Viability of the Asymptotic Safety scenario

beyond renormalizability ?

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$$\{\text{nonperturbative renormalizability}\}$$

$$\cap$$

$$\{\text{Background Independence}\}$$

$$\cap$$

$$\{\text{Hilbert space positivity}\}$$

$$\neq \qquad \varnothing \qquad \text{?}$$

Does the non-Gaussian fixed point of Quantum Einstein Gravity define a conformal field theory?

How is QEG in $2 + \varepsilon$ dimensions, $\varepsilon \to 0$, related to the strictly $2D$ approaches to quantum gravity ?

(Liouvilly theory, matrix models, stat. mech. simulations, etc.)

1. The two-dimensional limit of QEG taken at the level of action functionals

2. The NGFP as a conformal field theory

3. Unitarity vs. Stability vs. Locality

4. Outlook

Lit.: A. Nink and MR, JHEP 02 (2016) 167

and arXiv: 1512. 06805

Effective Einstein equation re-interpreted

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \qquad \text{(or similar)}$$

↳ matter-like field on classical spacetime with metric $\bar{g}_{\mu\nu}$

Generic EAA:

$$\Gamma_k[\Phi, \bar{\Phi}] \equiv \Gamma_k[\varphi; \bar{\Phi}]$$

$$\varphi^i \equiv \Phi^i - \bar{\Phi}^i \qquad = \langle \hat{\varphi}^i \rangle$$

$$\frac{1}{\sqrt{\bar{g}}} \frac{\delta \Gamma_k[\varphi; \bar{\Phi}]}{\delta \varphi^i} + (\mathcal{R}_k)_{ij} \varphi^j = J_i$$

Self-consistent backgrounds from tadpole condition:

$$\frac{\delta}{\delta \varphi^i} \Gamma_k[\varphi; \bar{\Phi}] \Bigg|_{\varphi = 0, \bar{\Phi} = \bar{\Phi}_k^{sc}} = 0$$

Or:

$$\frac{\delta}{\delta \Phi^i} \Gamma_k[\Phi, \bar{\Phi}] \Bigg|_{\Phi = \bar{\Phi} = \bar{\Phi}_k^{sc}} = 0$$

Example:

$$\Gamma_k = \Gamma_k^{grav}[g,\bar{g}] + \Gamma_k^{M}[g,A,\bar{g}] + S_{gf} + S_{gh}$$

<u>Tadpole eqs.</u> (@ vanishing ghosts):

$$0 = \frac{\delta}{\delta g_{\mu\nu}} \left\{ \Gamma_k^{grav}[g,\bar{g}] + \Gamma_k^{M}[g,A,\bar{g}] \right\} \Bigg|_{\substack{g=\bar{g}=\bar{g}_k^{sc} \\ A=\bar{A}=\bar{A}_k^{sc}}}$$

$$0 = \frac{\delta}{\delta A} \ \ldots\ldots$$

<u>Stress tensors</u> for A and $h_{\mu\nu}$:

$$T^{M}[\bar{g},A]^{\mu\nu} := \frac{2}{\sqrt{\bar{g}}} \frac{\delta}{\delta g_{\mu\nu}} \Gamma_k^{M}[g,A,\bar{g}] \Bigg|_{g=\bar{g}}$$

$$T^{grav}[\bar{g}]^{\mu\nu} := \frac{2}{\sqrt{\bar{g}}} \frac{\delta}{\delta g_{\mu\nu}} \Gamma_k^{grav}[g,\bar{g}] \Bigg|_{g=\bar{g}}$$

$$= \frac{2}{\sqrt{\bar{g}}} \frac{\delta}{\delta h_{\mu\nu}} \Gamma_k^{grav}[h;\bar{g}] \Bigg|_{h=0}$$

<u>Effective Einstein eq.</u> has zero LHS:

$$0 = T_{\mu\nu}^{grav}[\bar{g}_k^{sc}] + T_{\mu\nu}^{M}[\bar{g}_k^{sc}, \bar{A}_k^{sc}]$$

$\underline{d = 2 + \varepsilon} :$

QEG   in   the   Einstein - Hilbert   truncation

$$\Gamma_k^{grav}[g,\bar{g}] \equiv \Gamma_k^{grav}[g] = -\frac{1}{16\pi G_k} \int d^{2+\varepsilon}x \left(R - 2\Lambda_k \right)$$

$$g_k = k^\varepsilon G_k \quad , \quad \lambda_k = k^{-2} \Lambda_k$$

$$\partial_t g_k = \varepsilon g_k - b g_k^2 + O(g^3)$$

$\underline{NGFP :}$

$$g_* = \frac{\varepsilon}{b}$$

$$\lambda_* \sim \varepsilon$$

$\underline{RG \ trajectories}$ near $(g_*, \lambda_*) :$

$$\left(g_k , \lambda_k \right) =: \varepsilon \left(\breve{g}_k , \breve{\lambda}_k \right)$$

$$\left(G_k , \Lambda_k \right) =: \varepsilon \left(\breve{G}_k , \breve{\Lambda}_k \right)$$

$\underbrace{\qquad\qquad\qquad\qquad}$
finite for $\varepsilon \to 0$

Stress tensor of $h_{\mu\nu}$

$$T^{grav}_{\mu\nu}[\bar{g}] = \frac{1}{8\pi G_K}\left(\bar{G}_{\mu\nu} + \Lambda_K \, \bar{g}_{\mu\nu}\right)$$

Trace $\quad \bar{g}^{\mu\nu}\, T^{grav}_{\mu\nu}[\bar{g}] \equiv \Theta_K[\bar{g}] \quad :$

$$\Theta_K[\bar{g}] = \underbrace{\frac{1}{16\pi G_K}}_{= \varepsilon \breve{G}_K}\left[-\underbrace{(d-2)}_{=\varepsilon}\bar{R} + 2d\underbrace{\Lambda_K}_{=\varepsilon\breve{\Lambda}_K}\right]$$

has an <u>unambiguous</u> limit $\quad \varepsilon \to 0 \; :$

$$\Theta_K[\bar{g}] = \frac{1}{16\pi \breve{G}_K}\left[-\bar{R} + 4\breve{\Lambda}_K\right] + O(\varepsilon)$$

$$= \underbrace{\frac{1}{16\pi \breve{g}_K}\left[-\bar{R} + 4\breve{\lambda}_K k^2\right]}_{\text{"lives" in exactly 2 dim.!}} + O(\varepsilon)$$

For the theory defined by the constant NGFP trajectory:

$$\Theta^{NGFP}_K[\bar{g}] = \frac{b}{16\pi}\left[-\bar{R} + 4\lambda_* k^2\right]$$

$$\Theta^{NGFP}_{K=0}[\bar{g}] = -\frac{b}{16\pi}\bar{R}$$

Intrinsically 2D description of the limiting theory: $\Gamma_k^{grav, \, 2D}[g]$

Take limit $\varepsilon \to 0$ at the level of __actions.__

For spacetimes of arbitrary topological type:

Euler number $\searrow$ moduli $\searrow$

$$\frac{1}{\varepsilon} \int d^{2+\varepsilon}x \sqrt{g} \, R = \frac{4\pi}{\varepsilon} \chi + C(\{\tau\})$$

$$- \frac{1}{4} \int d^2x \sqrt{g} \, R \, \Box^{-1} R \quad + O(\varepsilon)$$

$\nearrow$ Polyakov (induced gravity) action: non-local !

$$\Gamma_k^{grav, \, 2D}[g] = \frac{1}{64\pi \, \breve{G}_k} \int d^2x \sqrt{g} \, R \, \Box^{-1} R$$

$$+ \frac{\breve{\Lambda}_k}{8\pi \, \breve{G}_k} \int d^2x \sqrt{g} \qquad + \text{topolog.}$$

Proof of a simple special case:

trivial spacetime topology allowing $g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$
globally, with $\hat{R} \equiv R(\hat{g}) = 0$.

$$R = e^{-2\sigma}\left[\hat{R} - (d-1)(d-2)\hat{D}_\mu\sigma\,\hat{D}^\mu\sigma - 2(d-1)\hat{\Box}\sigma\right]$$

$$\frac{1}{\varepsilon}\int d^d x\,\sqrt{g}\,R = \frac{1}{\varepsilon}\int d^d x\,\sqrt{\hat{g}}\,e^{\overbrace{(d-2)\sigma}^{\varepsilon}}\left[\underbrace{\hat{R}}_{=0} + \underbrace{(d-1)(d-2)(\hat{D}_\mu\sigma)^2}_{=\varepsilon}\right]$$

$$= \int d^2 x\,\sqrt{\hat{g}}\,\hat{D}_\mu\sigma\,\hat{D}^\mu\sigma \quad + O(\varepsilon)$$

$$= \int d^2 x\,\underbrace{\sigma}_{=-\frac{1}{2}\Box^{-1}R}\,\underbrace{\sqrt{\hat{g}}(-\hat{\Box})\sigma}_{=\frac{1}{2}\sqrt{g}R} \quad + O(\varepsilon)$$

$$= -\frac{1}{4}\int d^2 x\,\sqrt{g}\,R\,\Box^{-1}R \quad + \ldots$$

In $d=2$ for $\hat{R}=0$:

$$\sqrt{g}\,R = -2\,\sqrt{\hat{g}}\,\hat{\Box}\sigma$$

$$R = -2\,\Box\sigma$$

$$\leadsto \quad \sigma = -\frac{1}{2}\Box^{-1}R$$

Nonlocal EAA in exactly 2 dimensions

$$\Gamma_K^{\text{grav, 2D}}[g] = \left(\frac{3}{2}\frac{1}{\breve{g}_K}\right)\frac{1}{96\pi}\int d^2x \sqrt{g}\left(R\,\square^{-1}R + 8\,\breve{\lambda}_K\,k^2\right)$$

implies the following stress tensor:

$$T_{\mu\nu}^{\text{grav}}[g] = \left(\frac{3}{2}\frac{1}{\breve{g}_K}\right)\frac{1}{96\pi}\left[\,g_{\mu\nu}\,D_\gamma(\square^{-1}R)\,D^\gamma(\square^{-1}R)\right.$$

$$-2\,D_\mu(\square^{-1}R)\,D_\nu(\square^{-1}R) \;+\; 4\,D_\mu D_\nu(\square^{-1}R)$$

$$\left.-\,4\,g_{\mu\nu}R \;+\; 8\,\breve{\lambda}_K\,k^2\,g_{\mu\nu}\,\right]$$

reproduces $\quad \Theta_K[g] = \left(\frac{3}{2}\frac{1}{\breve{g}_K}\right)\frac{1}{24\pi}\left[-R + 4\,\breve{\lambda}_K\,k^2\right] \qquad \checkmark$

Effective field eq. for pure gravity:

$$\boxed{0 = T_{\mu\nu}^{\text{grav}}[\bar{g}_K^{\,sc}]} \quad\Rightarrow\quad \Theta_K[\bar{g}_K^{\,sc}] = 0$$

$$\Longleftrightarrow \quad \boxed{R(\bar{g}_K^{\,sc}) = 4\,\breve{\lambda}_K\,k^2}$$

Higher correlators $\langle T_{\mu\nu}^{grav}(x_1) \, T_{\alpha\beta}^{grav}(x_2) \cdots \rangle$

generated by multiple differentiation of $\Gamma_k^{grav, \, 2D}$

Example: Trace 2 pt. function at $K=0$ for $g_{\mu\nu} = \delta_{\mu\nu}$

$$\boxed{\langle \Theta_o(x) \, \Theta_o(y) \rangle = -\frac{1}{12\pi} \, c_{grav}^{NGFP} \, \partial^r \partial_r \, \delta(x-y)}$$

Sign of the Schwinger term is crucial:

Smeared operator $\Theta_o[f] := \int d^2x \, f(x) \, \Theta_o(x)$ satisfies

$$\underbrace{\langle \Theta_o[f]^2 \rangle}_{\text{formally positive}} = \frac{1}{12\pi} \, c_{grav}^{NGFP} \, \underbrace{\int d^2x \, (\partial_\mu f) \delta^{\mu\nu} (\partial_\nu f)}_{\text{positive}}$$

$\Longrightarrow$ It is impossible to define $\Theta_o[f]^2$ as a

positive comp. operator unless $c_{grav}^{NGFP} > 0$!

The "NGFP theory":

$$K \longmapsto (g_*, \lambda_*) \quad \forall k \in [0, \infty)$$

$$\Gamma_k^{\text{grav, 2D, NGFP}}[g] = \left(\tfrac{3}{2} b\right) \frac{1}{96\pi} \int d^2x \sqrt{g} \left(R \Box^{-1} R + 8 \lambda_* k^2\right)$$

Standard EA:

$$\boxed{\Gamma^{\text{grav, 2D, NGFP}}[g] = \left(\tfrac{3}{2} b\right) \frac{1}{96\pi} \int d^2x \sqrt{g} \; R \Box^{-1} R}$$

has vanishing cosmological constant;

self-consistent backgrounds are flat: $R(\bar{g}_k^{sc}) = 0$

Reminder :

Induced gravity by a generic CFT

Consider a 2D theory of matter fields x, couple it to gravity by diff. inv. action $S[x, g]$, and compute

$$e^{-S_{ind}[g]} = \int \mathcal{D}x \, e^{-S[x, g]}$$

Then the (conserved, symmetric) stress tensor

$$T[g]^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_{ind}[g]}{\delta g_{\mu\nu}}$$

has a trace of the form

$$T[g]^{\mu}{}_{\mu} = -c \frac{1}{24\pi} R \quad + const$$

corresponding to

$$S_{ind}[g] = +c \frac{1}{96\pi} \int d^2x \sqrt{g} \, R \, \square^{-1} R \quad + \cdots$$

iff the original theory is a CFT
and has central charge c.

NB: The modes of the (traceless, non-conserved) tensor

$$T'_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\gamma}{}_{\gamma} \quad \text{satisfy Virasoro algebra with}$$

central extension $\sim c$.

Unitary vs. non-unitary CFT's

A conformal field theory can be unitary only if its central charge is positive, $c > 0$.

unitary theories: Virasoro algebra represented on a true Hilbert space, no negative norm states

non-unitary models: no quantum mechanical probability interpretation !
$\longrightarrow$ stat. mech.

Asymptotic Safety: conformality and unitarity

(a) According to the NGFP theory, the "dynamics" of the metric fluctuations is governed by a <u>conformal</u> field theory.

(b) The central charge of this theory is

$$\boxed{c_{grav}^{NGFP} = \frac{3}{2} b = \frac{3}{2} \frac{1}{\overset{\circ}{g}_*}}$$

(c) In pure gravity (and gravity coupled to not too many matter fields) the 'grav' sector is described by a <u>unitary</u> CFT

    — thus complying with the requirement of Hilbert space positivity.

$$c_{grav}^{NGFP} = \begin{cases} 19 - N & \text{Linear param. } g = \bar{g} + h \\ 25 - N & \text{exponential p. } g = \bar{g}\, e^h \end{cases}$$

The stability — unitarity antagonism

$$\Gamma_{k=0}^{\text{grav, 2D, NGFP}}[g] = \frac{C_{\text{grav}}^{\text{NGFP}}}{96\pi} \underbrace{\int d^2x \sqrt{g}\, R \,\Box^{-1} R}_{\text{negative definite fctl. of } g_{\mu\nu}}$$

$$\left.\begin{array}{l} C_{\text{grav}}^{\text{NGFP}} < 0 \\[2em] C_{\text{grav}}^{\text{NGFP}} > 0 \end{array}\right\} \Longleftrightarrow \text{eff. action is} \left\{\begin{array}{l} \text{positive def.} \\[2em] \text{negative def.} \end{array}\right.$$

instability ?

no !

Liouville representation of the NGFP theory

fix topological class, moduli;

conformal gauge: $g_{\mu\nu} = e^{2\phi} \hat{g}_{\mu\nu}$

$$\Gamma_k^{grav,\,2D,\,NGFP}[e^{2\phi}\hat{g}] = \frac{c_{grav}^{NGFP}}{96\pi} \int \hat{R}\,\hat{\Box}^{-1}\hat{R} + \Gamma_k^{L}[\phi;\hat{g}]$$

due to the reference metric   Liouville action

$$\Gamma_k^{L}[\phi;\hat{g}] = \frac{c_{grav}^{NGFP}}{96\pi} \int d^2x\,\sqrt{\hat{g}}\left\{ -\tfrac{1}{2}\hat{D}_\mu\phi\,\hat{D}^\mu\phi \right.$$

$$\left. -\tfrac{1}{2}\hat{R}\phi + \breve{\lambda}_* k^2 e^{2\phi} \right\}$$

Kinetic term is negative iff central charge is positive !

"Wrong" sign analogous to 4D conformal factor "problem" :

physically correct (attractivity of gravity),

no real instability after imposing physical state conditions;

c.f. Gauss law: $\vec{\nabla}^2 A_0(t,\vec{x}) = (\psi^+\psi)(t,\vec{x})$

$\longrightarrow$ ϕ, A_0 fully constrained fields, determined by matter

Central charge of the Liouville – CFT per se

Γ^L_k implies eom $\boxed{\hat{\Box}\phi + (2\breve{\lambda}_* k^2)e^{2\phi} = \tfrac{1}{2}\hat{R}}$

and stress tensor $T^L_K[\phi;\hat{g}]^{\mu\nu} \equiv \dfrac{2}{\sqrt{\hat{g}}}\dfrac{\delta\Gamma^L_K}{\delta\hat{g}_{\mu\nu}}$

with the trace

$$\Theta^L_k[\phi;\hat{g}] \equiv \hat{g}_{\mu\nu}T^L_K[\phi;\hat{g}]^{\mu\nu} = \frac{C^{NGFP}_{grav}}{12\pi}\left[\hat{\Box}\phi + (2\breve{\lambda}_* k^2)e^{2\phi}\right]$$

$$\underset{\text{on shell}}{=} \frac{C^{NGFP}_{grav}}{24\pi}\hat{R}$$

$$=: -\frac{c^L}{24\pi}\hat{R}$$

$\Rightarrow$ Γ^L_K describes CFT, with central charge c^L:

$$\boxed{c^L = -C^{NGFP}_{grav}}$$

$\Rightarrow$ $\left\{\begin{array}{c} c^L<0 \\ c^L>0 \end{array}\right\}$ $\Longleftrightarrow$ pure Liouville CFT $\left\{\begin{array}{c} \text{not unitary} \\ \text{unitary} \end{array}\right\}$

$\Longleftrightarrow$ ϕ-mode $\left\{\begin{array}{c} \text{unstable} \\ \text{stable} \end{array}\right\}$

$\Longleftrightarrow$ C^{NGFP}_{grav} $\left\{\begin{array}{c} \text{positive} \\ \text{negative} \end{array}\right\}$

$\Longleftrightarrow$ full 'grav'- CFT $\left\{\begin{array}{c} \text{unitary} \\ \text{not unitary} \end{array}\right\}$

Pure a.s. gravity has $c^L<0$!

Combined Liouville + background system

Trace of the <u>total</u> stress tensor

from $\quad \Gamma_K^{grav, 2D, NGFP} \left[e^{2\phi} \hat{g} \right] \quad :$

$$\frac{2\hat{g}_{\mu\nu}}{\sqrt{\hat{g}}} \frac{\delta}{\delta \hat{g}_{\mu\nu}} \left(\frac{c_{grav}^{NGFP}}{96\pi} \int \sqrt{\hat{g}} \, \hat{R} \, \hat{\Box}^{-1} \hat{R} \right) \quad + \quad \Theta_K^L [\phi; \hat{g}]$$

$$= \frac{c_{grav}^{NGFP}}{24\pi} \left[-R(\hat{g}) + \underbrace{2\hat{\Box}\phi + (4\breve{\lambda}_* k^2) e^{2\phi}}_{\overset{o.s.}{=} + R(\hat{g})} \right]$$

explains the relative minus sign

$$= \frac{c_{grav}^{NGFP}}{24\pi} \left[\underbrace{- \left(R(\hat{g}) - 2\hat{\Box}\phi \right) e^{-2\phi} + 4\breve{\lambda}_* k^2}_{= R(e^{2\phi}\hat{g}) = R(g)} \right] e^{2\phi}$$

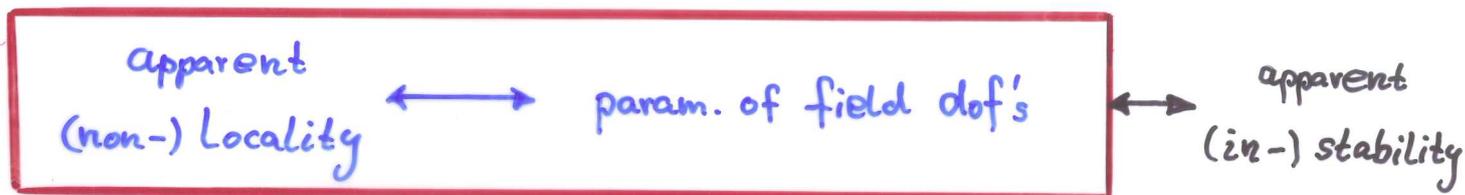$$= e^{2\phi} \cdot \Theta_K [g]$$

Same off-shell result as above!

Vanishes on-shell, i.e. for

$$g = \bar{g}_K^{sc} .$$

Locality

True degree of physical (non-) locality is in general
only weakly related to the naive appearance of the
action.

(⟶ correlators of observables)

apparent
(non-) locality ⟷ param. of field dof's ⟷ apparent
(in-) stability

Polyakov action $\int R \, \Box^{-1} R$ becomes local

(a) in certain gauges (conformal, light-cone, ···)

(b) by introducing additional fields in a covariant way;

example: Feigin - Fuks theory

$$e^{-\frac{25-N}{96\pi} \int R \Box^{-1} R}$$

$$= \int \mathcal{D}B \; e^{-\frac{24-N}{96\pi} \int d^2x \sqrt{g} \left\{ D_\mu B D^\mu B + 2 R B \right\}}$$

The reconstructed functional integral

There exists a **UV-regularized** functional integral with the **bare** ("classical") action

$$S^{grav}[g] = \frac{25-N}{96\pi} \, I[g] \quad , \quad I[g] \equiv \int d^2x \sqrt{g} \, R\,\Box^{-1}R \quad .$$

__Quantization à la Polyakov:__

$$Z = \int_{moduli} d\tau \int \mathcal{D}_{e^{2\phi}\hat{g}}\,\phi \; Z_{gh}[e^{2\phi}\hat{g}] \; Z_{matter}[e^{2\phi}\hat{g}] \; Y_{grav}[e^{2\phi}\hat{g}]$$

$$Y_{grav}[g] = e^{-S^{grav}[e^{2\phi}\hat{g}]} = Y_{grav}[\hat{g}] \, e^{+\frac{(25-N)}{12\pi}\Delta I}$$

$$Z_{matter}[g] = e^{-\frac{N}{96\pi} I[e^{2\phi}\hat{g}]} = Z_{matter}[\hat{g}] \, e^{+\frac{N}{12\pi}\Delta I}$$

$$Z_{gh}[g] = Z_{gh}[\hat{g}] \, e^{+\frac{(-26)}{12\pi}\Delta I}$$

$$\mathcal{D}_{e^{2\phi}\hat{g}}\,\phi = \mathcal{D}_{\hat{g}}\,\phi \; e^{+\frac{1}{12\pi}\Delta I}$$

$$\Delta I[\phi;\hat{g}] \equiv \frac{1}{2}\int d^2x \sqrt{\hat{g}} \left\{ \hat{D}_\mu\phi\,\hat{D}^\mu\phi + \hat{R}\phi \right\}$$

$$Z = \int d\tau \; Z_{gh}[\hat{g}] \, Z_{matter}[\hat{g}] \, Y_{grav}[\hat{g}] \int \mathcal{D}_{\hat{g}} \phi$$

Total anomaly budget:

$$C_{tot} \equiv \underset{NGFP}{(25-N)} + \underset{matter}{N} + \underset{Jacobian}{1} + \underset{bc-ghosts}{(-26)} = 0$$

In asym. safe quantum gravity, c_{tot} vanishes _always_,

i.e. for _any_ matter system, unlike in string theory.

$$N \xrightarrow[\text{safety}]{\text{asymptotic}} N_{eff} = N + (25-N) = 25$$

Cancellation of explicit matter contribution to the fctl. int.

and the *implicit matter dependence of the bare $\equiv$ NGFP*

gravity action.

$\Longrightarrow$ Quenching of KPZ scaling:

$N_{eff} = 25 \rightsquigarrow \phi$ decouples from gravitationally dressed

observables of matter CFT :

no quantum corrections to flat space - scaling dimensions

A second universality class ?

$$C_{grav}^{NGFP} = 25 \qquad g = \bar{g}e^{h}$$

weakly coupled
Liouville – DDK – KPZ th.

$$C_{grav}^{NGFP} = 19 \qquad g = \bar{g} + h \qquad \qquad ?$$

Perhaps more than a cooincidence:

The Virasoro algebra allows for unitary truncations
in the "critical dimensions of non-critical string theory"
advocated by Gervais:

$$D_{crit} = 0 , 7 , 13 , 19 , 25$$

$$\hat{=} \quad C_{grav} = 25 , 19 , 13 , 7 , 0$$

Summary and Outlook

What we found:

In the limit $d \to 2$, the $h_{\mu\nu}$-sector of BRST gauge fixed QEG, in the Einstein-Hilbert trunc., is described by a conformal and unitary quantum field theory. Its states inhabit a true Hilbert space, $\mathcal{H}_{grav}$.

After imposing the physical state conditions on the total space (computing the BRST cohomology), no physical states are left though:

$$\underbrace{\mathcal{H}_{grav}}_{\text{pos. !}} \oplus \mathcal{H}_{ghost} \longrightarrow \emptyset$$

What we must show ultimately:

In $d = 4$, the total space does contain physical states, and they form a Hilbert space with a positive definite inner product.