetry. However, we can build theories with multiple scalar fields in such a way so that we have extra (internal) global symmetries[a]*. Such theories have extra conserved currents and charges and they will be extremely important. One example that is relevant to condensed matter physics is the $O(N)$-model.*

Let us consider an example of a free theory with three real scalar fields (ϕ_1, ϕ_2, ϕ_3), organised as a triplet

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad , \quad \Phi^{\mathrm{T}} = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \quad . \tag{4.18}$$

One can build a relativistic theory of the "multi-field" Φ in complete analogy with the real scalar field theory by writing the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \Phi\right)^{\mathrm{T}} \left(\partial^\mu \Phi\right) - \frac{1}{2} m^2 \Phi^{\mathrm{T}} \Phi \equiv \sum_I \frac{1}{2} \left(\partial_\mu \Phi^{\mathrm{I}}\right) \left(\partial^\mu \Phi^{\mathrm{I}}\right) - \frac{1}{2} m^2 \Phi^{\mathrm{I}} \Phi^{\mathrm{I}} \tag{4.19}$$

a) Write the equations of motions for the "multi-field" components Φ^{I}, and convince yourselves that this theory is a theory of three free scalar fields.

b) Show that the Lagrangian is invariant under the field transformation $\Phi \to \mathrm{R}\Phi \equiv \mathrm{R}^{IJ}\Phi^J$, where R are operators symmetry group acting on the fields represented as 3×3-matrices, if $\mathrm{R}^{\mathrm{T}}\mathrm{R} = 1$. What is its infinitesimal transformation? Write it in terms of the generators of the symmetry group T_a, how many are there?

Apart from Poincaré invariance, we identified an extra **internal** global symmetry as the group of 3 dimensional rotations $O(3)$ in the space spanned by the fields (ϕ_1, ϕ_2, ϕ_3). These are not rotations in position space, but rotations in an internal space, i.e., they relate the three real scalar fields ϕ_1, ϕ_2 and ϕ_3 to each other in exactly the same way as a rotation in position space relates the three components of spatial vectors. This model is often called **vector model** or $O(3)$-model and it is a very important for condensed matter applications. For example, it describes magnetic systems where the spin variables can take any orientation, its discretized version is the Heisenberg model. We can now apply Noether's theorem and compute its associated conserved current and charge.

b) Show that the Noether currents corresponding to the $O(3)$ symmetry take the form

$$j^\mu_a = i \left(\partial^\mu \Phi\right) T_a \Phi \equiv i \left(\partial^\mu \Phi^I\right) \left(T_a\right)^{IJ} \Phi^J \quad \text{, with } \; I, J = 1, 2, 3 \tag{4.20}$$

and show that it is indeed conserved if the fields satisfy their equations of motion.

[a] Internal in the sense that they are not associated to some spacetime transformation.