N.4 Grover's search algorithm.

Given an oracle that gives

$$f(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases}$$

in the form

$$U_f \left(|x\rangle_m \otimes |y\rangle_1 \right) = |x\rangle_m \otimes |y + f(x)\rangle_1$$

$x \in \{0, \dots, N-1\}$, $N = 2^m$

Find the unknown number a with as few as possible calls of U_f.

Classically, the probability of finding a after n trials is $\frac{n}{N}$.

Grover: $\frac{\pi}{4}\sqrt{N}$ trials suffice on a QC.

Exercise: there is a very simple quantum network that implements U_f.

$$U_f \left(|x\rangle_m \otimes H|1\rangle \right) \overset{x,}{=} (-1)^{f(x)} \, |x\rangle_m \otimes H|1\rangle$$

$$=: \left(V|x\rangle_m \right) \otimes H|1\rangle .$$

U_f is unitary by definition $\longrightarrow$ V is unitary.

x_1 $x = a$

$$U_f \, |a\rangle_m \otimes H|1\rangle = U_f \left(|a\rangle_m \otimes \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle \right) \right)$$

$$= \frac{1}{\sqrt{2}} \left[|a\rangle_m \otimes |1\rangle - |a\rangle_m \otimes |0\rangle \right]$$

$$= |a\rangle_m \otimes \frac{1}{\sqrt{2}} \left(|1\rangle - |0\rangle \right)$$

$$= (-1)^1 \, |a\rangle_m \otimes H|1\rangle$$

Similarly, for $x \neq a$, get $(-1)^0 \, |x\rangle_m \otimes H|1\rangle$.

A general vector $|\psi\rangle$ is of the form $|\psi\rangle = \sum_x |x\rangle\langle x|\psi\rangle$

$$V|\psi\rangle = \sum_x |x\rangle\langle x|\psi\rangle \cdot \begin{cases} -1 & x = a \\ 1 & x \neq a \end{cases} \qquad -1 = 1 - 2$$

$$= |\psi\rangle - 2\,|a\rangle\langle a|\psi\rangle$$

$$= \left(1 - 2P_{|a\rangle} \right) |\psi\rangle \qquad \|V|\psi\rangle\|^2 = \langle\psi|\psi\rangle .$$

Let $|\phi\rangle = H^{\otimes m} |0\rangle_m = 2^{-m/2} \sum_{m=0}^{2^m - 1} |x\rangle_m$

and $W = 2|\phi\rangle\langle\phi| - 1$

Grover's algorithm: put in $|\phi\rangle$, then apply WV $\frac{\pi}{4}\sqrt{N}$ times.

Explanation. $\langle a|\phi\rangle = 2^{-m/2}$

Define $\Theta > 0$ by $\cos\left(\frac{\pi}{2} - \Theta\right) = \langle a|\phi\rangle = \frac{1}{\sqrt{N}}$

i.e. $\Theta = \arcsin \frac{1}{\sqrt{N}} = O\left(\frac{1}{\sqrt{N}}\right)$

We have

$$V|a\rangle = -|a\rangle$$

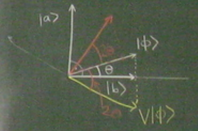
$$V|\phi\rangle = |\phi\rangle - 2^{-\frac{m}{2}+1} |a\rangle$$

$$W|a\rangle = 2^{-\frac{m}{2}+1} |\phi\rangle - |a\rangle$$

$$W|\phi\rangle = |\phi\rangle$$

V and W leave the plane $\mathcal{E} = \operatorname{span}\{|a\rangle, |\phi\rangle\}$ invariant.

$|b\rangle = |\phi\rangle - |a\rangle\langle a|\phi\rangle \qquad \langle b|a\rangle = 0$



$$WV|\phi\rangle = 2|\phi\rangle\langle\phi|V\phi\rangle - V|\phi\rangle$$

$\Rightarrow$ WV rotates $|\phi\rangle$ by an angle 2Θ in direction of $|a\rangle$ (in the plane $\mathcal{E}$)

$(WV)^2 \quad \dots \quad 5\Theta$

$(\quad)^3 \quad \dots \quad 7\Theta$

k iterations $(VW)^k \quad \dots \quad (2k+1)\Theta \overset{!}{\approx} \frac{\pi}{2}$

$$k = \frac{1}{2}\left(\frac{\pi}{2\Theta} - 1 \right) \approx \frac{\pi}{4\Theta} \approx \frac{\pi}{4}\sqrt{N}$$

D. System, Environment, State.

D.1 Density matrix

<u>Def.</u> $\mathcal{H} = \mathbb{C}^N$. $\rho \in M_N(\mathbb{C})$ is called a density matrix (or density operator)

$\Longleftrightarrow$ (i) $\rho = \rho^\dagger$

(ii) $\rho \geq 0$ $\left(\forall v \in \mathcal{H}: \langle v | \rho v \rangle \geq 0\right)$

(iii) $\operatorname{tr} \rho = 1$

By the spectral theorem, and (i), ρ has real eigenvalues $p_1, \dots, p_N$ and $\mathcal{H}$ has a corresponding ONB $\psi_1, \dots, \psi_N$ of EV of ρ, and

$$\rho = \sum_{k=1}^{N} p_k |\psi_k\rangle\langle\psi_k|$$

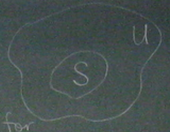
(ii) $\forall k \in \{1, \dots, N\}$: $p_k \geq 0$

(because $0 \overset{(ii)}{\leq} \langle \psi_k | \rho \psi_k \rangle = p_k \langle \psi_k | \psi_k \rangle = p_k$)

(iii) $\sum_{k=1}^{N} p_k = 1$

(because $\operatorname{tr}(\rho) = \sum_{k=1}^{N} \langle \psi_k | \rho \psi_k \rangle = \sum_{k=1}^{N} p_k \overset{(iii)}{=} 1$)

D.2 System and environment



$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_U$$

$|\psi\rangle = |\psi_S\rangle \otimes |\psi_U\rangle$ holds only for perfectly isolated systems (where we can forget about the environment)

$\leadsto$ In general $|\psi\rangle = \sum_{i,j} \underset{\in \mathbb{C}}{\psi_{i,j}}\, \underset{\text{ONB of } \mathcal{H}_S}{|i\rangle_S} \otimes \underset{\text{ONB of } \mathcal{H}_U}{|j\rangle_U}$

$$\sum_{i,j} |\psi_{i,j}|^2 = 1$$

How to describe a state, if S and U are coupled, but we make measurements only on S?

we take observables $A \otimes \mathbb{1}_{\mathcal{H}_U}$ $\quad A \in \mathcal{L}(\mathcal{H}_S)$, $A = A^\dagger$

$$\langle \psi | (A \otimes \mathbb{1}) \psi \rangle = \sum_{i,j,k,l} \bar{\psi}_{kl} \psi_{ij} \underbrace{\left(\langle k | \otimes \langle l |\right)(A \otimes \mathbb{1})\left(|i\rangle \otimes |j\rangle\right)}_{\langle k | A | i \rangle \cdot \underbrace{\langle l | j \rangle}_{= \delta_{lj}\ (\text{ONB!})}}$$

$\dim \mathcal{H}_S = M$

$\dim \mathcal{H}_U = N$

$$= \sum_{i,k=1}^{M} A_{ki} \underbrace{\sum_{l=1}^{N} \psi_{il} \bar{\psi}_{kl}}_{=: \rho_{ik}}$$

$$= \sum_{i,k=1}^{M} A_{ki} \rho_{ik} = \operatorname{tr}(A\rho) \qquad \rho \in M_M(\mathbb{C})$$

Claim: ρ is a density matrix on $\mathcal{H}_S$.

Proof. (i) exercise. $\quad \rho_{ik} = \sum_{l=1}^{N} \psi_{il} \bar{\psi}_{kl} = \overline{\rho_{ki}}$

(ii) $\langle v | \rho v \rangle = \sum_{i,k=1}^{M} \bar{v}_i \rho_{ik} v_k$

$$= \dots = \sum_{l=1}^{N} \underbrace{\left(\sum_{i=1}^{M} \overline{\bar{\psi}_{il} v_i}\right)}_{= \overline{w_l}} \underbrace{\sum_{k=1}^{M} \bar{\psi}_{kl} v_k}_{=: w_l}$$

$$= \sum_{l=1}^{N} |w_l|^2 \geq 0.$$

(iii) $\operatorname{Tr}(\rho) = \sum_{i=1}^{M} \rho_{ii} = \sum_{l=1}^{N} \sum_{i=1}^{M} \underbrace{\psi_{il} \bar{\psi}_{il}}_{|\psi_{il}|^2} = 1$ because $\|\psi\| = 1$

Conclusions: it is more natural to describe QM states by density operators.

[this is possible because density matrices are robust under "partial traces": $\operatorname{Tr}_{\mathcal{H}_S \otimes \mathcal{H}_U}\left(\rho \cdot (A \otimes \mathbb{1}_U)\right) = \operatorname{Tr}_{\mathcal{H}_S}\left(\tilde{\rho}_S \cdot A\right)$

and ρ density matrix on $\mathcal{H}_S \otimes \mathcal{H}_U$

$\Rightarrow \tilde{\rho}_S$ density matrix on $\mathcal{H}_S$]

D.3 Entropy.

$S(\rho) = -\underset{\text{Boltzmann's constant}}{k_B} \operatorname{Tr}(\rho \log \rho)$ von Neumann entropy

$= -k_B \sum_{k=1}^{n} p_k \log p_k$ $\longleftrightarrow$ Shannon information entropy (up to a factor

$S(\rho) \geq 0$, $S(\rho) = 0 \Leftrightarrow$ all p_k's except for one are zero