

Quantum Field Theory I

Assignment Week 4

Classroom Exercise 1: Coherent states of harmonic oscillators

Motivation: This exercise aims to introduce coherent states in quantum mechanics and quantum field theory. These states are specific superposition of n -particle states and they are the most “classical” states one can build from a quantum theory. These states have a long range of application in condensed matter physics from matter fields in superconductors to Bose-Einstein condensates.

In the classic quantum harmonic oscillator ($\omega = 1 = m$), the spectrum is organised in energy eigenstates through the ladder operators

$$\hat{a} |0\rangle = 0, \quad \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle. \quad (1.1)$$

Now we would like to build a state out of the energy eigenstates that mimics as much as possible the classical harmonic oscillator, whose classical position and momentum oscillate with a relative phase of $\frac{\pi}{2}$. Such a state is not stationary, so it cannot be an energy eigenstate. An operator such as the annihilation operator

$$\hat{a} = \frac{1}{\sqrt{2}} (\hat{X} + i\hat{P}) \quad (1.2)$$

has ingrained such a phase difference. Thus we are looking for eigenstates $|\alpha\rangle$ satisfying the relation:

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle. \quad (1.3)$$

a) Show that the coherent state $|\alpha\rangle$ can be written as an excitation of the vacuum as follows:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\alpha \hat{a}^\dagger} |0\rangle. \quad (1.4)$$

b) What is the probability to find E_n if we measure the energy of the state $|\alpha\rangle$? Do you recognize this probability distribution?

c) Using the definition of the variance

$$(\Delta \mathcal{O})^2 = \langle \hat{\mathcal{O}}^2 \rangle - \langle \mathcal{O} \rangle^2, \quad (1.5)$$

where $\hat{\mathcal{O}}$ is any observable operator, calculate the product of standard deviations $\Delta x \cdot \Delta p$ and compare it with Heisenberg's uncertainty principle.

Exercise 1: How to NOT quantize the real scalar field

Motivation: In the lecture, you quantized the real scalar field by imposing non-trivial commutation relations. This choice, although motivated, seems ad-hoc; can't we consistently quantize the scalar field by imposing anti-commutation relations? In other words, the question is how much freedom do we have when we quantize a real

scalar field classically satisfying the Klein-Gordon equation? Note that this choice would have crucial physical consequences, because it would mean that spin-0-particles could behave as fermions and obey the Pauli exclusion principle. This would follow because $|p_1 p_2\rangle = a_{p_1}^\dagger a_{p_2}^\dagger |0\rangle = -a_{p_2}^\dagger a_{p_1}^\dagger |0\rangle = -|p_2 p_1\rangle$.

You saw in the lecture that a real scalar field is described classically by the action

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{m^2}{2} \phi^2 \right), \quad (2.6)$$

and the field $\phi(\vec{x})$ and its conjugate momentum $\Pi(\vec{p})$, in the Schroödinger picture, have the following mode expansions

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right), \quad (2.7)$$

$$\Pi(\vec{x}) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right). \quad (2.8)$$

Now instead of non-trivial commutation relations between $\phi(\vec{x})$ and $\Pi(\vec{x})$, we are going to adopt anti-commutation relations and see how far we can get. To that end, we assume,

$$\{\phi(\vec{x}), \phi(\vec{y})\} = 0 = \{\Pi(\vec{x}), \Pi(\vec{y})\}, \quad (2.9)$$

$$\{\phi(\vec{x}), \Pi(\vec{y})\} = \delta^{(3)}(x - y). \quad (2.10)$$

- a) Note that Eqs. (2.9) are the anticommuting analog of the usual commutation relations, while the non-trivial relation (2.10) differs by a factor of i . This is important. Figure out why the relation

$$\{\phi(\vec{x}), \Pi(\vec{y})\} = i\delta^{(3)}(x - y) \quad (2.11)$$

is inconsistent.

- b) Using the (seemingly) consistent set of relations (2.9)-(2.10), show that the Fourier transforms of $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(\vec{p})$ satisfy the relation

$$\{\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})\} = (2\pi)^3 \delta^{(3)}(p + q) \quad (2.12)$$

- c) Using Eq. (2.12) derive the following relations^a

$$\{a_{\vec{p}}, a_{\vec{q}}\} = i(2\pi)^3 \delta^{(3)}(p - q) \quad (2.13)$$

$$\{a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger\} = -i(2\pi)^3 \delta^{(3)}(p - q) \quad (2.14)$$

$$\{a_{\vec{p}}, a_{\vec{q}}^\dagger\} = 0, \quad (2.15)$$

You see now that so far we haven't encountered any problem with adopting anti-commutation relations, even if they seem a bit unusual. Nevertheless, we should expect some non-trivial consequences since we had to change the right-hand side of Eq. (2.12).

- d) Why is this theory problematic? Hint: calculate the Hamiltonian in terms of creation/annihilation operators following the steps in the lecture notes.

^aThe form of the Fourier transforms of the field and its momentum in terms of creation/annihilation operators comes simply from the form of the Lagrangian/Hamiltonian which remains unchanged here.

Exercise 2: Quantization of the complex scalar field

Motivation: In this exercise we proceed with the quantization of the complex scalar field. This kind of theory is very important since it is the simplest theory that can describe charged particles. In the Standard Model of particle physics, the Higgs field is a complex scalar field^a. What is more, their consistency with causality naturally gives rise to the notion of anti-particles.

We start with the action of a free complex scalar field

$$S = \int d^4x [(\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi] . \quad (3.16)$$

If we decompose the field into its real and imaginary part, the action (3.16) reduces to an action of two independent free real scalar fields (which are exactly the real and imaginary part of the complex scalar). This theory, similarly to the $O(3)$ -model, is of course Poincaré invariant but there is also another (internal) symmetry with a conserved current and charge. To calculate these currents, instead of consider the real and imaginary part as the independent degrees of freedom, we equivalently consider ϕ and ϕ^* as independent.

- Calculate the canonically conjugate fields (sometimes called “momentum” fields) of ϕ and ϕ^* , and write down their equations of motion.
- Show that the transformation $\phi \rightarrow e^{ia}\phi$ with a constant, is a symmetry of the theory described by Eq. (3.16). This is called $U(1)$ -symmetry since the transformation $U(a) = e^{ia}$ is a representation of the $U(1)$ Lie group. What is its infinitesimal version for both field ϕ and ϕ^* ?
- Calculate the conserved current and charge associated to this symmetry. What is the conserved charge of the field ϕ ? What is the conserved charge of the field ϕ^* ?

Now we are ready to quantize this theory keeping in mind that it is really a theory of two independent fields with opposite charges^b. For the quantization, we promote the fields to operators, where complex conjugation corresponds to hermitian conjugation, i.e., in the Schrödinger picture,

$$\phi^*(\vec{x}) \rightarrow \phi^\dagger(\vec{x}) , \quad (3.17)$$

and the fields and its canonical conjugate “momenta” adopt non-trivial commutation relations:

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^{(3)}(x - y) \quad , \quad [\phi^\dagger(\vec{x}), \Pi^\dagger(\vec{y})] = i\delta^{(3)}(x - y) \quad (3.18)$$

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 = [\Pi(\vec{x}), \Pi(\vec{y})] \quad , \quad [\phi^\dagger(\vec{x}), \phi^\dagger(\vec{y})] = 0 = [\Pi^\dagger(\vec{x}), \Pi^\dagger(\vec{y})] \quad (3.19)$$

- Argue about why we need the following mode expansion for the complex field

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left\{ a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}} \right\} , \quad (3.20)$$

where $b_{\vec{p}}$ a second set of creation/annihilation operators which is different then $a_{\vec{p}}$. Calculate the mode expansion of the canonical conjugate fields in the Schrödinger picture and then calculate all commutation relations of the creation/annihilation operators such that they are compatible with Eqs. (3.18)-(3.19)^c, i.e. show that the only non-trivial commutation relations are the following:

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(p - q) \quad , \quad [b_{\vec{p}}, b_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(p - q) \quad (3.21)$$

e) Compute the Hamiltonian in terms of creation/annihilation operators and interpret the results. What are the energies of the states $a_{\vec{p}}^\dagger|0\rangle$ and $b_{\vec{p}}^\dagger|0\rangle$?

f) Switch to the Heisenberg picture for the field

$$\phi(x) \equiv \phi_H(t, \vec{x}) = e^{iHt} \phi(\vec{x}) e^{-iHt} . \quad (3.22)$$

Show that the creation/annihilation operators in the Heisenberg picture take the form

$$e^{iHt} a_{\vec{p}} e^{-iHt} = a_{\vec{p}} e^{-i\omega_{\vec{p}}t} , \quad e^{iHt} b_{\vec{p}} e^{-iHt} = b_{\vec{p}} e^{-i\omega_{\vec{p}}t} , \quad (3.23)$$

g) Show that the propagators $\langle 0|\phi(x)\phi^\dagger(y)|0\rangle$ and $\langle 0|\phi^\dagger(x)\phi(y)|0\rangle$ in their momentum integral representation take the form

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)} , \quad (3.24)$$

$$\langle 0|\phi^\dagger(y)\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{+ip(x-y)} . \quad (3.25)$$

Are these propagator vanishing for spacelike separation $(x-y)^2$? Give an interpretation of this propagators as transition amplitudes. Finally, calculate also the commutator $[\phi(x), \phi^\dagger(y)]$ and show that it can be written as

$$[\phi(x), \phi^\dagger(y)] = \langle 0|\phi(x)\phi^\dagger(y)|0\rangle - \langle 0|\phi^\dagger(y)\phi(x)|0\rangle . \quad (3.26)$$

When does the commutator vanish? The vanishing of the commutator gives us (micro)causality in Quantum Field Theory.

With all of the above you should realize the following amazing things!

1. A consistent quantization of a free complex scalar field leads to different types of 1-particles states that are distinguished by their charge. This charge is the associated Noether charge of the $U(1)$ global symmetry. In fact, these states have opposite charges, thus we have particle/antiparticle states.
2. The existence of charged particles is something that we are all familiar classically. After all, in classical electrodynamics for example, nobody talks about the positron but only the electron. Nevertheless we see here that anti-particles are crucial to ensure that the complex field is also causal. Intuitively, by (micro)causality here we mean that the vanishing of the commutator for spacelike separation or in other words, that the creation/annihilation of particles cannot influence the creation/annihilation of particles separated by spacelike distances.
3. Everything fits together! If we want charged particles in a relativistic quantum theory with conserved charges, we need some global symmetry like $U(1)$, which requires that the scalar field is complex. Therefore, we must also have antiparticles with respect to this charge. Now some last food for thought; What if we demand that the symmetry $U(1)$ is local, i.e. depends on the spacetime points? In a few weeks, we will learn in the lecture how to deal with such a local symmetries, and they are even more amazing!

^aActually the Higgs field is a bit more complicated, It is a doublet of two complex scalars transforming in the fundamental representation of $SU(2)$.

^bThese charges are associated with the global $U(1)$ symmetry and are not electrical charges. You will see how fields with electric charges are introduced when you will talk about local $U(1)$ symmetry, instead of global.

^cFollow the steps of the lecture for the real scalar field.