Condensed Matter Theory problem set 11

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Problem 29: Bogoliubov transformation

(a) The bosonic operators $\hat{a}$, $\hat{a}^\dagger$ satisfy the canonical commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{a}, \hat{a}] = 0$. Define a new pair of operators by the transformation

$$\hat{b} = u\hat{a} + v\hat{a}^\dagger, \qquad \hat{b}^\dagger = u^*\hat{a}^\dagger + v^*\hat{a} \tag{1}$$

with $u, v \in \mathbb{C}$. Under which condition is this transformation canonical, *i.e.*, $[\hat{b}, \hat{b}^\dagger] = 1$? Find a parametrization for u and v in terms of real numbers.

(b) The Bogoliubov Hamiltonian for spin-1/2 electrons ($\sigma = \uparrow, \downarrow$) with isotropic dispersion relation $\xi_{\boldsymbol{k}}$ and real energy gap $\Delta \in \mathbb{R}$ is defined as

$$\mathcal{H} = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \sum_{\boldsymbol{k}} \left(\Delta c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} + \Delta c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \right). \tag{2}$$

Show that this Hamiltonian can be brought into the matrix form

$$\mathcal{H} = \sum_{\boldsymbol{k}} \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} & c_{-\boldsymbol{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\boldsymbol{k}} & -\Delta \\ -\Delta & -\xi_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} + \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}}. \tag{3}$$

Diagonalize $\mathcal{H}$ with the help of the Bogoliubov transformation

$$\begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\boldsymbol{k}} & \sin\theta_{\boldsymbol{k}} \\ \sin\theta_{\boldsymbol{k}} & -\cos\theta_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} \gamma_{\boldsymbol{k}\uparrow} \\ \gamma^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix}; \tag{4}$$

determine $\theta_{\boldsymbol{k}}$ such that the anomalous terms with $\gamma\gamma$ and $\gamma^\dagger\gamma^\dagger$ vanish. Sketch the spectrum of fermionic quasiparticles $\gamma_{\boldsymbol{k}\sigma}$ as a function of $\boldsymbol{k}$ and of $\xi_{\boldsymbol{k}}$.

Problem 30: Energy of the superconducting ground state

Compute the condensation energy in the superconducting ground state ($\Delta > 0$) relative to the normal state ($\Delta = 0$, $E_k = |\xi_k|$),

$$\Delta E = \langle \mathcal{H} - \mu N \rangle_{\mathrm{S}} - \langle \mathcal{H} - \mu N \rangle_{\mathrm{N}}\,, \tag{5}$$

expressed in terms of the gap Δ, the volume and the electronic density of states ν_0, in the limit of weak coupling $\nu_0 g \ll 1$. One can, *e.g.*, compute the expectation value of $\mathcal{H} - \mu N$ with the BCS wave function $|\mathrm{BCS}\rangle$, or (recommended) evaluate the term for the ground state energy in the Bogoliubov Hamiltonian. Assume that the attractive interaction is constant in a shell of width $\hbar\omega_D$ around the Fermi surface (as in the lecture), and use particle-hole symmetry within this shell, $-\sum_{k<k_F} \xi_k = \sum_{k>k_F} \xi_k$.

Problem 31: Hubbard-Stratonovich transformation

The partition function of an electron-phonon system can be written as the path integral

$$\mathcal{Z} = \int D\psi^* D\psi \int D\phi^* D\phi \, \exp\left\{-S_{\mathrm{el}}[\psi^*, \psi] - S_{\mathrm{ph}}[\phi^*, \phi] - S_{\text{el-ph}}[\psi^*, \psi, \phi^*, \phi]\right\} \tag{6}$$

with Grassmann fields ψ^*, ψ representing the fermions and complex fields ϕ^*, ϕ for the phonons. The action for the phonons S_{ph} and for the electron-phonon interaction $S_{\text{el-ph}}$ have the form

$$S_{\mathrm{ph}}[\phi^*, \phi] = \sum_q \phi_q^* (-i\omega_m + \omega_{\boldsymbol{q}}) \phi_q \tag{7}$$

$$S_{\text{el-ph}}[\psi^*, \psi, \phi^*, \phi] = \sum_q M_{\boldsymbol{q}} \rho_q (\phi_q + \phi_{-q}^*) \tag{8}$$

with the electronic density operator $\rho_q = \sum_k \psi_{k+q}^* \psi_k$, electron-phonon coupling $M_{\boldsymbol{q}}$, and the usual kinetic term $S_{\mathrm{el}}[\psi^*, \psi]$ for the electrons. The multi-indices $q = (i\omega_m, \boldsymbol{q})$ are a short-hand notation for frequency and momentum indices, and the sum $\sum_q = \sum_{\boldsymbol{q}} \beta^{-1} \sum_m$ includes a Matsubara sum as well as a momentum sum.

Perform a Gaussian integration over the phonon fields ϕ^*, ϕ in $\mathcal{Z}$ and derive an effective action $S_{\mathrm{eff}}[\psi^*, \psi]$ for the electrons alone (neglect a constant term which does not depend on ψ^* and ψ):

$$\mathcal{Z} = \int D\psi^* D\psi \, e^{-S_{\mathrm{eff}}[\psi^*, \psi]} \tag{9}$$

$$e^{-S_{\mathrm{eff}}[\psi^*, \psi]} = \int D\phi^* D\phi \, \exp\left\{-S_{\mathrm{el}}[\psi^*, \psi] - S_{\mathrm{ph}}[\phi^*, \phi] - S_{\text{el-ph}}[\psi^*, \psi, \phi^*, \phi]\right\} \tag{10}$$

The effective action contains an interaction term between the electrons; compare this to the result of the Fröhlich transformation given in the lecture.