The Ward identity of scale transformations

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Functional Renormalization
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References

Based on:

R.P. and G.P. Vacca, “The background scale Ward identity in quantum gravity” Eur.Phys.J. C77 (2017) no.1, 52 arXiv:1611.07005 [hep-th]

and work in progress with G. P. Vacca and V. Skrinjar.

Scale transformations

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Rescaling of lenghts

$$\delta s = \epsilon s$$

can be realized either by $\delta x^\mu = \epsilon x^\mu$ or $\delta g_{\mu\nu} = 2\epsilon g_{\mu\nu}$.
Choose the latter.

Anomalous scale invariance

Scalar field in external metric with action $S(\phi; g_{\mu\nu})$.
Probe behavior under scale transformations.

$$\begin{aligned} \delta_\epsilon g_{\mu\nu} &= 2\epsilon g_{\mu\nu} \\ \delta_\epsilon \Phi &= -\frac{d-2}{2}\epsilon\Phi \end{aligned}$$

Assume

$$\delta_\epsilon S = 0$$

Scale invariance broken in the quantum theory.

The cutoff term

$$\Delta S_k(\phi; g_{\mu\nu}) = \int d^d x \sqrt{g}\, \phi\, R_k\, \phi\ .$$

$$R_k = k^2 r(y)\ ; \qquad y = \Delta/k^2$$

Since

$$\delta_\epsilon \Delta = -2\epsilon\Delta\ .$$

we have $\delta_\epsilon R_k = -2\epsilon k^2 y r'$.
On the other hand $\partial_t R_k = 2k^2 r - 2k^2 y r'$, so

$$\delta_\epsilon R_k = -2\epsilon R_k + \epsilon \partial_t R_k\ .$$

The cutoff term transforms as

$$\delta_\epsilon \Delta S_k(\phi; g_{\mu\nu}) = \epsilon \frac{1}{2} \int d^d x \sqrt{g}\, \phi\, \partial_t R_k\, \phi\ .$$

The EAA

$$W_k(J; g_{\mu\nu}) = \log \int (d\Phi) Exp\Big[- S - \Delta S_k + \int d^d x \, J\Phi \Big]$$

$$\Gamma_k(\varphi; g_{\mu\nu}) = -W_k(J; g_{\mu\nu}) + \int d^d x \, J\varphi - \Delta S_k(\varphi)$$

where $\varphi = \langle \phi \rangle = \frac{\delta W_k}{\delta J}$

The Ward identity

$$
\begin{aligned}
\delta_\epsilon W_k &= -\langle \delta_\epsilon \Delta S_k \rangle + \int d^d x \, J \langle \delta_\epsilon \Phi \rangle \\
&= \epsilon \Bigg[-\frac{1}{2} \mathrm{Tr}\, \partial_t R_k \frac{\delta^2 W_k}{\delta J \delta J} + \frac{1}{2} \int d^d x \sqrt{g} \, \frac{\delta W_k}{\delta J} \partial_t R_k \frac{\delta W_k}{\delta J} \\
&\quad + \frac{d-2}{2} \int d^d x \, J \frac{\delta W_k}{\delta J} \Bigg] ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_\epsilon \Gamma_k(\varphi) &= -\delta_\epsilon W_k + \int d^d x \, J \delta_\epsilon \varphi - \delta_\epsilon \Delta S_k(\varphi) \\
&= \epsilon \frac{1}{2} \mathrm{Tr}\, \partial_t R_k \frac{\delta^2 W_k}{\delta J \delta J} \\
&= \epsilon \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta \varphi \delta \varphi} + R_k \right)^{-1} k \frac{d R_k}{dk} = \epsilon \partial_t \Gamma_k
\end{aligned}
$$

Meaning

$$\delta_\epsilon \Gamma_k = \epsilon \partial_t \Gamma_k$$

thus defining

$$\delta_\epsilon^E = \delta_\epsilon - \epsilon k \frac{d}{dk}$$

we have

$$\delta^E \Gamma_k = 0$$

The trace anomaly

$$\epsilon \int d^d x \left[2 g_{\mu\nu} \frac{\delta \Gamma_k}{\delta g_{\mu\nu}} - \frac{d-2}{2} \varphi \frac{\delta \Gamma_k}{\delta \varphi} \right] = \epsilon k \frac{d\Gamma_k}{dk}$$

$$\int dx \sqrt{g} \, \langle T^{\mu}{}_{\mu} \rangle_k = \frac{d-2}{2} \int dx \, \varphi \frac{\delta \Gamma_k}{\delta \varphi} + k \frac{d\Gamma_k}{dk}$$

Local transformations

Assume S is invariant under Weyl transformations

$$\begin{aligned} \delta_\epsilon g_{\mu\nu} &= 2\epsilon(x) g_{\mu\nu} \\ \delta_\epsilon \Phi &= -\frac{d-2}{2}\epsilon(x)\Phi \end{aligned}$$

Plan

If we can write cutoff actions that are invariant under extended transformations where the fields transform as above and also

$$\delta k = -\epsilon(x) k$$

then we will find the same msWI as before.

Note that k cannot be constant!

Weyl calculus

Introduce a dilaton field χ and define flat abelian gauge field $b_\mu = -\chi^{-1}\partial_\mu\chi$ transforming as $\delta b_\mu = \partial_\mu\epsilon$.
For scalar field ϕ of weight w

$$D_\mu\phi = \partial_\mu\phi - wb_\mu\phi$$

More generally

$$\hat{\Gamma}_\mu{}^\lambda{}_\nu = \Gamma_\mu{}^\lambda{}_\nu - \delta^\lambda_\mu b_\nu - \delta^\lambda_\nu b_\mu + g_{\mu\nu}b^\lambda$$

is invariant under local Weyl transformations, hence for a tensor of weight w

$$D_\mu t = \hat{\nabla}_\mu t - wb_\mu t$$

is diffeomorphism and Weyl covariant.

Cutoff terms

Replacing ∇_μ by D_μ the cutoff terms now satisfy

$$\delta_\epsilon \Delta S_k = \int dx \, \epsilon \, k \frac{\delta}{\delta k} \Delta S_k$$

Local ERGE

$$\delta k \frac{\delta \Gamma_k}{\delta k} = \frac{1}{2}\mathrm{STr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + \mathcal{R}_k\right)^{-1} \delta k \frac{\delta \mathcal{R}_k}{\delta k}\right]$$

Modified Weyl WI

$$\delta_\epsilon \Gamma_k = \int dx\, \epsilon k \frac{\delta \Gamma_k}{\delta k}$$

$$\langle T^\mu{}_\mu \rangle_k(x) = \frac{d-2}{2} \varphi(x) \frac{\delta \Gamma_k}{\delta \varphi(x)} + k(x) \frac{\delta \Gamma_k}{\delta k(x)}$$

Note: where is the RG?

Assume $u = k/\chi$ is constant. Think of the EAA as

$$\Gamma_k(\phi; g_{\mu\nu}, \chi) = \Gamma_u(\phi; g_{\mu\nu}, \chi)$$

It satisfies

$$u\frac{d\Gamma_k}{du} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi} + R_k\right)^{-1} u\frac{\delta R_u}{du}$$

R. P., New J. Phys. **13** 125013 (2011) arXiv:1110.6758 [hep-th]
A. Codello, G. D'Odorico, C. Pagani, R. P., Class. Quant. Grav. 30 (2013), arXiv:1210.3284 [hep-th]
C. Pagani, R. P. Class. Quant. Grav. **31** (2014) 115005, arXiv:1312.7767 [hep-th]

Turn on gravity

Definition of EAA requires a background split

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Two generalizations

- “physical” scale transformation $\delta_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu}$
- “background” scale transformations $\delta_\epsilon g_{\mu\nu} = 0$

Split symmetry

Bare action is invariant under

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$\delta \bar{g}_{\mu\nu} = \epsilon_{\mu\nu} \ , \qquad \delta h_{\mu\nu} = -\epsilon_{\mu\nu} \ .$$

but the EAA $\Gamma_k(\mathbf{h}; \bar{\mathbf{g}})$ is not.

Same with exponential split

$$\mathbf{g} = \bar{\mathbf{g}} e^{\mathbf{X}} \ ; \qquad X^\rho{}_\nu = \bar{g}^{\rho\sigma} h_{\sigma\nu} \ .$$

$$\delta \bar{g}_{\mu\nu} = \epsilon_{\mu\nu} \ , \qquad \delta \mathbf{X} = -\frac{ad_{\mathbf{X}}}{e^{ad_{\mathbf{X}}} - 1} \bar{\mathbf{g}}^{-1} \delta \bar{\mathbf{g}}.$$

Plan

- Write the anomalous Ward identity for the split symmetry or a subgroup thereof
- Solve it to eliminate from the EAA a number of fields equal to the number of parameters of the transformation
- Write the flow equation for the EAA depending on the remaining variables

Transformations

Here I will consider the case of a rescaling of the background

$$\delta\bar{g}_{\mu\nu} = 2\epsilon\bar{g}_{\mu\nu}$$

Define $h^\mu{}_\nu = h^{T\mu}{}_\nu + \frac{1}{d}\delta^\mu{}_\nu h$, $h = h^\perp + \bar{h}$ with $\int dx\sqrt{\bar{g}}h^\perp = 0$

In exponential parametrization $g_{\mu\nu}$ is left invariant provided

$$\begin{aligned} \delta h^{T\mu}{}_\nu &= 0 \\ \delta h^\perp &= 0 \\ \delta\bar{h} &= -2d\epsilon \end{aligned}$$

(Note $\delta h^T_{\mu\nu} = 2\epsilon h^T_{\mu\nu}$)

Gauge fixing

$$S_{GF} = \frac{1}{2\alpha} \int d^d x \sqrt{\bar{g}}\, F_\mu Y^{\mu\nu} F_\nu ,$$

$$F_\mu = \bar{\nabla}_\rho h^\rho{}_\mu - \frac{\beta + 1}{d} \bar{\nabla}_\mu h$$

$$\delta F_\mu = 0$$

To compensate $\delta\sqrt{\bar{g}} = d\epsilon\sqrt{\bar{g}}$, choose

$$Y^{\mu\nu} = \bar{\Delta}^{\frac{d-2}{2}} \bar{g}^{\mu\nu} \; .$$

Since $\delta\bar{\Delta} = -2\epsilon\bar{\Delta}$, we have $\delta S_{GF} = 0$.

Ghost action

$$S_{gh}(C^*_\mu, C^\mu; \bar{g}_{\mu\nu}) = \int d^d x \sqrt{\bar{g}}\, C^*_\mu Y^{\mu\nu} \Delta_{FP\nu\rho} C^\rho$$

$$\delta^{(Q)}_\eta \bar{\mathbf{g}} = 0 \; ; \qquad \bar{\mathbf{g}} \delta^{(Q)}_\eta e^{\mathbf{X}} = \mathcal{L}_\eta \mathbf{g} = \mathcal{L}_\eta \bar{\mathbf{g}} e^{\mathbf{X}} + \bar{\mathbf{g}} \mathcal{L}_\eta e^{\mathbf{X}} \; .$$

$$\Delta_{FP\mu\nu} C^\nu = \bar{\nabla}_\rho \left((\delta^{(Q)}_C \mathbf{X})^\rho{}_\mu + \frac{1+\beta}{d} \delta^\rho{}_\mu \mathrm{tr}(\delta^{(Q)}_C \mathbf{X}) \right)$$

$$\begin{aligned} \delta^{(Q)}_C \mathbf{X} &= \frac{ad_{\mathbf{X}}}{e^{ad_{\mathbf{X}}} - \mathbf{1}} \left(\bar{\mathbf{g}}^{-1} \mathcal{L}_C \bar{\mathbf{g}} + \mathcal{L}_C e^{\mathbf{X}} e^{-\mathbf{X}} \right) \\ &= \bar{\mathbf{g}}^{-1} \mathcal{L}_C \bar{\mathbf{g}} + \mathcal{L}_C \mathbf{X} + \frac{1}{2} [\bar{\mathbf{g}}^{-1} \mathcal{L}_C \bar{\mathbf{g}}, \mathbf{X}] + O(C\mathbf{X}^2) \end{aligned}$$

Ghosts

Choosing

$$\delta C^*_\mu = 0 \; , \qquad \delta C^\mu = 0 \; .$$

one has

$$\delta \Delta_{FP\mu\nu} C^\nu = 0$$

and again $\delta S_{gh} = 0$

Finally

$$S_{aux} = \int dx \sqrt{\bar{g}}\, B_\mu Y^{\mu\nu} B_\nu$$

If $\delta B_\mu = 0$ then $\delta S_{aux} = 0$

Cutoff terms

$$\Delta S_k(h^\mu{}_\nu;\bar{g}_{\mu\nu}) = \frac{1}{2}\int d^dx\sqrt{\bar{g}}\,h^\mu{}_\nu\mathcal{R}_k(\bar{\Delta})h^\nu{}_\mu$$

$$\Delta S_k^{gh}(C^*_\mu, C^\mu;\bar{g}_{\mu\nu}) = \int d^dx\sqrt{\bar{g}}\,C^*_\mu\mathcal{R}_k^{gh}(\bar{\Delta})C^\mu$$

$$\Delta S_k^{aux}(B_\mu;\bar{g}_{\mu\nu}) = \int d^dx\sqrt{\bar{g}}\,B^*_\mu\bar{g}^{\mu\nu}\mathcal{R}_k^{aux}(\bar{\Delta})B_\nu$$

$\mathcal{R}_k = c\,k^d r(y)$
$\mathcal{R}_k^{gh}(\bar{\Delta}) = c_{gh}k^d r(y)$
$\mathcal{R}_k^{aux}(\bar{\Delta}) = c_{aux}k^{d-2}r(y)$
and $y = \bar{\Delta}/k^2$

Transformations

As before

$$\delta\mathcal{R}_k = \epsilon(-d\mathcal{R}_k + \partial_t\mathcal{R}_k) \ .$$

$$\begin{aligned}\delta\Delta S_k(h^\mu{}_\nu;\bar{g}_{\mu\nu}) &= \frac{1}{2}\epsilon\int d^dx\sqrt{\bar{g}}\left[h^{T\mu}{}_\nu\,\partial_t\mathcal{R}_k h^{T\nu}{}_\mu + h\,\partial_t\mathcal{R}_k h\right] \\ &\quad -2d\epsilon\int d^dx\sqrt{\bar{g}}\,\mathcal{R}_k h\,,\end{aligned}$$

$$\delta\Delta S_k^{gh}(C^*_\mu, C^\mu;\bar{g}_{\mu\nu}) = \epsilon\int d^dx\sqrt{\bar{g}}\,C^*_\mu\partial_t\mathcal{R}_k^{gh}C^\mu \ .$$

$$\delta\Delta S_k^{aux}(B_\mu;\bar{g}_{\mu\nu}) = \epsilon\int d^dx\sqrt{\bar{g}}\,B_\mu\bar{g}^{\mu\nu}\partial_t\mathcal{R}_k^{aux}B_\nu \ .$$

The generating functionals

$$e^{W_k(j_{T\mu}{}^{\nu}, j, J_*^\mu, J_\mu; \bar{g}_{\mu\nu})} = \int (dh dC^* dC dB) Exp\Big[- S - S_{GF} - S_{gh} - \Delta S_k - \Delta S_k^{gh} - \Delta S_k^{aux} + \int d^d x \left(j_{T\mu}{}^{\nu} h^{T\mu}{}_{\nu} + jh + J_*^\mu C_\mu^* + J_\mu C^\mu \right) \Big]$$

$$\Gamma_k(h^T_{\mu\nu}, h, C^*_\mu, C^\mu; \bar{g}_{\mu\nu}) = -W_k(j_{T\mu}{}^{\nu}, j, J_*^\mu, J_\mu; \bar{g}_{\mu\nu}) + \int d^d x \left(j_{T\mu}{}^{\nu} h^{T\mu}{}_{\nu} + jh + J_*^\mu C_\mu^* + J_\mu C^\mu \right) - \Delta S_k - \Delta S_k^{gh} - \Delta S_k^{aux}$$

The msWI

$$\delta_\epsilon \Gamma_k = \epsilon \partial_t \Gamma_k$$

Under finite transformations

$$\Gamma_k({h^T}^\mu{}_\nu, h^\perp, \bar{h}, C^*_\mu, C^\mu; \bar{g}_{\mu\nu}) = \Gamma_{\Omega^{-1}k}({h^T}^\mu{}_\nu, h^\perp, \bar{h} - 2d \log \Omega, C^*_\mu, C^\mu; \Omega^2 \bar{g}_{\mu\nu})$$

Solving the msWI

$$\Gamma_k({h^T}^{\mu}{}_{\nu}, h^{\perp}, \bar{h}, C^*_{\mu}, C^{\mu}; \bar{g}_{\mu\nu}) = \hat{\Gamma}_{\hat{k}}({h^T}^{\mu}{}_{\nu}, h^{\perp}, C^*_{\mu}, C^{\mu}; \hat{g}_{\mu\nu})$$

where e.g.

$$\hat{k} = \bar{V}^{1/d} k \; ; \qquad \hat{g}_{\mu\nu} = \bar{V}^{-2/d} \bar{g}_{\mu\nu}$$

We have eliminated *one* degree of freedom.

In linear parametrization

T. R. Morris, JHEP **1611** (2016) 160, arXiv:1610.03081 [hep-th]

N. Ohta, arXiv:1701.01506 [hep-th]

Local transformations

Can we generalize to local Weyl transformations?
If we can write gauge and cutoff actions that are invariant under extended transformations where the fields transform as above and also

$$\delta k = -\epsilon(x) k$$

then we will find the same msWI as before.

Weyl calculus

Choose a representative $\hat{g}_{\mu\nu}$ such that

$$\bar{g}_{\mu\nu} = e^{2\bar{\sigma}}\hat{g}_{\mu\nu} \; ; \qquad g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu} \; ; \qquad \sigma = \bar{\sigma} + \frac{1}{2d}h$$

Define flat abelian gauge field

$$b_\mu = -\partial_\mu \bar{\sigma}$$

transforming as $\delta b_\mu = \partial_\mu \epsilon$.
Proceed as with matter fields.

Gauge fixing and cutoff terms

Replacing $\bar{\nabla}_\mu$ by D_μ can write invariant GF and ghost terms.
Writing

$$Y^{\mu\nu} = e^{-(d-2)\bar{\sigma}} \bar{g}^{\mu\nu}$$

we do not need auxiliary fields anymore.

The cutoff terms now satisfy

$$\delta_\epsilon \Delta S_k = \int dx \, \epsilon \, k \frac{\delta}{\delta k} \Delta S_k$$

Modified background Weyl WI

$$\delta_\epsilon \Gamma_k = \int \epsilon k \frac{\delta \Gamma_k}{\delta k}$$

or

$$\delta^E_\epsilon \Gamma_k = 0$$

has solution

$$\Gamma_k(h^{T\mu}{}_\nu, h, C^*_\mu, C^\mu; \hat{g}_{\mu\nu}, \bar{\sigma}) = \hat{\Gamma}_{\hat{k}}(h^{T\mu}{}_\nu, \sigma, C^*_\mu, C^\mu; \hat{g}_{\mu\nu})$$

where

$$\hat{k} = e^{\bar{\sigma}} k$$

and we already defined the invariants

$$\sigma = \bar{\sigma} + \frac{1}{2d} h \, ; \qquad \hat{g}_{\mu\nu} = e^{-\bar{\sigma}} \bar{g}_{\mu\nu}$$

We have eliminated one function.

HOWEVER...

...the fiducial metric $\hat{g}_{\mu\nu}$ is an unphysical external element.
It introduces a new split symmetry

$$\delta\hat{g}_{\mu\nu} = 2\epsilon\hat{g}_{\mu\nu} \; ; \delta\bar{\sigma} = -\epsilon$$

that leaves the background metric invariant.

The chosen cutoff does not behave in a simple way under these transformations. Writing and solving the corresponding WI is possible but requires additional approximations.

Summary

- Anomalous WI for scale transformations, both global and local, is expression of dimensional analysis
- Background global scale WI can be solved in full generality. Will give correct dependence of EAA on volume.
- Background Weyl WI can probably be solved within some truncations.
- Simple scaling properties require certain choices in the gauge and cutoff terms

To do

- Apply WI to particle physics models with classical scale invariance
- Relation between local FRGE and Osborn's local RG
- Use pure cutoffs in study of gravitational truncated RG flows
- Solve the mWI of background Weyl transformations in some truncation
- WI of physical scale transformations in QG