MKTP3.1 — GENERAL RELATIVITY

GEOMETRIC STRUCTURE AND DYNAMICS OF SPACETIME

BJÖRN MALTE SCHÄFER*

typesetting by Rebecca Kuntz

version 1.1

Faculty for Physics and Astronomy, Heidelberg University

lecture notes in general relativity (2015, 2017, 2020)
published under license CC BY-NC-SA 4.0
lecture LaTeX-class: https://vhbelvadi.com/latex-lecture-notes-class/

CONTENTS

*bjoern.malte.schaefer@uni-heidelberg.de

A RELATIVITY AND GRAVITY

Surprisingly, the concepts behind general relativity as a theory of gravity are entirely geometrical and require differential geometry as a description of the geometry and dynamics of space time. In this script for the lecture on general relativity as a master-level course at Heidelberg University we will encounter gravity as a geometric effect of spacetime and the geometrisation of physical laws, understand the structure of the laws of Nature by drawing analogies between classical and relativistic mechanics of point particles, the theory of scalar and vectorial fields, touch on concepts like invariance, covariance and symmetries, and will develop an intuition about gravity. There are three great applications of general relativity: black holes, FLRW-cosmologies and gravitational waves, and in all these areas there have been major experimental and observational advances in the last couple of years.

In this script we will mostly use a coordinate-based description of tensors with explicit indices. For those, we adopt the summation convention, with Greek indices running over all spacetime coordinates and Latin indices over the spatial ones, should the coordinate choice allow this.

☠ *only coordinate choices aligned with the spatial hypersurfaces have this!*

A.1 *Why is classical Newtonian gravity insufficient?*

It is important to realise that at the time of A. Einstein's thinking about relativity, there was no actual need to abandon I. Newton's theory of gravity. The perihelion shift of Mercury could have easily have had systematic origins, and many of the arguments against Newton gravity to be the ultimate theory of gravity are purely conceptual.

First of all, there is no dynamics of the gravitational potential Φ in Newton's theory. According to the Poisson-equation as the field equation of Newton-gravity,

$$\Delta\Phi(x^i,t) = 4\pi G\,\rho(x^i,t), \tag{A.1}$$

the potential is source by the density field ρ in an instantaneous way as the Laplace-operator Δ can, unlike the d'Alembert-operator $\Box$, not generate any retardation. 👻

The missing retardation could be easily fixed, though. Motivated by the ideas of special relativity that there is no clear distinction between the t- and x^i-coordinates, one could make the replacement

$$\Delta = \delta^{ij}\,\partial_i\partial_j \;\rightarrow\; \Box = \eta^{\mu\nu}\,\partial_\mu\partial_\nu = \partial^2_{ct} - \Delta \tag{A.2}$$

which gives a Lorentz-covariant field equation,

$$\Delta\Phi(x^i) = 4\pi G\, \rho(x^i) \;\rightarrow\; \Box\, \Phi(x^\mu) = 4\pi G\, \rho(x^\mu) \tag{A.3}$$

with proper retarded (and advanced) potentials. We can quickly check that there is propagation of excitations of Φ with c along a light cone: Plane waves $\Phi \sim \exp(\pm i\eta_{\mu\nu} k^\mu x^\nu)$ yield $\Box\Phi = -\eta_{\mu\nu} k^\mu k^\nu \Phi = 0$ with a null-vector k^μ, $\eta_{\mu\nu} k^\mu k^\nu = k_\mu k^\mu = 0$. That this "covariantised" field equation already allows wave-like excitations of the gravitational field is foreshadowing the emergence of gravitational waves in proper relativistic theory.

Then, the (energy) density ρ should be the tt-component of the energy momentum tensor $T^{\mu\nu}$ as suggested by special relativity and follow a Lorentz-transformation rule when boosting from one Lorentz-frame to another: But the gravitational potential in Newton's theory is scalar and would necessarily be equal in all frames. In fact, there should be additional components of the gravitational field beyond Φ if it was to be sourced by the energy momentum-tensor $T^{\mu\nu}$.

A.2 *What would be the most general classical theory of gravity?*

It turns out that Newtonian gravity is not even the most general classical (i.e. non-relativisic, and of course non-quantum) theory of gravity! For seeing this, we would approach the construction of a field equation from a variational principle, by writing down an action integral $S = \int d^3x\, \mathcal{L}(\Phi, \partial_i\Phi)$ with a Lagrange-density $\mathcal{L}(\Phi, \partial_i\Phi)$ that is dependent on the potential Φ and the first derivative $\partial_i\Phi$. Hamilton's principle $\delta S = 0$ would then suggest that

$$\delta S = \delta \int d^3x \mathcal{L}(\Phi, \partial_i\Phi) = \int d^3x \frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial\partial_i\Phi}\delta\partial_i\Phi = \int d^3x \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\Phi}\right)\delta\Phi = 0 \tag{A.4}$$

by using $\delta\,\partial_i\Phi = \partial_i\delta\,\Phi$ and performing an integration by parts, so that we can extract the Euler-Lagrange-equation

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\Phi} = 0 \tag{A.5}$$

which determines the field equation once the Lagrange density $\mathcal{L}$ is chosen. Certainly, Newton would have wanted to have a linear field equation such that the superposition principle is valid, and the Ostrogradsky-theorem disallows terms of higher derivative order beyond second derivatives, and we would like an isotropic gravitational field around a spherically symmetric matter distribution. These arguments imply that there can be at most squares of the potential in the Lagrange-density as the Euler-Lagrange equation decreases the order by one through differentiation, and that there should be the invariant $\delta^{ij}\partial_i\Phi\partial_j\Phi$ (as a scalar product of two vectors it is invariant under rotations) as a kinetic term in the Lagrange-density: Therefore, the most general Lagrange-density would be

$$\mathcal{L}(\Phi, \partial_i\Phi) = \frac{1}{2}\,\delta^{ij}\,\partial_i\Phi\partial_j\Phi + 4\pi G\rho\Phi + \lambda\Phi + \frac{m^2}{2}\Phi^2, \tag{A.6}$$

with the Newtonian gravitational constant G and two new constants, m and λ. Of course, as Lagrange-densities only ever appear inside integrals, it is only determined up to an integration by parts, so the kinetic term can equally written as

$$\mathcal{L}(\Phi, \partial_i\Phi) = -\frac{1}{2}\,\Phi\,\delta^{ij}\,\partial_i\partial_j\Phi + \cdots, \tag{A.7}$$

with $\Phi\delta^{ij}\partial_i\partial_j\Phi = \Phi\Delta\Phi$. For carrying out the variation, one needs to substitute the Lagrange-density into the Euler-Lagrange equation. We obtain for

please always use new names for the indices in the variation!

$$\mathcal{L} = \frac{1}{2}\,\delta^{ab}\partial_a\Phi\partial_b\Phi \tag{A.8}$$

the derivative

$$\frac{\partial\mathcal{L}}{\partial\partial_i\Phi} = \frac{1}{2}\,\delta^{ab}\left(\frac{\partial\partial_a\Phi}{\partial\partial_i\Phi}\cdot\partial_b\Phi + \partial_a\Phi\frac{\partial\partial_b\Phi}{\partial\partial_i\Phi}\right) = \frac{1}{2}\left(\delta^{ab}\,\delta^i_a\,\partial_b\Phi + \delta^{ab}\,\partial_a\Phi\cdot\delta^i_b\right) = \partial^i\Phi \tag{A.9}$$

and finally

$$\partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\Phi} = \partial_i\partial^i\Phi = \Delta\Phi, \tag{A.10}$$

while the derivative with respect to the field Φ itself is very easy,

$$\frac{\partial\mathcal{L}}{\partial\Phi} = 4\pi G\rho + \lambda + m^2\,\Phi, \tag{A.11}$$

such that Newton's field equation should be of Yukawa-form, and being inhomogeneous even in vacuum,

$$\left(\Delta - m^2\right)\Phi = 4\pi G\,\rho + \lambda. \tag{A.12}$$

While the value of the gravitational constant $G \simeq 10^{-11}\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2$ is well known, the cosmological constant λ is in fact non-zero and plays a role on scales above 10^{25}m, but is completely irrelevant inside the Solar System. It would, however, have the funny consequence that there would be gravitational effects in empty space! Specifically for $\rho = 0$ the field equation becomes

Many people claim that the cosmological constant is a part of a relativistic theory of gravity, but this is just untrue.

$$\Delta\Phi = \lambda \quad \text{such that} \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \lambda \tag{A.13}$$

and the solution for the gravitational acceleration g_r reads

$$\frac{\partial\Phi}{\partial r} = -g_r = \frac{\lambda}{r^2}\int \mathrm{d}r\,r^2 = \frac{\lambda}{3}r, \tag{A.14}$$

increasing linearly with distance: This is in fact observed in cosmology on very large scales in the distance-redshift-relation of supernovae!

The additional constant m is very difficult to interpret classically, but we should see what effects it might have, by solving the resulting field equation. In three dimensions or more, and on scales below 10^{25}m, the case $m = 0$ reduces the field equation in vacuum to $\Delta\Phi = 0$, i.e. to

$$\Delta\Phi = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\Phi\right) = 0, \tag{A.15}$$

suggesting that $\left(r^{n-1}\frac{\partial}{\partial r}\Phi\right)$ should be constant for equation to be valid. Specifically in 3 dimensions on obtains a scale-invariant Coulomb-potential, $\Phi \propto 1/r$ whereas a nonzero value for m introduces a scale in the form of a Yukawa-potential $\Phi \propto \exp(-mr)/r$. For Newton, it must have been an empirical fact that m would be vanishing as there are perfectly Keplerian orbits in the Solar System.

☠ *Debye-screened electrostatic fields in polarisable media is a good example of Yukawa-potentials.*

Let us come to the fact that orbits of planets in the Solar System are almost perfectly Keplerian: A very instructive derivation of Kepler's third law is to use mechanical similarity transforms of the classical action S, in particular for power-law potentials just as the Coulomb-potential.

$$S = \int dt\, L(x_i, \dot{x}_i) \quad \text{with} \quad L(x_i, \dot{x}_i) = \frac{m}{2}\,\delta^{ij}\,\dot{x}_i\dot{x}_j - m\,\Phi \tag{A.16}$$

If we introduce a scaling of distance and time with the transformations $x \to \alpha x$ and $t \to \beta t$, the kinetic term transforms according to $T \to \frac{\alpha^2}{\beta^2}T$ and the potential term with $\Phi \to \alpha^n\,\Phi$ if the potential is in fact a power law of with exponent n, $\Phi \sim x^n$.

If the two scaling factors are related through $\frac{\alpha^2}{\beta^2} \sim \alpha^n$, L changes only by an overall factor, which can not matter because the Hamiltonian principle is invariant under affine transformations of the action (or equivalently, of the Lagrange function):

$$L \to aL + b \quad \text{implies} \quad S \to aS + b \quad \text{with} \quad S = \int dt\, L \tag{A.17}$$

such that

$$\delta S = 0 \;\to\; \delta\,(aS + b) = a\,\delta S = 0, \tag{A.18}$$

and a cancels. Therefore, the two scalings in length and time can not be independent and their relation must depend on the exponent of the power law of the potential: This is summarised by the similarity condition $t^2 \sim x^{2-n}$, which is sometimes referred to as classical similarity: Motion inside a potential with a given exponent is described by an equivalence class of Lagrange-functions (and their solutions), which get mapped onto each other by a similarity transform. The most basic choices of n correspond to well known problems in classical mechanics:

$$n = 2 \qquad t^2 \sim x^0 \quad \text{isochronism of a pendulum} \tag{A.19}$$

$$n = 1 \qquad t^2 \sim x \quad \text{inclined plane, constant acceleration} \tag{A.20}$$

$$n = 0 \qquad t^2 \sim x^2 \quad \text{inertial motion with constant velocity} \tag{A.21}$$

$$n = -1 \qquad t^2 \sim x^3 \quad \text{Kepler's third law} \tag{A.22}$$

Supposedly, the first case was discovered by G. Galilei himself, who noticed that the oscillation period of a pendulum (measured with the pulse on his wrist) did not depend on the amplitude. The last case, Kepler's third law of planetary motion, is necessarily a consequence of the $1/r$-form of the potential and any Yukawa-type contribution would break the scaling relation.

Besides, Kepler's third law provides a neat trick to remember the units of the gravitational constant, $G \sim 10^{-11}\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2$, where one can immediately recognise the three powers of length divided by the two powers of time! For our Sun, $GM_\odot \sim 10^{19}\ \mathrm{m}^3/\mathrm{s}^2 \sim (1\ \mathrm{AU})^3/(1\ \mathrm{yr})^2$. But mechanical similarity applied to the Solar System is a really powerful concept: All planetary orbits are scaled versions of each other, and for measuring distances one really only needs a calendar!

We have obtained $\Phi \propto 1/r$ from direct solution of Poisson's equation in the case $\rho = 0$, but there needs to be a fundamental argument why this is necessary, and this argument comes in the shape of Gauß's law. The gravitational acceleration g_i is given as the gradient $g_i = -\partial_i\Phi$, and has in a spherically symmetric case only a radial component, $g_r = -\partial_r\Phi$. It is linked to the Poisson equation by

$$\Delta\,\Phi = \delta^{ij}\,\partial_i\partial_j\Phi = -\delta^{ij}\,\partial_i g_j = -\mathrm{div}\boldsymbol{g} = 4\pi G\,\rho \tag{A.23}$$

suggesting that the divergence of the acceleration is $\Delta\Phi$ and proportional to ρ. Recasting the Poisson-equation into integral form yields

$$\int_V \mathrm{d}^3r\ \mathrm{div}\boldsymbol{g} = \int_{\partial V} \mathrm{d}\mathbf{A}\cdot\boldsymbol{g} = -4\pi G\cdot\int_V \mathrm{d}^3r\rho = -4\pi GM. \tag{A.24}$$

Here, $\int \mathrm{d}\mathbf{A}\cdot\boldsymbol{g}$ is the flux of the field through the surface $\partial V = 4\pi r^2$ for a spherical integration volume V appropriate for the isotropic case. As a consequence, the acceleration decreases $\propto 1/r^2$ as the flux needs to be the same at every distance and surfaces increase $\propto r^2$! Now we can set up an entire chain of arguments: The flux of $\vec{g}$ through surfaces ∂V is constant, so g needs to be $\propto 1/r^2$ and $\Phi \propto 1/r$. Then, mechanical similarity requires that $t^2 \propto r^3$. And in addition, Bertrand's theorem makes sure that the orbits are closed ellipses.

At this point it is a very large surprise that Mercury, the planet closest to the Sun where perhaps the gravitational field behaves unusual, shows a tiny violation of Kepler's third law and in fact of Bertrand's theorem, too: Neither is the orbit a closed ellipse nor is Kepler's law fulfilled. There is a small precession of the point of closest approach to the Sun, called perihelion precession, amounting to 43 arcseconds in about 1000 orbits (The number is usually stated as 43 arcseconds in 100 years, but this refers to Earth years!). By now, we know many systems that show pericentre precession, even much more pronounced than Mercury in the Solar System. For instance, PSR 1913+10 with 4 arcseconds per orbit, PSR J0737-3039 with 20 arcseconds per orbit, and the system OJ287 with 40° per orbit! For a precession to appear, the gravitational field needs to be stronger in the vicinity of a massive object compared to the Newtonian prediction, and neither m nor λ could achieve this: They both correspond to long-distance modifications of gravity: That would be a very strong argument for the necessity of a new theory of gravity. And we can see a tiny glimpse onto geometry. Combining the constant of gravity G with the speed of light c,

$$\frac{G}{c^2} \sim 10^{-28}\mathrm{m/kg} \tag{A.25}$$

which assigns a length scale to the field generating mass. With the specific value $M_\odot \simeq 10^{30}$kg for the mass of the Sun on obtains

$$\frac{GM_\odot}{c^2} \sim 10^2\mathrm{m} \tag{A.26}$$

which we will encounter later as the Schwarzschild radius of the Sun. Perhaps we can change how surfaces scale with r?

A.3 *Lorentz-geometry*

The foundational idea of general relativity is differential geometry, i.e. a varying geometry of spacetime, with locally Minkowskian properties, i.e. we will see that the laws of special relativity will be valid locally in freely falling reference frames. Lorentz-transforms and rotations apply locally to the transitions between frames with different orientation relative to each other or moving at constant velocities υ relative to each other. The homogeneity of spacetime should be respected by the coordinate choice, meaning that it should not single out certain spacetime points.

An observer looking at two coordinate choices could measure the rate at which the coordinates x^μ and x'^μ are drifting by as a function of her or his proper time τ, defining the velocity

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \text{const.}\ \text{, with } x^\mu = \begin{pmatrix} t \\ x^i \end{pmatrix} \sim \text{ 4-vector} \tag{A.27}$$

which is constant for inertial motion and suitably chosen coordinates, and the corresponding acceleration

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = 0, \text{ and identically in S}' : \ \frac{\mathrm{d}^2x'^\mu}{\mathrm{d}\tau^2} = 0 \tag{A.28}$$

which then vanishes. Then, the relation between the two velocities and accelerations is given by

$$\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{dx}^\nu}{\tau}, \text{ with Jacobian } \frac{\partial x'^\mu}{\partial x^\nu} \tag{A.29}$$

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}, \tag{A.30}$$

where

$$\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho} = 0 \tag{A.31}$$

for transformations between frames that are linear and therefore conserve homogeneity. The solution for $x'^\mu(x^\nu)$ follows then as

$$x'^\mu = \mathrm{A}^\mu{}_\nu\, x^\nu + a^\mu, \tag{A.32}$$

implying that the transformation between frames should be affine.

Let's construct this transform from the most general transition between two frames, where we align for simplicity the coordinate axes with the direction of relative motion, taken to be the x-axis. There is an event with coordinates $\begin{pmatrix} t \\ x^i \end{pmatrix}$ in S and $\begin{pmatrix} t' \\ x'^i \end{pmatrix}$ in S′, and the two frames move with a relative (constant) velocity υ. A linear transform would then the only one to respect the homogeneity of spacetime (nonlinear transforms would always single out certain spacetime points), so we make the ansatz:

$$x' = ax + bt,\;\; a, b \text{ arbitrary, but } x = vt \text{ must imply } x' = 0 \tag{A.33}$$

$$x' = 0 = avt + bt = (av + b)\,t \;\Rightarrow\; b = -av, \text{ and:} \tag{A.34}$$

$$x' = a(x - vt)\;(*) \tag{A.35}$$

Reversing the roles of S and S′ implies that

$$x = ax' + bt' \text{ but } x' = -vt \text{ must imply } x = 0 \tag{A.36}$$

$$x = 0 = -avt' + bt' = (-av + b)\,t' \;\Rightarrow\; b = +av, \text{ and:} \tag{A.37}$$

$$x = a(x' + vt')\;(**) \tag{A.38}$$

But this relation between x and x' is not yet fixed without an additional assumption that determines the value of a. Here, Nature would have in fact a choice! Either, Nature could work with a universal time coordinate (or rather, a parameter, as it does not participate in transforms unlike the other coordinates). A universal time parameter would require that $t = t'$, which is the defining property of Galilei-transforms. Then,

$$x' = a(x - vt) \tag{A.39}$$

$$x = a(x' + vt) = a(a(x - vt) + vt) = a^2x + (1 - a)\,vt = x \tag{A.40}$$

which can only be realised if $a = 1$. Nature chose instead, for very good reasons, the speed of light to be equal in all frames, $c = c'$, which requires Lorentz- instead of Galilei-transforms between frames. In this choice,

$$x' = ct' = a(ct - vt) \tag{A.41}$$

$$x = ct = a(ct' - vt') \tag{A.42}$$

$$\Rightarrow\; c^2tt' = a^2(c - v)(c + v)\cdot tt', \tag{A.43}$$

where the third equation was obtained by multiplying the first two. Dividing by tt' and solving for a yields the Lorentz-factor γ,

$$a = \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \text{with} \quad \beta = \frac{v}{c} \tag{A.44}$$

We should note that Lorentz-transformations, due to their linearity, do not 'mix' the spatial coordinates. The Lorentz-factor γ diverges at $\beta = 1$ and would indeed become imaginary for values $\beta > 1$. Taylor-expanding γ for small velocities β gives the result that

$$\gamma \sim 1 + \left.\frac{\partial^2\gamma}{\partial\beta^2}\right|_{\beta=0} \cdot \frac{\beta^2}{2} = 1 + \frac{\beta^2}{2}, \text{ with } \left.\frac{\partial\gamma}{\partial\beta}\right|_{\beta=0} = 1 \tag{A.45}$$

which is perfectly consistent with the fact that for low velocities $\beta \ll 1$ and $\gamma \simeq 1$, Lorentz- and Galilei-transforms are indistinguishable. Writing ct and arranging the temporal and spatial coordinates into a vector $x^\mu = \begin{pmatrix} ct \\ x \end{pmatrix}$ allows to use the standard matrix-form of the Lorentz-transformation by eliminating x' from (*) and (**):

$$x' = \gamma\,(x - vt) = \gamma\,(x - \beta c t) \tag{A.46}$$

$$ct' = \gamma\,(ct - \beta x), \tag{A.47}$$

so that one arrives at

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{A.48}$$

encapsulating the Lorentz-transform in a matrix $\Lambda^{\mu}{}_{\nu}$, with $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$.

We have seen that coordinates undergo a joint transformation and that any physical statement on coordinates of an event is only sensibly within a specified frame S. From that one might wonder if there is a way to make true statements about physical properties of a system independent from a specification of a frame: That is exactly the idea of a Lorentz-invariant. Similarly to rotations, where $r^2 = \delta_{ij} x^i x^j$ are invariant, which essentially corresponds to the statement $\cos^2\alpha + \sin^2\alpha = 1$ if the rotation is parameterised by an angle α, one can define invariants for Lorentz transforms and relate them to the rapidity ψ which is indicative of the relative velocity between the frames.

Setting $\cosh\psi = \gamma$ and $\sinh\psi = \beta\gamma$ (which is sensible if you look at the range of values of γ and $\beta\gamma$, and compare with the hyperbolic functions), one obtains the relation $\tanh\psi = \frac{\beta\gamma}{\gamma} = \beta$ between the rapidity and the dimensionless velocity $\beta = \upsilon/c$. Lorentz-transformations can then be written compactly as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}, \tag{A.49}$$

as a hyberbolic rotation, suggesting an invariant through $\cosh^2(\psi) - \sinh^2(\psi) = \gamma^2 - \beta^2\gamma^2 = \gamma^2(1 - \beta^2) = 1$, which we have already derived by direct calculation, $(ct')^2 - x'^2 = (ct)^2 - x^2$.

Analogous to rotations we write the Lorentz-invariant as $s^2 = (ct)^2 - x^2$ by introducing the Minkowski-metric,

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.50}$$

such that one can write $s^2 = \eta_{\mu\nu}\, x^{\mu} x^{\nu}$. In contrast to the invariant r^2 in Euclidean space, Lorentz-invariants can be positive, negative or zero, and as the sign of the Lorentz-invariant is of course conserved under transforms, too, the classification into timelike ($s^2 > 0$), spacelike ($s^2 < 0$) and lightlike (or null, $s^2 = 0$) is very suggestive.

Let's now imagine the motion of a point through spacetime: The Lorentz-invariant reads

$$s^2 = (ct)^2 - x^2 = (ct')^2 - x'^2 = (c\tau)^2 \tag{A.51}$$

in two frames S and S′, and the choice of comoving coordinates $x' = 0$ defines proper time $t' = \tau$, which is read off a clock in the rest frame S′. Rewriting the Lorentz-invariant for infinitesimal coordinate differences,

$$ds^2 = (c\,dt)^2 - dx^2 = (c\,d\tau)^2, \tag{A.52}$$

then shows that the passage of coordinate time $\mathrm{d}t$ and proper time $\mathrm{d}\tau$ differ by an inverse Lorentz-factor,

$$\mathrm{d}\tau = \sqrt{1-\beta^2}\mathrm{d}t \quad \text{with} \quad \beta = \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{A.53}$$

That implies that the length $\mathrm{d}s$ of the spacetime curve that a point takes is actually measure by the comoving clock displaying proper time $\mathrm{d}\tau$, at least for timelike motion with velocities $\beta < 1$!

At this point, by merging the temporal coordinate with the spatial coordinates we obtained $\mathbb{R}^4$ with a particular geometric structure, given by the Minkowski scalar product $\langle x, y\rangle = \eta_{\mu\nu}x^\mu y^\nu$, trading the positive definiteness of the Euclidean scalar product for the ability to define general invariants.

A.3.1 Lie-groups and the generation of the Lorentz-group

Rotations and Lorentz-boosts are the fundamental transforms that leave a Lorentzian spacetime invariant. Both transformations are (non-Abelian) groups and are parameterised by real numbers, the rotation angles in the first and the rapidities in the second case. One might ask now the question whether there is something analogous to a basis of these groups, such that all group elements can be addressed by a suitable choice of the rotation angle or the rapidity: It turns out that this presumption is true, and it brings us to the topic of Lie-groups. Lie-groups are continuously parameterised groups and are generated from a basic building block, called, well, a generator.

If we choose the set of Pauli-matrices,

☠ There are many different definitions of Pauli-matrices, we're using the real-valued ones here, which in a real-valued linear combination, are a basis of the space of 2×2-matrices.

$$\sigma^{(0)} = \begin{pmatrix} +1 & 0 \\ 0 & +1\end{pmatrix},\quad \sigma^{(1)} = \begin{pmatrix} +1 & 0 \\ 0 & -1\end{pmatrix},\quad \sigma^{(2)} = \begin{pmatrix} 0 & +1 \\ -1 & 0\end{pmatrix},\quad \sigma^{(3)} = \begin{pmatrix} 0 & +1 \\ +1 & 0\end{pmatrix}, \tag{A.54}$$

to begin with, we can investigate which type of transformation could be generated by substituting them into and exponential series, for instance

$$\Lambda = \exp\left(\Psi\cdot\sigma^{(3)}\right) = \sum_n \frac{1}{n!}\left(\psi\cdot\sigma^{(3)}\right)^n. \tag{A.55}$$

For evaluating the matrix-valued exponential series, one needs to know all powers of the matrix in question. In the case of the Pauli-matrices, it is easy to show that only ever other Pauli-matrices appear. Specifically for $\sigma^{(3)}$ one gets:

$$\left(\sigma^{(3)}\right)^0 = \sigma^{(0)},\ \left(\sigma^{(3)}\right)^1 = \sigma^{(3)},\ \left(\sigma^{(3)}\right)^2 = \sigma^{(0)},\ \left(\sigma^{(3)}\right)^3 = \sigma^{(3)}. \tag{A.56}$$

Then, the exponential series can be summed,

$$\Lambda = \sigma^{(0)}\cdot\sum_n \frac{\psi^{2n}}{(2n)!} + \sigma^{(3)}\sum_n \frac{\psi^{2n+1}}{(2n+1)!} = \sigma^{(0)}\cdot\cosh\psi + \sigma^{(3)}\cdot\sinh\psi = \begin{pmatrix}\cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi\end{pmatrix}, \tag{A.57}$$

and one recovers the expression for the Lorentz-transform as a hyperbolic rotation. T he invariant $\det\Lambda = \cosh^2\psi - \sinh^2\psi = 1$, which otherwise appears as a property of the hyperbolic function, comes out naturally like this: Using $\ln\det\Lambda = \mathrm{tr}\ln\Lambda$ with $\Lambda = \exp(\psi\cdot\sigma^{(3)})$ implies that $\ln\det\Lambda = \psi\cdot\mathrm{tr}\sigma^{(3)} = 0$, because Pauli-matrices (with the exception of $\sigma^{(0)}$) are traceless. Then, the determinant needs be equal to one.

Surely, boosts and rotations are groups, but how does one need to combine their continuous parameters? This question is readily answered by the tools that Lie-groups provide: For instance, two successive boosts

$$\Lambda(\phi) \cdot \Lambda(\psi) = \exp\left(\phi \cdot \sigma^{(3)}\right) \cdot \exp\left(\psi \cdot \sigma^{(3)}\right) = \exp\left(\left(\phi + \psi\right)\sigma^{(3)}\right) = \Lambda\left(\phi + \psi\right), \quad \text{(A.58)}$$

implying that rapidities (and not the velocities!) are in fact additive parameters for boosts. If one wants to revert to velocities, one can use the addition theorem for the hyperbolic tangent:

$$\tanh(\phi) + \tanh\psi = \tanh(\phi + \psi) \cdot \left[1 + \tanh(\phi) \cdot \tanh\psi\right] \quad \text{(A.59)}$$

We have just shown that rapidities add for boosts, and from the commutativity of the addition of real numbers one should then obtain the commutativity of the boosts into the same direction:

$$\Lambda(\phi) \cdot \Lambda(\psi) = \Lambda\left(\phi + \psi\right) = \Lambda\left(\psi + \phi\right) = \Lambda(\psi) \cdot \Lambda(\phi), \quad \text{(A.60)}$$

which implies for the inverse boost that

$$\Lambda(\psi) \cdot \lambda(-\psi) = \Lambda(\psi - \psi) = \Lambda(0) = \mathrm{id} \;\Rightarrow\; \Lambda(\psi)^{-1} = \Lambda(-\psi) \quad \text{(A.61)}$$

as a perfectly intuitive result: The inverse boost is that with the inverse velocity or rapidity. In complete analogy we would have obtained rotations by starting the constructing with $\sigma^{(2)}$ instead of $\sigma^{(3)}$.

☠ *Lorentz-transforms are orthogonal, but with respect to η, not δ.*

Finally, one could ask the question what happens if rotations around different axes and boosts into different directions are combined. If both transformations are generated by basis elements A and B in an exponential series, their successive application $\exp(A)\exp(B)$ is only equal to $\exp(A + B)$ if the generators commute, $[A, B] = AB - BA = 0$, which is not the case in every of our examples. The Rubik's cube demonstrates nicely that rotations in 3 dimensions do not commute, and neither do boosts: In fact, if one moves from one inertial frame into another by a combination of two boosts and moving back by interchanging the two boosts leaves you with a rotation! In the case of non-commuting generators, the two transformation need to be combined using the Baker-Hausdorff-Campbell-formula,

☠ *my most favourite formula of all of physics!*

$$\exp(A)\exp(B) = \exp(A + B) \cdot \exp\left(-\frac{1}{2}[A, B]\right), \quad \text{(A.62)}$$

to lowest order, or exactly if $[A, [A, B]] = 0$ and $[B, [B, A]] = 0$ is valid. In fact, one can define a set of generators in 4d for the group comprising rotations around all three axes and boosts into all 3 directions with a very rich algebra of generators, called the Lorentz-algebra.

A.4 *relativistic motion through spacetime*

It might come as a surprise that variational principles, being so typical of classical mechanics, only make sense in the context of relativity: Here, there is a well define geometric interpretation, the Lagrange-function and the action are measurable quantities, many properties such as their convexity and their affine invariance are made sure by geometry, they are naturally invariant under Lorentz-transforms and there is a natural pathway to include gravity.

A.4.1 variational principles for relativistic mechanics

The fundamental idea of variational principles (and which ironically is not present clearly in classical mechanics) is to link invariant quantities of a system in the form of the Lagrange-function with a covariant equation of motion. Specifically, the Lagrange function $L(x^i, \dot{x}^i)$ of classical mechanics

$$L(x^i, \dot{x}^i) = \frac{m}{2}\delta_{ij}\dot{x}^i\dot{x}^j - m\,\Phi \tag{A.63}$$

is invariant as the norm of $\dot{x}^i$ is unaffected by rotations of the coordinate systems and because the scalar potential Φ does not have any internal degrees of freedom. With Hamilton's principle $\delta S = 0$ for the variation of the action

$$S = \int \mathrm{d}t\; L(x^i, \dot{x}^i) \tag{A.64}$$

one obtains through the Euler-Lagrange equation a covariant equation of motion

$$\ddot{x}^i = -\partial^i\Phi, \tag{A.65}$$

which sets two vectors in relation to each other, namely the acceleration $\ddot{x}^i$ and the potential gradient $\partial^i\Phi$, which of course have the same transformation properties. While this is a perfectly valid example of covariance generated from an invariant Lagrange-function, one should note that while it is invariant unter rotations, it is not invariant under Galilei-transforms.

Let's take a leap of faith and replace the Lagrange-function by something relativistic, for instance the proper time $\tau = \int \mathrm{d}\tau$, which is measurable with a clock, fully Lorentz-invariant as

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - \gamma_{ij}\mathrm{d}x^i\mathrm{d}^j = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{A.66}$$

and has an intuitive geometric interpretation as the arc-length of the trajectory through spacetime in the geometry defined by the Minkowski-metric $\eta_{\mu\nu}$.

Suppose that a particle travels through spacetime along a trajectory $x^\mu(\tau)$. Then, we can define the 4-velocity u^μ as the rate at which the coordinates pass by the observer,

$$u^\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}x^\mu(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}x^\mu(\tau) = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}x^\mu = \gamma\cdot\begin{pmatrix} c \\ \upsilon^i \end{pmatrix} \tag{A.67}$$

with Lorentz-factor $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma$. The normalisation of the 4-velocity can be computed straightforwardly,

$$\eta_{\mu\nu}u^\mu u^\nu = u_\mu u^\mu = \gamma^2\cdot\left(c^2 - \upsilon_i\upsilon^i\right) = c^2, \tag{A.68}$$

because $\gamma^2\left(1-\beta^2\right) = 1$ for $\beta^i = \frac{\upsilon^i}{c}$. The 4-velocity, or the tangent to the trajectory $x^\mu(\tau)$ is therefore timelike $\eta_{\mu\nu}\,u^\mu u^\nu = c^2 > 0$ and the particle moves inside the light cone.

Inertial motion of a free particle should proceed along a straight line as a natural result of the relativistic variational principle. Indeed, starting with the arc-length s

$$S = \int_A^B \mathrm{d}s = \int_A^B c\mathrm{d}\tau = c \cdot \int_A^B \frac{\mathrm{d}t}{\gamma} \tag{A.69}$$

of a trajectory linking the spacetime points A to B we obtain the elapsed proper time τ (which can be measured by a clock carried along by the particle) or the integrated laboratory time $\int \mathrm{d}t/\gamma$, weighted by the Lorentz-factor, which is responsible for relativistic time dilation: $\mathrm{d}\tau = \mathrm{d}t \cdot \sqrt{1-\delta_{ij}\,\beta^i\beta^j} = \frac{1}{\gamma}\cdot \mathrm{d}t$, and because $\gamma \geq 1$, $\mathrm{d}\tau$ is always smaller than $\mathrm{d}t$ and proper time elapses slower.

This would imply that the Lagrange function of a free particle is $L(\dot{x}^i) = 1/\gamma$, and that the action S is in fact the arc-length of a trajectory. In the slow-motion limit $|\beta| \ll 1$ one should recover the classical Lagrange-function: Taylor-expanding yields

$$S \simeq -mc^2 \int_A^B \mathrm{d}t \cdot \left(1 - \frac{\delta_{ij}}{2}\beta^i\beta^j\right) = +mc^2 \int_A^B \mathrm{d}t \cdot \delta_{ij}\,\beta^i\beta^j + \mathrm{const.} = m \cdot \int_A^B \mathrm{d}t \cdot \delta_{ij}\,\upsilon^i\upsilon^j \tag{A.70}$$

with irrelevant prefactors, as affine transformations $S \rightarrow aS + b$ with two constants a, b drop out in the Euler-Lagrange-equation. Effectively, the non-relativistic limit yields the kinetic energy as the leading-order term of the proper time integral.

Funnily, Lorentz-covariance is lost in the non-relativistic limit and Galilei-invariance is not generated: If one carries out a Galilei-transform by setting $x^i \rightarrow x^i + \upsilon^i t$, and $\dot{x}^i \rightarrow \dot{x}^i + \upsilon^i$ one obtains:

$$L = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j \rightarrow \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \delta_{ij}\dot{x}^i\upsilon^j + \frac{1}{2}\delta_{ij}\upsilon^i\upsilon^j = L + \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta_{ij}x^i\upsilon^j + \delta_{ij}\upsilon^i\upsilon^j \cdot t\right), \tag{A.71}$$

where the additional term is a total time derivative with no influence on the variational principle: We find ourselves in the weird situation that we need a new concept to remedy the error made by classical Galilei-invariance!

Is inertial motion really proceeding along a straight line? Hamilton's principle requires that $\delta S = 0$, so

$$\delta S = -mc^2\,\delta \int_A^B \mathrm{d}\tau = -mc^2 \int_A^B \frac{\eta_{\mu\nu}}{2\mathrm{d}\tau} \cdot \left[\mathrm{d}x^\mu \cdot \delta\mathrm{d}x^\nu + \delta\mathrm{d}x^\mu \cdot \mathrm{d}x^\nu\right] = -mc^2 \int_A^B \eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\delta\mathrm{d}x^\nu, \tag{A.72}$$

where we have used the symmetry of the integrand to get rid of the factor 1/2. For continuing, we interchange variation and differentiation, $\delta\mathrm{d}x^\nu = \mathrm{d}\delta x^\nu$ and perform an integration by parts

$$\delta S = +mc^2 \int_A^B \mathrm{d}\tau\,\eta_{\mu\nu}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}\mathrm{d}x^\nu, \tag{A.73}$$

where $\mathrm{d}(\mathrm{d}x^\mu/\mathrm{d}\tau) = (\mathrm{d}^2x^\mu/\mathrm{d}\tau^2)\,\mathrm{d}\tau$, and with the assumption of vanishing variation on the boundary. The result then is that the 4-acceleration needs to vanish,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = 0 \tag{A.74}$$

for fulfilling Hamilton's principle, and the equation of motion is solved to yield $x^\mu(\tau) = a^\mu\tau + b^\mu$ with two integration constants: In fact, the solution is a straight line through spacetime.

A.4.2 Legendre-transforms and Hamilton-functions

We have seen in the last chapter that the Lagrange-function is much more a statement of causal motion in spacetime and has little to do with energies: Those appear after Legendre-transform, which is always well defined because the Lagrange function is a convex functional in $\dot{x}$ - this is, incidentally, the same reason why the variation yields a unique result and finds a unique extremum. In fact, the relativistic Lagrange-function $\mathrm{L} = 1/\gamma$ is perfectly convex as it always lies above its tangent: To visualise this, one can write $1/\gamma = \sqrt{c^2 - \upsilon^2}$, whose graph is a semi-circle!

Not only do convex functions have uniquely defined Legendre-transforms, but the Legendre-transformed function is again convex, making sure that the inverse transform is possible, too. Starting with the relativistic Lagrange-function

$$\mathrm{L}(\dot{x}) = \frac{1}{\gamma} = \sqrt{\mathrm{c}^2 - \dot{x}_i\dot{x}^i} \tag{A.75}$$

we can define the canonical momentum

$$p^i = \frac{\partial\mathcal{L}}{\partial\dot{x}^i} = \frac{\dot{x}^i}{\sqrt{\mathrm{c}^2 - \dot{x}_j\dot{x}^j}} \tag{A.76}$$

which we need to convert into a relation for $\upsilon(p)$: Let's do this in one dimension for simplicity.

$$p^2\left[\mathrm{c}^2 - \upsilon^2\right] = \upsilon^2,\; p^2\mathrm{c}^2 = \upsilon^2\left(1 + p^2\right) \;\rightarrow\; \upsilon = \frac{\mathrm{c}p}{\sqrt{1 + p^2}} \tag{A.77}$$

Then, the Legendre-transform, replacing $\dot{x} = \upsilon$ by p can be carried out and the Hamilton-function $\mathcal{H}$ can be obtained:

$$\mathcal{H}(p) = \dot{x}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \mathcal{L}(\dot{x}(p)) = v\cdot\frac{v}{\sqrt{\mathrm{c}^2 - v^2}} + \sqrt{\mathrm{c}^2 - v^2} = vp + \frac{v}{p} = \mathrm{c}\cdot\sqrt{1 + p^2}, \tag{A.78}$$

and if we include the prefactor $m\mathrm{c}^2$:

$$\mathcal{H}(p) = \sqrt{(m\mathrm{c}^2)^2 + \mathrm{c}^2p^2} \simeq m\mathrm{c}^2 + \frac{p^2}{2m} + \cdots, \tag{A.79}$$

where $m\mathrm{c}^2$ is the rest mass and $\frac{p^2}{2m}$ is the kinetic energy which appear in a Taylor-expansion in the last step.

For massive particles the energy-momentum-relation $\mathcal{H}$ allows statements about dispersion: In fact, phase and group velocities can not be equal for massive particles,

$\mathcal{H}/p \neq \mathrm{d}\mathcal{H}/\mathrm{d}p$, but one can show that

$$\upsilon_{\mathrm{ph}} \cdot \upsilon_{\mathrm{gr}} = \frac{\mathcal{H}}{p}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}p} = \frac{cp}{\sqrt{1+p^2}}\frac{c\sqrt{1+p^2}}{p} = c^2 \tag{A.80}$$

i.e. that the geometric mean of phase and group velocity is the speed of light. That in turn implies that the phase velocity of massive particles needs to be $\upsilon_{\mathrm{ph}} > c$ if their group velocity is subluminal, $\upsilon_{\mathrm{gr}} < c$. And, as a shortcut,

$$c^2 = \frac{\mathcal{H}}{p}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}p} = \frac{\mathrm{d}(\mathcal{H}^2)}{\mathrm{d}(p^2)} \tag{A.81}$$

which can be integrated to give $\mathcal{H}^2 = (cp)^2 + \mathrm{const}$, with the rest mass as the integration constant.

We have already encountered the classification of Lorentz-vectors in timelike, spacelike and lightlike, and we saw that 4-velocities u^μ are normalised according to $\eta_{\mu\nu}u^\mu u^\nu = c^2 > 0$ with the associated motion inside the light cone. Clearly, that normalisation is conserved under Lorentz-transforms, but one might be curious as to the possibility whether forces could accelerate a particle to super-luminous speeds: A classical argument would be that this would be energetically impossible due to relativistic mass increase (which is really only a consequence of proper time dilation), but there is a more elegant, geometric argument. Acting on a charged, massive particle with a Lorentz-force leads to the equation of motion

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{q}{m}F^{\mu\nu}u_\nu. \tag{A.82}$$

Multiplying both sides with u_μ then gives a relation how the normalisation of u^μ would change under the influence of a Lorentz-force:

$$u_\mu\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u_\mu u^\mu\right) = \frac{q}{m}F^{\mu\nu}u_\mu u_\nu = 0, \tag{A.83}$$

where the last term is vanishing as a contraction between an antisymmetric tensor $F^{\mu\nu}$ and a symmetric one, $u_\mu u_\nu$, making sure that the normalisation $u_\mu u^\mu = c^2$ is conserved and the motion of a massive particle is restricted to the inside of the light cone: Electromagnetic forces can therefore not push a particle outside the light cone and it is impossible to achieve superluminal speeds.

At this point, the Lorentz-geometry arises because of the requirement that the speed of light was equal in al inertial frames, but one might ask if there is a more fundamental reason: As it is, the constancy of c might just be an empirical observation. The truth is very far from that as the Lorentz-geometry is a natural way for Nature to construct hyperbolic partial differential equations as her field equations (where Maxwell's equations or even the gravitational field equation are just examples). Hyperbolic (partial) differential equations are peculiar because they (*i*) realise a unique time evolution for specified initial conditions, (*ii*) are perfectly time-invertible and (*iii*) show causal propagation: There is a finite speed (in our case c) at which excitations of the fields travel, and the Lorentzian structure of spacetime makes sure that the light cones are in fact identical in all frames: In this way one can be sure that the initial conditions for the evolution of the fields at a given coordinate are identical in all frames!

That implies that the fundamental Lorentzian structure of spacetime is in fact compatible with the hyperbolicity of the field equations. This is reached by defining a partial differentiation ∂^μ with respect to the coordinates,

$$\partial^\mu = \begin{pmatrix} \partial^{ct} \\ -\partial^i \end{pmatrix} \tag{A.84}$$

where the minus-sign is added to make sure that the divergence of x^μ is equal to the dimensionality, i.e. 4:

$$\partial_\mu x^\mu = \frac{\partial x^\mu}{\partial x^\mu} = 4 = \partial_{ct}(ct) + \partial_i x^i = 1 + 3 = \eta_{\mu\nu}\,\partial^\mu x^\nu, \tag{A.85}$$

and the corresponding linear form is given by $\partial_\mu = \eta_{\mu\nu}\partial^\nu = (\partial_{ct}, \partial_i)$. Then, the d'Alembert-operator would be naturally Lorentz-invariant because it is defined as a Lorentz-scalar,

$$\Box = \partial_\mu \partial^\mu = \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_{ct}^2 - \Delta, \tag{A.86}$$

and typical wave equations like $\Box\Phi = 0$ would generate a light cone, as propagation of excitation proceeds with velocities $\pm c$:

$$\Box\,\Phi = \left(\partial_{ct}^2 - \partial_x^2\right) = \left(\partial_{ct} + \partial_x\right)\left(\partial_{ct} - \partial_x\right)\Phi = 0. \tag{A.87}$$

The same property is reflected by the wave vectors being null: $\Phi = \exp\left(\pm\,\mathrm{i}k_\alpha x^\alpha\right)$ solves the wave equation

$$\Box\,\Phi = \eta^{\mu\nu}\,\partial_\mu\partial_\nu\Phi = 0 \tag{A.88}$$

only of $\eta_{\mu\nu}k^\mu k^\nu = 0$, which holds again in every frame. Giving the components of the wave vector k^μ the interpretation of the angular frequency ω and the spatial wave vector k^i,

$$k^\mu = \begin{pmatrix} \frac{\omega}{c} \\ k^i \end{pmatrix} \tag{A.89}$$

shows first of all the dispersion-free propagation along the light cone, as the normalisation $\eta_{\mu\nu}k^\mu k^n u = 0$ implies that $\omega^2/c^2 - k^2 = 0$ and therefore a proportionality $\omega = \pm ck$, such that the phase velocity ω/k and the group velocity $\mathrm{d}\omega/\mathrm{d}k$ are identical and dispersion is not taking place. Secondly, the (relativistic) Doppler-effect can be derived by projecting k^μ onto an observer's 4-velocity u^μ. At rest, u^μ has only a temporal nonzero component of c, such that $\omega = \eta_{\mu\nu}u^\mu k^\mu$, but for a moving observer with u'^μ one obtains

☠ *k^μ being a null-vector and dispersion-free propagation are equivalent.*

$$\omega' = \eta_{\mu\nu}u'^\mu k^\nu = \gamma(\omega - \upsilon_i k^i). \tag{A.90}$$

A.4.3 non-relativistic motion in weak gravitational potentials

In anticipation of general relativity we should have a look at changing the geometry of spacetime and to move away from a Lorentzian space. And we need to make sure that the relativistic line element is in fact the relativistic generalisation of the classical Lagrange-function.

Weak gravitational potentials $\Phi = -GM/r$ sourced by a mass M at distance r perturb the line element

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij}\, dx^i dx^j \tag{A.91}$$

such that one recovers in the Minkowski-metric at large distances $r \gg 2GM/c^2$. If that is the case, the passage of proper time of a stationary observer where $dx^i = 0$ would be dilated

$$ds^2 = c^2 d\tau^2 \simeq \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 \tag{A.92}$$

and proper time would in fact depend on the presence of gravitational potentials! That would then imply that the variational principle should find a different trajectory if Φ is nonzero compared to the case $\Phi = 0$. The action would again be given as the line element, but now derived from the actual perturbed metric $g_{\mu\nu}$ instead of the Minkowski-metric $\eta_{\mu\nu}$:

$$S = -mc \int_A^B ds = -mc \int_A^B d\tau \cdot \sqrt{g_{\mu\nu}\, u^\mu u^\nu} \quad \text{using} \quad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \tag{A.93}$$

Substituting the 4-velocity u^μ with the spatial component $\gamma \upsilon^i$ then yields for the action

$$S = -mc \int_A^B d\tau \cdot \gamma \sqrt{\left(1 + \frac{2\Phi}{c^2}\right) c^2 - \left(1 - \frac{2\Phi}{c^2}\right) \cdot \delta_{ij}\, \upsilon^i \upsilon^j} \tag{A.94}$$

which is then approximated to give

$$S \simeq -mc \int_A^B dt \sqrt{c^2 \cdot \left(1 + \frac{2\Phi}{c^2} - \delta_{ij}\, \beta^i \beta^j\right)} \tag{A.95}$$

and finally Taylor-expanded to yield

$$S = \simeq -mc^2 \int_A^B dt \left(1 + \frac{\Phi}{c^2} - \frac{\delta_{ij}\, \beta^i \beta^j}{2}\right) \tag{A.96}$$

so that we finally arrive at

$$S = \int_A^B dt \left(m\delta_{ij}\, \frac{1}{2} \upsilon^i \upsilon^j - m\Phi\right) = \int_A^B dt\, L \tag{A.97}$$

where we recognise the classical Lagrange function in the integrand, with a kinetic and a potential term.

A.4.4 photon propagation on the Lorentzian spacetime

Up to this point, have shown that the archetypical hyperbolic wave equation $\Box\Phi = 0$ is solved in fact by plane waves $\Phi \sim \exp(\pm i\eta_{\mu\nu}\, k^\mu x^\nu)$ with a wave vector k^μ which is null, $\eta_{\mu\nu}k^\mu k^\nu = 0$. The same should be true for the propagation of electromagnetic waves, so we need to make sure that Maxwell's equations provide a pathway to obtain a hyperbolic wave equation for the field tensor $F^{\mu\nu}$. Specifically, the homogenous Maxwell-equation (or the Bianchi-identiy) should be the relevant here, as electromagnetic waves are vacuum solutions.

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0, \tag{A.98}$$

to which one can apply the differentiation ∂_λ to obtain

$$\partial_\lambda\partial^\lambda F^{\mu\nu} + \partial_\lambda\partial^\mu F^{\nu\lambda} + \partial_\lambda\partial^\nu F^{\lambda\mu} = 0. \tag{A.99}$$

Identifying the d'Alembert operator $\partial_\lambda\partial^\lambda = \Box$ and using commutativity of partial derivatives, $\partial_\lambda\partial^\mu = \partial^\mu\partial_\lambda$ as well as the antisymmetry of the field tensor, $F^{\nu\lambda} = -F^{\lambda\nu}$ and $\partial_\lambda\partial^\nu = \partial^\nu\partial_\lambda$ this becomes

$$\Box F^{\mu\nu} - \partial^\mu\partial_\lambda F^{\lambda\nu} + \partial^\nu\partial_\lambda F^{\lambda\mu} = 0. \tag{A.100}$$

Now in vacuum, i.e. in the absence of a source $\jmath^\mu = 0$, the field equation is $\partial_\mu F^{\mu\nu} = 0$ and in fact there is a wave-equation with for the field tensor,

$$\Box F^{\mu\nu} = \eta^{\alpha\beta}\, \partial_\alpha\partial_\beta\, F^{\mu\nu} = 0. \tag{A.101}$$

Analogously to the case of a scalar field one expects a plane wave of the type $F^{\mu\nu} \simeq \exp(\pm i\eta_{\gamma\delta}\, k^\gamma x^\delta)$ to solve this equation. Doing that, it is a good idea to use different indices for the differentiation and for the quadratic form $\eta_{\gamma\delta}\, k^\gamma x^\delta$ and to rename the indices with the Kronecker-δ appearing through $\partial_\alpha x^\mu = \partial x^\mu/\partial x^\alpha = \delta^\mu_\alpha$.

$$\Box F^{\mu\nu} = \eta^{\alpha\beta}\, \partial_\alpha\partial_\beta \exp(\pm i\eta_{\gamma\delta}\, k^\gamma k^\delta) = (\pm i)^2 \cdot \exp\left(\pm i\eta_{\gamma\delta}\, k^\gamma x^\delta\right) \eta_{\gamma\beta}\, k^\gamma k^\beta = 0, \tag{A.102}$$

recovering the null-condition $\eta_{\gamma\beta}\, k^\gamma k^\beta = 0$, confirming that excitations of the electromagnetic field do in fact travel along null-lines, which implies that the Maxwell-equations respect the fundamental Lorentzian structure of spacetime.

The field equation makes sure that the excitations of the fields are perpendicular to the propagation direction and that the wave is indeed transverse: Again, using the ansatz $F^{\mu\nu} = F^{(0),\mu\nu} \exp(\pm i\eta_{\gamma\delta}\, k^\gamma x^\delta))$ one immediately convinces oneself that

$$\partial_\mu F^{\mu\nu} = F^{(0),\mu\nu} \cdot \partial_\mu \exp(\pm i\eta_{\gamma\delta}\, k^\gamma x^\delta) = (\pm i) \exp(\pm i\eta_{\gamma\delta}\, k^\gamma x^\delta) \cdot \eta_{\gamma\mu} F^{(0),\mu\nu} k^\gamma = 0 \tag{A.103}$$

and therefore $\eta_{\gamma\mu} F^{(0),\mu\nu} k^\gamma = 0$. In terms of the field components of the electric field E^i this means that $\delta_{ij} k^i E^j = 0$, and the analogous statement for the magnetic field B^i would be obtained from the dual field tensor $\eta_{\gamma\mu}\tilde{F}^{(0),\mu\nu} k^\gamma = 0$, as a consequence of electromagnetic duality in vacuum.

Photons move along null-lines, so the arc length measured along their trajectory x^μ will always come out as zero: That means that one can not work with the proper time τ. Using a new affine parameter λ to address the points along the trajectory $x^\mu(\lambda)$ suggests the definition of the wave vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, because

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}\cdot \mathrm{d}\lambda^2 = \eta_{\mu\nu}\,k^\mu k^\nu \mathrm{d}\lambda^2 = 0 \tag{A.104}$$

At this point, we should start to be careful not to link the Lorentz-geometry to any particular coordinate choice. When considering light cone coordinates, $\mathrm{d}u = c\mathrm{d}t + \mathrm{d}x$ and $\mathrm{d}v = c\mathrm{d}t - \mathrm{d}x$ the line element is given by

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = c^2\mathrm{d}t^2 - \mathrm{d}x^2 = (c\mathrm{d}t + \mathrm{d}x)(c\mathrm{d}t - \mathrm{d}x) = \mathrm{d}u\cdot\mathrm{d}v, \tag{A.105}$$

and the corresponding Lorentzian metric is represented by the matrix

$$\eta_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.106}$$

in these coordinates. Surely, the geometry is identical and has not been changed by the new definition of coordinates, and the spectrum of eigenvalues of the new metric is identical.

A.4.5 photon propagation through weak gravitational fields

At this point we should derive a puzzling result, which was in fact the first proper prediction of general relativity: that gravitational fields have a stronger effect on the motion of relativistic particles such as photons compared to non-relativistic particles. We start by introducing a weak perturbation to the Minkowski-metric and define the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{A.107}$$

with $g_{\mu\nu}$ being the metric tensor. For fixed Cartesian coordinates and a weak gravitational potential Φ with $|\Phi| \ll c^2$ the line element becomes

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j. \tag{A.108}$$

It will be the case that for slow motion $\left|\dot{x}^i\right| \ll c$ the classical equation of motion is valid and will come out as $\ddot{x}^i = -\partial^i\Phi$, as expected, by variation of $\int \mathrm{d}\tau$. There is some intuition to this result because a non-relativistic particle moves essentially only along the ct-axis of the coordinate frame, so that $\mathrm{d}\tau$ is approximately equal to $(1 + \Phi/c^2)\mathrm{d}t$.

Photons need an entirely different argumentation, because they always follow null-lines, $\mathrm{d}s^2 = 0$. For that case we can define an effective speed of propagation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c\cdot\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \simeq c\cdot\left(1 + \frac{2\Phi}{c^2}\right) \tag{A.109}$$

such that we can define an index of refraction, which is proportional to 2Φ instead of Φ! That realisation prompted A. Eddington in 1919 to measure gravitational light deflection during a Solar eclipse and the deflection angle was indeed twice as large as expected from a Newtonian theory.

B TOPOLOGICAL AND METRIC STRUCTURE OF SPACETIME

General relativity requires that the idea of a vector-space (with Lorentzian geometry) as a model for spacetime is given up. From the example of the perihelion precession of the planet Mercury we saw that the gravitational field around massive objects like the Sun is stronger compared to the prediction of a Newtonian theory: This is surprising, because the $1/r$-form of the potential is a direct consequence of the fact that surfaces of spheres scale $\propto r^2$, so typical for a Euclidean vector-space. The new model for spacetime that was pioneered by Albert Einstein and by David Hilbert was that of a manifold: A topological space with a metric and a differential structure, and ultimately, curvature as an expression of the gravitational field. The decisive property of curved manifolds is a locally defined, varying geometry, encapsulated by the metric, which becomes dependent on the coordinates.

The topological structure explains the connectivity of sets of spacetime points and introduces open sets, which are used to construct continuous mappings of the spacetime points onto their coordinates. Changes from one coordinate choice to another need to be invertible and differentiable (which is called a diffeomorphism). Adding a metric structure to the manifold allows the measurement of norms of vectors and the angle between them, and the construction of invariants. Finally, the construction of parallel transport and that of a covariant derivative allows statements about variations of vector- and tensor-fields defined on the manifold. We need to make sure that all these structures are compatible with each other.

B.1 *metric structure of manifolds and coordinate transforms*

We have already encountered weak perturbations to the Minkowski-metric $\eta_{\mu\nu}$ mediated by the gravitational field in the limit of weak fields $|\Phi| \ll c^2$ (which is only valid in a particular coordinate choice!). A general metric tensor $g_{\mu\nu}$ defines an infinitesimal contribution $\mathrm{d}s^2$ to the line element,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{B.110}$$

between two points that have an infinitesimal coordinate difference $\mathrm{d}x^\mu$. With this definition, the metric tensor is symmetric as $\mathrm{d}s^2$ would not pick up any antisymmetric contribution in the contraction with $\mathrm{d}x^\mu\mathrm{d}x^\nu$.

The line element $\mathrm{d}s^2$ is scalar, under coordinate transformations we should obtain:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu = g_{\mu\nu}\cdot\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}\mathrm{d}x'^\rho\mathrm{d}x'^\sigma \tag{B.111}$$

isolating the transformation rule for the metric to be

$$g_{\mu\nu}\cdot\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \equiv g'_{\rho\sigma}, \tag{B.112}$$

and is naturally inverse to that of vectors like $\mathrm{d}x^\mu$

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu, \tag{B.113}$$

making sure that the coordinate transformation by the Jacobian and its inverse cancel each other,

$$\frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\rho} = \frac{\partial x^\mu}{\partial x^\rho} = \delta^\mu_\rho \tag{B.114}$$

The interpretation of $\mathrm{d}s^2$ as the arc-length of a trajectory through spacetime is still that of proper time, $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2$, measured on a clock comoving with a massive particle. We will show that photons would follow null-lines, $\mathrm{d}s^2 = 0$, so that the definitions of light-cones and their associated causal structure is valid on metric manifolds in exactly the same way.

A metric defines a geometry by defining distances and angles: It is a mapping of a pair of vectors x, y onto a positive number obeying the three metric axioms:

1. $g(x,y) \geq 0$, if $g(x,y) = 0 \leftrightarrow x = y$ positive definiteness
   Because from a physical motivation, the classification of vectors into timelike, spacelike and lightlike is incredibly important, we will soften this axiom and allow negative values for $\mathrm{d}s^2 = g(\mathrm{d}x, \mathrm{d}x) = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$: This, ultimately, defines a pseudo-Riemannian geometry.

In relativity we're dealing with pseudo-metrics: $\mathrm{d}s^2$ is negative for space-like vectors, but those lie outside the light cone and do not correspond to causal processes.

2. $g(x,y) = g(y,x)$ symmetry
   This axiom is fulfilled by $g_{\mu\nu}$ being a symmetric, real valued tensor defining a quadratic form

3. $g(x,y) + g(y,z) \geq g(x,z)$ triangle inequality
   Again, there might be physical situations, where a "detour" is shorter than the direct path, and the classical example for this is the twin paradoxon: Lightlike vectors have smaller norms than timelike vectors.

You would not believe how much I'd like at this point to go off on a tangent about the necessity of a metric structure and the possibility of having geometries that are defined in different ways, for instance avoiding scalar products. Instead, I would just like to emphasise that the only metric geometry allowing for hyperbolic evolution of the field equations along an invariant light cone is the Lorentzian one.

It is important to clarify the relation between an arbitrary geometry $g_{\mu\nu}$ and the Lorentz-geometry $\eta_{\mu\nu}$: If one zooms in onto a single point of spacetime, it should have a locally Minkowskian shape and allow for the local choice of Cartesian coordinates (called normal coordinates in this context). Clearly, with a coordinate transform one can transform the metric

$$g'_{\rho\sigma}(x) = g_{\mu\nu}(x) \cdot \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} \tag{B.115}$$

at one point in such a way that it becomes diagonal with eigenvalues λ_μ, because it is symmetric. A rescaling of the coordinates $x^\mu \to x^\mu\sqrt{\lambda_\mu}$ would then make $g_{\mu\nu}$ identical to $\eta_{\mu\nu}$.

But it should not be possible to bring the entire manifold to a Lorentzian shape and to choose globally Cartesian coordinates: To show this, we need to overcome the idea that an arbitrary coordinate transform would be able to define just the right transform to ensure $g_{\mu\nu} = \eta_{\mu\nu}$ at every point. Let's consider a general coordinate transform $x^\mu(x'^\nu)$ at a point P:

$$x^\mu(x') = x^\mu\Big|_\mathrm{P} + \frac{\partial x^\mu}{\partial x'^\nu}\Big|_\mathrm{P}\left(x'^\nu - x'^\nu_\mathrm{P}\right) \tag{B.116}$$

$$+ \frac{1}{2}\frac{\partial^2 x^\mu}{\partial x'^\nu \partial x^\rho}\Big|_\mathrm{P} \cdot \left(x'^\nu \cdot x'^\nu_\mathrm{P}\right)\left(x'^\rho - x'^\rho_\mathrm{P}\right) \tag{B.117}$$

$$+ \frac{1}{3!}\frac{\partial^3 x^\mu}{\partial x'^\nu \partial x'^\rho \partial x'^\sigma}\Big|_\mathrm{P}\left(x'^\nu - x'^\nu_\mathrm{P}\right)\left(x'^\rho - x'^\rho_\mathrm{P}\right)\left(x'^\sigma - x'^\sigma_\mathrm{P}\right) + \cdots \tag{B.118}$$

and count the number of degrees of freedom that is provided at every order of the Taylor-expansion and see if they suffice to have $g = \eta$ and to make all derivatives of g appear at arbitrary order. If that would be the case, a coordinate transform could be found that diagonalises the metric at every point and makes it globally Minkowskian, across the entire manifold.

1. At lowest order, there are are more degrees of freedom provided by the coordinate transform to diagonalise the metric $g_{\mu\nu}$ and have unit diagonal entries: We can adjust the coordinate transform to make $g_{\mu\nu} = \eta_{\mu\nu}$ at the point P, because counting the degrees of freedom yields

$$\frac{\partial x^\mu}{\partial x'^\nu} \sim n^2 \tag{B.119}$$

   because there are n choices for x and n independent choices for x'

$$g'_{\mu\nu} \sim \frac{n(n+1)}{2} \tag{B.120}$$

   because the metric is a symmetric, real-valued $n \times n$ matrix. The counting shows that $n^2 > n(n+1)/2$ for every number of dimensions n, so there are enough degrees of freedom to adjust $g_{\mu\nu} = \eta_{\mu\nu}$ locally at P.

2. At second order, the number of degrees of freedom provided by the coordinate transform is exactly that needed to make the first derivatives of the metric vanish at P.

$$\frac{\partial^2 x^\mu}{\partial x'^\nu \partial x'^\rho} \sim \frac{n^2(n+1)}{2} \tag{B.121}$$

   because the differentiations should not be counted twice for $\nu = \rho$, and

$$\frac{\partial g'_{\mu\nu}}{\partial x'^\rho} \sim \frac{n^2(n+1)}{2} \tag{B.122}$$

   because there are n possible differentiations of a symmetric matrix. Surprisingly, the degrees of freedom provided by the coordinate transform suffice exactly to have the derivatives $\partial_\rho g_{\mu\nu}$ disappear locally at P.

3. At third order, the number of degrees of freedom provided by the coordinate transform falls short of the number needed to make the second derivatives of the metric at the point P disappear.

$$\frac{\partial^3 x^\mu}{\partial x'^\nu \partial x'^\rho \partial x'^\sigma} \sim n \cdot \frac{n(n+1)(n+2)}{6} \tag{B.123}$$

because all derivatives must be different, while

$$\frac{\partial g'_{\mu\nu}}{\partial x'^{\rho}\partial x'^{\sigma}} \sim \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2} \tag{B.124}$$

because both the metric and the double partial are symmetric. As $n^2(n+1)(n+2)/6 > n^2(n+1)^2/4$, the second derivatives of the metric can not be made to vanish at P in the general case.

Continuing this line of reasoning shows that the problem exacerbates: The numbers of degrees of freedom provided by the coordinate transforms always falls short of the degrees of freedom needed to make higher order derivatives of the metric vanish. From that we conclude that there can only be two cases: Either the manifold is already Lorentzian but with an unfortunate coordinate choice, in which case there is a global construction of normal coordinates, or the manifold has new properties expressed by the non-vanishing second derivatives of the metric: This is in fact the curvature, as a new intrinsic property of the manifold that exists in any coordinate choice. But even if that is the case, our argument shows that the spacetime structure is locally Lorentzian with a Minkowski-metric.

B.2 *locally Minkowskian structure and the equivalence principle*

While this argument is elegant, we might ask if the coordinate choice that achieves a locally flat structure has a particular physical meaning: This is in fact the case, as an expression of the equivalence principle which stipulates that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu} = 0$ in a freely falling frame of reference. In such a freely falling frame, one recovers (locally!) perfectly Lorentzian geometries and the laws of special relativity are valid, for instance Maxwell's equations as defined on a flat, Minkowskian spacetime. The "size" r of the freely falling laboratory in which special relativity applies at least approximatively is given by the requirement that curvature effects associated with the second derivatives of the metric can not be dominant:

$$\frac{1}{r^2} = \left|\frac{\partial^2 g}{\partial x^2}\right| \longrightarrow r = \left|\frac{\partial^2 g}{\partial x^2}\right|^{-\frac{1}{2}} \tag{B.125}$$

And we will see in a second that the Christoffel-symbols $\Gamma^{\alpha}{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left[\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right]$ will be zero, due to their proportionality to ∂g, and that the covariant derivative $\nabla_\mu \upsilon^\alpha = \partial_\mu \upsilon^\alpha + \Gamma^{\alpha}{}_{\mu\beta}\,\upsilon^\beta$ reverts back to the partial derivative $\partial_\mu \upsilon^\alpha$.

B.3 *vectors and fields on manifolds*

Let us start with the picture that a manifold as the continuum of spacetime points has been given coordinates by a suitable mapping, so every point P has coordinates, x^μ. Changing from one coordinate set x^μ to a new set x'^ν should be done in an invertible, differentiable way. The manifold itself is not a vector space, but we can define abstract fields on the manifold: If they have internal degrees of freedom, their components can be expressed in the local set of basis vectors spanning the tangent space (or cotangent space, if their degrees of freedom rather correspond to linear forms instead of vectors).

One of the easiest geometric objects we can define is a curve $C(\lambda) = x^\mu(\lambda)$ visiting the spacetime points x^μ as the (possibly affine) parameter λ evolves. If there is a scalar field $\phi(x^\mu)$ defined on the manifold, the rate at which the field amplitude changes along the curve $x^\mu\lambda$ would be given by

$$\frac{d\phi}{d\lambda} = \frac{d}{d\lambda}\phi(x^\mu(\lambda)) = \frac{dx^\mu}{d\lambda}\frac{\partial\phi}{\partial x^\mu} = u^\mu\frac{\partial\phi}{\partial x^\mu} \tag{B.126}$$

and we recognise in the last term the scalar multiplication of the field gradient $\partial\Phi/\partial x^\mu$ into the tangent vector $u^\mu = dx^\mu/d\lambda$. In this sense, one can think of the tangent u^μ and of dx^μ as vectors. In transforming from on set of coordinates to another set shows that the vector u^μ and the linear forms $\partial_\mu\phi$ transform consistently:

$$\frac{d\phi}{d\lambda} = u^\mu\frac{\partial\phi}{\partial x^\mu} = u^\nu\delta^\mu_\nu\cdot\frac{\partial\phi}{\partial x^\mu} = u^\nu\frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial\phi}{\partial x^\mu} = u'^\alpha\frac{\partial\phi}{\partial x'^\alpha} \tag{B.127}$$

so that in fact the vector u^μ transforms with the Jacobian and the linear form $\partial_\mu\phi$ with the inverse Jacobian of the coordinate transform.

We can run all possible curves through the point x^μ and get a complete set of tangent vectors which would ultimately constitute, after proper orthonormalisation, a local basis set: the basis of the tangent space $T_P M$ at the point P with the coordinates x^μ: It is important to realise that the tangent space's basis set exists for a given choice of coordinates and that a different coordinate choice would induce a new basis set. In particular, a neighbouring point Q can have a different tangent space $T_Q M$. That implies that if we take the same abstract vector $\boldsymbol{\upsilon}$ and express it with the basis sets at $T_P M$ and $T_Q M$ in coordinates υ^μ, the tuples will in general differ, and that one needs a more elaborate concept of differentiating vectors than just partial derivatives: the covariant derivative.

Up to this point, the manifold has two structures: the topological structure which defines open sets and allows the definition of continuous coordinate mappings, and the metric structure which defines the geometry through a scalar product. The two structures are compatible with each other, as the definition of open sets with the metric is never in contradiction with the topology. The next step is the definition of a differentiable structure constructed with parallel transport.

B.4 *parallel transport and the covariant derivative*

Parallel transport generates a perfect copy of an abstract vector at a different spacetime position. After defining coordinates and therefore entries of a vector tuple, the parallel transported copy $\upsilon^\mu_\parallel(x+\delta x)$ of the vector $\upsilon^\mu(x)$ at a new, infinitesimally distant position $x+\delta x$ is given by

$$\upsilon^\mu_\parallel(x+\delta x) = \upsilon^\mu(x) - \Gamma^\mu{}_{\alpha\beta}\,\upsilon^\alpha(x)\cdot\delta x^\beta + \cdots \tag{B.128}$$

at lowest order. It is conventional to use a minus-sign in front of the Christoffel-symbol $\Gamma^\mu{}_{\alpha\beta}$, which generates the transformation rule for the vector υ^μ, because we have a different set of tangent vectors at $x+\delta x$ compared to the point x, and therefore different expansions of the same vector into two different basis sets. In fact, the best way to visualise the Christoffel-symbol is to think of $\Gamma^\mu{}_{\alpha\beta}$ as a transformation matrix in the indices α and μ acting on the components υ^α for shifts in any possible direction δx^β.

There are cases where the connection is trivially zero, that is when index by index the components of the parallel-transported vector are identical to the original vector, which would be the case in a vector-space or a flat manifold with Cartesian coordinates. In the general case, the tangent spaces at x and $x + \delta x$ are not identical and have a different set of basis vectors, so the expansion of the abstract vector, although it is in fact identical at x and $x + \delta x$ under perfect parallel transport, needs to be different.

With this definition of parallel transport we can ask whether a vector field υ has changed moving from x to $x + \delta x$, or equivalently, if it has a derivative. It is senseless just to compare the entries of the vectors as they exist in different tangent spaces, rather, we need to compare the vector field at $x + \delta x$ with a parallel transported version of υ taken from x to $x + \delta x$.

Taking the limit $\delta x^\beta \to 0$ to get the differential rate of change yields

$$\nabla_\beta \upsilon^\mu = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu_\parallel(x+\delta x)}{\delta x^\beta} = \lim_{\delta x^\beta \to 0} \frac{\upsilon^\mu(x+\delta x) - \upsilon^\mu(x)}{\delta x^\beta} + \Gamma^\mu{}_{\alpha\beta} \cdot \upsilon^\alpha(x) \cdot \frac{\delta x^\beta}{\delta x^\beta} \tag{B.129}$$

such that the covariant derivative is given by

$$\nabla_\beta \upsilon^\mu = \partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha \tag{B.130}$$

if we substitute the partial derivative as the index-by-index comparison of the entries υ^μ at the two infinitesimally separated points. For scalar fields Φ there is no distinction between the covariant derivative and the conventional partial derivative, $\nabla_\mu = \partial_\mu \phi$ because there are no internal degrees of freedom whose entries would change if the set of basis vectors is different, hence the field can only have a derivative if it assumes a different value. Using Cartesian coordinates on a flat manifold allows the usage of the connection $\Gamma^\mu{}_{\alpha\beta} = 0$, because all tangent spaces are identical (or aligned) and vectors do not change their entries moving from one tangent space to another, therefore $\upsilon^\mu_\parallel(x + \delta x) = \upsilon^\mu(x)$ in parallel transport and consequently, $\nabla_\beta \upsilon^\mu = \partial_\beta \upsilon^\mu$.

Higher-order tensors require a Christoffel-symbol for every index

$$\nabla_\beta T^{\mu\nu} = \partial_\beta T^{\mu\nu} + \Gamma^\mu{}_{\alpha\beta}\, T^{\alpha\nu} + \Gamma^\nu{}_{\alpha\beta}\, T^{\mu\alpha} \tag{B.131}$$

because their basis set is the Cartesian product of the basis of $T_P M$, one factor for each index.

The covariant differentiation can be constructed for linear forms (or covariant vectors) in a way that is compatible with the differentiation of (contravariant) vectors: Because a product $\upsilon^\mu w_\mu = g_{\mu\nu} \upsilon^\mu w^\mu$ would be scalar, the covariant derivative reverts back into a partial one:

$$\nabla_\beta\left(\upsilon^\mu w_\mu\right) = \partial_\beta\left(\upsilon^\mu w_\mu\right) = \partial_\beta \upsilon^\mu \cdot w_\mu + \upsilon^\mu \cdot \partial_\beta w_\mu. \tag{B.132}$$

If we require the covariant differentiation to obey a Leibnitz-rule, the last term can be written as:

$$\nabla_\beta\left(\upsilon^\mu w_\mu\right) = \nabla_\beta \upsilon^\mu \cdot w^\mu + \upsilon^\mu \nabla_\beta w_\mu = \left(\partial_\beta \upsilon^\mu + \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha\right) w_\mu + \upsilon^\mu \nabla_\beta w_\mu \tag{B.133}$$

Then, the term $\partial_\beta \upsilon^\mu \cdot w_\mu$ drops out and renaming the indices $\mu \leftrightarrow \alpha$

$$\upsilon^\mu\left(\nabla_\beta w_\mu\right) = \upsilon^\mu \cdot \partial_\beta w_\mu - \Gamma^\mu{}_{\alpha\beta}\, \upsilon^\alpha w_\mu = \upsilon^\mu \partial_\beta w_\mu - \Gamma^\alpha{}_{\mu\beta}\, \upsilon^\mu w_\alpha = \upsilon^\mu\left(\partial_\beta w_\mu - \Gamma^\alpha{}_{\mu\beta}\, w_\alpha\right) \tag{B.134}$$

gives the final result

$$\nabla_\beta w_\mu = \partial_\beta w_\mu - \Gamma^\alpha{}_{\mu\beta}\, w_\alpha \tag{B.135}$$

for the covariant derivative of a linear form, with a minus-sign instead of a plus-sign.

Up to this point, the connection has been arbitrary but we will now focus on Levi-Civita-connections: Those are metric-compatible and torsion-free, and can therefore be computed from the metric and its derivatives. A metric manifold with such a connection and the corresponding covariant derivative is referred to as a Riemannian geometry. It is important to achieve the compatibility between the metric and the differentiable structure of the manifold so that we can compute the connection coefficients from the metric itself. Scalar products $\upsilon^\mu w_\mu = g_{\mu\nu}\, \upsilon^\mu w^\nu$ between two vectors υ and $\mathbf{w}$ that are parallel transported should be identical: The parallel transport of two abstract vectors only changes the tuples υ^μ and w^μ because the tangent spaces change and a different basis set is provided at every point. The scalar product is an abstract measure of the lengths and relative orientations of the two vectors and that statement should be invariant:

$$g(\upsilon(x), \mathbf{w}(x)) = g(\upsilon_{\|}(x + \delta x), \mathbf{w}_{\|}(x + \delta x)) \tag{B.136}$$

For that to be conserved, parallel transport by δx^β should not change anything, neither the length nor the relative orientation of the two vectors, $\upsilon_{\|}(x+\delta x) = \upsilon(x+\delta x)$, and $\delta x^\beta \nabla_\beta \upsilon^\mu$ is necessarily zero. Stating that the scalar product of parallel-transported vectors remains constant is equivalent to

$$\delta x^\beta\, \nabla_\beta\, g = \delta x^\beta\, \nabla_\beta \left(\upsilon^\mu w_\mu\right) = \delta x^\beta\, \nabla_\beta \left(g_{\mu\nu} \upsilon^\mu w^\nu\right) = 0 \tag{B.137}$$

As the covariant derivatives obeys a Leibnitz-rule, one can write

$$\delta x^\beta\, \nabla_\beta \left(g_{\mu\nu}\, \upsilon^\mu w^\nu\right) = \delta x^\beta \left(\nabla_\beta\, g_{\mu\nu} \cdot \upsilon^\mu w^\nu + g_{\mu\nu}\, \nabla_\beta\, \upsilon^\mu \cdot w^\nu + g_{\mu\nu}\, \upsilon^\mu\, \nabla_\beta w^\nu\right) \tag{B.138}$$

and therefore, as $\delta x^\beta \nabla_\beta \upsilon^\mu = 0$ and $\delta x^\beta \nabla_\beta w^\nu = 0$ as an expression of parallel transport,

$$\delta x^\beta\, \nabla_\beta\, g_{\mu\nu} \cdot \upsilon^\mu w^\nu = 0. \tag{B.139}$$

Because that statement must be valid for every index choice, we can isolate the metric compatibility condition

$$\nabla_\beta\, g_{\mu\nu} = 0, \tag{B.140}$$

stating that the covariant derivative of the metric must be zero. On the other hand, the metric is a covariant tensor, so its covariant derivative is explicitly given by

$$\nabla_\beta\, g_{\mu\nu} = \partial_\beta\, g_{\mu\nu} - \Gamma^\alpha{}_{\beta\mu}\, g_{\alpha\nu} - \Gamma^\alpha{}_{\beta\nu}\, g_{\mu\alpha} = 0. \tag{B.141}$$

As a second condition, we require symmetry of the Christoffel-symbol in the lower two indices,

$$\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}, \tag{B.142}$$

which is called the torsion-free condition of the connection. With that, we can write out eqn. B.141 with cyclically permuted indices (μ, ν, β):

$$\nabla_\mu\, g_{\nu\beta} = \partial_\mu\, g_{\nu\beta} - \Gamma^\alpha_{\ \mu\nu}\ g_{\alpha\beta} - \Gamma^\alpha_{\ \mu\beta}\ g_{\nu\alpha} = 0 \tag{B.143}$$

as well as

$$\nabla_\nu g_{\mu\beta} = \partial_\nu\, g_{\beta\mu} - \Gamma^\alpha_{\ \nu\mu}\ g_{\alpha\beta} - \Gamma^\alpha_{\ \nu\beta}\ g_{\mu\alpha} = 0 \tag{B.144}$$

and combine all three by computing B.143 +B.144 − B.141 = 0:

$$\partial_\mu\, g_{\beta\nu} - \Gamma^\alpha_{\ \mu\nu}\ g_{\alpha\beta} - \Gamma^\alpha_{\ \beta\mu}\ g_{\alpha\nu} + \partial_\nu\, g_{\mu\beta} - \Gamma^\alpha_{\ \mu\nu}\ g_{\alpha\beta} - \Gamma^\alpha_{\ \beta\nu}\ g_{\mu\nu} - \partial_\beta\, g_{\mu\nu} + \Gamma^\alpha_{\ \beta\mu}\ g_{\alpha\nu} + \Gamma^\alpha_{\ \beta\nu}\ g_{\mu\nu} = 0. \tag{B.145}$$

Finally, we solve for the Christoffel-symbol $\Gamma^\alpha_{\ \mu\nu}$:

$$\partial_\mu\, g_{\beta\nu} + \partial_\nu\, g_{\mu\beta} - \partial_\beta\, g_{\mu\nu} = 2\Gamma^\alpha_{\ \beta\nu}\, g_{\alpha\beta} \tag{B.146}$$

and isolate $\Gamma^\gamma_{\ \mu\nu}$ by multiplication with the inverse metric $g^{\beta\gamma}$,

$$\Gamma^\alpha_{\ \mu\nu}\ g_{\alpha\beta}g^{\beta\gamma} = \Gamma^\gamma_{\ \mu\nu} = \frac{g^{\beta\gamma}}{2}\Big(\partial_\mu\, g_{\beta\nu} + \partial_\nu\, g_{\mu\beta} - \partial_\beta\, g_{\mu\nu}\Big) \tag{B.147}$$

by using $g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$. Please keep in mind that

$$\upsilon^\alpha = \delta^\alpha_\beta\ \upsilon^\beta = g^{\alpha\beta}\ v_\beta = g^{\alpha\beta}g_{\beta\gamma}\ \upsilon^\gamma\ \rightarrow\ g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma \tag{B.148}$$

as the defining equation for the inverse metric $g^{\mu\nu}$ for any metric $g_{\mu\nu}$. It is a standard exercise to show that the Christoffel-symbol $\Gamma^\alpha_{\ \mu\nu}$ is **not** a tensor, but that the covariant derivatives $\nabla_\beta\ \upsilon^\mu$ and $\nabla_\beta\ w_\mu$ are.

B.5 *geodesics as autoparallel curves*

A curve $x^\mu(\lambda)$ parameterised by λ can be autoparallel in the sense that the tangent $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ does not change, or equivalently, that the tangent vector u^μ is always a parallel transported version of itself along the curve. Then, writing $\dot{x}^\mu = u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ for simplicity,

$$\dot{x}^\beta\nabla_\beta\dot{x}^\alpha = 0 \tag{B.149}$$

because $\dot{x}^\mu_\parallel(x+\delta x) = \dot{x}^\mu(x+\delta x)$. We can substitute the explicit form of the covariant derivative to get

$$\dot{x}^\beta\nabla_\beta\dot{x}^\nu = \dot{x}^\beta\Big[\partial_\beta\dot{x}^\alpha + \Gamma^\alpha_{\ \beta\mu}\,\dot{x}^\mu\Big] = \dot{x}^\beta\cdot\partial_\beta\dot{x}^\alpha + \Gamma^\alpha_{\ \beta\mu}\,\dot{x}^\beta\dot{x}^\mu = 0 \tag{B.150}$$

Rewriting the first term as a differentiation along λ yields

$$\dot{x}^\beta\partial_\beta\dot{x}^\alpha = \frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\frac{\partial\dot{x}^\alpha}{\partial x^\beta} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big(\dot{x}^\alpha\Big) = \ddot{x}^\alpha \tag{B.151}$$

which defines the standard form of the geodesic equation,

$$\ddot{x}^\alpha + \Gamma^\alpha_{\ \mu\nu}\,\dot{x}^\mu\dot{x}^\nu = 0. \tag{B.152}$$

A curve that obeys this equation of motion and follows an autoparallel line is called a geodesic. Geodesics generalise the concept of a straight line through Euclidean space to manifolds, where straight and autoparallel are equivalent. One would already suspect at this point that inertial motion, where no accelerations are felt, corresponds to motion along an autoparallel line. But at the same time, freely falling motion through a gravitational field would likewise be characterise by a feeling of perfect weightlessness and the absence of inertial forces: And one is correct in guessing that geodesics are in fact trajectories through spacetime followed by freely falling particles.

Because the rate at which particle pass by the coordinates does not need to be constant for inertial motion (imagine a particle drifting off-centre through Euclidean space with polar coordinates) we should not use the statement $\ddot{r} = \ddot{\varphi} = 0$ as a characterisation of inertial motion, possibly motivated by Newtonian thinking. Instead, autoparallelity condition would be the proper thing to do. And as the connection has been defined to be metric compatible, we immediately see that the modulus of the velocity, defined as the scalar product $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, is conserved.

It is possible to reverse-engineer Newton's equation of motion in a gravitational field with our knowledge of relativity and to rediscover the geodesic equation, adding perhaps some support for the idea on the connection between autoparallelity and geodesic motion: Newton's equation of motion reads

$$\ddot{x}^i + \partial^i\Phi = 0 \tag{B.153}$$

for a particle falling through the gravitational potential, where no accelerations can be felt. The dot denotes the derivative with respect to laboratory time, which for small velocities is equal to the proper time, $t = \tau$. Because we already suspect that the potential is measured in units of c^2 as suggested by the weak field-metric, one can write:

$$\ddot{x}^i + \partial^i\frac{\Phi}{c^2}\cdot c\cdot c = 0. \tag{B.154}$$

Perhaps the two cs are just the t-component of the 4-velocity in the slow motion limit,

$$\ddot{x}^i + \partial^i\frac{\Phi}{c^2}\dot{x}^t\dot{x}^t = 0 \tag{B.155}$$

with coordinates (a tuple!) and velocities (a vector!)

$$x^\mu = \begin{pmatrix} ct \\ x^i \end{pmatrix}, \quad \dot{x}^\mu = \begin{pmatrix} c \\ \upsilon^i \end{pmatrix}, \tag{B.156}$$

where the difference between coordinate time and proper time vanishes, and $\gamma = 1$. If we identify the Christoffel-symbol

$$\Gamma^i_{tt} = \partial^i\frac{\Phi}{c^2} \tag{B.157}$$

with a suitable derivative of the metric, one gets

$$\ddot{x}^i + \Gamma^i_{tt}\,\dot{x}^t\dot{x}^t = 0 \tag{B.158}$$

Finally, making everything covariant by replacing i with α and reinstating τ instead of t

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\ \mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{B.159}$$

one obtains the geodesic equation, with the affine parameter τ. This immediately poses the question if these statements are only true for a particular choice of the affine parameter. This is not the case, as geodesics are invariant under affine reparameterisations $\lambda \to \lambda'$!

We have seen that autoparallelity of the tangent vector is equivalent to the geodesic equation,

$$\dot{x}^\beta\nabla_\beta\dot{x}^\mu = 0 \quad \to \quad \ddot{x}^\beta + \Gamma^\beta_{\ \mu\nu}\ \dot{x}^\mu\dot{x}^\nu = 0 \tag{B.160}$$

where

$$\dot{x}^\beta = \frac{d\lambda}{d\lambda'}\frac{dx^\beta}{d\lambda} \quad \text{and} \quad \ddot{x}^\beta = \frac{d}{d\lambda'}\Big(\frac{d\lambda}{d\lambda'}\frac{dx^\beta}{d\lambda}\Big) \tag{B.161}$$

yielding the following conversion

$$\frac{d}{d\lambda'}\Big(\Big(\frac{d\lambda}{d\lambda'}\Big)\cdot\frac{dx^\beta}{d\lambda}\Big) + \Gamma^\beta_{\ \mu\nu}\ \frac{d\lambda}{d\lambda'}\frac{dx^\mu}{d\lambda}\cdot\frac{d\lambda}{d\lambda'}\frac{dx^\nu}{d\lambda} = 0 \tag{B.162}$$

from the chain rule, and by applying the Leibnitz-rule,

$$\frac{d^2\lambda}{d\lambda'^2}\cdot\frac{dx^\beta}{d\lambda} + \Big(\frac{d\lambda}{d\lambda'}\Big)^2\frac{d^2x^\beta}{d\lambda^2} + \Gamma^\beta_{\ \mu\nu}\ \frac{d\lambda}{d\lambda'}\frac{dx^\mu}{d\lambda}\cdot\frac{d\lambda}{d\lambda'}\cdot\frac{dx^\nu}{d\lambda} = 0 \tag{B.163}$$

such that

$$\frac{d^2x^\beta}{d\lambda^2} + \Gamma^\beta_{\ \mu\nu}\frac{dx^\mu}{d\lambda}\cdot\frac{dx^\nu}{d\lambda} = -\frac{d^2\lambda}{d\lambda'^2}\cdot\Big(\frac{d\lambda'}{d\lambda}\Big)^2\cdot\frac{dx^\beta}{d\lambda} \tag{B.164}$$

If there is now a linear relationship between λ and λ', the derivative $d^2\lambda/d\lambda'^2$ vanishes, making sure that one recovers the geodesic equation in both parameters: In fact, there seems to be an entire class of affine parameters which are all equally suited to be used to define autoparallelity or the geodesic equation, all related by affine transformations $\lambda' = a\lambda + b$.

In classical mechanics with $\ddot{x}^i + \partial^i\Phi = 0$ as the equation of motion, this looks like nothing particular beyond mechanical similarity: $t \to at + b$ implies that $\ddot{x}$ acquires a factor a^{-2}, but Φ has units of velocity^2, so that it will have a factor of a^{-2}, too, which cancels. But we can make an interesting statement about the relativistic Doppler-effect, which arises as a projection of a photon's wave vector k^μ onto the observer's world line with the tangent u^μ, $\omega = g_{\mu\nu}u^\mu k^\nu$. Clearly, reparameterisation of u^μ brings in a factor of a^{-1}, but the photon wave vector should not change, such that the frequency only changes by a single factor of a^{-1}: We can not work with the same affine parameter for photons and massive particles.

In fact, the wave vector as the tangent to the photon geodesic is normalised to zero, $g_{\mu\nu}k^\mu k^\nu = 0$, while there is a particular choice of the affine parameter for massive particles such that the tangent is normalised to c^2. With the proper time τ and tangents $u^\mu = dx^\mu/d\tau$ one always obtains the normalisation $g_{\mu\nu}u^\mu u^\nu = c^2$. And, in both cases, geodesic motion conserves this normalisation as a consequence of metric compatibility $\nabla_\alpha g_{\mu\nu}$ and the autoparallelity condition $u^\alpha \nabla_\alpha u^\mu = 0$:

$$u^\alpha \nabla_\alpha \left(g_{\mu\nu}\, u^\mu u^\nu \right) = u^\alpha \nabla_\alpha\, g_{\mu\nu} \cdot u^\mu u^\nu + g_{\mu\nu}\, u^\alpha \nabla_\alpha\, u^\mu \cdot u^\nu + g_{\mu\nu}\, u^\mu u^\alpha \nabla_\alpha u^\nu = 0 \quad \text{(B.165)}$$

B.6 *geodesic motion through a variational principle*

Relativity surprises with the idea that the variational principles of classical mechanics have a clear geometric meaning: Particles move along trajectories in spacetime with extremised arc lengths. The central result of the last chapter was that autoparallelity leads to the geodesic equation and that autoparallel lines are straight in a general sense: But is straight equivalent to shortest? Writing down the action as the integrated arc length gives

$$S = \int_A^B \mathrm{d}s = \int_A^B \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda = \int_A^B \mathrm{d}\lambda\, L\left(x^\mu, \dot{x}^\mu, g_{\mu\nu}\right) \quad \text{(B.166)}$$

with L being the generalised Lagrange function. A variation of the trajectory $x^\mu(\lambda) \rightarrow x^\mu(\lambda) + \delta x^\mu(\lambda)$ by $\delta x^\mu(\lambda)$ generates a variation δS of the arc length,

$$\delta S = \int_A^B \mathrm{d}\lambda \left[\frac{\partial L}{\partial x^\alpha} \delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha} \delta \dot{x}^\alpha \right] \quad \text{with } \delta \dot{x}^\alpha = \frac{\mathrm{d}}{\mathrm{d}\lambda} \delta x^\alpha \quad \text{(B.167)}$$

which can be recast into

$$\delta S = \int_A^B \mathrm{d}\lambda \left[\frac{\partial L}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^\alpha} \right] \delta x^\alpha \quad \text{(B.168)}$$

through an integration by parts, where no variation is done at the end points A and B. Then, the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha} \quad \text{(B.169)}$$

can be isolated, as it applies to the generalised Lagrange function

$$L\left(x^\mu, \dot{x}^\mu, g_{\mu\nu}\right) = \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \quad \text{(B.170)}$$

which depends on the trajectory and its tangent, apart from the metric itself defining the geometry. The derivatives can be directly computed, keeping in mind that the metric itself is a function of the coordinates, for the derivative with respect to the coordinates,

$$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2L} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \cdot \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} \quad \text{(B.171)}$$

and for the derivative with respect to the velocities,

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{1}{2L} \cdot g_{\mu\nu} \left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} \cdot \dot{x}^\nu + \dot{x}^\mu \cdot \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha} \right) = \frac{1}{2L} \left(g_{\alpha\nu}\, \dot{x}^\nu + g_{\mu\alpha}\, \dot{x}^\mu \right) = \frac{1}{L} g_{\alpha\mu}\, \dot{x}^\mu \tag{B.172}$$

Substitution into the Euler-Lagrange-equation yields

$$\frac{d}{d\lambda} \left(\frac{1}{L} \cdot g_{\alpha\mu}\, \dot{x}^\mu \right) = -\frac{\dot{L}}{L^2} g_{\alpha\mu}\, \dot{x}^\mu + \frac{1}{L} \dot{g}_{\alpha\mu}\, \dot{x}^\mu + \frac{1}{L} g_{\alpha\mu}\, \ddot{x}^\mu = \frac{1}{L} \left[-\frac{\dot{L}}{L} g_{\alpha\mu}\, \dot{x}^\mu + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} \cdot \dot{x}^\mu \dot{x}^\nu + g_{\alpha\mu} \ddot{x}^\mu \right] \tag{B.173}$$

where the derivative of the metric is given by the chain rule, $\dot{g}_{\alpha\mu} = \partial_\nu\, g_{\alpha\mu} \cdot \dot{x}^\nu$, so that one arrives at

$$\frac{d}{d\lambda} \left(\frac{1}{L} \cdot g_{\alpha\mu}\, \dot{x}^\mu \right) = \frac{1}{L} \left[-\frac{\dot{L}}{L^2} g_{\alpha\mu}\, \dot{x}^\mu + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} \cdot \dot{x}^\mu \dot{x}^\nu + g_{\alpha\mu}\, \ddot{x}^\mu \right], \tag{B.174}$$

which leads to

$$-\frac{\dot{L}}{L} g_{\alpha\mu}\, \dot{x}^\mu + \frac{\partial}{\partial x^\nu} g_{\alpha\mu}\, \dot{x}^\mu \dot{x}^\nu + g_{\alpha\mu}\, \ddot{x}^\mu = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \dot{x}^\mu \dot{x}^\nu \tag{B.175}$$

with a symmetrisation $\frac{1}{2}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} \right)$ of the second term one then obtains

$$\ddot{x}_\alpha + \frac{1}{2} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \dot{x}^\mu \dot{x}^\nu = \frac{\dot{L}}{L} \cdot \dot{x}^\alpha \tag{B.176}$$

Multiplying this relation with the inverse metric $g^{\beta\alpha}$ shows the emergence of the Christoffel symbol,

$$\ddot{x}^\beta + \frac{g^{\beta\alpha}}{2} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \dot{x}^\mu \dot{x}^\nu = \frac{\dot{L}}{L} \cdot \dot{x}^\beta \tag{B.177}$$

which one can replace in the equation,

$$\ddot{x}^\beta + \Gamma^\beta_{\ \mu\nu}\, \dot{x}^\mu \dot{x}^\nu = \frac{\dot{L}}{L} \cdot \dot{x}^\beta = \frac{\ddot{S}}{\dot{S}} \dot{x}^\beta. \tag{B.178}$$

The arc length $S = \int d\lambda\, L$ has the derivatives $L = \dot{S}$ and $\dot{L} = \ddot{S}$. If in particular an affine parameter is chosen, then $\ddot{S} = 0$, and one obtains the classic geodesic equation,

$$\ddot{x}^\beta + \Gamma^\beta_{\ \mu\nu}\, \dot{x}^\mu \dot{x}^\nu = 0. \tag{B.179}$$

Conceptually, the geodesic equation joins straight, autoparallel motion and with the principle of external proper time or minimal arc length to arbitrary geometry, as the proper time is a preferred affine parameter because it has a measurable physical meaning.

There is a number of interesting properties of gravity: Firstly, all objects experience the same acceleration irrespective of their mass; with acceleration being meant as the rate of the rate at which the coordinates pass by the object, not as a physical acceleration which is always absent in free fall. This is very much different for e.g. electrically charged particles experiencing electromagnetic fields. In this case, the arc length is computed with

$$S = \int_A^B \left(\mathrm{d}\tau + \frac{q}{m} g_{\mu\nu} A^\mu \mathrm{d}x^\nu\right) \tag{B.180}$$

with a vector potential A^μ. Clearly, the decisive quantity here is the specific charge q/m, and particles with different specific charge will follow different trajectories through the same field A^μ.

This specific charge for gravitational fields would correspond to the ratio between the gravitational mass as the coupling strength of massive particle to the gravitational field and the inertial mass. This ratio has been found to be unity at the level of 10^{-11}, giving a strong empirical indication of the universality of gravity. In fact, variation $\delta S = 0$ of (**) gives

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\ \mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{q}{m} F^\alpha_{\ \mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{B.181}$$

so that any deviation from freely-falling motion must be proportional to the specific charge $\frac{q}{m}$, with $F^\alpha_{\ \mu} = g^{\alpha\beta} F_{\beta\mu}$.

Gravitational lensing is naturally explained by the geodesic equation even the photon has a vanishing mass, $m_\gamma = 0$. It is sufficient to use the geodesic equation for the wave vector k^μ of the photon as the force-free, gravitational left hand side of the geodesic equation allows for phenomena like gravitational lensing, effectively through

$$\frac{\mathrm{d}k^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha_{\ \mu\nu} k^\mu k^\nu = 0, \tag{B.182}$$

for the wave vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ for the affine parameter $\lambda \neq \tau$ parameterising the photon trajectory $x^\mu(\lambda)$.

Lastly, inertial motion through a vector space with Cartesian coordinates suggest a Euclidean straight line: $\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = 0 \rightarrow x^\alpha = a^\alpha \tau + b^\alpha$, because in Cartesian coordinates the metric is constant and the Christoffel-symbol vanishes.

Geodesic, autoparallel motion corresponds to freely falling particles, generalising the idea of inertial motion to curved manifolds, as a representation of gravitational fields. One should be careful, however, to associate $g_{\mu\nu} \neq \eta_{\mu\nu}$ or $\Gamma^\alpha_{\ \mu\nu} \neq 0$ to gravitational fields, as both statements can be true locally in a certain coordinate choice. Rather, one should think of geodesic motion as taking care of the coordinate choice by establishing autoparallelity of a straight line, irrespective of the presence of curvature or gravity. Both inertial motion and freely falling motion are, in addition, both characterised by a sensation of perfect weightlessness of an observer moving along with the particle, and are therefore, a priori indistinguishable.

B.7 *equivalence and the relativistic origin of Newton's axioms*

The geodesic equation is a description of a straight line (in the autoparallel sense) through spacetime and should, as such, be a generalisation of the law of inertia and the Newtonian equation of motion. In fact, Newton's inertial law states that force-free motion proceeds at constant speed along a straight line, which is perfectly fulfilled by the geodesic equation: Straight actually means autoparallel, as the proper concept for more complicated coordinate choices, and the normalisation $g_{\mu\nu}u^\mu u^\nu = c^2$ of the velocity u^μ is conserved. Force-free in the Newtonian sense might pertain to both inertial motion through a flat spacetime or freely-falling motion through a curved spacetime: There is no fundamental difference between these two cases. To take things to extremes, one could say that Newton's first axiom is the definition of the word "straight": As soon as there are no accelerations measured, the trajectory is necessarily autoparallel.

We have already seen that the Newtonian equation of motion $\ddot{x}^i + \partial^i\Phi = 0$ with the gravitational potential Φ is hidden in geodesic equation for small velocities and weak fields, exactly the limit Newton could have been aware of. Writing $\ddot{x}^i + \partial^i\Phi = 0$ to allude at force-free motion is in the spirit of the geodesic equation $\ddot{x}^\alpha + \Gamma^\alpha_{\ \mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, and only non-gravitational forces would replace the zero on the right hand side, for instance an electromagnetic force,

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\ \mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{q}{m}g_{\mu\nu}\,F^{\alpha\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{B.183}$$

for a particle with specific charge q/m experiencing electromagnetic fields $F^{\alpha\nu}$. With this idea in mind, I personally don't like to speak about gravitational forces: Rather, I would call them gravitational accelerations which get modified by non-gravitational accelerations that are computed from the actual field with the specific charge q/m as the coupling constant, to yield an actual acceleration.

Ultimately, the third axiom actio = -reactio (with a minus-sign, as actio and reactio take place in opposing directions!) is the most interesting in view of relativity. It concerns non-geodesic motion with the appearance of inertial forces (reactio), which are opposed to the actual forces (actio) acting on a particle. To understand where this might come from we should first have a look at the way how classical inertial forces like the centrifugal force or the Coriolis-force are contained in the geodesic equation. In the slow-motion limit with $\tau = t$, $\gamma = 1$, fixed $u^t = c$ and $u^i = \upsilon^i$ we get

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{\ \mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{\ tt}\,c^2 + \Gamma^i_{\ mt}\,\upsilon^m c + \Gamma^i_{\ tn}\,c\upsilon^n + \Gamma^i_{\ mn}\,\upsilon^m\upsilon^n = \frac{q}{m}f^i, \tag{B.184}$$

with a non-gravitational acceleration $\frac{q}{m}f^i$. The Christoffel symbol is symmetric in the lower two indices $\Gamma^i_{\ mn} = \Gamma^i_{\ nm}$ because of the torsion-free condition and $\Gamma^i_{\ tt} = \partial^i\frac{\Phi}{c^2}$ specifically would be the gradient of a classical gravitational potential. Then, fundamentally, there are two terms in the geodesic equation, $\Gamma^i_{\ mn}\,\upsilon^m\upsilon^n \sim (\boldsymbol{\Omega}\times\boldsymbol{\upsilon})\times\boldsymbol{\upsilon}$ corresponding to a centrifugal acceleration which is quadratic in the velocities, and the Coriolis acceleration $2\Gamma^i_{\ mt}\,\upsilon^m c \sim 2\boldsymbol{\Omega}\times\boldsymbol{\upsilon}$ with the factor 2 appearing naturally out of the two identical terms linear in the velocity υ.

This is in fact a surprising result: The velocity-dependent inertial accelerations appear as the non-relativistic limit of the geodesic equation, up to terms $\propto \upsilon^2$ because of the term $\Gamma^i_{\ \mu\nu}\,u^\mu u^\nu$. It seems to be the case that the velocity dependence of accelerations is natural, similar to the Lorentz force $\propto \boldsymbol{\upsilon}\times\mathbf{B}$. Here, υ^1 is the highest power that can be generated by $\frac{q}{m}g_{\mu\nu}F^{\alpha\nu}u^\mu$. The term $\Gamma^i_{\ tt} = \partial^i\frac{\Phi}{c^2}$ is an eternal source of confusion: The geodesic equation with such a term clearly refers to autoparallel motion along a straight line, but one tends to think of a curved trajectory, for instance when thinking about throwing a ball along a parabolic curve. But please keep in mind that there is a second definition of straightness corresponding to the Minkowski-space with the metric $\eta_{\mu\nu}$ that one might use instinctively instead of $g_{\mu\nu}$. Balls and planets follow autoparallel lines through spacetimes, and parabolas and elliptical orbits (besides, the parabola that is followed by a ball is only the second order Taylor-expansion of an elliptical orbit around the Earth's centre) are straight, otherwise Newton's first axiom could not be fulfilled.

If there is really the equivalence between inertial accelerations and gravitational accelerations, as made clear by Einstein's elevator argument, there should be a deep connection between the two. First of all, inertial motion in e.g. rotating or accelerating coordinate frames is the uninteresting case, because the geodesic equation makes the job of computing the rate of change of the passage of the coordinates perfectly and all we see are coordinate effects. It becomes more interesting if there is a non-gravitational force acting on a particle such that inertial forces appear as a consequence of, well, the change of the state of motion, but relative to what? At this point Mach's principle comes in and clarifies that inertial frames are defined in by the large-scale distribution of matter in the Universe. If there is a perfect equivalence between inertial and gravitational forces, we should be able to ask how inertial accelerations are sourced and what their gravitational origin is, after having thought of gravitational accelerations to be inertial: They vanish in freely falling frames and affect all objects in exactly the same way irrespective of their mass. Coming back to Newton's third axiom we should suspect that the inertial reactio is in fact gravitationally induced, because the state of motion changes relative to the masses in the Universe, and because there is an additional velocity dependent gravitational force acting on the particle.

A second striking example is the rotational flattening of the Sun, whose diameter at the equator is larger than the diameter taken at the poles, as a consequence of the centrifugal force acting on it due to its rotation. But how would you interpret the same observation from a frame co-rotating with the Sun? There, the entire universe rotates in the opposite direction and there is an additional component of the gravitational field which pulls on the Sun's equator.

There is an interesting remainder of the idea that accelerated frames and gravitational potentials are equivalent left in classical mechanics: A boost into a frame with constant acceleration a^i is defined by

$$x'^i = x^i + \frac{1}{2}a^i t^2 \quad \rightarrow \quad \dot{x}'^i = \dot{x}^i + a^i t, \tag{B.185}$$

such that the Lagrange-function L transforms accordingly,

$$\mathrm{L}' = \frac{m}{2}\delta_{ij}\,\dot{x}'^i\dot{x}'^j = \mathrm{L} + \frac{m}{2}\left(2\delta_{ij}\,\dot{x}^i a^j t + \delta_{ij}\,a^i a^j t^2\right) \tag{B.186}$$

The last term can be written as a total derivative, $t^2 = \mathrm{d}(t^3/3)/\mathrm{d}t$ and does not matter in the variation, as total derivatives of functions that only depend on time and coordinate (but not velocity) never have an influence on the variational principle. The second term, however, can be rewritten using the fundamental theorem of calculus, as a differentiation of an integral,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}t\,\dot{x}^i \cdot t = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}t\,x^i = x^i \tag{B.187}$$

so that we can apply an integration by parts in the last step. Collecting these results yields

$$\mathrm{L}' = \mathrm{L} - m\,\delta_{ij}\,x^i a^j, \tag{B.188}$$

so that the Lagrange-function has acquired a new term that corresponds to a potential with a constant slope: This is classical equivalence between a linear potential and a frame accelerated at a constant rate.

B.8 *geodesic deviation, curvature and gravity*

Geodesics as autoparallel lines through spacetime are the trajectories of freely falling particles. The geodesic equation computes the rates of change $\ddot{x}^\mu$ of the passage $\dot{x}^\mu$ of the coordinates past the particle, completely independent from the presence of curvature. Actually, neither the metric $g_{\mu\nu}$ nor the Christoffel-symbol $\Gamma^\alpha{}_{\mu\nu}$ do contain information about gravity, and neither does the covariant derivative ∇_μ: They are all constructed to deal with the arbitrariness of coordinate choices. In addition, we already know that the gravitational field does not exist at a single point, because both conditions $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\alpha{}_{\mu\nu} = 0$ can always be achieved **locally** by a coordinate transform.

☠ *$g_{\mu\nu} \neq \eta_{\mu\nu}$ does not imply that there is gravity, and neither does $\Gamma^\alpha{}_{\mu\nu} \neq 0$, and neither does $\partial_\mu \neq \nabla_\mu$!*

A possible idea would be to look at the relative motion of freely falling particles. Locally, every particle experiences perfect weightlessness, but that does not imply that the relative acceleration must be zero. Imagine two astronauts holding hands and falling through space(time) and following Keplerian orbits around the Earth. The astronaut on the lower orbit moves with a higher velocity according to Kepler's first law and would actually accelerate away from the astronaut in the higher orbit. Such an experiment could serve as an experiment to determine whether gravitational fields (or spacetime curvature) is present, because it is non-local and because it would be sensitive to the second derivatives $\partial^2 g$ of the metric, which partially resist coordinate transforms as they can not be made to vanish.

The quantity determining the relative acceleration between two freely falling particles is the Riemann-tensor,

$$R^\alpha{}_{\mu\nu\beta} = \left(\frac{\partial \Gamma^\alpha{}_{\mu\nu}}{\partial x^\beta} - \frac{\partial \Gamma^\alpha{}_{\mu\beta}}{\partial x^\nu} + \Gamma^\alpha{}_{\rho\beta}\Gamma^\rho{}_{\mu\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\mu\beta}\right). \tag{B.189}$$

It is through $\partial\Gamma \sim \partial(g\partial g)$ composed of second derivatives of the metric which shows that it contains information about the manifold that can not be made to vanish by a coordinate transform. We will see in the next chapter that it contains all information about curvature of the manifold and the deviation from a Lorentzian geometry. In particular, the geodesic deviation equation

$$\frac{d^2\upsilon^\alpha}{d\lambda^2} = R^\alpha{}_{\mu\nu\beta} \cdot u^\mu u^\nu \upsilon^\beta \tag{B.190}$$

defines the experiment one can test for the presence of gravitational fields. If there is no relative acceleration $d^2\upsilon^\alpha/d\lambda^2 = 0$ for every index choice one must conclude that the Riemann curvature vanishes, $R^\alpha{}_{\mu\nu\beta} = 0$ and that the motion of the two test particles takes place in Minkowskian space, but possibly with a weird coordinate choice.

C DIFFERENTIAL STRUCTURE OF SPACETIME AND CURVATURE

C.1 *Riemann curvature tensor*

The connection, which establishes parallel transport of vectors and tensors across a manifold, defines the covariant derivative of these quantities because a proper rate of change can be measured through the comparison of e.g. a vector with the parallel transported counterpart. The Levi-Civita connection is singled out among all possible connections as the (i) metric compatible $\nabla_\alpha g_{\mu\nu} = 0$ and (ii) torsion-free $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$ one, in which case the connection coefficients can be computed from the metric $g_{\mu\nu}$ and its first derivatives $\partial_\alpha g_{\mu\nu}$ alone. The metric structure of a manifold $g_{\mu\nu}$, with an additional differential structure ∇_α, defines the Riemann-geometry.

All these ideas and concepts are independent from actual curvature and are rather an expression of the choice of coordinates as they only use the metric and its first derivatives, for which there is always a coordinate transform to make them vanish locally, and because only second derivatives would contain information about curvature, we should use them to quantify it. Additionally, we would like to have a covariant quantification of curvature in the form of a tensor: the Riemann curvature. Only if the Riemann-curvature is nonzero, $R_{\alpha\beta\mu\nu} = 0$ as a properly covariant expression, the manifold is flat. None of the statements $g_{\mu\nu} = \eta_{\mu\nu}$, $\nabla_\alpha = \partial_\alpha$, or $\Gamma^{\alpha}{}_{\mu\nu} = 0$ are able to make a statement about curvature.

C.1.1 Riemann curvature in parallel transport

The order of parallel transport of vectors and tensors matters in shifts along different directions. Starting with the expression for parallel transport by $\delta\bar{x}^\beta$,

$$\upsilon^\mu(x + \delta\overline{x}) = \upsilon^\mu(x) - \Gamma^{\mu}{}_{\alpha\beta} \cdot \upsilon^\alpha(x)\,\delta\overline{x}^\beta \tag{C.191}$$

we can define two paths: first a shift by $\delta\bar{x}$ followed by a shift by δx,

$$\upsilon^\mu(x + \delta\overline{x}) + \delta x) = \upsilon^\mu(x + \delta\overline{x}) - \Gamma^{\mu}{}_{\alpha\beta}\,(x + \delta\overline{x}) \cdot \upsilon^\alpha(x + \delta\overline{x}) \cdot \delta x^\beta \tag{C.192}$$

which evaluates to

$$= \upsilon^\mu(x) - \Gamma^{\mu}{}_{\alpha\beta}\,(x)\cdot\upsilon^\alpha(x)\,\delta\overline{x}^\beta - \left[\Gamma^{\mu}{}_{\alpha\beta} + \frac{\partial\Gamma^{\mu}{}_{\alpha\beta}}{\partial x^\gamma}\cdot\delta\overline{x}^\gamma\right]\cdot[\upsilon^\alpha(x) - \Gamma^{\alpha}{}_{\gamma\delta}\,(x)\,\upsilon^\gamma(x)\,\delta\overline{x}^\delta]\cdot\delta x^\beta \tag{C.193}$$

with $\Gamma^{\mu}{}_{\alpha\beta}\,(x+\delta\bar{x}) = \Gamma^{\mu}{}_{\alpha\beta}\,(x) + \frac{\partial\Gamma^{\mu}{}_{\alpha\beta}}{\partial x^\gamma}(x)\cdot\delta\overline{x}^\gamma$ being the Taylor-expansion of the Christoffel-symbol at $x + \delta\bar{x}$. Alternatively, the two shifts can be interchanged, for a parallel transport first by δx and then by $\delta\bar{x}$.

$$\upsilon^\mu((x + \delta x) + \delta\overline{x}) = \upsilon^\mu(x + \delta x) - \Gamma^{\mu}{}_{\alpha\beta}\,(x + \delta x) \cdot \upsilon^\alpha(x + \delta x) \cdot \delta\overline{x}^\beta \tag{C.194}$$

yielding

$$\upsilon^\mu(x) - \Gamma^{\mu}{}_{\alpha\beta}\,(x)\cdot\upsilon^\alpha(x)\,\delta x^\beta - \left[\Gamma^{\mu}{}_{\alpha\beta}\,(x) + \frac{\partial\Gamma^{\mu}{}_{\alpha\beta}}{\partial x^\gamma}\delta x^\gamma\right]\left[\upsilon^\alpha(x) - \Gamma^{\alpha}{}_{\gamma\delta}\,(x)\,\upsilon^\gamma(x)\,\delta x^\delta\right]\cdot\delta\overline{x}^\beta \tag{C.195}$$

with an equivalent Taylor-expansion. Then, the change $\delta\upsilon^\mu$ in parallel transport to the point $x+\delta x+\delta\bar{x}$ along two different paths is given by

$$\delta\upsilon^\mu = \upsilon^\mu((x+\delta\bar{x})+\delta x) - \upsilon^\mu((x+\delta x)+\delta\bar{x}) = R^\mu{}_{\alpha\beta\gamma}\cdot\upsilon^\alpha\,\delta x^\beta\,\delta\bar{x}^\gamma \tag{C.196}$$

where we can isolate the Riemann-curvature,

$$R^\mu{}_{\alpha\beta\gamma} = \frac{\partial}{\partial x^\beta}\Gamma^\mu{}_{\alpha\gamma} - \frac{\partial}{\partial x^\gamma}\Gamma^\mu{}_{\alpha\beta} + \Gamma^\mu{}_{\delta\beta}\,\Gamma^\delta{}_{\alpha\gamma} - \Gamma^\mu{}_{\delta\gamma}\,\Gamma^\delta{}_{\alpha\beta}\,, \tag{C.197}$$

after renaming $\gamma\leftrightarrow\beta$ in the second expression to have $\delta x^\beta\,\delta\bar{x}^\gamma$). Flat manifolds with vanishing Riemann curvature $R^\mu{}_{\alpha\beta\gamma}=0$ would necessarily exhibit no change at all of the transported vector, i.e. $\delta\upsilon^\mu=0$.

In a flat manifold the Riemann-tensor is zero in **every** *coordinate choice.*

Of course, the contravariant index ν can be lowered with a contraction,

$$R_{\mu\alpha\beta\gamma} = g_{\mu\nu}R^\nu{}_{\alpha\beta\gamma\delta}. \tag{C.198}$$

And it is important to memorise the antisymmetry of the Riemann tensor in every index pair,

$$R_{\mu\alpha\beta\gamma} = -R_{\alpha\mu\beta\gamma} = -R_{\mu\alpha\gamma\beta} = +R_{\alpha\mu\gamma\beta} \tag{C.199}$$

as well as the algebraic Bianchi-identity,

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0 \tag{C.200}$$

with cyclic index permutation of the last three indices while keeping the first index fixed.

C.1.2 Riemann-curvature from covariant derivatives

Covariant derivatives (into different direction) in contrast to partial derivatives, do not commute.

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\upsilon^\alpha = \left[\nabla_\mu\,,\nabla_\nu\right]\upsilon^\alpha = -R^\alpha{}_{\beta\mu\nu}\upsilon^\beta, \tag{C.201}$$

and the commutator defines, as before, the Riemann curvature $R^\alpha{}_{\beta\mu\nu}$. Concerning the index structure, it is best to remember that for every choice of μ and ν there is an transformation in α and β acting on the vector υ^β. As vectors are rotated in parallel transport with a Levi-Civita connection, α and β are an antisymmetric index pair because they effectively encode a rotation matrix. μ and ν are likewise an antisymmetric index pair, due to the commutator in the definition of the Riemann curvature, $[\nabla_\mu,\nabla_\nu] = -[\nabla_\nu,\nabla_\mu]$.

Acting on a vector υ^μ with covariant differentiation ∇_β yields

$$\nabla_\beta\upsilon^\mu = \partial_\beta\upsilon^\mu + \Gamma^\mu{}_{\beta\delta}\,\upsilon^\delta = t_\beta{}^\mu \tag{C.202}$$

with a tensor $t_\beta{}^\mu$ as a result. In further covariant differentiation ∇_γ one needs to watch out for co- and contravariant indices, with different signs in their respective Christoffel-symbols:

$$\nabla_\gamma\,t_\beta{}^\mu = \partial_\gamma\,t_\beta{}^\mu - \Gamma^\alpha{}_{\gamma\beta}\,t_\alpha{}^\mu + \Gamma^\mu{}_{\gamma\alpha}\,t_\beta{}^\alpha \tag{C.203}$$

Substituting eqn. C.202 into eqn. C.203 gives:

$$\nabla_\gamma(\nabla_\beta \upsilon^\mu) = \partial_\gamma\partial_\beta \upsilon^\mu - \partial_\gamma \Gamma^\mu{}_{\beta\delta} \cdot \upsilon^\delta - \Gamma^\mu{}_{\beta\delta}\, \partial_\gamma\, \upsilon^\delta - \Gamma^\alpha{}_{\gamma\beta} \cdot \left[\partial_\alpha \upsilon^\mu + \Gamma^\mu{}_{\alpha\delta}\, \upsilon^\delta\right] \tag{C.204}$$

If one interchanges the order of differentiation and builds the antisymmetric combination $\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu$ one can isolate the Riemann tensor,

$$R^\mu{}_{\alpha\beta\gamma} = \frac{\partial}{\partial x^\beta}\Gamma^\mu{}_{\alpha\gamma} - \frac{\partial}{\partial x^\gamma}\Gamma^\mu{}_{\alpha\beta} + \Gamma^\mu{}_{\delta\beta}\,\Gamma^\delta{}_{\alpha\gamma} - \Gamma^\mu{}_{\delta\gamma}\,\Gamma^\delta{}_{\alpha\beta}\,. \tag{C.205}$$

as the partial derivatives of υ^μ drop out, according to $\partial_\gamma\partial_\beta \upsilon^\mu = \partial_\beta\partial_\gamma \upsilon^\mu$.

The two approaches are related to each other as parallel transport of a vector υ_α is performed using the covariant derivative as an operator, $\delta x^\beta\, \nabla_\beta$. One can think about extending this infinitesimal parallel transport to parallel transport operator for finite distances by exponentiation. Then, parallel transports with shift operators

$$\exp(\delta x^\beta\, \nabla_\beta)\upsilon^\mu = \upsilon^\mu(x + \delta x) \text{ shift vector} \tag{C.206}$$

would follow the Baker-Hausdorff-Campbell formula,

$$\exp(\delta x^\beta\, \nabla_\beta)\exp(\delta x^\gamma\, \nabla_\gamma) \simeq \exp(\delta x^\beta\, \nabla_\beta + \delta \overline{x}^\gamma\, \nabla_\gamma)\exp\left(-\frac{1}{2}\cdot \delta x^\beta\, \delta \overline{x}^\gamma\, [\nabla_\beta, \nabla_\gamma]\right) \tag{C.207}$$

where translations into different directions would not notice about the presence of curvature in the case $[\nabla_\beta\,, \nabla_\gamma] \sim R^\mu{}_{\alpha\beta\gamma} \neq 0$.

Tensors that are derived from the Riemann-curvature by contraction with the metric include the Ricci-curvature $R_{\mu\beta}$

$$R_{\mu\beta} = g^{\alpha\nu}\, R_{\alpha\mu\nu\beta}, \tag{C.208}$$

where the contraction over the first and third index is the only sensible one, given the antisymmetry of the Riemann-tensor in the first and last index pair. Further contraction yields the Ricci-scalar R

$$R = g^{\mu\beta}\, R_{\mu\beta} = g^{\mu\beta}\, g^{\alpha\nu}\, R_{\alpha\mu\nu\beta} \tag{C.209}$$

which is a quantification of the (local) curvature, similarly to the Kretschmann-scalar K,

$$K = R^{\alpha\mu\nu\beta}\, R_{\alpha\mu\nu\beta} = g^{\alpha\gamma}\, g^{\mu\rho}\, g^{\nu\sigma}\, g^{\beta\delta}\, R_{\alpha\mu\nu\beta}\, R_{\gamma\rho\sigma\delta} \tag{C.210}$$

Both curvature scalars are independent from the coordinate choice and are a convenient quantification of curvature.

The Ricci-tensor and the Ricci-scalar define the Einstein-tensor $G_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \tag{C.211}$$

which is surprisingly the only rank-2 tensor with vanishing covariant divergence, $g^{\alpha\mu}\nabla_\alpha G_{\mu\nu} = 0$ (the other one being the metric itself, $g^{\alpha\mu}\nabla_\alpha g_{\mu\nu} = 0$, due to metric compatibility), as will become relevant in the next chapter.

C.1.3 What happens to vectors in parallel transport?

Levi-Civita connections are constructed to be metric-compatible which will imply that vectors, if transported around a closed loop, will conserve their norm. Then, the only way in which they can be affected in by curvature is a rotation: One can in fact make that determination because the transported vector is brought back into the original tangent space if the connection is torsion-free.

We can compute explicitly that the norm of a vector $\boldsymbol{\upsilon}$ does not change, expressing parallel transport by δx^μ with the covariant derivative $\delta x^\mu \nabla_\mu$ as an operator acting on a geometric object like a vector or a scalar product. Bringing in the commutator of ∇_μ is a convenient way of interchanging the order of parallel transport from the starting point to the destination and to subtract the two results from each other: If the norm is conserved, the result should be zero.

$$g_{\alpha\beta}\upsilon^\alpha\upsilon^\beta \rightarrow \delta x^\mu \delta\bar{x}^\nu [\nabla_\mu, \nabla_\nu]\left(g_{\alpha\beta}\, \upsilon^\alpha\upsilon^\beta\right) = \delta x^\mu \delta\bar{x}^\nu \Big(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\Big)\Big(g_{\alpha\beta}\, \upsilon^\alpha\upsilon^\beta\Big) \quad \text{(C.212)}$$

Metric compatibility ensures that $\nabla g = 0$, so we obtain, dropping the common prefactor $\delta x^\mu \delta\bar{x}^\nu$,

$$= g_{\alpha\beta}\, \nabla_\mu\nabla_\nu \left(\upsilon^\alpha\upsilon^\beta\right) - g_{\alpha\beta}\, \nabla_\nu\nabla_\mu\Big(\upsilon^\alpha\upsilon^\beta\Big) \quad \text{(C.213)}$$

Expanding the expression with the Leibnitz-rule yields

$$= g_{\alpha\beta}\left(\nabla_\mu\nabla_\nu\, \upsilon^\alpha \cdot \upsilon^\beta + \nabla_\nu\, \upsilon^\alpha \cdot \nabla_\mu\, \upsilon^\beta + \nabla_\mu\, \upsilon^\alpha \cdot \nabla_\nu\, \upsilon^\beta + \upsilon^\alpha\, \nabla_\mu\nabla_\nu\, \upsilon^\beta\right) -$$
$$g_{\alpha\beta}\left(\nabla_\nu\nabla_\mu\, \upsilon^\alpha \cdot \upsilon^\beta + \nabla_\mu\, \upsilon^\alpha\, \nabla_\nu\, \upsilon^\beta + \nabla_\nu\, \upsilon^\alpha\, \nabla_\mu\, \upsilon^\beta + \upsilon^\alpha\, \nabla_\nu\nabla_\mu\, \upsilon^\beta\right) \quad \text{(C.214)}$$

and reordering the terms

$$= g_{\alpha\beta}\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)\upsilon^\alpha \cdot \upsilon^\beta + g_{\alpha\beta}\, \upsilon^\alpha\Big(\nabla_\mu\nabla_n u - \nabla_\nu\nabla_\mu\Big)\, \upsilon^\beta \quad \text{(C.215)}$$

Finally, identifying the Riemann curvature and renaming the indices in the second term gives:

$$= g_{\alpha\beta}\, R^\alpha{}_{\gamma\mu\nu}\, \upsilon^\gamma\upsilon^\beta + g_{\alpha\beta}\, \upsilon^\alpha\, R^\beta{}_{\gamma\mu\nu}\, \upsilon^\gamma = 2\, R_{\alpha\gamma\mu\nu}\, \upsilon^\alpha\upsilon^\gamma = 0, \quad \text{(C.216)}$$

which is zero as a consequence of the antisymmetry of the Riemann-tensor in the first index pair: The norm of υ^α is conserved.

In exactly the same way one can show that the scalar product $g_{\alpha\beta}\upsilon^\alpha w^\beta$ between two vectors υ^α and w^β is conserved. Indeed, repeating the entire calculation shows that

$$\Big[\nabla_\mu\,, \nabla_\nu\Big]\Big(g_{\alpha\beta}\, \upsilon^\alpha w^\beta\Big) = \Big(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\Big)\Big(g_{\alpha\beta}\, \upsilon^\alpha w^\beta\Big) = \ldots = R_{\alpha\gamma\mu\nu}\left(\upsilon^\alpha w^\gamma + \upsilon^\gamma w^\alpha\right) = 0, \quad \text{(C.217)}$$

keeping in mind that the tensor $\left(\upsilon^\alpha w^\gamma + \upsilon^\gamma w^\alpha\right)$ is perfectly symmetric.

With these results, we can revisit the defining equation of Riemann-curvature:

$$[\nabla_\mu, \nabla_\nu]\upsilon^\alpha = R^\alpha{}_{\beta\mu\nu}\upsilon^\beta \quad \text{(C.218)}$$

where the antisymmetry in the $\mu\nu$-index pair is obvious because of the commutator,

$[\nabla_\mu, \nabla_\nu] = -[\nabla_\nu, \nabla_\mu]$. If a vector υ^α is transported in a loop and compared to the original vector, it can not have changed its norm because of metric compatibility, and it exists (if the manifold is torsion-free) at the same point and can be decomposed in terms of the basis of the same tangent space. The only possible difference between the vectors is a rotation, and this is exactly the meaning of the Riemann-tensor (and which gives you a great way to memorise its antisymmetry in the index pair $\alpha\beta$): It is essentially a rotation matrix in $\alpha\beta$ for every $\mu\nu$-pair.

D SOURCES OF THE GRAVITATIONAL FIELD

D.1 *gravity and matter*

The source of gravity in the Poisson equation $\Delta\Phi = 4\pi G\rho$ as the field equation is the matter density ρ. As a scalar potential Φ is identical in all frames. To make the source consistent with the field, we need to assume in Newtonian gravity that the density ρ is identical in all frames, too, in contradiction with relativistic effects like mass increase and length contraction that would affect the matter density, and with the fact that from a moving frame of reference ρ would be perceived as a momentum density rather than a matter density. For Newtonian gravity this is all irrelevant as the Poisson equation states a relation between two absolute quantities. The continuity equation for the matter density

$$\partial_t\, \rho + \partial_i(\rho\, \upsilon^i) = 0 \tag{D.219}$$

is phenomenological and expresses the idea that matter is not arbitrarily created or annihilated, and the partial derivatives refer to spacetime as being Euclidean, but in the spirit of Galilean relativity, but weirdly with a static relation between ρ and Φ.

Electrodynamics building on Lorentzian relativity does things better: The source of the electromagnetic field $F^{\mu\nu}$ in Maxwell's equation $\partial_\mu F^{\mu\nu} = 4\pi/c\, \jmath^\nu$ is the 4-current density as a Lorentz-vector $\jmath^\mu$. Neither the charge density nor the current density are absolute but depend on the state of motion of the observer relative to the charge. As $\jmath^\mu$ is a timelike vector (because charges are tied to massive particles), it is always possible to boost into the rest-frame of a charge with a suitable Lorentz-transform, $\jmath^\mu \rightarrow \Lambda^\mu{}_\alpha \jmath^\alpha$. There should be a consistent transformation between all terms of a formula, so Maxwell's field equation

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} \jmath^\nu \tag{D.220}$$

implies, that the Faraday-tensor $F^{\mu\nu}$ should transform, too, $F^{\mu\nu} \rightarrow \Lambda^\mu{}_\alpha \Lambda^\mu{}_\beta F^{\alpha\beta}$ as well as the partial derivative $\partial_\mu = \Lambda_\mu{}^\alpha \partial_\alpha$, which inherits its transformation property from the coordinates. The relativistic charge density $\jmath^\mu$ is conserved,

$$\partial_\mu \jmath^\mu = \partial_{ct}(c\rho) + \partial_i \jmath^i = 0, \tag{D.221}$$

consistently in all Lorentz-frames, because $\partial_\mu \jmath^\mu$ is a scalar.

It is fun to notice that the Maxwell equation and the Lorentz-equation introduce a nice consistency between the fields and the charges: Multiplying the Lorentz-equation for the acceleration of a charge

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} = \frac{q}{m}\eta^{\beta\gamma}\,\mathrm{F}_{\alpha\beta}u_\gamma \tag{D.222}$$

with $\eta^{\alpha\delta}\,u_\delta$ yields the conservation of the normalisation of the velocity as a timelike vector

$$\eta^{\alpha\delta}\,u_\delta\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta^{\alpha\delta}\,u_\alpha u_\delta\right) = \frac{q}{m}\eta^{\alpha\delta}\eta^{\beta\gamma}\,\mathrm{F}_{\alpha\beta}u_\gamma u_\delta = 0 \tag{D.223}$$

while simultaneously acting on the Maxwell equation with the differentiation $\eta^{\gamma\nu}\,\partial_\gamma$

$$\eta^{\beta\mu}\,\partial_\beta\,\mathrm{F}_{\mu\nu} = \frac{4\pi}{\mathrm{c}}j_\nu \tag{D.224}$$

shows that the charge is conserved $\partial_\mu j^\mu = 0$

$$\eta^{\beta\mu}\,\eta^{\gamma\nu}\,\partial_\beta\partial_\gamma\,\mathrm{F}_{\mu\nu} = \frac{4\pi}{\mathrm{c}}\,\eta^{\gamma\nu}\,\partial_\gamma j_\nu = 0 \tag{D.225}$$

In both cases, the contraction of the antisymmetric tensor $\mathrm{F}^{\mu\nu}$ with the symmetric tensors $\partial_\mu\partial_\nu$ and $\upsilon_\mu\upsilon_\nu$ implies the conservation.

We would like these ideas to be realised for gravity as well: There should be a source of gravity with a proper covariant conservation law and a consistent transformation between the source and the field, all of course consistent with the Poisson equation in the limit of static sources and weak gravitational fields. With the knowledge of special relativity one notices a decisive difference between ρ as a charge density and ρ as a matter density: One can imagine that a cloud of charge gets Lorentz-contracted by a factor of γ as seen from an observer moving relative to the charge, implying that the charge density ρ is indeed the ct-component of a time-like Lorentz-vector. A cloud of matter seen from an observer moving relative to it would experience the same Lorentz-contraction, but there is relativistic mass increase in addition to it, introducing two instead of a single power of γ. This transformation property can not be reconciled with a single-indexed quantity like j^μ but requires a double indexed quantity: In fact, we will introduce the energy-momentum tensor $\mathrm{T}^{\mu\nu}$ with $\mathrm{T}^{tt} = \rho c^2$ in accordance with this idea.

D.2 *(relativistic) fluids as sources of gravity*

Fluids are a continuum description of matter, i.e. a field where at every point the density and the velocity are defined: It is a valid picture to think of the fluid as being composed of small fluid elements across which the gradients of the fields do not vary strongly and linearisations apply. Fluid elements react to forces exerted by the surrounding fluid if their size is changed or if their shapes are distorted by gradients of the velocity field across the fluid element; in general there is a force $\mathrm{F}^i = \sigma^{ij}\mathrm{dA}_j$ acting on the surface element dA_j, parameterised by the shear tensor σ_{ij}, which is necessarily symmetric, $\sigma_{ij} = \sigma_{ji}$. While this relation is in general tensorial, the separation $\sigma_{ij} \rightarrow \sigma_{ij} + p\delta_{ij}$ would define a traceless anisotropic stress tensor σ_{ij} and the isotropic pressure p. Effects in the relation of anisotropic stress are parameterised by the shear viscosity and if in addition there are no viscous effects in relation to the change of volume of fluid element parameterised by the bulk viscosity, the fluid is ideal and only shows dynamic effects in relation with pressure p.

An ideal fluid is therefore characterised by density, pressure and velocity, and these quantities are assembled into the energy momentum tensor $T_{\mu\nu}$,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p\, g_{\mu\nu} \tag{D.226}$$

and we will convince ourselves retrospectively that this is the correct quantity, by showing the equivalence of covariant conservation of $T_{\mu\nu}$ by means of a continuity equation $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ and the equations of relativistic fluid mechanics. The components of the energy momentum tensor,

$$T_{\mu\nu} = \begin{pmatrix} T_{tt} & T_{ti} \\ T_{jt} & T_{ij} \end{pmatrix} \tag{D.227}$$

contain the energy density T_{tt}, the energy flux in i–direction, T_{jt} being the component j of momentum density and T_{ij} the projection of the i–momentum in j-direction. In the local rest frame with Cartesian coordinates one would obtain $g_{\mu\nu} = \eta_{\mu\nu}$ as well as $u^\mu = (c, 0)^t$ such that

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \tag{D.228}$$

The trace $g^{\mu\nu}T_{\mu\nu}$ has the value $\rho c^2 - 3p$, which likewise is true in any frame and any geometry: $g^{\mu\nu}T_{\mu\nu} = (\rho + p/c^2)g^{\mu\nu}u_\mu u_\nu - p g^{\mu\nu}g_{\mu\nu} = \rho c^2 - 3p$ because $g^{\mu\nu}u_\mu u_\nu = c^2$ and $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4$. Many fluids are characterised by a fixed relation between pressure p and energy density ρc^2, which is referred to as the equation of state parameter $w = p/(\rho c^2)$. With the equation of state, the trace becomes $g^{\mu\nu}T_{\mu\nu} = (1 - 3w)\rho c^2$. A good way to remember this is the realisation that for photons the relationship $p = \rho c^2/3$ holds, implying that $g^{\mu\nu}T_{\mu\nu} = 0$ as $w = +1/3$, in accordance with a direct computation of the energy-momentum tensor from the Maxwell-Lagrange-density.

The conservation law $g^{\alpha\mu}\,\nabla_\alpha T_{\mu\nu} = 0$ for the energy momentum tensor is vectorial (in the index ν), in contrast to the corresponding law for the charge density $g^{\alpha\mu}\nabla_\alpha \jmath_\mu = 0$, which is a scalar expression. To make sense of it nonetheless, one can project the vector $g^{\alpha\mu}\,\nabla_\alpha T_{\mu\nu} = 0$ onto the velocity u^μ and a plane perpendicular to it. Computing the gradient $g^{\alpha\mu}\,\nabla_\alpha T_{\mu\nu} = 0$ yields

$$g^{\alpha\mu}\nabla_\alpha\left[\left(\rho+\frac{p}{c^2}\right)u_\mu u_\nu - p\,g_{\mu\nu}\right] = g^{\alpha\mu}\left[\nabla_\alpha\left(\rho+\frac{p}{c^2}\right)\cdot u_\mu u_\nu + \left(\rho+\frac{p}{c^2}\right)\nabla_\alpha(u_\mu u_\nu) - \nabla_\alpha p\cdot g_{\mu\nu}\right] = 0 \tag{D.229}$$

keeping in mind that metric compatibility states that $\nabla_\alpha g_{\mu\nu} = 0$ and that the product of velocities in the second term resolves to $g^{\alpha\mu}\,\nabla_\alpha(u_\mu u_\nu) = g^{\alpha\mu}\,\nabla_\alpha u_\mu \cdot u_\nu + g^{\alpha\mu}\,u_\mu \cdot \nabla_\alpha u_\nu$.

Computing $u^\nu g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ as the projection of the covariant conservation law onto u^ν yields, if applied to the form of eqn. D.229 :

$$= g^{\alpha\mu}\,\nabla_\alpha\left(\rho + \frac{p}{c^2}\right)\cdot u_\mu u^\nu u_\nu + g^{\alpha\mu}\left(\rho + \frac{p}{c^2}\right)\cdot \nabla_\alpha u_\mu \cdot u^\nu u_\nu - \\ g^{\alpha\mu}\left(\rho + \frac{p}{c^2}\right)\cdot u_\mu u^\nu\,\nabla_\alpha u_\nu \; - g^{\alpha\mu}\,\nabla_\alpha\, p\cdot g_{\mu\nu}\, u^\nu \tag{D.230}$$

where we can carry out a number of simplifications: $u_\nu u^\nu = c^2$ in the first and second term. Then, $\nabla_\alpha(u_\nu u^\nu) = 0 = u_\nu \nabla_\alpha u^\nu + \nabla_\alpha u_\nu \cdot u^\nu = 2 \cdot u_\nu \nabla_\alpha u^\nu$ implies that the third term vanishes, and finally $g^{\alpha\mu}\,\nabla_\alpha\, p \cdot g_{\mu\nu}\, u^\nu = g^{\alpha\mu}\, g_{\mu\nu} \nabla_\alpha p\, u^\nu = \delta^\alpha_\nu\, \nabla_\alpha\, p \cdot u^\nu = \nabla_\alpha p \cdot u^\alpha$. Therefore, one arrives at

$$u^\nu g^{\alpha\mu} \nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu}\, \nabla_\alpha \left(\rho + \frac{p}{\mathrm{c}^2}\right) \cdot \mathrm{c}^2\, u_\mu + g^{\alpha\mu} \left(\rho + \frac{p}{\mathrm{c}^2}\right) \nabla_\alpha u_\mu \cdot \mathrm{c}^2 - \nabla_\alpha\, p \cdot u^\alpha = 0 \quad \text{(D.231)}$$

and lastly

$$u^\nu g^{\alpha\mu} \nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu} \left[\nabla_\alpha\!\left(\rho\, \mathrm{c}^2 \cdot u_\mu\right) + p \nabla_\alpha u_\mu\right] = 0 \quad \text{(D.232)}$$

which is exactly the relativistic continuity equation. The non-relativistic limit is recovered by setting $\rho\,\mathrm{c}^2 \gg p$ as well as $u^\mu = (c, \upsilon^i)^t$ with $\gamma = 1$, and using Cartesian coordinates implies $g^{\alpha\mu} = \eta^{\alpha\mu}$ as well as $\nabla_\alpha = \partial_\alpha$:

$$g^{\alpha\mu} \left[\nabla_\alpha(\rho \mathrm{c}^2 \cdot u_\mu) + p\, \nabla_\alpha u_\mu\right] = \eta^{\alpha\mu} \left[\partial_\alpha\, \rho \mathrm{c}^2 \cdot u_\mu + \rho \mathrm{c}^2 \partial_\alpha u_\mu + p\, \partial_\alpha u_\mu\right] \simeq \mathrm{c}^2 \cdot \eta^{\alpha\mu}\, \partial_\alpha(\rho\, u_\mu) = 0 \quad \text{(D.233)}$$

where the last term in the brackets reads

$$\partial_t\, \rho + \partial_i(\rho\, \upsilon^i) = 0 \quad \text{(D.234)}$$

in the preferred coordinate frame, which is exactly the continuity equation from classical continuum mechanics: But unlike classical mechanics, where continuity is an empirical finding, it results in relativity from the covariant conservation of $\mathrm{T}_{\mu\nu}$.

We can resubstitute the conservation law eqn. D.232 into the divergence D.229 and see how we can isolate a statement about the conservation of momentum density. Again writing out $g^{\alpha\mu} \nabla_\alpha \mathrm{T}_{\mu\nu} = 0$ for the energy momentum tensor of an ideal fluid and writing out the expression fully gives:

$$g^{\alpha\mu} \nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu}\Big[\nabla_\alpha(\rho u_\mu) u_\nu + \frac{p}{\mathrm{c}^2} \nabla_\alpha u_\mu \cdot u_\nu + \nabla_\alpha p \cdot \frac{u_\mu u_\nu}{\mathrm{c}^2} + \frac{p}{\mathrm{c}^2} u_\mu \nabla_\alpha u_\nu + \rho \cdot u_\mu \nabla_\alpha u_\nu - \nabla_\alpha p\, g_{\mu\nu}\Big] \quad \text{(D.235)}$$

where the sum of the first two terms correspond exactly to the continuity equation D.232 (up to a pre-factor of c^2), and are therefore zero. Consequently,

$$g^{\alpha\mu} \nabla_\alpha \mathrm{T}_{\mu\nu} = g^{\alpha\mu}\Big[\Big(\frac{u_\mu u_\nu}{\mathrm{c}^2} - g_{\mu\nu}\Big) \cdot \nabla_\alpha\, p + \Big(\rho + \frac{p}{\mathrm{c}^2}\Big) \cdot u_\mu \nabla_\alpha u_\nu\Big] = 0 \quad \text{(D.236)}$$

In this way, one arrives at the relativistic Euler equation as an expression of momentum conservation:

$$g^{\alpha\mu}\Big[\Big(\frac{u_\mu u_\nu}{\mathrm{c}^2} - g_{\mu\nu}\Big) \nabla_\alpha\, p + \Big(\rho + \frac{p}{\mathrm{c}^2}\Big) u_\mu \nabla_\alpha\, u_\nu\Big] = 0 \quad \text{(D.237)}$$

First, we see that only pressure gradients perpendicular to the velocity are ever relevant,

$$g^{\alpha\mu} \Big(\frac{u_\mu u_\nu}{\mathrm{c}^2} - g_{\mu\nu}\Big) \nabla_\alpha p = \nabla^\perp p \quad \text{(D.238)}$$

because one applies a projection operator on the gradient in pressure, projecting out the component of ∇_μ p perpendicular to u^μ, and secondly, if the motion of a fluid element proceeds along a geodesic with autoparallelity $u^\mu \nabla_\mu u_\nu = 0$ given,

$$\left(\rho + \frac{p}{c^2}\right)g^{\alpha\mu}u_\mu\nabla_\alpha u_\nu = \left(\rho + \frac{p}{c^2}\right)u^\mu\,\nabla_\mu u_\nu \tag{D.239}$$

that those pressure gradients must be zero! Pressure gradients would push a fluid element away from the geodesic that characterises free fall.

The nonrelativistic limit can be constructed by approximating the autoparallelity condition,

$$u^\mu\nabla_\mu u^\nu \simeq u^\mu\partial_\mu u^\nu = c\partial_{ct}u^j + u^i\partial_i u^j \tag{D.240}$$

which shows that the nonlinearity of the Euler-equation has a relativistic origin, and furthermore for a flat background where $\nabla_\mu = \partial_\mu$ that

$$\rho\left(\partial_t u^j + u^i\partial_i u^j\right) = -\partial^j p \tag{D.241}$$

or equivalently, that

$$\partial_t u^j + u^i\partial_i u^j = -\frac{\partial^j p}{\rho} \tag{D.242}$$

which is the classical Euler-equation for ideal fluid mechanics. Allowing for weak, static Newtonian gravity one work with the approximation that pressure is scalar (actually it is only a partial trace of the energy momentum tensor!), so $\nabla_\alpha p = \partial_\alpha p$ and we obtain for the covariant derivative

$$u^\mu\,\nabla_\mu u_\nu = u^\mu\left(\partial_\mu u_\nu - \Gamma^\alpha_{\mu\nu}\,u_\alpha\right) \tag{D.243}$$

while Newtonian gravity is a weak and static perturbation to the line element,

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}\,dx^i dx^j \tag{D.244}$$

from which we isolate the two metric functions

$$g_{tt} = \left(1 + \frac{2\Phi}{c^2}\right) \quad \text{and} \quad g_{ii} = -\left(1 - \frac{2\Phi}{c^2}\right) \tag{D.245}$$

Working towards the nonrelativistic limit we would replace $g^{\alpha\beta} = \eta_{\alpha\beta}$ but keep the derivative $\partial_\mu g_{\alpha\beta}$ with the exception $\partial_{ct}g_{\mu\nu} = 0$ as Newtonian fields are necessarily static. The derivatives of the metric then reflect potential gradients, $\partial_i\, g_{\mu\nu} = \pm\frac{2}{c^2}\,\partial_i\Phi\delta_{\mu\nu}$ which become the Christoffel-symbol $\Gamma^i_{tt} \sim +\partial^j\Phi$. So ultimately, we arrive at the Euler-equation of classical ideal fluid mechanics including a gravitational potential Φ,

$$\partial_t u^j + (u^i\partial_i)u^j = -\frac{\partial^j p}{\rho} - \partial^j\Phi, \tag{D.246}$$

from the covariant divergence $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$. Alternative to resubstituting we can take the vector $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ and project it straight away onto a plane perpendicular to u^ν, by means of a projection operator $P_\perp^{\nu\rho} = u^\nu u^\rho/c^2 - g^{\nu\rho}$, to arrive at the Euler-equation.

D.3 *fields as sources of gravity*

Relativistic gravity should be compatible with relativistic fields as well as fluids, similarly to electrodynamics which is equally valid for a classical charge density as a source or a charge density that is computed from the probability determined by the wave functions of the particles according to the Born-postulate: This is made sure by the fact that fields can be assigned an energy-momentum tensor as an expression of local energy density, momentum density and stress, which obeys automatically relativistic conservation laws as soon as the Lagrange-density $\mathcal{L}$ of the fields does not explicitly depend on the coordinate, meaning that the working principle of the fields should be identical everywhere and at every time.

A scalar field ϕ on an arbitrary, possibly curved spacetime with metric $g_{\mu\nu}$ for instance would be described by the Lagrange-function

$$\mathcal{L} = \mathcal{L}(\phi, \nabla_\alpha\phi, g_{\mu\nu}) \tag{D.247}$$

if its dynamics is universal, so that $\mathcal{L}$ depends on the field ϕ and its derivative $\nabla_\alpha\phi$ (which would of course be $= \partial_\alpha\phi$ as ϕ is scalar, but let's use the covariant formalism), but not explicitly on the coordinates x^μ. The action integral would read

$$S = \int d^4x\,\sqrt{-\det g}\,\mathcal{L} \tag{D.248}$$

where the additional factor $\sqrt{-\det g}$ makes sure that the volume element is invariant under coordinate transforms (we come to this in the next chapter). The field equation follows from variation according to Hamilton's principle $\delta S = 0$. Specifically,

$$\delta S = \int d^4x\,\sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\,\delta\nabla_\alpha\phi\right) \tag{D.249}$$

Using the interchangeability $\delta\nabla_\alpha\phi = \nabla_\alpha\delta\phi$ and integration by parts while keeping the variation on the boundary fixed gives

$$\delta S = \int d^4x\,\sqrt{-\det g}\left(\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\alpha\,\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\right) = 0 \tag{D.250}$$

from which we extract the Euler-Lagrange equation, now in a covariant formulation ready to work on a curved background,

$$\nabla_\alpha\,\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \tag{D.251}$$

Next, it'd be great if an expression for the energy momentum tensor $T_{\mu\nu}$ would directly follow from the coordinate independent Lagrange-function $\mathcal{L}$, possibly along with a covariant conservation law in the form $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$. In fact, if $\mathcal{L}(\phi, \nabla_\alpha\phi)$ does not depend on position the variation $\delta\mathcal{L}$ is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\,\delta\nabla_\alpha\phi = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\,\delta\phi\right] + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\cdot\delta\phi \tag{D.252}$$

where in the last step the Leibnitz-rule was used to introduce the derivative of the product $\partial\mathcal{L}/\partial\nabla_\alpha\phi\,\delta\phi$, which suggests that the Euler-Lagrange-equation should be substituted,

$$\delta\mathcal{L} = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\delta\phi\right] + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\alpha\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\right)}_{=0}\delta\phi \quad\rightarrow\quad \delta\mathcal{L} = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\delta\phi\right] \tag{D.253}$$

Next, we need to write the variation in $\mathcal{L}$ from an infinitesimal translation of the field $\delta\phi$ (because the Lagrange-density does not change itself as a function of coordinate, it can only change if the fields themselves are different!), i.e. to think of a way of actually generating the variation from an infinitesimal shift in the coordinates:

$$\phi\left(x^\mu + \delta x^\mu\right) = \phi\left(x^\mu\right) + \nabla_\nu\phi\left(x^\mu\right)\cdot\delta x^\nu + \ldots \tag{D.254}$$

again using covariant derivatives for generality. Then, the field variation $\delta\phi$ is given by

$$\delta\phi = \phi(x^\mu + \delta x^\mu) - \phi(x^\mu) = \nabla_\nu\phi\cdot\delta x^\nu = g^{\mu\nu}\,\nabla_\mu\phi\,\delta x_\nu \tag{D.255}$$

On the other hand, shifting the Lagrange function $\mathcal{L}$ by an amount δx_β is easily achieved by the displacement defined through the covariant derivative, $\delta x^\beta\nabla_\beta = g^{\alpha\beta}\delta x_\beta\nabla_\alpha$:

$$\delta\mathcal{L} = g^{\alpha\beta}\,\nabla_\alpha\mathcal{L}\cdot\delta x_\beta. \tag{D.256}$$

Combining both yields

$$\delta\mathcal{L} = g^{\alpha\beta}\,\nabla_\alpha\mathcal{L}\cdot\delta x_\beta = \nabla_\alpha\left[\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\cdot g^{\mu\nu}\,\nabla_\mu\phi\,\delta x_\nu\right] \tag{D.257}$$

As the same covariant derivative ∇_α acts on both terms, they can be combined to give

$$\nabla_\alpha\left[\mathcal{L}\cdot\delta x^\alpha - \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}g^{\mu\nu}\,\nabla_\mu\phi\,\delta x^\nu\right] = 0 \tag{D.258}$$

This equation would be perfect if it was independent of the shift δx, but it appears with different indices in the two terms. A possible remedy is a renaming $\delta x^\alpha = g^{\alpha\mu}\,g_{\mu\nu}\,\delta x^\nu = \delta^\alpha_\nu\,\delta x^\nu$, so that the formula becomes

$$\nabla_\alpha\left[g^{\alpha\mu}\,g_{\mu\nu}\,\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi}\nabla_\nu\phi\right]\delta x^\nu = g^{\alpha\mu}\,\nabla_\alpha\left[\mathcal{L}\,g_{\mu\nu} - \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\nabla_\nu\phi\right]\delta x^\nu = 0 \tag{D.259}$$

where we can identify the energy momentum tensor as computed for the field ϕ from its Lagrange-function $\mathcal{L}(\phi, \nabla_\alpha\phi, g_{\mu\nu})$,

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\nabla_\nu\phi - \mathcal{L}g_{\mu\nu} \tag{D.260}$$

including the conservation law $g^{\alpha\mu}\,\nabla_\alpha T_{\mu\nu} = 0$ in a covariant formulation. The idea, that the energy-momentum tensor $T_{\mu\nu}$ mediates between the field and the gravitational field equation is very interesting: As soon as the dynamics of the fields are universal, $T_{\mu\nu}$ is defined , covariantly conserved, and computable from $\mathcal{L}$, irrespective of the actual substance. In this sense, general relativity is the gravitational theory of systems with conserved energy and momentum in the same way as Maxwell-electrodynamics is the electromagnetic theory for systems with conserved charges.

E GRAVITATIONAL FIELD EQUATION

E.1 *what should be realised in a gravitational field equation?*

The field equation for gravity should first of all be a tensorial relationship between curvature and the energy-momentum tensor the source of gravity, with a symmetric curvature tensor isolated from the full Riemann curvature. Tensorial relationships are necessary to have a consistent and well-defined transformation property of all terms in the field equation. The field equation should operate on a 4-dimensional background and allow for wave-like propagating solutions.

The field equation should obey covariant energy-momentum conservation. As a second order partial differential equation (because the Riemann-curvature is made from the second derivatives of the metric) it should be hyperbolic and allow modes to propagate on the light cone. In contrast to our first attempts at constructing a generalisation of the Poisson-equation within special relativity, there should be a natural explanation why $m = 0$ but why $\lambda \neq 0$. But nevertheless, the limit of the field equation for weakly perturbed, static spacetimes should fall back on the classical Poisson equation $\Delta\Phi = 4\pi G\rho + \lambda$ (I'm trying to make a point that the cosmological constant λ was always part of a classical theory).

It is a surprising result found by D. Lovelock that general relativity is unique as a relativistic theory of gravity for conserved energy and momentum in 4 dimensions with a second order hyperbolic and local field equation with a single dynamical field, the metric $g_{\mu\nu}$. It is an astonishing fact that the field equation of general relativity is as fundamental as the Maxwell equations with nothing more fundamental from which it could be derived. So all we can hope is to go through arguments why the equation is sensible and how physical concepts are realised. I should mention that there are ideas in relation to constructive gravity with the central idea that the theory for the material fields (like A^μ) already fixes the dynamics of the metric $g_{\mu\nu}$ up to the point that the gravitational field equation can be constructed from the Lagrange-density of the Maxwell-field.

E.2 *construction of the field equation*

The first issue in the quest to link the Riemann curvature $R_{\alpha\beta\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ is the different rank of the two tensors. The Ricci-curvature $R_{\beta\nu} = g^{\alpha\mu}R_{\alpha\beta\gamma\mu}$ would be (up to an overall sign) the only non-vanishing contraction of the Riemann-curvature and it would be symmetric as well, as can be shown with the algebraic Bianchi-identity,

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0 \tag{E.261}$$

for the cyclic permutation of $\beta, \mu\nu$ while keeping the first index α fixed. Applying a contraction with $g^{\alpha\mu}$ to the algebraic Bianchi-identity gets rid of the second term due to the antisymmetry of $R_{\alpha\mu\nu\beta}$ in $\alpha\mu$. Then,

$$g^{\alpha\mu}R_{\alpha\beta\mu\nu} + g^{\alpha\mu}R_{\alpha\nu\beta\mu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu} - g^{\alpha\mu}R_{\alpha\nu\mu\beta} = R_{\beta\nu} - R_{\nu\beta} = 0 \tag{E.262}$$

again using antisymmetry, this time of $R_{\alpha\nu\beta\mu}$ in the second index pair, which shows the symmetry of the Ricci tensor, $R_{\beta\nu} = R_{\nu\beta}$. Then, the Ricci-scalar $R = g^{\beta\nu}R_{\beta\nu} = g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}$ as a contraction of the Ricci-tensor $R_{\beta\nu}$ is well-defined and not fixed to zero by any index exchange symmetry.

Would $R_{\mu\nu} \propto T_{\mu\nu}$ be viable field equation? Covariant energy-momentum conservation requires that $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$, but one can show that the divergence $g^{\alpha\mu}\nabla_\alpha R_{\mu\nu} \neq 0$, so that the field equation would be inconsistent. Instead, one needs a more elaborate curvature quantity: the Einstein-tensor $G_{\mu\nu}$. Starting from the differential Bianchi-identity

$$\nabla_\tau R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\tau} + \nabla_\nu R_{\alpha\beta\tau\mu} = 0 \tag{E.263}$$

with cyclic permutation in $\tau, \mu\nu$ and α, β fixed, one can make the substitution $R_{\alpha\beta\tau\mu} = -R_{\alpha\beta\mu\tau}$ in the last term with the index antisymmetry in the second pair. Contraction with $g^{\alpha\mu}$ yields:

$$g^{\alpha\mu}\nabla_\tau R_{\alpha\beta\mu\nu} + g^{\alpha\mu}\nabla_\mu R_{\alpha\beta\nu\tau} - g^{\alpha\mu}\nabla_\nu R_{\alpha\beta\mu\tau} = 0 \tag{E.264}$$

Using metric compatibility $\nabla_\alpha g_{\mu\nu} = 0$ in the last term, followed by a contraction with $g^{\beta\tau}$ then introduces the Ricci-scalar R, because $g^{\alpha\mu}g^{\beta\tau}R_{\alpha\beta\mu\tau} = R$. The first term gives $g^{\alpha\mu}g^{\beta\tau}\nabla_\tau R_{\alpha\beta\mu\nu} = g^{\beta\tau}\nabla_\tau R_{\beta\nu}$, which is the divergence of the Ricci-tensor. The most complicated term is the middle one: Starting from the algebraic Bianchi-identity $R_{\alpha\beta\nu\tau} + R_{\alpha\nu\tau\beta} + R_{\alpha\tau_\beta\nu} = 0$ one can construct the argument that $R_{\beta\alpha\nu\tau} = R_{\alpha\nu\tau\beta} + R_{\alpha\tau\beta\nu}$ using the antisymmetry in the first index pair of the first term, followed by the contraction of $\nabla_\mu R_{\beta\alpha\nu\tau}$, over $\beta\tau$ and $\alpha\mu$, which comes out as $g^{\beta\tau}g^{\alpha\mu}[R_{\alpha\nu\tau\beta} + R_{\alpha\tau\beta\nu}]$, where the first term vanishes due to the (anti)symmetry of the indices and only $g^{\alpha\mu}\nabla_\mu R_{\alpha\nu}$ is left over, with an additional overall minus-sign. Realising that this term is, like the first one, the divergence of the Ricci-tensor albeit with different (internal) indices, the final result is:

$$2g^{\alpha\mu}\nabla_\mu R_{\alpha\nu} - \nabla_\nu R \quad \rightarrow \quad g^{\alpha\mu}\nabla_\mu [2R_{\alpha\nu} - Rg_{\alpha\nu}] = 0 \tag{E.265}$$

indicating that this particular combination of the Ricci-tensor, the Ricci-scalar and the metric is divergence-free and could appear in the field equation. Commonly, one defines the Einstein-tensor $G_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} \tag{E.266}$$

for this purpose, which inherits its symmetry from $R_{\mu\nu}$ and $g_{\mu\nu}$. It is a memorable result that the trace of $G_{\mu\nu}$

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{R}{2}g^{\mu\nu}g_{\mu\nu} = R - \frac{R}{2}\delta^\mu_\mu = R - 4\frac{R}{2} = -R. \tag{E.267}$$

is just the negative Ricci-scalar R.

Realising that the metric is the second rank-2 tensor with vanishing divergence due to metric compatibility suggests as a possible gravitational field equation

☠ *There is no "derivation" of the field equation, it is so fundamental that we don't know any more fundamental principle from which it could originate!*

dimension	1	2	3	4
$T_{\mu\nu}$	1	3	6	10
$R_{\mu\nu}$	0	1	6	10
$R_{\alpha\beta\mu\nu}$	0	1	6	20

Table 1: Numbers of entries of $T_{\mu\nu}$, $R_{\mu\nu}$ and $R_{\alpha\beta\mu\nu}$ as a function of the dimensionality n of spacetime: While the number of entries of $T_{\mu\nu}$ is simply determined by symmetry $T_{\mu\nu} = T_{\nu\mu}$, the entries of $R_{\mu\nu}$ and $R_{\alpha\beta\mu\nu}$ must be derived from the index exchange symmetries, for instance, that there can not be any curvature in 1 dimension, because the non-commutativity of the covariant derivative in different directions never arises: there is only one direction. In 3 dimensions, there can not be any curvature beyond Ricci-curvature and the gravitational field would only exist at locations where the energy-momentum tensor is nonzero. Only in 4 dimensions or more there are components of curvature beyond Ricci curvature and the gravitational field can exist away from the source.

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{E.268}$$

with two gravitational constants G and Λ. It is a second-order nonlinear hyperbolic partial differential equation which respects the local covariant energy-momentum conservation and constitutes 10 independent relations in 4 dimensions, due to the symmetry of $R_{\mu\nu}$, $g_{\mu\nu}$ and $T_{\mu\nu}$. Hyperbolicity of the field equation is a consequence of the sign-change in the signature $(+,-,-,-)$ of the metric $g_{\mu\nu}$, which falls back onto the Minkowskian-metric in freely-falling frames, $g_{\mu\nu} = \eta_{\mu\nu}$, and ultimately, hyperbolicity will allow for wave-type solutions: gravitational waves!

One issue needs considerable explanation: The Riemann-curvature as a complete characterisation of the spacetime curvature has 20 entries in 4 dimensions (reduced from $4^4 = 256$ to 20 by the index exchange symmetries), but the field equation only fixes half of the curvature, similarly to the Poisson equation $\Delta\Phi = 4\pi G\rho$, where only the trace $\Delta\Phi = \delta^{ij}\partial_i\partial_j\Phi$ of the tidal field tensor $\partial_i\partial_j\Phi$ is determined by the field equation. In electrodynamics, the field equation $\Box A^\mu = 4\pi/c\, \jmath^\mu$ in Lorentz-gauge $\partial_\mu A^\mu = 0$ fixes 4 of the 10 derivatives $\partial_\alpha\partial_\beta A^\mu$, so this is really a common feature for all field theories. If this was not the case, we could have only Ricci-curvature, and it could only exist at places where the energy-momentum tensor is nonzero, $T_{\mu\nu} \neq 0$. Clearly, this would be a weird theory of gravity, as the field should be free to propagate away from the source into spacetime.

E.3 *Ricci- and Weyl-curvature*

In classical gravity, $\Delta\Phi = \delta^{ij}\partial_i\partial_j\Phi$ is invariant as the trace of the tidal field $\partial_i\partial_j\Phi$: It does not change under rotations of the coordinate system and links the potential to the source $4\pi G\rho$. Starting with Φ one obtains the gravitational acceleration $g_j = -\partial_j\Phi$, of which one can compute the divergence $\mathrm{div}\boldsymbol{g} = \delta^{ij}\partial_i g_j = -\delta^{ij}\partial_i\partial_j\Phi$ which tells you about a nonzero ρ at the point where Δ acts on Φ. Vice versa, however, does $\Delta\Phi = 0$ not imply that there is no gravitational field, it only implies that at that particular location there is no source, and clearly can gravity exist at locations outside the field generating matter, for instance on the surface of the Earth. This suggests that one would like to separate $\Delta\Phi$ from $\partial_i\partial_j\Phi$ and define the traceless shear

$$\tilde{\Phi}_{ij} = \partial_i\partial_j\Phi - \frac{\Delta\Phi}{3}\delta_{ij}. \tag{E.269}$$

$\tilde{\Phi}_{ij}$ are the components of the tidal shear that are sourced elsewhere and propagate to the point where the derivatives of Φ are computed.

E.4 *curvature invariants*

There are two possible ways to quantify geometric properties of manifolds, or, in fact, tensorial or vectorial fields: Either, one is able to write down a relation between tensors of compatible rank and index structure, in which case all terms in an equation transform covariantly under coordinate transforms, or one can construct invariants by a full index contraction. Then, one obtains a scalar which is necessarily invariant under coordinate transforms and has to assume an identical value in all frames. That is the reason why scalars are so convenient: Their entries do not only for a given coordinate choice but are universally true. What one gives up, however, is a significant part of the information that gets lost in contraction. But sometimes, scalars have a physical interpretation and can isolate important information on a tensor.

In classical gravity, we can compute the tidal field $\partial_i\partial_j\Phi$ as the curvature analogue and build contractions of this quantity, for instance with the Euclidean metric: $\delta^{ij}\partial_i\partial_j\Phi = \Delta\Phi$ is rotationally invariant (reflecting the fundamental properties of Euclidean spaces, and proportional to $4\pi G\rho + \lambda$). Or, one constructs the quadratic quantity $\delta^{ai}\delta^{bj}\partial_a\partial_b\Phi\,\partial_i\partial_j\Phi$, which corresponds to the Frobenius-norm of Phi which is positive definite: We can conclude $\Phi = 0$ from a vanishing Frobenius-norm, but we can not do that from $\Delta\Phi = 0$, which only means that at that particular location no source of the field exists.

The central quantity for curvature in relativity is the Riemann-tensor $R_{\alpha\beta\mu\nu}$, with a range of possibilities to form a scalar. For instance, the Ricci-scalar $R = g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta\mu\nu}$ would be a quantity analogous to $\Delta\Phi$, as it is proportional to the trace of the energy momentum tensor $T = g^{\mu\nu}T_{\mu\nu}$, minus 4Λ if the cosmological constant is included. That's clearly only the curvature that is generated locally by $T_{\mu\nu}$ (and by Λ), but not the complete curvature. In analogy to the Frobenius-norm one could think of Kretschmann-scalar $K = R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}R_{\alpha\beta\gamma\delta}R_{\mu\nu\rho\sigma}$.

The Weyl-tensor $C_{\alpha\beta\mu\nu}$ would correspond to the traceless tidal shear $\tilde{\Phi}_i$, because the locally generated part of the curvature has been eliminated. Then, clearly both $\delta^{ij}\tilde{\Phi}_{ij}$ and $g^{\alpha\mu}g^{\beta\nu}C_{\alpha\beta\mu\nu}$ vanish. But $\delta^{ai}\delta^{bj}\,\tilde{\Phi}_{ab}\tilde{\Phi}_{ij}$ is not required to be zero by $\Delta\Phi = 0$, and neither is the Weyl-scalar $C = C^{\alpha\beta\mu\nu}C_{\alpha\beta\mu\nu} = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}C_{\alpha\beta\gamma\delta}C_{\mu\nu\rho\sigma}$: It would serve as an invariant quantification of the curvature at a point of all gravitational fields that are sourced elsewhere. As such, the Weyl-curvature $C_{\alpha\beta\mu\nu}$ is a covariant generalisation of the traceless tidal tensor $\tilde{\Phi}_{ij}$.

E.5 *weak and static gravity*

General relativity needs to be consistent with classical gravity in the limit of weak curvature and static gravitational fields consistent with a non-relativistic matter distribution at rest. The trace of the field equation is given by

$$g^{\mu\nu}R_{\mu\nu} - \frac{R}{2}g^{\mu\nu}g_{\mu\nu} = -R = -\frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu} = -\frac{8\pi G}{c^4}T \tag{E.270}$$

using $R = g^{\mu\nu}R_{\mu\nu}$, $T = g^{\mu\nu}T_{\mu\nu}$ and $g^{\mu\nu}g_{\mu\nu} = \delta^{\mu}_{\mu} = 4$, while the trace of the energy momentum tensor is given by

$$T = g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}\left[\left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu - p\cdot g_{\mu\nu}\right] = \left(\rho + \frac{p}{c^2}\right)g^{\mu\nu}\,u_\mu u_\nu - p g^{\mu\nu}g_{\mu\nu} = \rho c^2 - 3p \simeq \rho c^2 \tag{E.271}$$

if the matter is non-relativistic, $p \ll \rho c^2$, so that the Ricci-scalar just depends on the matter density,

$$R = -\frac{8\pi G}{c^2}\rho. \tag{E.272}$$

A weak perturbation an otherwise Minkowskian spacetime by a static gravitational potential Φ has the form

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}\,dx^i dx^j \tag{E.273}$$

where the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with the condition $|h_{\mu\nu}| \ll 1$ is only valid in that particular Cartesian coordinate choice. Then, the inverse metric can be approximated to be $g^{\mu\nu} \simeq \eta^{\mu\nu}$ with an error of the order h^2. The tt-component of the Ricci-tensor in general given by

$$R_{tt} = \partial_t \Gamma^{\mu}{}_{t\mu} - \partial_\mu \Gamma^{\mu}{}_{tt} + \Gamma^{\nu}{}_{t\mu}\Gamma^{\mu}{}_{\nu t} - \Gamma^{\nu}{}_{tt}\Gamma^{\mu}{}_{\mu\nu} \tag{E.274}$$

where the first term $\partial_t \Gamma^{\mu}{}_{t\mu} = 0$ for static fields, and the squared Christoffel-symbols $+\Gamma^{\nu}{}_{t\mu}\Gamma^{\mu}{}_{\nu t} - \Gamma^{\nu}{}_{tt}\Gamma^{\mu}{}_{\mu\nu}$ would contribute at order h^2, so we neglect them. The only contributing term is then

$$R_{tt} = -\partial_\mu \Gamma^{\mu}{}_{tt} = -\partial_i \Gamma^{i}{}_{tt} = -\partial_i\left(\frac{\delta^{ij}}{2}\left(-\partial_j h_{tt}\right)\right) = \frac{1}{2}\delta^{ij}\partial_i\partial_j h_{tt} = \frac{\Delta\Phi}{c^2} \tag{E.275}$$

because $h_{tt} = 2\Phi/c^2$. Collecting the results on the traces and the weak field, static limit then yields

$$R_{tt} = \frac{\Delta\Phi}{c^2} = \frac{4\pi G}{c^4}\rho c^2 \quad\rightarrow\quad \Delta\Phi = 4\pi G\rho, \tag{E.276}$$

which one recognises as the classic Poisson field equation.

E.6 *Weyl-curvature*

There is a very good physical reason to decompose the Riemann tensor $R_{\alpha\beta\mu\nu}$ as full quantification of curvature into two parts: The Ricci-curvature $R_{\beta\nu} = g^{\alpha\mu}R_{\alpha\beta\mu\nu}$, which appears in field equation as $R_{\beta\nu} - R/2\; g_{\beta\nu}$ and which is proportional to the energy-momentum tensor $T_{\beta\nu}$, and the remaining curvature components, which form the Weyl-tensor $C_{\alpha\beta\mu\nu}$ describing the curvature that has been sourced by energy and momentum elsewhere and has propagated to the spacetime point under consideration.

As already discussed, the field equation should not fully fix the curvature and set it to be proportional to the source of the field, which is a typical structure in all field equations. Electrodynamics, for instance, equates only 4 components of the $24 = 6 \times 4$ possible derivatives $\partial^\beta F^{\mu\nu}$ of $F^{\mu\nu}$ to be equal to the source $\jmath^\nu$ according to $\eta_{\beta\mu}\partial^\beta F^{\mu\nu} = 4\pi/c\;\jmath^\nu$. But is there a constraint on the remaining 20 components? Yes, in fact through the Bianchi-identity,

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\nu} = 0 \quad \text{or, equivalently} \quad \eta^{\beta\mu}\partial_\beta \tilde{F}_{\mu\nu} = 0 \tag{E.277}$$

with the dual tensor $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$.

Similarly, the off-trace parts of the curvature form the Weyl-tensor which obeys an analogous differential Bianchi-identity. In fact, it obeys the same antisymmetry relations as the Riemann-tensor, i.e.

$$C_{\alpha\beta\mu\nu} = -C_{\beta\alpha\mu\nu} = -C_{\alpha\beta\nu\mu} \tag{E.278}$$

as well as an algebraic Bianchi-identity

$$C_{\alpha\beta\mu\nu} + C_{\alpha\mu\nu\beta} + C_{\alpha\nu\beta\mu} = 0 \tag{E.279}$$

and

$$g^{\alpha\mu}\, C_{\alpha\beta\mu\nu} = 0 \tag{E.280}$$

and finally a differential Bianchi-identity

$$\nabla_\tau C_{\alpha\beta\mu\nu} + \nabla_\mu C_{\alpha\beta\nu\tau} + \nabla_\nu C_{\alpha\beta\tau\mu} = 0. \tag{E.281}$$

Let's construct a systematic decomposition of the Riemann curvature $R_{\alpha\beta\mu\nu}$: From any symmetric tensor $X_{\alpha\beta}$ one can derive the quantity $\tilde{X}_{\alpha\beta\mu\nu}$

$$\tilde{X}_{\alpha\beta\mu\nu} = A_{\alpha\mu}\, g_{\beta\nu} + A_{\beta\nu}\, g_{\alpha\mu} - A_{\alpha\nu}\, g_{\beta\mu} - A_{\beta\mu}\, g_{\alpha\nu}. \tag{E.282}$$

This definition of $\tilde{X}_{\alpha\beta\mu\nu}$ makes sure that the quantity fulfils the properties

$$\tilde{X}_{\alpha\beta\mu\nu} = -\tilde{X}_{\alpha\beta\mu\nu} = -\tilde{X}_{\beta\alpha\mu\nu} \quad \text{and} \quad \tilde{X}_{\alpha\beta\mu\nu} + \tilde{X}_{\alpha\mu\nu\beta} + \tilde{X}_{\alpha\nu\beta\mu} = 0 \tag{E.283}$$

i.e. effectively the index exchange symmetries of the Riemann-tensor, suggesting the ansatz

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + a \cdot \tilde{R}_{\alpha\beta\mu\nu} + b\, R \cdot \tilde{g}_{\alpha\beta\mu\nu} \tag{E.284}$$

with the Ricci-scalar R, and $\tilde{g}_{\alpha\beta\mu\nu}$ and $\tilde{R}_{\alpha\beta\mu\nu}$ from the metric $g_{\mu\nu}$ and the Ricci-tensor $R_{\mu\nu}$, respectively. Then, the two factors a and b can be determined through contraction.

This decomposition can be used to show an extremely interesting algebraic property of the Weyl-curvature $C_{\alpha\beta\mu\nu}$ as the part of curvature that propagates: The tensor can only be nonzero in more than four dimensions, suggesting that gravity can only exist at locations where the energy momentum tensor is zero in less than four dimensions, entirely defeating the purpose of a field theory:

- $n = 1$

  no Riemann-curvature, $R_{\alpha\beta\mu\nu} = 0$, because of the exchange symmetry in e.g. the last two indices: There can't be any curvature in one dimensions, because the covariant derivatives always commute, as they apply only to a single direction.

- $n = 2$

Riemann-curvature is always proportional to the Ricci-scalar and the metric, as two-dimensional manifolds are always maximally symmetric,

$$R_{\alpha\beta\mu\nu} = \frac{R}{2}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{E.285}$$

- $n = 3$

  Riemann-curvature is proportional to the Ricci-tensor and the Ricci scalar, but the Weyl-tensor vanishes identically,

$$R_{\alpha\beta\mu\nu} = \left(g_{\beta\mu}\,R_{\alpha\nu} + g_{\alpha\nu}\,R_{\beta\mu} - g_{\beta\nu}\,R_{\alpha\mu} - g_{\alpha\mu}\,R_{\beta\nu}\right) + \frac{R}{2}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{E.286}$$

  That implies that the full Riemann-curvature needs to vanish if $T_{\mu\nu}$ is linked to the Ricci-curvature as in the conventional field equation: There would not be vacuum solutions in 2 or 3 dimensions.

- $n = 4$

  Ricci- and Weyl-curvature can simultaneously exist

$$R_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu} + \frac{1}{2}\left(g_{\beta\mu}\,R_{\alpha\nu} + g_{\alpha\nu}\,R_{\beta\mu} - g_{\beta\nu}\,R_{\alpha\mu} - g_{\alpha\mu}\,R_{\beta\nu}\right) \tag{E.287}$$

  and $R_{\alpha\beta\mu\nu}$ can be nonzero even if $R_{\mu\nu}$ is zero as a consequence of $T_{\mu\nu} = 0$.

Spacetimes without Weyl-curvature, $C_{\alpha\beta\mu\nu} = 0$ (as for instance FLRW-spacetimes) are conformally flat and their metric can always be written as

$$g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu} \tag{E.288}$$

i.e. as originating with a (coordinate-dependent) conformal factor $\Omega^2(x) > 0$ from the flat Minkowski-metric: This implies that the light cone structure of these spacetimes is identical perfectly Minkowskian light cones: The conformal factor drops out in the condition $\mathrm{d}s^2 = g_{\mu\nu}k^\mu k^\nu = \Omega(x)^2\eta_{\mu\nu}k^\mu k^\nu = 0$. That would be automatically the case in 2 and 3 dimensions.

A direct computation (which is very tedious) shows that Weyl-curvature is invariant under conformal transformations $g_{\mu\nu} \to \Omega(x)g_{\mu\nu}$ of the metric and that the Weyl-tensor maps onto itself: $C_{\alpha\beta\mu\nu} \to C_{\alpha\beta\mu\nu}$.

The differential Bianchi-identity is the dynamical equation for the Riemann curvature:

$$\nabla_\tau\,R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\tau} + \nabla_\nu\,R_{\alpha\beta\tau\mu} = 0 \tag{E.289}$$

Contraction with $g^{\alpha\mu}$ then yields:

$$g^{\alpha\mu}\,\nabla_\tau\,R_{\alpha\beta\mu\nu} + g^{\alpha\mu}\,\nabla_\mu\,R_{\alpha\beta\nu\tau} + g^{\alpha\mu}\,\nabla_\nu\,R_{\alpha\beta\tau\mu} = 0 \tag{E.290}$$

Identifying the Ricci-scalar in the first (and after an index swap) in the last term yields:

$$g^{\alpha\mu}\,\nabla_\mu\,R_{\alpha\beta\nu\tau} = \nabla^\alpha\,R_{\alpha\beta\nu\tau} = \nabla_\nu\,R_{\beta\tau} - \nabla_\tau\,R_{\beta\nu}. \tag{E.291}$$

As the same differential Bianchi-identity applies to the Weyl-tensor $C_{\alpha\beta\mu\nu}$ as well, one obtains a very similar result

$$g^{\alpha\mu}\,\nabla_\mu\,R_{\alpha\beta\nu\tau} = \nabla^\alpha\,C_{\alpha\beta\nu\tau} = \nabla_\nu\,S_{\beta\tau} - \nabla_\tau\,S_{\beta\nu} = C_{\beta\nu\tau} \tag{E.292}$$

with the Schouten-tensor

$$S_{\beta\tau} = \frac{R_{\beta\tau}}{2} - \frac{R}{6}g_{\beta\tau} \tag{E.293}$$

and the Cotton-tensor

$$C_{\beta\nu\tau} = \nabla_\nu\,S_{\beta\tau} - \nabla_\tau\,S_{\beta\nu} \tag{E.294}$$

such that the differential Bianchi-identity assumes a shape that is in fact reminiscent of the field equation in Maxwell-electrodynamics! For vacuum both $R_{\beta\tau}$ and R vanish, such that the $S_{\beta\tau}$ is necessarily zero, implying that

$$g^{\alpha\mu}\nabla_\mu C_{\alpha\beta\nu\tau} = 0 \quad \text{invacuum.} \tag{E.295}$$

If there are is a field-generating energy momentum content $T_{\beta\tau} \neq 0$, one would obtain in a non-vacuum situation

$$g^{\alpha\mu}\,\nabla_\mu\,C_{\alpha\beta\nu\tau} = \nabla^\alpha\,C_{\alpha\beta\nu\tau} = C_{\beta\nu\tau} = \frac{4\pi G}{c^4}\cdot\left[\nabla_\nu T_{\beta\tau} - \nabla_\tau T_{\beta\nu} - \frac{1}{3}\left(\nabla_\tau T\cdot g_{\beta\nu} - \nabla_\nu T\cdot g_{\beta\tau}\right)\right] \tag{E.296}$$

similar to $g^{\alpha\mu}\nabla_\alpha\,F_{\mu\nu} = 4\pi/c\,\jmath_\nu$.

E.7 *Raychaudhuri-equation*

The Raychaudhuri-equation gives a very pictorial and intuitive impression of the effects of the two types of curvature (Ricci and Weyl). It's even possible to apply the concept to classical gravity, so let's do this first: A bundle of geodesics $x^i(t)$ with relative velocities υ^i

$$x'^i = x^i + \upsilon^i t \tag{E.297}$$

would exhibit relative motion

$$\frac{\partial x'^i}{\partial x^j} = \delta^i_j + \frac{\partial \upsilon^i}{\partial x^j}\cdot t \simeq \frac{\partial x'^i}{\partial x^j} = \exp\left(\frac{\partial \upsilon^i}{\partial x^j}\cdot t\right) \tag{E.298}$$

at order t, in the spirit of a Lie-generated transformation. The change of the volume elements from d^3x to d^3x' is given by

$$d^3x' = \det\left(\frac{\partial x'}{\partial x}\right)d^3x \tag{E.299}$$

with the Jacobian determinant of the coordinate change. Using my third most favourite formula,

$$\ln\,d^3x' = \ln\,\det\left(\frac{\partial x'}{\partial x}\right) + \ln\,d^3x \tag{E.300}$$

following from $\ln\det A = \ln\prod_i \lambda_i = \sum_i \ln\lambda_i = \mathrm{tr}\ln A$ for any non-singular matrix A one arrives at

$$\ln\det\left(\frac{\partial x'}{\partial x}\right) = \mathrm{tr}\ln\left(\frac{\partial x'}{\partial x}\right) = \mathrm{tr}\ln\exp\left(\frac{\partial \upsilon}{\partial x}t\right) = t\cdot\mathrm{tr}\left(\frac{\partial \upsilon}{\partial x}\right) \tag{E.301}$$

with the identification

$$\mathrm{tr}\frac{\partial \upsilon}{\partial x} = \delta^{ij}\,\partial_j \upsilon_i = -\delta^{ij}\,\partial_j\partial_i\,\Phi = -\Delta\Phi = -4\pi G\rho \tag{E.302}$$

such that the matter density ρ (appearing through the substitution of the Poisson equation $\Delta\Phi = 4\pi G\rho$) inside a cloud of freely falling test particles (made sure by the Newtonian equation of motion $\dot{\upsilon}^i + \partial_i\Phi = 0$) causes a negative change of the volume. Interestingly, the appearance of a cosmological constant λ would likewise contribute to the volume evolution, and we witness this actually in cosmology.

The same intuition applies to a relativistic theory of gravity, as the Ricci-curvature is responsible to the volume change of a spacetime volume. The picture that emerges is that Ricci-curvature changes volumes while keeping their shape intact, and that Weyl-curvature changes shapes while conserving their volumes (at least to lowest order). In all theories this distinction is made by a decomposition into the trace and the traceless part of the curvature.

E.8 *nonlinearity and locality*

The field equation of general relativity are nonlinear partial differential equations with the important consequence that the superposition principle does not apply, which was such a convenient tool in classical gravity for solving the Poisson equation $\Delta\Phi = 4\pi G\rho + \lambda$. There, it's always possible to separate the problems one faces when determining the potential Φ from ρ: the inversion of the differential operator, to account for boundary conditions (as the Poisson-equation is an elliptical partial differential equation) and the possibly complicated geometry of the source ρ. In the case of linear field theories one achieves that by means of a Green-function $G(\boldsymbol{r},\boldsymbol{r}')$ as a solution to the field equation for a point charge $\delta_D(\boldsymbol{r}-\boldsymbol{r}')$, for simplicity on small scales where $\lambda = 0$ in a good approximation:

$$\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = 4\pi G\delta_D(\boldsymbol{r}-\boldsymbol{r}') \tag{E.303}$$

Using linearity, the equation can be multiplied with $\rho((r))$ and integrated over d^3r'. Effectively, this is exactly the expression of the superposition principle as one adds up the contributions to Φ at $\boldsymbol{r}$ from the source distribution $\rho(\boldsymbol{r}')$:

$$\int d^3r'\,\Delta\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \Delta\int d^3r'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \Delta\Phi = 4\pi G\int d^3r'\,\rho(\boldsymbol{r}')\delta_D(\boldsymbol{r}-\boldsymbol{r}') = 4\pi G\rho(\boldsymbol{r}) \tag{E.304}$$

so that

$$\Phi(\boldsymbol{r}) = \int d^3r'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{E.305}$$

	Schwarzschild-solution	grav. waves	FLRW-cosmologies	white dwarfs
homogeneous	t	$r \pm ct$	r	t
isotropic	yes	yes	yes	yes
varies along	r	r,t	t	r
gravity	strong	weak	strong	weak...strong
scales	$r_S = \frac{2GM}{c^2}$	linear physics	$\rho_{crit} = \frac{3H_0^2}{8\pi G}$	eqn. of state
curvature	Weyl	Weyl	Ricci	Weyl + Ricci
sources	vacuum solution	vacuum solution	p, ρ (ideal fluid)	p, ρ (ideal fluid)

Table 2: Compilation of the simplest solutions of general relativity together with their symmetries and peculiar physical properties. It should be emphasised that a coordinate choice has been taken which is particularly suited to the symmetry of the respective spacetimes.

is the required solution for the potential. Effectively, going from ρ to Φ relies on linearity, and going from Φ to ρ uses the locality of the equation as it determines the classical equivalent of Ricci-curvature. General relativity, however, is nonlinear, because pictorially the Christoffel-symbols contain terms of the type $g\partial g$, the Riemann curvature $(g\partial g)^2$ and $\partial(g\partial g)$, and finally the Ricci-curvature terms of the type $g(g\partial g)^2$ and $g\partial(g\partial g)$. Despite the nonlinearities, the field equation is still local, as it links the Ricci-part of the curvature to the energy-momentum tensor, as exemplified by the consideration of the change in volume of freely falling clouds of test particles in the Raychaudhuri-equation.

And I would like to mention, that the field equation of general relativity is a hyperbolic differential equation: Therefore, the solution is already unique if initial conditions are specified, while boundary conditions are not necessary. Hyperbolicity makes sure that excitations of the gravitational field are propagating along the light cones defined differentially by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = 0$. The nonlinearities of the field equation make it very difficult to find solutions for arbitrary $T_{\mu\nu}$, as one can not use the Green-method which would require linear superposition. But there are solutions for reasonable simple and symmetric cases, which are listed in Table. E.8 and which will be discussed in Sects. F, G and H.

F BLACK HOLES

F.1 *Schwarzschild black holes*

The Schwarzschild geometry refers to the geometry outside of a spherically symmetric static matter distribution as a generalisation to the Newtonian gravitational potential $\Phi = -GM/r$. Just like the latter follows from the solution of the vacuum ($\rho = 0$) Poisson-equation $\Delta\Phi = 0$ in the spherically symmetric case, K. Schwarzschild obtained his solution from the gravitational field equation in vacuum. $T_{\mu\nu} = 0$ implies directly $T = g^{\mu\nu}T_{\mu\nu} = 0$, such that the trace of the field equation becomes

$$g^{\mu\nu}R_{\mu\nu} - \frac{R}{2}g^{\mu\nu}g_{\mu\nu} = -R = -\frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu} = 0 \quad \rightarrow \quad R = 0 \qquad \text{(F.306)}$$

restricting ourselves to scales $\ll 1/\sqrt{\Lambda}$, meaning that the cosmological constant can be neglected. The trace relation implies that the Ricci-scalar R vanishes for vacuum solutions as a general result. Then, what remains from the field equation is

$$R_{\mu\nu} = 0 \tag{F.307}$$

which is to be solved for a spherically symmetric, static case. It would be wrong to conclude from $R_{\mu\nu} = 0$ that there could not be any curvature: There can not be any Ricci-curvature $R_{\mu\nu}$ in a vacuum case, but the field equation does not restrict the Weyl-curvature $C_{\alpha\beta\mu\nu}$, in the same way as the classical Poisson-equation only restricts $\Delta\Phi = 0$ but not the traceless tidal field $\partial_i\partial_j\Phi - \Delta\Phi/3\,\delta_{ij}$. Although there is no parallel in Newtonian theory as there is no notion of general covariance, the Weyl-curvature must obey the differential Bianchi-identity, which acts as the dynamic equation of $C_{\alpha\beta\mu\nu}$.

Guided by isotropy and staticity as symmetries, a suitable ansatz for the metric could be

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right] \tag{F.308}$$

for an intuitive coordinate choice. Clearly, one would like to work with spherical coordinates r, θ, φ, augmented by a temporal ct-coordinate. But there is some fineprint attached to this: The ct-coordinate would be the conventional coordinate time at the location of an infinitely distant observer, where $A(r) \to 1$ asymptotically to recover Minkowskian space. In this asymptotically flat space, the coordinate time would be identical to the proper time of observers at rest relative to the black hole. The radial coordinate r has the same limit $B(r) \to \infty$ as $r \to \infty$ to make the spatial submanifold appear as a flat Euclidean space. Clearly, the dependence of A and B on the radial coordinate is there to encode curvature effects in the measurement of time intervals and radial distances, and these curvature effects do not depend on time, as a reflection of staticity.

The typical scaling $\propto 4\pi r^2$ of spheres of radius r is obtained by integration over the two angles at fixed r, which in turn is actually defining the radial coordinate! The area element is $\mathrm{d}A = \sqrt{g_{\theta\theta}g_{\phi\phi}} = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ such that

$$\int_{4\pi} \mathrm{d}A = r^2 \int_0^{\pi} \mathrm{d}\theta\,\sin\theta \int_0^{2\pi} \mathrm{d}\phi = 4\pi\,r^2 \tag{F.309}$$

i.e. the radial coordinate r is chosen in such a way that the scaling of surfaces of spheres with radius is defined just like in flat Euclidean space, despite the fact that there are curvature effects present. At least in this single coordinate direction, the effects of curvature have disappeared through a suitable coordinate choice.

Writing the metric in this coordinate choice in matrix form

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{F.310}$$

makes it apparent that it is diagonal, and the inverse can be found quickly using the determinant $\det(g_{\mu\nu}) = -A(r)B(r)\cdot r^4\sin^2\theta$, such that

$$g^{\mu\nu} = \begin{pmatrix} A^{-1}(r) & 0 & 0 & 0 \\ 0 & -B^{-1}(r) & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -(r^2\sin^2\theta)^{-1} \end{pmatrix} \tag{F.311}$$

Many of the Christoffel-symbols vanish due to the high degree of symmetry. If the metric does not change in a certain coordinate direction along x^α, $\partial_\alpha g_{\mu\nu}$ is zero and does not contribute to the Christoffel-symbol. It is a technical exercise to show that the nonzero $\Gamma^{\alpha}{}_{\mu\nu}$ are:

$$\Gamma^{t}{}_{tr} = \frac{A'}{2A} \qquad \Gamma^{r}{}_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \tag{F.312}$$

$$\Gamma^{r}{}_{tt} = \frac{A'}{2B} \qquad \Gamma^{r}{}_{\phi\phi} = -\frac{r\sin^2\theta}{B} \qquad \Gamma^{\phi}{}_{r\phi} = \frac{1}{r} \tag{F.313}$$

$$\Gamma^{r}{}_{rr} = \frac{B'}{2B} \qquad \Gamma^{\theta}{}_{r\theta} = \frac{1}{r} \qquad \Gamma^{\phi}{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \tag{F.314}$$

together with torsion-free condition $\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}$, for switching the order of the two covariant indices. From those, we can compute the Riemann-tensor and determine the Ricci-tensor $R_{\beta\nu} = g^{\alpha\mu} R_{\alpha\beta\mu\nu}$ as its contraction: Again, the nonzero elements of $R_{\beta\nu}$ are:

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = 0 \tag{F.315}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} \tag{F.316}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \tag{F.317}$$

$$R_{\phi\phi} = R_{\theta\theta} \cdot \sin^2\theta \tag{F.318}$$

which comes out diagonal but not proportional to the metric, as a reflection of the presence of Weyl-curvature. Setting $R_{\mu\nu} = 0$ yields differential equations (the fourth involving $R_{\phi\phi} = 0$ is redundant because it is proportional to the third equation) whose solution will fix the two functions $A(r)$ and $B(r)$. Adding $B/A\times$ eqn. F.315 and adding it to eqn. F.316 yields

$$-\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} + \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = 0 \tag{F.319}$$

which immediately simplifies to

$$\frac{1}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \tag{F.320}$$

The term in brackets needs to vanish exactly for any choice of r, so one can determine

$$\frac{A'}{A} + \frac{B'}{B} = 0 \,\Big|\cdot AB, \quad BA' + AB' = \frac{d}{dr}(AB) = 0 \tag{F.321}$$

such that the product AB needs to be constant, $AB = \alpha$. Substituting $B = \alpha/A$ and the derivative $B' = -\frac{\alpha}{A^2}A'$ into eqn. F.317,

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0 \tag{F.322}$$

from which one can isolate a differential equation for A,

$$A + rA' = \frac{d}{dr}(rA) = \alpha, \tag{F.323}$$

and therefore $rA = \alpha r + k$ with an integration constant k. In this way, we have obtained the two metric functions

$$A(r) = \alpha\left(1 + \frac{k}{r}\right) \quad \text{and} \quad B(r) = \left(1 + \frac{k}{r}\right)^{-1} \tag{F.324}$$

where the two constants α and k need to be identified by comparison with a Minkowski-metric that is weakly perturbed by a potential: In this way, we match up the two solutions in the weak field limit and make them consistent with each other.

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right)c^2dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi\right) \tag{F.325}$$

is the weakly perturbed line element with a potential Φ, in our case $\Phi = -GM/r$ generated by an isotropic matter distribution with mass M according to the Poisson-equation. Comparison with the Schwarzschild line element yields for the metric functions

$$\left(1 + \frac{2\Phi}{c^2}\right) = A(r) \tag{F.326}$$

if $\alpha = c^2$ and $k = -2GM/c^2$. With this identification, the Schwarzschild line element becomes

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2\,dt^2 - \frac{1}{1 - \frac{2GM}{c^2}}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi\right) \tag{F.327}$$

where one can read off the Schwarzschild-radius

$$r_S = \frac{2GM}{c^2} \tag{F.328}$$

which assigns a length scale to the gravitational field generated outside a spherically symmetric and static matter distribution of mass M. $G/c^2 \simeq 10^{-28}$m, so a good number to remember is a few hundred meters for the Schwarzschild radius of the Sun with $M_\odot \simeq 10^{30}$kg. The Earth, with a considerably lower mass of $M_\oplus \simeq 10^{24}$kg has a Schwarzschild radius smaller by a factor 10^6. But please be careful: The gravitational field outside of every spherically symmetric matter distribution is of Schwarzschild form, there is no requirement that the mass would be somehow concentrated to $r < r_S$.

A subtle but very interesting point is that indirectly through the Newtonian solution, we have introduced a boundary condition: Spacetime becomes flat and Minkowskian at very large distances. This is necessary because by deactivating time-evolution of the gravitational field (because of the assumption of staticity) the field equation as a hyperbolic partial differential equation falls back onto an elliptical partial differential equation, which has only unique solutions if boundary conditions are specified. In our case, this would be a Dirichlet boundary condition. The situation is similar to the transition from $\Box\phi = 0$ as a hyperbolic PDE to $\Delta\phi = 0$ as an elliptical PDE if $\partial_{ct}\Phi = 0$.

I would like to emphasise that, as technically straightforward the comparison of the two line elements may seem, there is quite a lot happening from a conceptual point of view: With increasing radial distance one would expect the metric functions to approach one, but also the definition of the radial coordinate r approaches the Euclidean regime, so there is a smooth interpolation from the curved spacetime to a flat one. It is interesting that through this matching of the two models the mass gains a significance: Before, we only had two functions A and B, and the mass of the field-generating object was nowhere to be found. As long as one deals with spacetime as a curved manifold, the choice of coordinates is arbitrary and bears no physical significance. Curvature varies with changing r, but r is not an indication of distance, so one actually can not know how far away the black hole is, and neither how curvature, mass and distance are related. In the asymptotically Minkowskian spacetime, which is a vector space with normal coordinates, distance as coordinate difference has an absolute sense, so the decrease of field strength with distance is indicative of the mass.

Funnily enough, the same problem also arises in Newtonian gravity. For vacuum solutions, $\Delta\Phi = 0$ is to be solved, yielding $\Phi \propto 1/r$ in the spherically symmetric case, which is perfectly scale invariant. As the superposition principle holds here (for the Poisson-equation as a linear field equation), the scaling of Φ with M is natural, and G is there to fix the units. But nowhere there is a moment where the prefactors are determined by requiring Φ to have a specific value at a given distance, which would effective amount to a Dirichlet boundary condition.

At this point one should make clear that it'd be very wrong to say that "gravity becomes strong" at $r = r_S$, or that curvature would start to dominate, or that classical gravity would need to get replaced by relativity. As a statement involving coordinates this can not be universally true. There are certain things that the infinitely distant observer can not compute at $r = r_S$, but this is a consequence of the coordinate choice, as spacetime is curved but perfectly regular, as none of the curvature invariants diverges. Certainly one would notice an increase in e.g. the Kretschmann-scalar $K = 48r_S^2/r^6$ moving towards smaller r from the Minkowski-regime where $K = 0$, but this is a relative statement between $r \to \infty$ and finite r.

As discussed in the chapter about the equivalence principle, any freely falling frame recovers a perfectly Minkowskian spacetime with a coordinate choice making sure that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu} = 0$ locally. The amount of curvature the defines the size of this laboratory in which special relativity holds for instance by $\delta = (r_S^2/K)^{1/6}$. Even at r_S, motion of particles separated by less than δ is unaffected by curvature to first order. Of course, δ becomes less as r decreases.

F.2 *Birkhoff's theorem*

Up to this point we chose metric functions $A(r)$ and $B(r)$ of the Schwarzschild-metric to be functions of the radial coordinate only

$$ds^2 = A(t,r)\,dt^2 - B(t,r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{F.329}$$

Surprisingly, the result would have been identically the same if we had started with an allowed time-dependence $A(t, r)$ and $B(t, r)$, as spherical symmetry disallows time-dependences in vacuum. One might have noticed that up to now we used three equations from the diagonal vacuum field equation $R_{\mu\nu} = 0$ to fix just two functions A and B; please be reminded that $R_{\phi\phi} = 0$ is automatically already fulfilled by $R_{\theta\theta} = 0$. If a time-dependence of the two metric functions is introduced, the Christoffel-symbols and the Ricci-tensor become more complicated and contain time derivatives, but the vacuum field equation enforces then staticity with the additional third differential equation that is unused if one restricts A and B to be functions of r only: Somehow, the assumption of static gravitational fields is superfluous if one deals with a spherically symmetric vacuum solution.

This result is known as the Birkhoff-theorem: The fields outside spherically symmetric matter distributions need to be static and to be of the Schwarzschild-type. For instance, a radially pulsating spherically symmetric matter distribution would generate a perfectly static curved spacetime, and all that matters is the total mass M. A remainder of the Birkhoff-theorem is present in Newtonian gravity: There, the field of a spherically symmetric matter distribution was always computed as if the matter was concentrated at the central point, which arose as a peculiarity of the Poisson-equation. Many students ask at this point how a black hole can grow by accreting matter if there is no dynamical evolution of the gravitational field, which naively would be in contradiction with intuition, as a larger mass black hole should show a stronger gravitational field. Accretion and black hole growth is only possible if spherical symmetry is broken, though, even the case of accreting a spherically symmetric spherical shell of matter onto the black hole would not change the physical situation: Outside of the shell, the field stays static and corresponds to the combined masses of the shell and the black hole itself.

F.3 *conformal scaling of the Schwarzschild solution*

The Schwarzschild-geometry is a spherically symmetric vacuum solution: As such, it possesses only Weyl-curvature and no Ricci-curvature. Weyl-curvature is invariant unter conformal transformations of the metric, $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$ with a conformal factor $\Omega^2(x) > 0$, so the question naturally arises, what the physical significance of a conformally rescaled Schwarzschild solution actually might be. Conformal rescaling is present in classical gravity too, $\Delta\,\Phi = 0$ is invariant under $\Phi \to \; \Omega^2\,\Phi$, but we would rather call this mechanical similarity of a scale-free potential: Increasing the mass M can always be absorbed in Φ by going to larger distances r.

Applied to the Schwarzschild geometry, the metric transforms under conformal transformations as

$$g_{\mu\nu} \to \Omega^2(r) \cdot g_{\mu\nu} \quad \text{and consequently} \quad \mathrm{d}s^2 \to \Omega^2(r)\mathrm{d}s^2 \tag{F.330}$$

where only $\Omega^2(r)$ would respect the fundamental symmetry. Applied to the line element in Schwarzschild coordinates we get:

$$dds^2 = \left(1 - \frac{r_S}{r}\right) c \left(\Omega\,\mathrm{d}t^2\right)^2 - \frac{1}{1 - \frac{r_S}{r}}\left(\Omega\,\mathrm{d}r^2\right) - \left(\Omega r\right)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}^2\phi\right) \tag{F.331}$$

with Schwarzschild radius $r_S = \frac{2GM}{c^2}$. Absorbing the conformal factor in a redefinition of the coordinates $\left(\Omega\,\mathrm{d}r^2\right) = \mathrm{d}R$ and $\left(\Omega r\right)^2 = R$ then gives

$$ds^2 = \left(1 - \frac{\Omega r_S}{R}\right) c^2 d\tau^2 - \frac{1}{1 - \frac{\Omega r_S}{R}} dR^2 - R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{F.332}$$

if the conformal factor is constant, $R = \Omega r \rightarrow dR = \Omega$, and $d\tau = \Omega\, dt$. That is just the Schwarzschild line element for an upscale mass,

$$r_S = \Omega \cdot r_S = \frac{2G}{c^2}\left(\Omega M\right), \tag{F.333}$$

i.e. the Schwarzschild solution is invariant under conformal transforms; $M \rightarrow \Omega M$ and $r_S \rightarrow \Omega r_S$ is absorbed by $r \rightarrow \Omega r$ and $t \rightarrow \Omega t$ as coordinate choices, so we have recovered a similarity transform and the class of Schwarzschild solutions for different masses is just related by a constant stretching of the spacetime by a factor of Ω, which is perhaps a bit surprising keeping in mind that r is not the Euclidean distance but a geometrically constructed radial coordinate. Of course, in classical gravity the same result would just be the scale-invariance of the potential, $\Phi = -\frac{GM}{r}$.

Photon geodesics are invariant under conformal transforms:

$$ds^2 = g_{\mu\nu}\, k^\mu k^\nu \rightarrow \Omega^2\, g_{\mu\nu}\, k^\mu k^\nu = 0 \tag{F.334}$$

as the conformal factor Ω^2 drops out, so we can explore the causal structure of Schwarzschild spacetimes in their generality even though there is no scale invariance as in the case of a classical Newtonian gravitational field $\Phi \propto -1/r$, because conformal invariance replaces that particular concept.

Constructing radial photon geodesics for the Schwarzschild geometry sets $d\theta = 0$, $d\phi = 0$ such that the photons only propagate along r as time passes: The geodesic equation does not predict any deviation as $d^2\theta/d\lambda^2 = 0$ as well as $d^2\phi/d\lambda^2 = 0$, perfectly in agreement with intution - there should not be any acceleration in the angular directions for a radially moving photon in a spherically symmetric field. The Schwarzschild line element

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \frac{1}{1 - \frac{r_S}{r}} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) = 0 \tag{F.335}$$

with $big(d\theta^2 + \sin^2\theta\, d\phi^2\big) = 0$. Therefore, the photon visits the coordinates r as measured by the passage of time t as measured by an infinitely distant observer as

$$\frac{dr}{dt} = \pm c - \left(1 - \frac{r_S}{r}\right) \tag{F.336}$$

with + for outgoing and − for infalling photons. This can be solved for the trajectory $t(r)$,

$$\pm c\, dt = \frac{dr}{1 - \frac{r_S}{r}} \tag{F.337}$$

and integrated to give

$$\pm ct = r + r_S \cdot \ln(r - r_S) + \text{const} \quad \text{using} \quad \int \frac{dr}{1 - \frac{r_S}{r}} = r + r_S \ln(r - r_S) + \text{const} \tag{F.338}$$

At large distances $r \to \infty$, $r + r_S \ln(r - r_S)$ approaches r and the light cone becomes Minkowskian $\pm ct = r$, but at $r = r_S$, the light cone collapses as the effective speed of propagation of the photon approaches zero: It is unable to change the radial distance (into any direction!) as time (of the infinitely distant observer) passes:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm c\left(1 - \frac{r_S}{r}\right) \to 0 \tag{F.339}$$

Therefore, photons would be unable to propagate away from a source at $r = r_S$ and would certainly never reach an observer at $r > r_S$: That's the reason why black holes are black. Please keep in mind that nowhere one would need concepts of energy loss or redshifting of photons; instead, it is much clearer to think of the null-condition $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu/\mathrm{d}\lambda \mathrm{d}x^\nu/\mathrm{d}\lambda = 0$ as being generally true, and the effective speed of propagation $\mathrm{d}r/\mathrm{d}t$ being dependent on the particular coordinate choice.

F.4 *coordinate singularity at the Schwarzschild radius*

The Schwarzschild geometry has a diverging line element at $r = r_S$, but that divergence only concerns the metric and has no physical implication: Firstly, it is not present in other coordinate choices and secondly, all curvature invariants stay finite at the Schwarzschild-radius, for instance $R = 0$ for the Ricci-scalar and $K = 48r_S^2/r^6$ for the Kretschmann-scalar.

All apparent irregularities at r_S are only a consequence of the coordinate choice, and so is this curious switch between timelike and spacelike distances at r_S: The Schwarzschild line element in Schwarzschild coordinates assumes the form

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right)c^2\mathrm{d}t^2 - \frac{1}{1 - \frac{r_S}{r}}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.340}$$

where the first prefactor shows this behaviour,

$$\left(1 - \frac{r_S}{r}\right)\begin{cases} > 0 \text{ if } r > r_S \\ < 0 \text{ if } r < r_S \end{cases} \tag{F.341}$$

whereas the second prefactor shows exactly the opposite behaviour,

$$-\frac{1}{1 - \frac{r_S}{r}}\begin{cases} < 0 \text{ if } r > r_S \\ > 0 \text{ if } r < r_S \end{cases} \tag{F.342}$$

so that $c\mathrm{d}t > 0$ and $\mathrm{d}r = 0$ imply a positive $\mathrm{d}s^2$ at $r > r_S$ and a negative $\mathrm{d}s^2$ at $r < r_S$, while $\mathrm{d}r > 0$ with $c\mathrm{d}t = 0$ would cause $\mathrm{d}s^2$ to be negative at $r > r_S$ and positive at $r < r_S$, interchanging the classification of timelike and spacelike vectors.

F.5 *Painlevé-Gullstrand-coordinates*

Solving the geodesic equation for a massive particle that is initially at rest at infinity, $\mathrm{d}r/\mathrm{d}\tau = 0$ and $\mathrm{d}t/\mathrm{d}\tau = 1$ by isolating the metric from the line element,

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right)c^2\mathrm{d}t^2 - \frac{1}{1 - \frac{r_S}{r}}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.343}$$

and computing the necessary Christoffel-symbols shows that massive particles do not cross the horizon as viewed by an observer at infinity, in fact their velocity defined as the rate at which they change the radial coordinate r in terms of t approaches zero:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\mathrm{c}\left(1-\frac{r_\mathrm{S}}{r}\right)\cdot\sqrt{\frac{r_\mathrm{S}}{r}} \to 0 \quad \text{at} \quad r = r_\mathrm{S} \tag{F.344}$$

At this point, Painlevé and Gullstrand came up with this idea: As one is completely free in choosing coordinates (as long as there is an invertible and differentiable way of changing between them), one can have a non-uniform time-coordinate $\mathrm{T}(r) = t - a(r)$ with the differential $\mathrm{dT} = \mathrm{d}t - a'\,\mathrm{d}r$, prime denoting a differentiation with respect to the radial coordinate r. Using this new coordinate in the Schwarzschild line element yields

$$\mathrm{d}s^2 = \left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}^2\left(\mathrm{dT} + a'(r)\,\mathrm{d}r\right)^2 - \frac{1}{1-\frac{r_\mathrm{S}}{r}}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.345}$$

with a non-diagonal term appearing in the metric,

$$\mathrm{d}s^2 = \left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}^2\mathrm{dT}^2 + 2a'\left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}\,\mathrm{dT}\,\mathrm{d}r + \left[\left(1-\frac{r_\mathrm{S}}{r}\right)a'^2 - \frac{1}{1-\frac{r_\mathrm{S}}{r}}\right]\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.346}$$

Up to here, the function $a(r)$ was unspecified, so we might set the term in front of the $\mathrm{d}r$-differential to unity,

$$\left[\left(1-\frac{r_\mathrm{S}}{r}\right)a'^2 - \frac{1}{1-\frac{r_\mathrm{S}}{r}}\right] \equiv -1 \tag{F.347}$$

provided that the differential equation

$$a' = -\frac{1}{1-\frac{r_\mathrm{S}}{r}}\cdot\sqrt{\frac{r_\mathrm{S}}{r}} \tag{F.348}$$

has a solution. This is in fact the case as it is solved by

$$a(r) = r_\mathrm{S}\cdot\ln\left(\frac{y+1}{y-1} - 2y\right) \quad \text{with} \quad y = \sqrt{\frac{r}{r_\mathrm{S}}} \tag{F.349}$$

Then, the final form of the Schwarzschild line element in Painlevé-Gullstrand coordinates is given by:

$$\mathrm{d}s^2 = \left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}^2\,\mathrm{dT}^2 + 2a'\left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{c}\,\mathrm{dT}\,\mathrm{d}r - \mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.350}$$

which is perfectly regular at $r = r_\mathrm{S}$ and allows tracking of particles through the Schwarzschild horizon: Solving a radial geodesic for a massive particle with $\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0$ and $\frac{\mathrm{dT}}{\mathrm{d}\tau} = 1$ as initial conditions shows that the particle approaches $\frac{\mathrm{d}r}{\mathrm{dT}} = -\mathrm{c}\sqrt{\frac{r_\mathrm{S}}{r}} = -\mathrm{c}$ at $r = r_\mathrm{S}$. It is weird to look at history and see that Painlevé and Gullstrand were criticised for their coordinate construction because they ”assigned too much significance to the coordinates" when in fact they showed that coordinate choices adapted to a physical problem at hand were possible and sensible.

F.6 *propagation of fields on a curved spacetime*

Light propagation on a curved spacetime differs technically in a very important point from Minkowski spacetimes with Cartesian coordinates. There, when the wave equation was formulated in terms of partial derivatives which required to compute differentiations of a wave-type ansatz $\phi = \exp(\pm i k_\mu x^\mu)$, leading to $\partial x^\mu / \partial x^\alpha = \delta^\mu_\alpha$. The situation is very different on a manifold, where partial differentiations are replaced by covariant ones, and while $\partial_\alpha x^\mu$ remains well defined, $\nabla_\alpha x^\mu$ is a senseless operation: Covariant differentiations can only be applied to vectors and tensors (well, and scalars, $\nabla_\mu\phi = \partial_\mu\phi$), but the coordinates form only a tuple! Only transformation of infinitesimal coordinate differences $\mathrm{d}x^\mu$ is well defined in terms of a Jacobian, as $\mathrm{d}x^\mu$ has the properties of a vector, but the coordinates themselves do not have this property. In addition, even an expression like the scalar product $k_\mu x^\mu$ in $\exp(\pm i k_\mu x^\mu)$ is highly doubtful on a manifold, as it does not combine two vectors.

☠ *Please never try to apply covariant derivatives to coordinate tuples, $\nabla_\alpha x^\mu$ is not defined!*

Formulating a wave equation for a free scalar field ϕ on a manifold starts inevitably at the action integral

$$S = \int \mathrm{d}^4x\, \sqrt{-g}\, g^{\mu\nu}\, \nabla_\mu\phi\nabla_\nu\phi \tag{F.351}$$

and substitution into the Euler-Lagrange formula yields the covariant wave equation,

$$\nabla_\alpha \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \quad\rightarrow\quad g^{\mu\nu}\,\nabla_\mu\nabla_\nu\phi = 0 \tag{F.352}$$

where the d'Alembert-operator is scalar, $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. Defining the vector $\upsilon_\nu = \nabla_\nu\phi = \partial_\nu\phi$ which points into the direction of the field gradients of ϕ, gives

$$g^{\mu\nu}\,\nabla_\mu\left(\nabla_\nu\phi\right) = g^{\mu\nu}\,\nabla_\mu\upsilon_\nu = \nabla_\mu\, g^{\mu\nu}\,\upsilon_\nu = \nabla_\mu\upsilon^\mu = 0 \tag{F.353}$$

using metric compatibility, so that we can formulate the covariant divergence, with the suitable Christoffel-symbol, where two of the indices become equal.

$$\nabla_\mu\upsilon^\mu = \partial_\mu\upsilon^\mu + \Gamma^\mu{}_{\mu\alpha}\,\upsilon^\alpha \tag{F.354}$$

In particular, a Levi-Civita connection would have

$$\Gamma^\mu{}_{\mu\alpha} = \frac{g^{\mu\beta}}{2}\cdot\left[\partial_\mu\, g_{\beta\alpha} + \partial_\alpha\, g_{\mu\beta} - \partial_\beta\, g_{\mu\alpha}\right] = \frac{1}{2}\left[g^{\mu\beta}\,\partial_\mu\, g_{\beta\alpha} + g^{\mu\beta}\,\partial_\alpha\, g_{\mu\beta} - g^{\mu\beta}\,\partial_\beta\, g_{\mu\alpha}\right] \tag{F.355}$$

i.e. essentially

$$\Gamma^\mu{}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta}\,\partial_\alpha\, g_{\mu\beta} \tag{F.356}$$

There is a curious relation between the covariant divergence and the covolume $g = \det(g_{\mu\nu})$. My third most favourite formula in theoretical physics says that

$$g = \det(g_{\mu\nu}) = \exp\ln\,\det(g_{\mu\nu}) = \exp\,\mathrm{tr}\,\ln(g_{\mu\nu}) \tag{F.357}$$

relating the logarithm of the determinant with the trace of the matrix-valued logarithm, which is easily checked in the principal axis frame. Then,

$$\partial_\alpha g = g \cdot \partial_\alpha \mathrm{tr}\ \ln(g_{\mu\nu}))\, g \cdot \mathrm{tr}\, \partial_\alpha\ \ln(g_{\mu\nu})) = g \cdot \mathrm{tr}\left(g^{-1} \cdot \partial_\alpha g_{\mu\nu}\right) = g \cdot g^{\mu\nu} \cdot \partial_\alpha g_{\mu\nu} \quad \text{(F.358)}$$

using the linearity of the derivative as well as the inverse metric. With the derivative of the square root one then obtains

$$g^{\mu\nu}\, \partial_\alpha g_{\mu\nu} = \frac{1}{g}\partial_\alpha g, \quad \text{and therefore} \quad \frac{1}{2}g^{\mu\nu}\, \partial_\alpha g_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\alpha \sqrt{-g}. \quad \text{(F.359)}$$

With this result one can write for the contracted Christoffel-symbol

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\, \partial_\alpha \sqrt{-g} \quad \text{(F.360)}$$

and finally for the covariant divergence

$$\nabla_\mu \upsilon^\mu = \partial_\mu \upsilon^\mu + \Gamma^{\mu}{}_{\mu\alpha}\, \upsilon^\alpha = \partial_\mu \upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g} \cdot \upsilon^\alpha$$
$$\overset{\mu\leftrightarrow\alpha}{=} \partial_\mu \upsilon^\mu + \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} \cdot \upsilon^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, \upsilon^\mu\right) \quad \text{(F.361)}$$

using the Leibnitz-rule. With the covariant divergence, the wave equation becomes

$$g^{\mu\nu}\, \nabla_\mu\nabla_\nu\, \phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, \partial^\mu\, \phi\right) = 0 \quad \text{(F.362)}$$

which is obviously not just $\partial_\mu\partial^\mu\phi = 0$; there is clearly an influence from the background onto wave propagation, for instance from the spacetime around a black hole.

We can solve the wave equation in the eikonal or geometric optics approximation, where the changes in the geometric properties of spacetime take place on spatial scales much larger than the wavelength, and temporal changes much larger than the frequency. Again, we need to navigate through the fact that the coordinate tuple x^μ is not a vector: Writing $\Phi = \mathrm{A}\exp(\mathrm{iS}/\epsilon)$ with an amplitude A, a phase $\mathrm{S}(x^\mu)$ as a function of the coordinates and a parameter ϵ which controls the rate of phase change (which must be high in comparison to the scales on which spacetime changes its geometry) one derives by heavy application of the Leibnitz-rule

$$\partial_\nu\, \Phi = \partial_\nu \mathrm{A} \exp\left(\frac{\mathrm{iS}}{\epsilon}\right) + \frac{\mathrm{i}}{\epsilon}\mathrm{A}\exp\left(\frac{\mathrm{iS}}{\epsilon}\right)\partial_\nu\, \mathrm{S} \quad \text{(F.363)}$$

as well as for the second derivative

$$\nabla_\mu\, \partial_\nu\, \Phi = \nabla_\mu\partial_\nu \mathrm{A}\exp\left(\frac{\mathrm{iS}}{\epsilon}\right) + \frac{\mathrm{i}}{\epsilon}\partial_\nu\, \mathrm{A}\exp\left(\frac{\mathrm{iS}}{\epsilon}\right)\nabla_\mu \mathrm{S} + \frac{1}{\epsilon}\nabla_\mu\, \mathrm{A}\exp\left(\frac{\mathrm{i}}{\epsilon}\mathrm{S}\right)\partial_\nu \mathrm{S} +$$
$$\frac{\mathrm{i}}{\epsilon}\mathrm{A}\exp\left(\frac{\mathrm{iS}}{\epsilon}\right)\nabla_\mu\partial_\nu \mathrm{S} + \left(\frac{\mathrm{i}}{\epsilon}\right)^2 \mathrm{A}\exp\left(\frac{\mathrm{iS}}{\epsilon}\right)\nabla_\mu\, \mathrm{S}\partial_\nu\, \mathrm{S} \quad \text{(F.364)}$$

Sorting the terms by powers in ϵ leads to $1/\epsilon^2$ as the dominating term for high ϵ (making sure the phase changes are fast) which exactly corresponds to geometric optics,

$$g^{\mu\nu}\,\nabla_\mu\nabla_\nu\,\Phi = \left(\frac{\mathrm{i}}{\epsilon}\right)^2 \Phi g^{\mu\nu}\,\partial_\mu\,\mathrm{S}\,\partial_\nu\,\mathrm{S} = 0 \tag{F.365}$$

keeping in mind that the phase function is scalar. Defining the wave vector k^μ as the gradient in S one recovers the null-condition

$$g^{\mu\nu}\,k_\mu k_\nu = 0. \tag{F.366}$$

At order $1/\epsilon$ on obtains

$$g^{\mu\nu}\,\nabla_\mu\nabla_\nu\,\Phi = \left(\frac{\mathrm{i}}{\epsilon}\right)\left[g^{\mu\nu}\left(k_\mu\,\partial_\nu\,\mathrm{A} + \partial_\mu\,\mathrm{A}\,k_\nu\right)\exp\left(\frac{\mathrm{iS}}{\epsilon}\right) + g^{\mu\nu}\,\Phi\nabla_\mu k_\nu\right] = 0 \tag{F.367}$$

which suggest a relation how the amplitude of the wave is transported through spacetime,

$$2\,k_\mu\partial^\mu\,\mathrm{A} + \mathrm{A}\nabla_\mu\,k^\mu = 0 \tag{F.368}$$

Although we solved a wave-equation for a massless scalar field ϕ on a curved background, the essential results are applicable to the Maxwell-field A^μ as well.

F.7 *causal structure of black holes*

The Schwarzschild-geometry is the unique solution for a spherically symmetric gravitational field in vacuum, but the particular choice of Schwarzschild coordinates, motivated by the passage of time for the infinitely distant observer for t and the Euclidean scaling of surfaces with radius r is unsuited to parameterise the metric at $r_\mathrm{S} = 2\mathrm{GM}/c^2$: there exists a coordinate singularity. The issue is really only an unfortunate choice of coordinates as all curvature invariants stay finite for every finite r, and there is really only a divergence of the curvature invariants as convenient coordinate-independent quantifications of curvature at $r = 0$. Additionally, because the mass of the black hole in the Schwarzschild solution was injected into the derivation only at the stage of embedding the spacetime into an asymptotically flat Minkowski-spacetime with a weak perturbation caused by a Newtonian potential $\Phi = -\mathrm{GM}/r$, the pecularity of the Schwarzschild radius only applies to the infinitely distant observer.

Instead of using Schwarzschild coordinates (ct, r, θ, φ) one can find much better coordinates by looking at the radial motion of null-geodesics, corresponding to ingoing or outgoing light rays. That is the foundational idea of Eddington-Finkelstein coordinates, and we need two sets of coordinate as required by differential geometry: non-flat manifolds need to be covered by at least two sets of coordinate maps. Null-geodesics are of course the expression of the causal structure of spacetime, as hyberbolic differential equations cause massless fields to propagate along the light cones and restrict massive fields to propagate strictly within the light cones.

The coordinates ct, r of photons in radial motion where $\mathrm{d}\phi = \mathrm{d}\theta = 0$ fulfil the relation

$$\mathrm{c}t = -r - r_\mathrm{S}\cdot\ln(r - r_\mathrm{S}) + \mathrm{const.} \tag{F.369}$$

obtained by direct integration of the Schwarzschild line element $\mathrm{d}s^2 = 0$ for massless particles. The integration constant is defined to be a new coordinate p

$$p = \mathrm{c}t + r + r_\mathrm{S} \cdot \ln(r - r_\mathrm{S}) \tag{F.370}$$

with the differential $\mathrm{d}p$

$$\mathrm{d}p = \mathrm{c}\,\mathrm{d}t + \frac{\mathrm{d}r}{1 - \frac{r_\mathrm{S}}{r}} \tag{F.371}$$

that will be used to replace ct in the Schwarzschild line element,

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right) \mathrm{c}^2\,\mathrm{d}t^2 - \frac{1}{1 - \frac{r_\mathrm{S}}{r}} \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.372}$$

to yield

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right)\left[\mathrm{d}p^2 - 2 \cdot \frac{1}{1 - \frac{r_\mathrm{S}}{r}} \mathrm{d}p\,\mathrm{d}r + \frac{1}{(1 - \frac{r_\mathrm{S}}{r})^2} \mathrm{d}r^2\right] - \frac{1}{1 - \frac{r_\mathrm{S}}{r}} \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.373}$$

finally arriving at

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right) \mathrm{d}p^2 - 2\,\mathrm{d}p\,\mathrm{d}r - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{F.374}$$

with $\mathrm{d}s^2 = 0$ for photons. Clearly, any divergent behaviour of $\mathrm{d}s^2$ at $r = r_\mathrm{S}$ is avoided. The null-condition for radially moving photons suggests

$$\left(1 - \frac{r_\mathrm{S}}{r}\right)\left(\frac{\mathrm{d}p}{\mathrm{d}r}\right)^2 = 2\frac{\mathrm{d}p}{\mathrm{d}r} \tag{F.375}$$

with two distinct solutions: $\mathrm{d}p/\mathrm{d}r = 0$, i.e. $p = \mathrm{const}$, and

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{2}{1 - \frac{r_\mathrm{S}}{r}} \;\rightarrow\; \frac{p}{2} = r + r_\mathrm{S} \ln(r - r_\mathrm{S}) + \mathrm{const} \tag{F.376}$$

With p one can define a new time coordinate t':

$$t' \equiv p - r = \mathrm{c}t + r_\mathrm{S} \cdot \ln(r - r_\mathrm{S} \tag{F.377}$$

such that the line element reads

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right) \mathrm{c}^2\,\mathrm{d}t'^2 - 2\,\frac{r_\mathrm{S}}{r}\,\mathrm{d}t'\,\mathrm{d}r - \left(1 + \frac{r_\mathrm{S}}{r}\right) \mathrm{d}r^2 - r^2\left(\;\right) \tag{F.378}$$

It is perfectly regular for the entire Schwarzschild geometry and has for null-lines $\mathrm{d}s^2 = 0$ the two branches

$$\begin{cases} \mathrm{c}t' = -r + \mathrm{const.} \\ \mathrm{c}t' = r + 2r_\mathrm{S} \cdot \ln(r - r_\mathrm{S}) \end{cases} \tag{F.379}$$

In analogy, one can define retarded Eddington-Finkelstein coordinates q instead of advanced ones for outward moving radial photons,

$$\mathrm{d}q = \mathrm{c}\,\mathrm{d}t - \frac{\mathrm{d}r}{1-\frac{r_\mathrm{S}}{r}} \quad \text{(instead of + sign)} \tag{F.380}$$

It is important to realise that the Eddington-Finkelstein coordinates approach Minkowski-light cones at large distances from the black hole, and that they always consist of a linear branch and a nonlinear one, making the light cone tilt towards $r = r_\mathrm{S}$: After the coordinate change, this replaces the closing up of the light cones in Schwarzschild coordinates. It illustrates the geometric origin of the event horizon. For large distances, the outward travelling photon can reach even larger distances, but at the Schwarzschild radius the outward travelling photon has to stay at $r = r_\mathrm{S}$, and at smaller radii, the ”outward” travelling photon actually moves towards smaller radii.

F.8 *Kruskal-coordinates*

Kruskal-coordiantes combine retarded and advanced Eddington-Finkelstein coordinates to construct effectively Minkowskian light cones.

$$\begin{cases} \mathrm{d}p = \mathrm{c}\,\mathrm{d}t + \alpha\cdot\mathrm{d}r \;\rightarrow\; \mathrm{c}\,\mathrm{d}t = \mathrm{d}p - \alpha\,\mathrm{d}r \\ \mathrm{d}q = \mathrm{c}\,\mathrm{d}t - \alpha\,\mathrm{d}r \;\rightarrow\; \mathrm{c}\,\mathrm{d}t = \mathrm{d}q + \alpha\mathrm{d}r \end{cases} \tag{F.381}$$

with $\alpha = 1/(1-\frac{r_\mathrm{S}}{r})$. As we need $c^2\mathrm{d}t^2$ in the Schwarzschild line element, we could try out to substitute each of the two relations, each one providing one power of $c\mathrm{d}t$:

$$\mathrm{d}s^2 = \frac{1}{\alpha}\mathrm{c}^2\,\mathrm{d}^2 - \alpha\mathrm{d}r^2 = \frac{1}{\alpha}\mathrm{d}p\,\mathrm{d}q + \mathrm{d}p\,\mathrm{d}r - \mathrm{d}q\,\mathrm{d}r - \alpha\mathrm{d}r^2 - \alpha\mathrm{d}r^2 = \frac{1}{\alpha}\mathrm{d}p\,\mathrm{d}q \tag{F.382}$$

after substitution of $\mathrm{d}p\,\mathrm{d}r - \mathrm{d}q\,\mathrm{d}r = (\mathrm{d}p - \mathrm{d}q)\,\mathrm{d}r = 2\alpha\mathrm{d}r^2$, which is obtained by differencing both equations in eqn. F.381. Therefore, the Schwarzschild line element reads

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right)\mathrm{d}p\,\mathrm{d}q \tag{F.383}$$

which is obviously a modified line element for light cone coordinates, and in Minkowskian space at $r \rightarrow \infty$ one would recover $\mathrm{d}s^2 = \mathrm{d}p\mathrm{d}q$. In fact, reintroducing new spatial and temporal coordinates $(c\bar{t}, \bar{r})$ through the conversion

$$\begin{cases} \mathrm{c}\mathrm{d}\bar{t} = \frac{1}{2}(\mathrm{d}p + \mathrm{d}q) \\ \mathrm{d}\bar{r} = (\mathrm{d}p - \mathrm{d}q) \end{cases} \tag{F.384}$$

gives a line element that is even more reminiscent of Minkowski-space, again with a prefactor approaching unity as $r \rightarrow \infty$.

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right)\left[\mathrm{c}^2\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{r}^2\right] \tag{F.385}$$

At this point it might appear very surprising that one finds light cone coordinates with a conformal factor $1 - r_\mathrm{S}/r$ for a non-conformally flat spacetime with clearly present Weyl-curvature $C_{\alpha\beta\mu\nu}$! This contradiction is cleared up by realising that the argument only concerns the 2-dimensional submanifold in (ct, r), and there is no problem arising as 2-dimensional manifolds are always conformally flat as they are unable to support Weyl-curvature: The Riemann-tensor can be written in terms of the Ricci-curvature alone.

Kruskal-coordinates with effectively Minkowski-light cones only exist for radially moving photons in the (ct, r)-submanifold, which is 2-dimensional and therefore necessarily conformally flat!

Apart from that, there is a technical issue: The conformal factor $\Omega^2(r)$ is $1-\frac{r_S}{r}$ is zero at $r = R_S$, but conformal factors are supposed to be strictly positive. To reach conformal flatness we can introduce yet another coordinate transform $(p, q) \rightarrow (P, Q)$,

$$ds^2 = \left(1-\frac{r_S}{r}\right)dp\,dq = \left(1-\frac{r_S}{r}\right)\frac{dp}{dP}\frac{dq}{dQ}\,dP\,dQ \tag{F.386}$$

If the coordinate transformation can be constructed in a way that the conformal factor $1-\frac{r_S}{r}$ is absorbed into the coordinates, the line element would simply be $ds^2 = dPdQ$, and one would have reached conformal flatness in the submanifold:

$$\left(1-\frac{r_S}{r}\right)\frac{dp}{dP}\frac{dq}{dQ} \sim 1 \tag{F.387}$$

Kruskal's really bright idea was the choice

$$P = +\exp\left(\frac{P}{2r_S}\right),\; Q = -\exp\left(\frac{-q}{2r_S}\right) \;\rightarrow\; \frac{dP}{dp} = \frac{P}{2r_S},\; \frac{dQ}{dq} = +\frac{Q}{2r_S} \tag{F.388}$$

Then, the line element becomes

$$ds^2 = -\left(1-\frac{r_S}{r}\right)\cdot 4r_S^2\cdot\frac{dP}{P}\frac{dQ}{Q} = \left(1-\frac{r_S}{r}\right)4r_S^2\cdot\exp\left(-\frac{p}{2r_S}+\frac{q}{2r_S}\right)dP\,dQ \tag{F.389}$$

with the consistency condition

$$\frac{1}{2}(p-q) = r + r_S\cdot\ln(r-r_S) \tag{F.390}$$

With this coordinate transform, the line element reads

$$ds^2 = 4\left(1-\frac{r_S}{r}\right)r_S^2\cdot\exp\left(-\frac{r}{r_S}\right)\cdot\exp(-\ln(r-r_S))\,dP\,dQ \tag{F.391}$$

Further simplification with $\exp(-\ln(r-r_S)) = \frac{1}{r-r_S} = \frac{1}{r}\cdot\frac{1}{1-\frac{r_S}{r}}$ then yields the Kruskal line element

$$ds^2 = 4\exp\left(-\frac{r}{r_S}\right)\frac{r_S^3}{r}\cdot dP\,dQ \tag{F.392}$$

where one power of r_S has been added for consistency, as the line element has the unit of a squared length and the coordinates (P, Q) are dimensionless. The conformal factor is then

$$\Omega^2(r) = 4\exp\left(-\frac{r_S}{r}\right)\frac{r_S^3}{r} \tag{F.393}$$

which is strictly positive and nonsingular everywhere with the exception of $r = 0$.

F.9 *Reissner-Nordström black holes*

Clearly, the Schwarzschild black hole with its highly symmetric spacetime as a solution to the vacuum field equation $R_{\mu\nu}$ is a very attractive starting point to find solutions for the relativistic field equation, which is simplified dramatically due to the symmetries and the absence of a source. There is, perhaps a bit surprisingly, an analytic solution for the gravitational field outside of a spherically symmetric matter distribution which is electrically charged: the Reissner-Nordström black hole. In this case, spherical symmetry and staticity is maintained, but the electric field emanating from the charge distribution can propagate and its own energy content can contribute to spacetime curvature in addition to the central mass. As such, the solution is not a pure vacuum solution and possesses Ricci-curvature, sourced by the nonzero energy-momentum-tensor of the electric field, alongside the Weyl-curvature propagating away from the matter distribution.

With the same symmetry assumptions of isotropy and staticity, which suggests the line element to be of the Schwarzschild type,

$$\mathrm{d}s^2 = A(r)\, c^2\, \mathrm{d}t^2 - B(r)\, \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right) \tag{F.394}$$

with two (possibly different) metric functions $A(r)$ and $B(r)$, one does not only to solve the gravitational field equation but also the vacuum Maxwell-equation

$$g^{\alpha\mu}\, \nabla_\alpha F_{\mu\nu} = 0 \quad \leftrightarrow \quad R_{\mu\nu} = -\frac{8\pi G}{c^4}\, T_{\mu\nu}(F) \tag{F.395}$$

in a self-consistent way: The first equation provides the field $F_{\mu\nu}$ in vacuum for a given spacetime geometry and defines source $T_{\mu\nu}$, which in turn sources the Ricci-curvature $R_{\mu\nu}$ and fixes the geometry. At this point, please keep in mind that the Maxwell-field, due to the masslessness of the photon, has a vanishing trace T of the energy momentum tensor.

The scalars T and R are zero here, but that does not imply that $T_{\mu\nu}$ and $R_{\mu\nu}$ are zero!

Let's begin with the Maxwell-equation in vacuum, $g^{\alpha\mu}\, \nabla_\alpha F_{\mu\nu} = 0$ or equivalently, $\nabla_\mu F^{\mu\nu} = 0$. Writing out the covariant divergence

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^{\mu}{}_{\mu\beta}\, F^{\beta\nu} + \Gamma^{\nu}{}_{\mu\beta}\, F^{\mu\beta} \tag{F.396}$$

shows that one term drops out, as a contraction of the symmetric Christoffel-symbol $\Gamma^{\nu}{}_{\mu\beta}$ with the antisymmetric field tensor $F^{\mu\beta}$. Then, the index structure suggests that we can bring in the divergence formula for the index μ, yielding

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^{\mu}{}_{\mu\beta}\, F^{\beta\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) \tag{F.397}$$

The covolume is readily computed to be $-g = ABr^4 \cdot \sin^2\theta$, implying that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{AB}\, r^2 F^{rt}\right) = 0 \tag{F.398}$$

as all other derivatives vanish as a consequence of the assumed symmetries, with $F^{rt} = \partial^r A^t - \partial^t A^r$ being the only nonzero field component. To make things specific, we make the ansatz $A_\mu = \left(\Phi, 0, 0, 0\right)$ with the electrostatic potential Φ, where none of the entries of A_μ can depend on time. The field tensor with contravariant indices is then given by

$$F^{rt} = g^{r\mu}g^{tr}\, F_{\mu\nu} = g^{rr}g^{tt}\, F_{rt} = -\frac{E}{AB} \tag{F.399}$$

directly from $\partial_t F_{rt} = 0$ and the radial electric field $\partial_r F_{rt} = E$, along with the metric coefficients g^{rr} and g^{tt} (no summation is implied in the second last term!), finally suggesting the differential equation

$$\frac{d}{dr}\Big(\frac{r^2 E}{\sqrt{AB}}\Big) = 0, \quad \text{solved by } E(r) = \sqrt{AB}\cdot\frac{k}{r^2} \tag{F.400}$$

At infinity one recovers Minkowskian geometry, sp $A \to 1$, $B \to 1$ and k should be equal to $\frac{Q}{4\pi c}$ in Gaussian units to yield the static Coulomb-potential:

$$E(r) = \sqrt{AB}\cdot\frac{Q}{4\pi c r^2} \tag{F.401}$$

That would be the solution for the electric field for the - apart from symmetries - yet unknown background spacetime. The energy momentum tensor $T_{\mu\nu}$ of the Coulomb-field is given by

$$T_{\mu\nu} = g^{\rho\sigma}F_{\mu\rho}\cdot F_{\nu\sigma} - \frac{1}{4}g_{\mu\nu}\, F_{\rho\sigma}F^{\rho\sigma} \tag{F.402}$$

from the construction of $T_{\mu\nu}$ from the Maxwell-Lagrange density $S = \int d^4x\, \sqrt{-g}\cdot g^{\alpha\mu}g^{\beta\nu}\, F_{\alpha\beta}F_{\mu\nu}$ it comes out as naturally traceless,

$$T = g^{\mu\nu}\, T_{\mu\nu} = g^{\rho\sigma}g^{\mu\nu}\, F_{\mu\rho}F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}g^{\mu\nu}\, F_{\rho\sigma}F^{\rho\sigma} = 0 \tag{F.403}$$

with $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu = 4$.

The energy momentum-tensor $T_{\mu\nu}$ now acts as the source of the gravitational field: The vanishing trace implies that the Ricci-scalar is zero, too (that is in fact identical to the Schwarzschild case as a vacuum solution), but the Ricci-scalar is otherwise liked to the energy momentum-tensor through the field equation:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{F.404}$$

The expressions for the Ricci-tensor are identical to those in the Schwarzschild case, as the symmetries are identical:

$$R_{tt} = -4\pi G\cdot\frac{E^2}{B} \tag{F.405}$$

$$R_{rr} = +4\pi G\cdot\frac{E^2}{A} \tag{F.406}$$

$$R_{\theta\theta} = -4\pi G\cdot r^2\frac{E^2}{AB} \tag{F.407}$$

only that the right side is nonzero due to the presence of the source $T_{\mu\nu}$. Specifically, the source components read:

$$T_{tt} == g^{\rho\sigma} F_{t\rho}F_{t\sigma} - \frac{1}{4} \cdot F_{\rho\sigma}F^{\rho\sigma} = -\frac{E^2}{2B} \tag{F.408}$$

$$T_{rr} = g^{\rho\sigma} F_{r\rho}F_{r\sigma} - \frac{1}{4}g_{rr} F_{\rho\sigma}F^{\rho\sigma} = \frac{E^2}{2A} \tag{F.409}$$

$$T_{\theta\theta} = g^{\rho\sigma} F_{\theta\rho}F_{\theta\sigma} - \frac{1}{4}g_{\theta\theta} F_{\rho\sigma}F^{\rho\sigma} = -r^2\frac{E^2}{2AB} \tag{F.410}$$

using the form of the metric tensor and its inverse, the antisymmetry $F_{\mu\nu} = -F_{\nu\mu}$ of the field tensor, and the expression $F_{\rho\sigma}F^{\rho\sigma} = F_{rt}F^{rt} + F_{tr}F^{tr}$ for the trace. In both the Ricci-tensor and the energy momentum tensor, the information contained in the ϕ, ϕ-components is redundant with that in the θ, θ-component.

Proceeding with solving the field equation we obtain for the Ricci-tensor components specifically the same result as in the Schwarzschild case. Starting with B× eqn. F.406 + A× eqn. F.406 needs to vanish, from which one arrives at:

$$A'B + B'A = 0 \quad \rightarrow \quad \frac{d}{dr}(AB) = 0 \quad \rightarrow \quad AB = c^2. \tag{F.411}$$

Then, eqn. F.407 together with eqn. F.401 suggests that

$$A + rA' = c^2 \cdot \left(1 - \frac{G \cdot Q^2}{4\pi c^4} \cdot \frac{1}{r^2}\right) \tag{F.412}$$

which can be simplified using $A + rA' = \frac{d}{dr}(rA)$ to give

$$A(r) = c^2 \cdot \left[1 - \frac{2GM}{c^2 r} + \frac{GQ}{4\pi c^4 r^2}\right] \tag{F.413}$$

in analogy to the Schwarzschild case. Collecting all results and defining $q = \frac{GQ}{4\pi^2 c^4}$ yields the Reissner-Nordström line element,

$$ds^2 = \left(1 - \frac{r_S}{r} + \frac{q^2}{r^2}\right) c^2\, dt^2 - \frac{1}{1 - \frac{r_S}{r} + \frac{q^2}{r^2}} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{F.414}$$

as the spacetime geometry outside of a spherically symmetric matter and charge distribution. There are a couple of interesting observations to make: Firstly, the electric field $E(r)$ is really a Coulomb-field in Schwarzschild coordinates, as $AB = c^2$ one arrives at

$$E(r) = \frac{Q}{4\pi r^2} \tag{F.415}$$

which is perhaps not too surprising since the radial coordinate r in the Schwarzschild geometry is constructed to keep the scaling of surfaces $\propto 4\pi r^2$ fixed. Thinking of the Gauß-theorem applied to electrostatics one realises that the conservation of electric flux is made sure by diluting the field over larger and larger surfaces at increasing distance, such that the product remains constant. It is the particular construction of the Schwarzschild radial coordinate that this argument applies exactly despite curvature effects being present. Secondly, the new term proportional to q corresponds to the gravitational effect of the Coulomb field through its own energy content $\propto E^2$. Thirdly, all arguments on coordinate singularities apply likewise, as the Reissner-Nordström geometry has a finite curvature everywhere (except $r = 0$).

There is, however, a surprising result concerning the coordinate singularities: The line element implies a very interesting coordinate singularity structure:

$$ds^2 = \left(1 - \frac{r_S}{r} + \frac{q^2}{r^2}\right) c^2 \, dt^2 - \frac{1}{1 - \frac{r_S}{r} + \frac{q^2}{r^2}} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \tag{F.416}$$

shows that what matters are solutions to the quadratic equation $r^2 - r r_S + q^2 = 0$. If zeros exist, the metric function in front of the dr^2-differential can diverge. The existence of solutions of a quadratic equation is decided by the value of the discriminant, $Delta = 1 - \frac{r_S}{r} + \frac{q^2}{r^2}$. If it is negative, there are no solutions and the spacetime is regular everywhere. If it is zero, then there is a single solution of the quadratic equation and a single singularity arises. The interesting case is a positive discriminant: Then, there are two zeros and consequently, coordinate singularities at two different radii. From a physical point of view, the solution of

$$r^2 - r r_S + q^2 = 0 \quad \rightarrow \quad r_\pm = \frac{r_S \pm \sqrt{r_S^2 - 4q^2}}{2} \tag{F.417}$$

is determined by the comparison of mass and charge,

$$\begin{cases} r_S^2 > 4q^2 \text{ 2 horizons} \\ r_S^2 = 4q^2 \text{ 1 horizon at } r_S \\ r_S^2 < 4q^2 \text{ no real valued solution} \rightarrow \text{interpretation unclear} \end{cases} \tag{F.418}$$

as the Schwarzschild radius increases with mass. Interestingly, a highly charged black hole has no horizons at all. There are analogies to Eddington-Finkelstein and Kruskal-coordinates that can deal with the double horizon structure, but their construction is very technical.

F.10 *escape from a black hole*

Almost every student asks the question, after the causal structure of black holes is discussed, together with the impossible escape of photons form black hole if they are emitted inwards of $r_S = 2GM/c^2$, whether a sufficiently powerful spaceship can do that. Clearly, the spaceship is not in a state of freely falling motion but has non-gravitational accelerations acting on it. A short answer would be that light cones form the convex hull of all time-like geodesics, so the spaceship can at most travel inside the light cones, for which we have derived the causal structure, most clearly in e.g. Kruskal-coordinates.

Additionally, the causal structure is respected by electrodynamic forces: If they are added to the geodesic equation, they constitute a source term on the right hand side of the equation,

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = -\frac{q}{m} F^{\alpha\beta} u_\beta \tag{F.419}$$

with the velocity $u^\alpha = dx^\alpha/d\tau$, using proper time τ as an affine parameter, which is perfectly admissible as an affine parameter. u^α is time-like, $g_{\mu\nu}u^\mu u^\nu = c^2$, and this normalisation is conserved. Using autoparallelity, the equation of motion is rewritten as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial u^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\ \mu\nu}u^\mu u^\nu = u^\mu\left(\frac{\partial u^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\ \mu\nu}\right) = u^\mu\nabla_\mu u^\alpha = -\frac{q}{m}\mathrm{F}^{\alpha\beta}u_\beta. \tag{F.420}$$

Multiplying both sides with u^α makes the electromagnetic term vanish, as a contraction of an antisymmetric tensor $\mathrm{F}^{\alpha\beta}$ with a symmetric tensor $u_\alpha u_\beta$, while parallel transport conserves the normalisation of u^α because of metric compatibility; in a sense the two concepts are completely independent and do not interfere with each other. Explicitly,

$$u^\mu\nabla_\mu(u_\alpha u^\alpha) = u^\mu\nabla_\mu u_\alpha\cdot u^\alpha + u_\alpha u^\mu\nabla_\mu u^\alpha = u^\mu u^\alpha\left(\partial_\mu u_\alpha - \Gamma^\beta_{\ \mu\alpha}u_\beta\right) + u^\mu u_\alpha\left(\partial_\mu u^\alpha + \Gamma^\alpha_{\ \mu\beta}u^\beta\right) \tag{F.421}$$

and finally

$$u^\mu\nabla_\mu(u_\alpha u^\alpha) = u^\mu\partial_\mu\left(u_\alpha u^\alpha\right) = 0 \tag{F.422}$$

because of the normalisation $u_\alpha u^\alpha = c^2$.

G FRIEDMANN-UNIVERSES

G.1 *Friedmann-Lemaître-Robertson-Walker cosmologies*

Friedmann-Lemaître-Robertson-Walker spacetimes are highly symmetric solutions of the gravitational field equation for a particular matter distribution: Even though there is the cosmic large scale structure in the distribution of galaxies and strong inhomogeneities, fluctuations in the matter distribution are thought to subside approaching scales above a few hundred Mpc. This is summarised by the cosmological principle, which postulates that the matter and consequently the geometric properties of spacetime are homogeneous (they don't change as a function of position in the Universe) and isotropic (independent of the direction in which one observes the dynamics of spacetime). The high degree of symmetry in the matter distribution allows to find a non-vacuum solution to the gravitational field equation, and homogeneous and isotropic geometries sourced by ideal fluids constitute the class of FLRW-cosmologies. Observations of distant objects show that spacetime on these very large scales is dynamic.

Fundamental observers in a FLRW-spacetime are thought to be freely falling and are stationary with respect to their surrounding matter distribution. Their relative motion can be described by geodesic deviation, but every observer would naturally center a coordinate system on her or his position (allowed by symmetry) and perceive the properties of spacetime isotropically at every point. Is is perfectly possible that the world lines have intersected in the past (this was in fact the case!) and they might intersect in the future (which won't be the case according to our understanding). The first intersection point is called the Big Bang, and we'll come to the dynamic of congruences of geodesics at a later time.

A natural choice of the time coordinate is then the proper time τ of those observers, which need to be identical for every world line, again as a consequence of homogeneity, motivating the definition of synchronous time t. Spatial coordinates are defined to be comoving, meaning that every freely falling object stays at its respective coordinate. This defines a slicing of spacetime into spatial hypersurfaces of constant time, and a threading of spacetime in terms of world lines with a common passage of synchronous coordinate time.

The metric defines for an arbitrary set a measurable spacetime distance in form of the line element,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{c}^2\,\mathrm{d}t^2 - g_{ij}\,\mathrm{d}x^i \mathrm{d}x^j \tag{G.423}$$

The orthogonality of spacetime slices and threads suggests the separation into the temporal and spatial part of the metric. The line element measures the length of a world line which is perceived by the observer as her or his elapsed proper time, $c^2\mathrm{d}\tau^2 = \mathrm{d}s^2 = c^2\mathrm{d}t^2$ if $\mathrm{d}x^i = 0$ for comoving observers, and therefore $\tau = t$: Synchronous, physical time is measured by clocks of the fundamental observers, and elapses identically for everyone.

A particle at rest follows a world line defined by

$$x^\alpha = \begin{pmatrix} \mathrm{c}t \\ 0 \end{pmatrix} \quad \rightarrow \quad u^\alpha = \frac{\mathrm{d}}{\mathrm{d}\tau} x^\alpha = \frac{\mathrm{d}}{\mathrm{d}t} x^\alpha = \begin{pmatrix} \mathrm{c} \\ 0 \end{pmatrix} \tag{G.424}$$

and the tangent u^μ needs to fulfil the geodesic equation - otherwise the particle could not be freely falling. In fact,

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + \Gamma^\alpha_{\ \mu\nu}\; u^\mu u^\nu = 0 \tag{G.425}$$

is fulfilled because the specific form of the tangents u^μ requires just a single Christoffel-symbol,

$$\Gamma^\alpha_{\ tt} \; = \frac{g^{\alpha\beta}}{2}\left(\partial_t\, g_{t\beta} + \partial_t\, g_{\beta t} - \partial_\beta\, g_{tt}\right) = 0 \tag{G.426}$$

which is necessarily zero: The changes of u^μ vanish and the particles stay at their comoving coordinates.

Symmetry requires that the metric g_{ij} can only be a function of t. In a frame where g_{ij} is diagonal isotropy must hold, too, so all three eigenvalues must be identical:

$$\mathrm{d}s^2 = \mathrm{c}^2\,\mathrm{d}t^2 - a^2(t)\tilde{g}_{ij}\,\mathrm{d}x^i \mathrm{d}x^j \tag{G.427}$$

where $\tilde{g}_{ij}$ can be Euclidean, $\tilde{g}_{ij} = \delta_{ij}$, but it might as well be possible that the spatial submanifold has a constant (otherwise homogeneity would not hold) spatial curvature. Allowing for this case, the FLRW-line element assumes the shape

$$\mathrm{d}s^2 = \mathrm{c}^2\;\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right] \tag{G.428}$$

with the scale factor $a(t)$. The Euclidean case is recovered by $k = 0$. Spatially non-flat universes would have $k = +1$ if they are spherical with a positive curvature, and $k = -1$ if they are hyperbolical with a negative curvature.

G.2 *FLRW-cosmologies as maximally symmetric spacetimes*

FLRW-cosmologies are maximally symmetric spacetimes in what concerns the spatial part (also called a maximally symmetric 3-space), as one can write the Riemann-curvature as a function of the Ricci-scalar and the metric alone:

$$R_{\alpha\beta\mu\nu} = \frac{R}{12}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{G.429}$$

which is traced back to the fact that there is no Weyl-curvature $C_{\alpha\beta\mu\nu} = 0$ and that the Ricci-tensor comes out proportional to the the metric, $R_{\beta\nu} = R/4\ g_{\beta\nu}$, self-consistent with $g^{\beta\nu}R_{\beta\nu} = R/4\ g^{\beta\nu}g_{\beta\nu} = R/4\ \delta^{\beta}_{\beta} = R$.

The physical reason for the absence of Weyl-curvature is not only that FLRW-solutions are non-vacuum solutions, but also that there are absolutely no propagation effects of gravity, as the densities on every spatial hypersurface are constant. Absence of Weyl-curvature implies conformal flatness and Minkowski-light cones in conformal coordinates, and it is the case that the scale factor $a(\eta)$ is exactly the conformal factor $\Omega(\eta)$.

G.3 *conformal flatness of FLRW-cosmologies*

FLRW-cosmologies are systems with pure Ricci-curvature, and as their density on any spatial hypersurface is constant, the Weyl-tensor is necessarily zero: There are no propagation effects of gravity. As the Weyl-tensor vanishes, $C_{\alpha\beta\mu\nu} = 0$ the FLRW-spacetime is conformally flat and coordinates can be found where the metric can be written as

$$g_{\mu\nu} = \Omega^2(t)\eta_{\mu\nu}, \tag{G.430}$$

where the conformal factor is in this particular case only a function of time; and the suitable coordinate choice are conformal coordinates, the spatial part of which is usually called comoving. Specifically, the line element for a spatially flat FLRW-cosmology in physical time t and comoving coordinates r reads

$$ds^2 = c^2\,dt^2 - a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \tag{G.431}$$

If one defines conformal time $d\eta = \frac{dt}{a(t)} \rightarrow \eta = \int \frac{dt}{a(t)} \neq t$ one obtains a new temporal coordinate different from physical time. While the length of the world line of a particle at rest is measured in terms of proper time τ,

$$c^2 d\tau^2 = ds^2 = c^2 dt^2 \quad \rightarrow \quad \tau = t \tag{G.432}$$

such that proper time τ and coordinate time t come out equal, and must be equal everywhere due to the cosmological principle, conformal time intervals $d\eta = dt/a(t)$ have been short in the past and slow down as $a(t)$ expands, and catch up with dt today. In fact, the scale factor $a(t)$ plays the role of the conformal factor $\Omega(t)$, as in these coordinates the line element reads

$$ds^2 = a^2(t)\left[c^2\,d\tau^2 - dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \tag{G.433}$$

For radial (which can always be achieved using the cosmological principle, that allows to centre the coordinate frame on the observer such that $d\theta = d\phi = 0$ along the photon trajectory) light-like geodesics, one obtains

$$ds^2 = a^2(t)\left[c^2 dt^2 - dr^2\right] = 0 \tag{G.434}$$

and the scale factor as the conformal factor does not have any influence on light propagation, if measured in terms of comoving radial coordinate r and conformal time τ, in fact, in these coordinates one has perfectly conventional Minkowskian light cones, $c\mathrm{d}\eta = \pm\mathrm{d}r$ and from that, $c\eta = \pm r$. Whether the light cones expand to positive or negative infinity in terms of physical time instead of conformal time, depends on the relation $\mathrm{d}\eta = \mathrm{d}t/a(t)$ which might be divergent in which case a horizon appears.

G.4 *spatial curvature of FLRW-cosmologies*

Perhaps a bit surprisingly, spatial curvature $k \neq 0$ which affects the scaling of the surface of spheres with their comoving radii, does not imply deviations from conformal flatness as spacetime property: Homogeneity and isotropy as symmetries are still present, requiring the absence of Weyl-curvature, which in turn ensures conformal flatness. A general FLRW line element including spatial curvature is

$$\mathrm{d}s^2 = a^2\left[\mathrm{c}^2\mathrm{d}\eta^2 - \frac{1}{1-kr^2}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.435}$$

where $k = -1$ corresponds to negative, hyperbolic curvature, for which we define a new radial coordinate $r = \sinh\chi$ with the derivative $\mathrm{d}r/\mathrm{d}\chi = \cosh\chi$, implying

$$\mathrm{d}s^2 = a^2\left[\mathrm{c}^2\,\mathrm{d}\eta^2 - \frac{\cosh^2\chi}{1+\sinh^2\chi}\mathrm{d}\chi^2 - \sinh^2\chi\cdot\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.436}$$

with $\cosh^2\chi \equiv 1 + \sinh^2\chi$, and one finds again Minkowski light cones for radial photon geodesics, $\mathrm{d}s^2 = a^2\cdot\left[\mathrm{c}^2\,\mathrm{d}\eta^2 - \mathrm{d}\chi^2\right]$.

Similarly $k = +1$ corresponds to a spacetime with positive, spherical curvature. Definition of a new coordinate $r = \sin\chi$ with the derivative $\mathrm{d}r/\mathrm{d}\chi = \cos\chi$ then suggests for the line element,

$$\mathrm{d}s^2 = a^2\left[\mathrm{c}^2\,\mathrm{d}\eta^2 - \frac{\cos^2\chi}{1-\sin^2\chi}\mathrm{d}\chi^2 - \sin^2\chi\cdot\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{G.437}$$

with $\cos^2\chi = 1 - \sin^2\chi$. Then again, radial photon geodesics will come out as Minkowskian. It is even possible to redefine conformal coordinates as light cone coordinates,

$$\mathrm{d}u = \frac{1}{2}(\mathrm{c}\,\mathrm{d}\eta - \mathrm{d}\chi) \tag{G.438}$$

$$\mathrm{d}v = \frac{1}{2}(\mathrm{c}\,\mathrm{d}\eta + \mathrm{d}\eta) \tag{G.439}$$

where the line element would read $\mathrm{d}s^2 = 4a^2\mathrm{d}u\,\mathrm{d}v$, implying $\mathrm{d}u\mathrm{d}v = 0$ for photons.

The geometric interpretation of spatial curvature $k \neq 0$ is a non-Euclidean scaling of areas and volumes of spheres with their comoving radius (at a fixed time). Embedding spatial part of a spherically curved FLRW-spacetime into a 4d Euclidean space can be done with the transformation

$$x = R\sin\chi\,\sin\theta\,\cos\phi \tag{G.440}$$

$$y = R\sin\chi\,\sin\theta\,\sin\phi \tag{G.441}$$

$$z = R\sin\chi\,\cos\theta \tag{G.442}$$

$$w = R\cos\chi \tag{G.443}$$

with the constraint $x^2+y^2+z^2+w^2 = R^2$, defining the manifold. With this embedding, one can compute the area A of a sphere with radius R,

$$A = \int d\theta\, R\sin\chi \int d\phi R\sin\chi\,\cos\theta = 4\pi R^2\sin^2\chi \tag{G.444}$$

as well as the volume V,

$$V = \int d\chi\, R\cdot\int d\theta\, R\sin\chi\int d\phi\, R\sin\chi\cos\theta = 2\pi^2 R^3 \tag{G.445}$$

Because $\sin^2\chi \leq 1$ always, one obtains for positively curved spherical FLRW-cosmologies surfaces that are smaller than that in a Euclidean space. Repeating the exercise for hyperbolic, negatively curved cosmologies yields $A = 4\pi R\sinh^2\chi$, and systematically larger areas, as well as a divergent volume V, both as $\chi \to \infty$.

2do: redo with induced metric

G.5 *cosmological redshift*

The dynamics of the FLRW-spacetime has the effect that photons arrive at an observer redshifted lower frequency (or higher wavelength), caused by the changing geometry between emission and observation. To make the point that the lower frequency caused by the increase in scale factor is a transformation effect, we can try the following: Photon propagation is most conveniently described in conformal coordinates, where absolutely no property of the photon changes with time. What changes, however, is the definition of the scalar product that is needed to project the wave vector of the photon k^μ onto the world lines of the emitter and observer represented by the tangent u^μ, thereby defining the frequency $\omega = g_{\mu\nu}u^\mu k^\nu$. ω is a physical observable and comes out, as a scalar, independent of any coordinate choice for $g_{\mu\nu}$, u^μ and k^μ.

In conformal coordinates metric reads

$$g_{\mu\nu} = \begin{pmatrix} +a^2 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \tag{G.446}$$

such that $g_{\mu\nu} = \eta_{\mu\nu}$ at $a = 1$, i.e. today, and provides the scalar product for projecting k^μ onto u^μ.

The motion of galaxies is purely timelike along the ct- or $c\eta$-direction, and in conformal coordinates every galaxy and every observer stays at their comoving coordinate: $dr = 0$. The tangent $u^\mu = dx^\mu/d\tau = dx^\mu/dt$ is normalised to c^2, so we get:

☠ *Conformal time η is not an affine parameter, so we can't directly parameterise the world line with η. But it's perfectly permissible to take $u^\mu = dx^\mu/dt$ as a vector and do a coordinate transform switching from physical to conformal time.*

$$g_{\mu\nu}\, u^\mu u^\nu = c^2 = g_{\eta\eta}\, u^\eta u^\eta \quad \rightarrow \quad u^\eta = \frac{c}{\sqrt{g_{\eta\eta}}} = \frac{c}{a} \tag{G.447}$$

because of the motion along the $c\eta$-direction only the η-entry of u^μ is nonzero. Photons follow null geodesics, so

$$c^2\, \mathrm{d}\eta^2 - \mathrm{d}r^2 = 0 \quad \rightarrow \quad c\eta = \pm r \tag{G.448}$$

with a wave vector $k^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, using an affine parameter λ. k^μ is normalised to zero, $g_{\mu\nu}k^\mu k^\nu = 0$ and has the entries $k^\mu = (\omega/c, k)^t$. Then, the projection of the wave vector k^μ onto the tangent of the world lines of comoving systems u^μ is given by

$$\omega' = g_{\mu\nu}k^\mu u^\nu = a^2 k^\eta u^\eta = a^2 \cdot \frac{\omega}{c}\frac{c}{a} = a\omega \tag{G.449}$$

ω' is the frequency today, where $a = 1$ by convention. Reformulating the result in terms of wave length with the dispersion $\omega = ck$ (coming from $g_{\mu\nu}k^\mu k^\nu = (\omega/c)^2 - k^2 = 0$) and $k = 2\pi/\lambda$ then implies

$$\lambda' = \frac{\lambda}{a} \tag{G.450}$$

such that the redshift z is defined as

$$z = \frac{\lambda' - \lambda}{\lambda} = \frac{1}{a} - 1 \quad \rightarrow \quad a = \frac{1}{1+z}. \tag{G.451}$$

G.6 *cosmological horizons and causal structure*

The introduction of conformal coordinates brushes over the fact that depending on the cosmology photons are only given a finite time to propagate and can only reach finite physical distances, both coming from the finite past or traveling into a possibly finite future. Effectively, we ask the question whether there are limits to the light cones, which are not apparent in terms of conformal coordinates. The particle horizon is the limitation of the past light cone caused by a finite age of the Universe. The maximum distance a photon could have traveled since $a = 1$ is given by

☠ *Please keep in mind that for a vanilla model with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ this is in fact case! But arbitrary FLRW-models could realise anything.*

$$r_{\mathrm{PH}} = c\int_{t_i}^{t_0} \frac{\mathrm{d}t}{a} = c\int_{-\infty}^{0} \mathrm{d}\eta \tag{G.452}$$

where for an actual computation one needs $H = \dot{a}/a$. The origin of the conformal coordinate system in time is conveniently chosen to be $\eta = 0$ today. The event horizon is the maximum distance that light emitted today could possibly cover in the future:

$$r_{\mathrm{EH}} = c\int_{t_0}^{t_f} \frac{\mathrm{d}t}{a} = c\int_{0}^{+\infty} \mathrm{d}\eta \tag{G.453}$$

where it is clear that the behaviour of $1/a(t)$ is the decisive quantity that causes the integrals to converge or to diverge, while the $a(t)$ relation itself as a solution to the Friedmann-equation depends on all gravitating fluids and their properties ρ and w.

G.7 *Friedmann-equations*

Substituting the energy-momentum tensor $T_{\mu\nu}$ into the gravitational field equation and solving for $g_{\mu\nu}$ which in turn is needed for the motion of the fluid according to $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu}$ is a nice example of how gravity, geometry and motion work together. Our starting point is the FLRW-metric

$$ds^2 = c^2\,dt^2 - a^2(t)\cdot\left[\frac{1}{1-kr^2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \tag{G.454}$$

with the choice of using physical time t (identical to proper time τ of comoving observers) and comoving distance r as coordinates: this is referred to as the synchronous gauge, as the metric is constant on a spatial hypersurface defined through a constant value of t. We have already made the point that the FLRW-spacetime is conformally flat and has only Ricci-curvature. There is a single parameter, k, which determines the spatial geometry on a spatial hypersurface, and the only dynamic degree of freedom is the scale factor $a(t)$, which changes the distance definition on each hypersurface, moving from t to another time t'. It is a convention to set $a = 1$ today - there is a priori no particular instant in time defined singled out by the FLRW-metric, so we may bring in this human element.

Substitution of this metric into the field equation

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{G.455}$$

with a homogeneous and isotropic ideal fluid (the symmetries of the fluid need to be consistent with the symmetries of the metric, and the fluid can only be ideal as it otherwise would not obey local energy momentum conservation) yields the Friedmann-equations as dynamical equations for $a(t)$.

Turning to the energy-momentum tensor $T_{\mu\nu}$ as the source of the gravitational field and its covariant energy momentum conservation $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ (which we have already shown to be equivalent to the equations of relativistic fluid mechanics on a possibly curved background), one realises that the cosmological principle requiring a homogeneous and isotropic fluid makes sure that the Euler-equation is trivially fulfilled: There are no spatial gradients in p that would accelerate the fluid by non-gravitational forces. In fact,

$$\left(\rho + \frac{p}{c^2}\right)g^{\alpha\mu}\,u_\mu\nabla_\alpha\,u_\nu = -g^{\alpha\mu}\left(\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\right)\nabla_\alpha p \tag{G.456}$$

suggests that the relevant driving gradient in p gets projected onto a plane perpendicular to u^μ. The FLRW-symmetries disallow $\partial_i p$ in this hyperplane, but do not restrict $\partial_t p$. That component however, is in our coordinate choice perpendicular to the hyperplane, so it can not affect the motion of the fluid. From that we conclude that $g^{\alpha\mu}\,u_\mu\nabla_\alpha\,u_\nu = 0$, which is just the autoparallelity condition: The fluid elements follow geodesics.

The continuity equation, however, is not trivial and reads

$$g^{\alpha\mu}\left[\nabla_\alpha\left(\rho c^2 u_\mu\right) + p\nabla_\alpha u_\mu\right] \tag{G.457}$$

Rewriting it in terms of a divergence

$$\nabla_\mu\left(\rho c^2 u^\mu\right) + p\nabla_\mu u^\mu = 0 \tag{G.458}$$

using metric compatibility and using the divergence formula bringing in the covolume $\sqrt{-g}$

$$\partial_\mu\left(\sqrt{-g}\left(\rho c^2 u^\mu\right)\right) + p\partial_\mu\left(\sqrt{-g}u^\mu\right) = 0 \tag{G.459}$$

reduces to a considerably more simple shape using comoving coordinates: There are only derivatives ∂_t and only $u^t = c$, while $\sqrt{-g} = ca^3$:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{G.460}$$

There is an intuitive but potentially misleading reinterpretation of the continuity equation in the form of the adiabatic equation: While the mathematics is certainly correct, the physical interpretation is a bit problematic. Establishing the relation between the dp and da differentials,

$$\mathrm{d}\rho + 3\left(\rho + \frac{p}{\mathrm{c}^2}\right)\frac{\mathrm{d}a}{a} = 0 \tag{G.461}$$

and multiplying with a^3 one can use the Leibnitz-rule to write

$$\mathrm{d}\left(\rho \mathrm{c}^2 a^3\right) = -p\,\mathrm{d}(a^3) \tag{G.462}$$

which seems to suggest that the change in energy, given by the energy density multiplied with the volume a^3 is equal to the work done by changing the volume against the pressure p, reminiscent of the first law of thermodynamics. Please keep in mind, however, that pressure enters the field equations as a source of gravity and that there are no gradients in p that could perform work.

But the argument suggests a new question: Where is the limitation in the relation between pressure and energy density? Taking the trace of the field equation yields for the Ricci-scalar R

$$\mathrm{R}(t) = \frac{8\pi\mathrm{G}}{\mathrm{c}^4}\,\mathrm{T} + 4\Lambda \tag{G.463}$$

with the trace of the energy-momentum tensor T

$$\mathrm{T} = g^{\mu\nu}\cdot\left[\left(\rho + \frac{p}{\mathrm{c}^2}\right)u_\mu u_\nu - p\,g_{\mu\nu}\right] = \rho c^2 - 3p \tag{G.464}$$

such that one arrives at

$$\mathrm{R}(t) = \frac{8\pi\mathrm{G}}{c^4}\left(\rho c^2 - 3p\right) + 4\Lambda \tag{G.465}$$

which suggests that the Ricci-curvature is positive for all fluids with equation of state $w < +1/3$, and in the absence of a cosmological a fully radiation dominated Universe would have a vanishing Ricci-scalar, R = 0! Of course that is a direct consequence of the masslessness of the photon that already makes sure that T = 0, and would not imply that there is no Ricci curvature at all: The Ricci tensor would still be non-vanishing.

Then, one needs the Ricci-tensor $R_{\mu\nu}$ as well as the Ricci-scalar $R(t)$ for the field equation, following the chain $g_{\mu\nu} \rightarrow \Gamma^{\alpha}{}_{\mu\nu} \rightarrow R_{\alpha\beta\mu\nu} \rightarrow R_{\beta\nu} \rightarrow R$, for which there is really no shortcut (apart from the Cartan-formalism). It's important to realise that the Ricci-tensor comes out proportional to the metric, as required for maximally symmetric spacetimes, and therefore diagonal in our choice of coordinates,

$$R_{tt} = 3\frac{\ddot{a}}{a} \tag{G.466}$$

$$R_{rr} = \frac{-c^2}{1-kr^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2k\right) \tag{G.467}$$

$$R_{\theta\theta} = -\frac{c}{r^2}\left(a\ddot{a} + 2\dot{a}^2 + 2c^2k\right) \tag{G.468}$$

$$R_{\phi\phi} = R_{\theta\theta} \cdot \sin^2\theta \tag{G.469}$$

such that contraction $g^{\mu\nu}R_{\mu\nu} = R$ yields the Ricci-scalar,

$$R(t) = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{ck}{a^2}\right]. \tag{G.470}$$

Substitution into the gravitational field equation $G_{\mu\nu} = -8\pi G/c^4\, T_{\mu\nu} - \Lambda g_{\mu\nu}$ and separating $\dot{a}$ from $\ddot{a}$ then yields the standard form of the Friedmann-equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1+3w)\,\rho + \frac{\Lambda c^2}{3} \tag{G.471}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = +\frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{c^2k}{a^2} \tag{G.472}$$

which relate the evolution of the scale factor $a(t)$ to the presence of gravitating fluids, curvature and the cosmological constant. In parallel, covariant energy momentum conservation $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$ yields in these coordinates the adiabatic equation

☠ *Covariant energy-momentum conservation is already built into the gravitational field equation, so the adiabatic equation is not independent of the Friedmann equations.*

$$\frac{\dot{\rho}}{\rho} + 3(1+w)\frac{\dot{a}}{a} = 0, \tag{G.473}$$

from which the evolution of standard fluids with constant equation of state w can directly be read off: The equation is equivalent to $\partial_t \ln\rho = -3(1+w)\partial_t \ln a$, which is solved to be $\rho \propto a^{-3(1+w)}$, so one obtains naturally $\rho \propto a^{-3}$ for matter, $\rho \propto a^{-4}$ for radiation and a constant ρ for the cosmological constant.

For a single dominating fluid at critical density it is possible to relate the equation of state directly to the deceleration

$$q = -\frac{\ddot{a}a}{\dot{a}^2} \tag{G.474}$$

with this funky relationship:

$$3(1+w) = 2(1+q) \tag{G.475}$$

such that the following picture emerges:

$$w \qquad q \tag{G.476}$$

$$+\frac{1}{3} \quad +1 \quad \text{relativistic particles, e.g. photons} \tag{G.477}$$

$$\pm 0 \quad +\frac{1}{2} \quad \text{non-relativistic matter} \tag{G.478}$$

$$-\frac{1}{3} \quad 0 \quad \text{pure curvature, empty universe, like a fluid } w = -\frac{1}{3} \tag{G.479}$$

$$-1 \quad -1 \quad \Lambda \sim \text{like a fluid with eos } w = -1 \tag{G.480}$$

Fluids with positive equation of state have an attractive effect and slow down the expansion of the Universe, but as a increases, they get diluted: $\rho \propto a^{-3(1+w)}$ is a decreasing function for all w strictly larger than -1. Therefore any expanding Universe will work its way towards smaller values equation of state as time passes. But as soon as $w < -1/3$ something interesting happens as the deceleration changes its sign: $q > 0$ for all fluids with $w < -1/3$, such that the expansion of the Universe gets accelerated if the Universe has gotten large enough, that the densities are sufficiently small. Weirdly, an empty and therefore maximally hyperbolically curved universe, expands at a constant velocity: $q = 0$ for $w = -1/3$, and therefore $\ddot{a} = 0$, from which one integrates $\dot{a}$ to be constant and a to be a linear function in time: There is no gravity that changes the state of motion. While this may seem as an odd result, please keep in mind that in a completely empty (and therefore hyperbolic universe) there is no matter content that could by its gravitational action change the state of motion of spacetime! Or, if you prefer a fancy argument, one can invoke the Birkhoff-theorem: There is no gravitational dynamic outside a spherically symmetric matter distribution: Surely, FLRW-universes are isotropic, and because there is nothing inside, one deals with a vacuum solution, and therefore, the universe is in a state of inertial motion.

The logarithmic derivative of the scale factor as a function of time defines the Hubble-Lemaître-function

$$H(a) = \frac{\dot{a}}{a} \tag{G.481}$$

which defines the critical density ρ_{crit} as a scale. Multiplying the first Friedmann-equation G.472 with $1 = H_0^2/H_0^2$ yields

$$\left(\frac{\dot{a}}{a}\right)^2 = +\frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{c^2 k}{a^2} \tag{G.482}$$

so that we can identify

$$\rho_{\mathrm{crit}} = \frac{8\pi G}{3H_0^2} \tag{G.483}$$

with a numerical value of about 10^{-26}kg/m^3, roughly a few ten atoms per cubic metre. Equivalent to the density scale is the Hubble-length

$$\chi_H = \frac{c}{H_0} \tag{G.484}$$

roughly 10^{25}m in size. Redefining the terms in the Friedmann-equation by introducing the density parameters Ω_X

$$\Omega_\rho = \frac{\rho}{\rho_{crit}}, \quad \Omega_\Lambda = \frac{\Lambda}{3}\Big(\frac{c}{H_0}\Big)^2, \quad \Omega_k = -k\Big(\frac{c}{H_0}\Big)^2 \tag{G.485}$$

(please watch out for the minus-sign in the definition of Ω_k: negative curvature $k < 0$ has a positive Ω_k!) brings the first Friedmann-equation in the standard shape

$$H^2(t) = H_0^2 \cdot \Big[\frac{\Omega_\rho}{a^3} + \Omega_\Lambda + \frac{\Omega_k}{a^2}\Big] \tag{G.486}$$

which helps us to understand the meaning of critical density: As $H(t) = H_0$ at $a = 1$ necessarily,

$$\Omega_m + \Omega_k + \Omega_\Lambda = 1 \tag{G.487}$$

so that spatial curvature can only arise if the densities do not add up to the critical density. It seems natural that the gravitational field equation links the dynamics of the metric and therefore geometric properties of spacetime to the gravitating effect of all substances, but interestingly, we can use the field equation as well to assign properties of material substances such as ρ and p (or equivalently w) to a geometric property (curvature) or a phenomenon of gravity Λ.

G.8 *cosmological constant* Λ

The numerical value of the cosmological constant $\Lambda = 10^{-50}\text{m}^{-2}$ implies that it can only play a substantial role on scales of 10^{25}m an above, corresponding to the size c/H_0 of the observable Universe. The first Friedmann-equation shows that ultimately a continued expansion will necessarily lead to a Λ-dominated Universe,

$$\frac{\dot{a}}{a} = H_0\sqrt{\frac{\Omega_m}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\Lambda} \rightarrow H_0\sqrt{\Omega_\Lambda} \tag{G.488}$$

substantiating the idea that the cosmological fluids dominate in the order of decreasing equation of state w if the expansion is monotonic, $\dot{a} > 0$, i.e. if there is no recollapse of the Universe. Similarly, the second Friedmann-equation shows that the dynamics will be dominated by Λ because ρ is increasingly diluted, $\rho \rightarrow 0$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \frac{\Lambda c^2}{3} \tag{G.489}$$

in a way that $\ddot{a}$ becomes proportional to a as well as $\dot{a}$ as the expansion becomes exponential,

$$a(t) \sim \exp\Big(\sqrt{\frac{\Lambda c^2}{3}}\,t\Big) \tag{G.490}$$

leading to a deceleration of $q = -\ddot{a}a/\dot{a}^2 = -1$.

G.9 *size and age of FLRW-universes*

It is a funny realisation that the age of the Universe as the elapsed time between $a = 0$ and $a = 1$ can be finite or infinite, depending on the cosmological model; in fact, whether the point $a = 0$ is reached in a finite past is determined by the matter and energy content of the FLRW-cosmology, and usually high densities of matter or radiation cause that time to be finite.

To be exact, the age of the Universe would be the elapsed coordinate time (and hence the proper time) of a comoving observer, who has the right to center the coordinate frame onto herself or himself. Then, $\mathrm{d}r = 0$ and the age of the Universe corresponds to the length of the observer's world line. $H = \dot{a}/a$ implies $\mathrm{d}t = \mathrm{d}a/(aH)$ from the Hubble-Lemaître-function, and therefore

$$t = \int \mathrm{d}t = \int_0^1 \frac{\mathrm{d}a}{aH(a)} = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}a}{a\sqrt{\frac{\Omega_\gamma}{a^4} + \frac{\Omega_m}{a^3} + \Omega_\Lambda}} \tag{G.491}$$

where the inverse Hubble-Lemaître constant $1/H_0 \simeq 10^{17}$s determines the scale of the age of the Universe. While fluids with an equation of state $w > -1/3$ tend to make the integral converge, very negative equation of state parameters $w < -1/3$ will cause infinite ts. A good example is a pure Λ-dominated Universe, where the Hubble-Lemaître-function is constant. Then,

$$t = \int \mathrm{d}t = \int_0^1 \frac{\mathrm{d}a}{aH(a)} = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}a}{a} = \frac{1}{H_0}\int_0^1 \mathrm{d}\ln a \tag{G.492}$$

diverges logarithmically.

A related question is whether the Universe will exist an infinite time into the future. Coming back to the example with a Λ-dominated Universe as ours, the scale factor will increase exponentially in time, $a(t) = \exp(\sqrt{\Lambda}t)$, such that there is a finite a given at every time. The integral

$$t = \int \mathrm{d}t = \int_1^\infty \frac{\mathrm{d}a}{aH(a)} = \frac{1}{H_0}\int_1^\infty \frac{\mathrm{d}a}{a} = \frac{1}{H_0}\int_1^\infty \mathrm{d}\ln a \tag{G.493}$$

is divergent, too. In contrast, high values of the equation of state parameter w will make the integral convergent. A weird example is an empty, hyperbolically curved universe with $\Omega_k = 1$ and $w = -1/3$. Then, $t = 1/H_0 \int_0^1 \mathrm{d}a$ is exactly $1/H_0$, so the age is finite and the Universe will continue to exist into the infinite future.

G.10 *quintessence: dynamical fluids with varying w*

Up to this point, the equation of state $w = p/(\rho c^2)$ was a property of the fluid and expressed an intrinsic, unaltering property of the substance sourcing the gravitational field, for instance relativistic matter with $w = +1/3$ or nonrelativistic matter with $w = 0$. Interestingly, it was possible to map curvature as a property of spacetime onto a fluid with $w = -1/3$ or to think of the cosmological constant as a substance with $w = -1$. It is even possible to design an artificial fluid with a given Ω_X and an equation of state w_X that reproduces any expansion history $H(t)$ that one might think of, if one has the freedom to choose a function $w_X(t)$. Vice versa, it is an interesting question if one could take this one step further and not only generate any Hubble function $H(t)$ with the freedom to choose $w_X(t)$, but to set up a field that changes by interaction its gravitational properties such that it can vary its own equation of state: That is the foundational idea behind quintessence, the fifth substance after radiation, matter, curvature and the cosmological constant, substance meant here of course in a gravitational sense.

The quintessence construction starts with a scalar field ϕ which can only depend on t in accordance with the cosmological principle. ϕ can interact with itself in the sense of particle physics through the potential $V(\phi)$, a suitable Lagrange-function would be

$$\mathcal{L}(\phi, \nabla_\alpha\phi, g_{\mu\nu}) = \frac{1}{2}\, g^{\alpha\beta}\nabla_\alpha\phi\,\nabla_\beta\phi - V(\phi). \tag{G.494}$$

Apart from direct self-interaction the scalar field ϕ sources a gravitational field as it provides a nonzero energy momentum tensor, so as it evolves dynamically it does that in a varying geometry; in fact, it is best to think of the dynamical equations for ϕ and for $g_{\mu\nu}$ (or $a(t)$, which is the only degree of freedom in the metric if the FLRW-symmetries apply) as a coupled system with a joint solution.

Substitution into the corresponding Euler-Lagrange-equation for a scalar field ϕ on an arbitrary and possibly curved background yields a wave equation with a source term

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = -\frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{G.495}$$

which is effectively a Klein-Gordon-equation with a driving term. It can be interpreted as the covariant divergence of the vector $\upsilon_\alpha = \nabla_\alpha\phi = \partial_\alpha\phi$,

$$g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\,\phi = g^{\alpha\beta}\,\nabla_\alpha\,\upsilon_\beta = \nabla_\alpha\left(g^{\alpha\beta}\,\upsilon_\beta\right) = \nabla_\alpha\,\upsilon^\alpha = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\,\partial^\alpha\phi\right) \tag{G.496}$$

The covolume $\sqrt{-g}$ is quickly derived for the FLRW-metric to be $\sqrt{-g} = ca^3(t)$ and cosmological principle makes sure that there are only variations along the ct-direction, such that $\partial_\mu \to \partial_t$:

$$\frac{1}{\sqrt{-g}}\,\partial_\alpha\!\left(\sqrt{-g}\,\partial^\alpha\phi\right) = \frac{1}{a^3}\cdot\left(3a^2\,\dot{a}\,\dot{\phi} + a^3\ddot{\phi}\right) = \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} \tag{G.497}$$

such that the final Klein-Gordon-equation on a FLRW-background with scale factor $a(t)$ reads

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{G.498}$$

where we recognise the Hubble-Lemaître-function $H(t) = \dot{a}/a$. The energy-momentum tensor $T_{\mu\nu}$ as the source of the gravitational field can be derived from the Lagrange-function and is covariantly conserved as $\mathcal{L}$ does not explicitly depend on the coordinates, $g^{\alpha\mu}\nabla_\alpha T_{\mu\nu} = 0$,

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\nabla^\mu\phi}\nabla_\nu\,\phi - g_{\mu\nu}\,\mathcal{L} \tag{G.499}$$

specifically for the particular Lagrange-function,

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\,\frac{1}{2}\,g^{\alpha\beta}\,\nabla_\alpha\phi\nabla^\beta\phi + V(\phi)g_{\mu\nu} \tag{G.500}$$

which we view in terms of the energy-momentum tensor of an ideal fluid,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu - p g_{\mu\nu} \tag{G.501}$$

in order to be able to identify terms involving the field ϕ and its derivative $\dot{\phi}$ with the fluid-mechanical quantities ρ and p.

The actual entries of $T_{\mu\nu}$ can then be computed by enforcing the FLRW-symmetries, where only t-derivatives are present; we identify the tt-component with the energy density ρc^2 of an ideal fluid

$$T_{tt} = \left(\rho + \frac{p}{c^2}\right)u_t u_t - p \cdot g_{tt} = \rho c^2 \tag{G.502}$$

as $u_\mu = (c, 0)$ in the comoving frame, and the trace $g^{ij}T_{ij}$ with the pressure,

$$g^{ij}T_{ij} = \left(\rho + \frac{p}{c^2}\right)g^{ij}u_i u_j - p\,g^{ij}g_{ij} = +3a^2 p, \tag{G.503}$$

where the trace $g^{ij}g_{ij}$ only encompasses the diagonal elements of the metric and yields $-3a^2$, while the first term $g^{ij}u_i u_j$ does not contribute, as the spatial components of u_μ are zero: the fluid is at rest in the comoving frame. Comparing these two expressions with the energy-momentum tensor $T_{\mu\nu}$ of the field ϕ then yields for the tt-component

$$T_{tt} = \partial_t\phi \cdot \partial_t\phi - g_{tt} \cdot \frac{1}{2}g^{tt} \cdot \partial_t\phi\,\partial_t\phi + V(\phi)g_{tt} = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \rho c^2 \tag{G.504}$$

and correspondingly for the trace over the spatial components

$$g^{ij}T_{ij} = -g^{ij}g_{ij}\frac{1}{2}g^{tt}\partial_t\phi\partial_t\phi + g^{ij}g_{ij}V(\phi) = 3a^2\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) = 3a^2 p \tag{G.505}$$

Collecting the results yields for the equation of state w:

$$w = \frac{p}{\rho c^2} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{G.506}$$

which is an amazingly interesting result: The coupled system of the Klein-Gordon-equation and the Friedmann-equation allows a simultaneous evolution of ϕ and a. ϕ and $\dot{\phi}$ determine the energy-momentum tensor $T_{\mu\nu}$ and define the two fluid properties ρc^2 and p that enter the Friedmann-equation as source properties. The Friedmann-equation in turn provides the solution for $a(t)$ for a given fluid, and $a(t)$ enters the Klein-Gordon-equation as $\dot{a}/a$. Phenomenologically, one obtains a time-varying equation of state w from the dynamics of the field ϕ: If the field is static, $\dot{\phi} = 0$ and the equation of state w is equal to -1. In this case, ϕ mimicks a cosmological constant. But it would be natural that ϕ is accelerated by the gradient in $V(\phi)$, as determined through the Klein-Gordon-equation, so $\dot{\phi}$ increases at the expense of $V(\phi)$, and w will move away from -1 towards less negative numbers. In summary, the coupled system of differential equations for $\phi(t)$ and $a(t)$ allow the construction of a Friedmann-universe with a dynamical fluid; the freedom to choose the equation of state function $w(t)$ is mapped onto the choice of the potential $V(\phi)$ and initial conditions for ϕ. Effectively, one obtains repulsive gravity in the limit $\dot{\phi} \ll V(\phi)$, making quintessence a possible explanation of dark energy.

H WEAK FIELD GRAVITY AND GRAVITATIONAL WAVES

H.1 *weak field gravity and gravitational waves*

The gravitational field equation is a nonlinear, hyperbolic, partial differential equation. We have already encountered that the nonlinearity forbids the usage of a Green-function method for finding constructing solutions for a given $T_{\mu\nu}$, so that we can only hope to find solutions for very simple matter distributions such as the black hole solutions or the FLRW-cosmologies. While the gravitational field equation is certainly compatible with the Poisson-equation in the limit of small spacetime curvature for static matter distributions, it is a sensible question whether (i) there are gravitational effects that can be attributed to motion in the sourcing matter distribution, and (ii) the gravitational field can show dynamical behaviour on its own, in the form of wave-type propagating excitations: This would be natural for a hyperbolic PDE. If one replaces the Laplace operator $\Delta = \delta_{ij}\,\partial^i\partial^j$ in the Poisson-equation $\Delta\Phi = 4\pi G\rho$ (setting λ to zero for that instance) with the d'Alembert-operator $\Box = \eta_{\mu\nu}\,\partial^\mu\partial^\nu$ as the relativistic invariant constructed from ∂_μ, one obtains a typical wave equation $\Box\,\Phi = \partial_{ct}^2\Phi - \Delta\,\Phi = -4\pi G\rho$ with excitations travelling at the speed c away from the source ρ, irrespective of the frame: this is exactly the expression of hyperbolicity, i.e. the notion of a relativistically invariant light cone with wave-type excitations propagating along null-lines: Substitution of a plane wave $\Phi \propto \exp(\pm i k_\mu x^\mu)$ shows that $k_\mu k^\mu = 0$ and that $\omega = \pm ck$.

Incidentally (and I thank T. Baumgarte for this argument), requiring the matter distribution ρ to be homogeneous cancels the position-dependence of Φ, yielding $\partial_{ct}^2\Phi = -4\pi G\rho$, reminiscent of the second Friedmann-equation! This underlines the reasoning that depending on symmetry, black hole solutions, FLRW-solutions and wave-type solutions should naturally come out of the gravitational field equation at similar levels of symmetry (which deactivates certain derivatives), and that only in the limit of weak gravity one can expect to recover a pure wave equation.

H.2 *nonlinearities in the field equation*

The gravitational field equation is naturally nonlinear due to the construction of the Ricci-curvature from the metric. This is pictorially summarised in the schematic

$$g_{\mu\nu} \quad\rightarrow\quad \Gamma^{\alpha}{}_{\mu\nu} \quad\rightarrow\quad R_{\alpha\beta\mu\nu} \quad\rightarrow\quad R_{\beta\nu} \quad\rightarrow\quad R \tag{H.507}$$

$$g\partial g \qquad \Gamma^2 \sim (g\partial g)^2 \qquad g^2\partial g \qquad g^3\partial g \tag{H.508}$$

$$\partial\Gamma \sim \partial(g\partial g) \qquad g\partial(g\partial g) \qquad g^2\partial(g\partial g) \tag{H.509}$$

where clearly contractions between the metric and its derivatives are needed for computing the curvature. If symmetries are present, the complexity is significantly reduced because in a suitably aligned coordinate system, the partial derivative of the metric with respect to the coordinate direction in which a symmetry is present, would be zero: We have encountered this in the case of the Schwarzschild solution and the FLRW-cosmologies. Additionally, both these solutions have defined natural scales, the Schwarzschild radius $r_s = 2\mathrm{G}\mathrm{M}/c^2$ and the Hubble distance c/H_0 (or, equivalently, the critical density $\rho_\mathrm{crit} = 3\mathrm{H}_0^2/(8\pi\mathrm{G})$). In contrast, classical gravity in more than three dimensions is scale free, as the potential follows a power law, as long as effects of the cosmological constant are neglected on small scales, $\ll 1/\sqrt{\Lambda}$, reiterating the argument that the cosmological constant is a perfectly admissible feature of classical gravity.

From a conceptual point of view, we will formally and not just by analogy join weak perturbations $h_{\mu\nu}$ of the otherwise Minkowskian metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad \left|h_{\mu\nu}\right| \ll 1 \tag{H.510}$$

with the gravitational potential Φ, the gravitomagnetic field A^i and the gravitational shear h_{ij}. It should be emphasised that in this process one loses general covariance as this decomposition with weak perturbations makes statements about individual entries of $h_{\mu\nu}$, and their smallness compared to one can only be made in a preferred coordinate system. There is, however, residual Lorentz-covariance pertaining to non-accelerated frames of reference, i.e. a transformation law of the form

$$h_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta\, h_{\alpha\beta} \tag{H.511}$$

with Lorentz-transforms $\Lambda_\mu{}^\alpha$.

H.3 *gauging of the metric*

Transitions from one coordinate choice to another

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \tag{H.512}$$

where the differential function $\xi^\mu(x)$ defines the transform. The corresponding Jacobian is given by

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \frac{\partial \xi^\mu}{\partial x^\nu} + \mathcal{O}\left(\partial^2\xi\right) \tag{H.513}$$

with its inverse Jacobian

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \frac{\partial \xi^\mu}{\partial x'^\nu} + \mathcal{O}\left(\partial^2\xi\right) \tag{H.514}$$

so that

$$\frac{\partial x'^\mu}{\partial x^\nu} \cdot \frac{\partial x^\nu}{\partial x'^\beta} = \left(\delta^\mu_\nu + \frac{\partial \xi^\mu}{\partial x^\nu}\right) \cdot \left(\delta^\nu_\beta - \frac{\partial \xi^\nu}{\partial x'^\beta}\right) \simeq \underbrace{\delta^\mu_\nu\,\delta^\nu_\beta}_{\delta^\mu_\beta} - \underbrace{\delta^\mu_\nu \frac{\partial \xi^\nu}{\partial x'^\beta}}_{\frac{\partial \xi^\mu}{\partial x'^\beta}} + \underbrace{\delta^\nu_\beta \frac{\partial \xi^\mu}{\partial x^\nu}}_{\frac{\partial \xi^\mu}{\partial x^\beta}} = \delta^\mu_\beta \tag{H.515}$$

implying that we should not distinguish $\frac{\partial\xi^\mu}{\partial x^\beta}$ and $\frac{\partial\xi^\mu}{\partial x'^\beta}$ at this order. With this definition of a coordinate change, the metric transforms as

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\cdot\frac{\partial x^\beta}{\partial x'^\nu}\,g_{\alpha\beta} = \left(\delta^\alpha_\mu - \partial_\mu\xi^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu\xi^\beta\right)\left[\eta_{\alpha\beta} + h_{\alpha\beta}\right] \simeq$$
$$\eta_{\mu\nu} + h_{\mu\nu} - \delta^\alpha_\mu\,\partial_\nu\,\xi^\beta\cdot\eta_{\alpha\beta} - \partial_\mu\,\xi^\alpha\,\delta^\beta_\nu\,\eta_{\alpha\beta} = \eta_{\mu\nu} + h_{\mu\nu} - \partial_\nu\,\xi_\mu - \partial_\mu\,\xi_\nu \quad \text{(H.516)}$$

from which we isolate the transformation rule of the perturbation

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\,\xi_\nu - \partial_\nu\,\xi_\mu \quad \text{(H.517)}$$

The inverse metric obeys $g^{\mu\beta}g_{\beta\nu} = \delta^\mu_\nu$ by definition, such that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \text{(H.518)}$$

is a good enough approximation at that order and correct to $\mathcal{O}(h^2)$, with the inverse Minkowski-metric being $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\,h_{\alpha\beta}$.

$$g^{\mu\beta}g_{\beta\nu} = \left(\eta^{\mu\beta} - h^{\mu\beta}\right)\left(\eta_{\beta\nu} + h_{\beta\nu}\right) = \eta^{\mu\beta}\eta_{\beta\nu} + \eta^{\mu\beta}\cdot h_{\beta\nu} - h^{\mu\beta}\cdot\eta_{\beta\nu} + \mathcal{O}(h^2) \quad \text{(H.519)}$$

with $\eta^{\mu\beta}\eta_{\beta\nu} = \delta^\mu_\nu$, $\eta^{\mu\beta}\cdot h_{\beta\nu} = h^\mu{}_\nu = 0$ and $h^{\mu\beta}\eta_{\beta\nu} = h^\mu{}_\nu = 0$ at lowest order, $h^\mu{}_\nu = g^{\mu\alpha}h_{\alpha\nu} = \left(\eta^{\mu\alpha} - h^{\mu\alpha}\right)h_{\alpha\nu} \cong \eta^{\mu\alpha}h_{\alpha\nu}$. Effectively this implies that raising and lowering of indices is done with $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$, and that derivatives are replaced $\partial_\alpha g_{\mu\nu} = \partial_\alpha h_{\mu\nu}$ as $\eta_{\mu\nu}$ is constant in Cartesian coordinates.

H.4 *linearised gravitational field equation*

So far we have set up the metric as weak perturbation of the Minkowski-metric in Cartesian coordinates, determined the transformation properties and suitable approximations for the inverse metric. In this preferred frame with a particular coordinate choice we can continue to find a linearisation for curvature tensors, which are all ultimately computed from partial derivatives of the metric and by contractions with the metric.

The first step would be the Christoffel-symbols, where the inverse metric is replaced by the inverse Minkowski-metric,

$$\Gamma^\alpha{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu\,g_{\beta\nu} + \partial_\nu\,g_{\mu\beta} - \partial_\beta\,g_{\mu\nu}\right) \simeq$$
$$\frac{\eta^{\alpha\beta}}{2}\left(\partial_\mu\,h_{\beta\nu} + \partial_\nu\,h_{\mu\beta} - \partial_\beta\,h_{\mu\nu}\right) = \frac{1}{2}\left(\partial_\mu\,h^\alpha{}_\nu + \partial_\nu\,h_\mu{}^\alpha - \partial^\alpha\,h_{\mu\nu}\right), \quad \text{(H.520)}$$

renaming one of the indices, by writing $\partial_\alpha = \eta^{\alpha\beta}\,\partial_\beta$.

The Riemann-tensor is then derived in the limit that the dominating terms are the derivatives of the Christoffel symbols (in turn with the inverse Minkowski metric instead of the inverse actual metric), while the squared Christoffel-symbols are discarded,

$$R^\mu{}_{\alpha\beta\gamma} = \partial_\gamma\,\Gamma^\mu{}_{\alpha\beta} - \partial_\beta\,\Gamma^\mu{}_{\alpha\gamma} + \Gamma^\mu{}_{\delta\gamma}\,\Gamma^\delta{}_{\alpha\beta} - \Gamma^\mu{}_{\delta\beta}\,\Gamma^\delta{}_{\alpha\gamma} \quad \text{(H.521)}$$

Applying all simplifications then yields the final result for the Riemann-tensor,

$$R^{\mu}{}_{\alpha\beta\gamma} = \frac{1}{2}\partial_{\gamma}\left(\partial_{\alpha}\, h^{\mu}{}_{\beta} + \partial_{\beta}\, h_{\alpha}{}^{\mu} - \partial^{\mu}\, h_{\alpha\beta}\right) - \frac{1}{2}\partial_{\beta}\left(\partial_{\alpha}\, h^{\mu}{}_{\gamma} + \partial_{\gamma}\, h_{\alpha}{}^{\mu} - \partial^{\mu}\, h_{\alpha\gamma}\right) \quad \text{(H.522)}$$

so that finally one arrives at

$$R^{\mu}{}_{\alpha\beta\gamma} = \frac{1}{2}\left[\partial_{\gamma}\partial_{\alpha}\, h^{\mu}{}_{\beta} - \partial_{\beta}\partial_{\alpha}\, h^{\mu}{}_{\gamma} + \partial_{\beta}\partial^{\mu}\, h_{\alpha\gamma} - \partial_{\gamma}\partial^{\mu}\, h_{\alpha\beta}\right] \quad \text{(H.523)}$$

The contraction of the Riemann-tensor with the metric yields in a first step the Ricci-tensor, where we will use in this approximation the inverse Minkowski metric $\eta^{\mu\nu}$ as in the case of the Christoffel-symbols,

$$R_{\alpha\gamma} = \frac{1}{2}\left[\partial_{\alpha}\partial_{\gamma}\, h + \Box\, h_{\alpha\gamma} - \partial_{\gamma}\partial^{\mu}\, h_{\alpha\mu} - \partial_{\mu}\partial_{\alpha}\, h^{\mu}{}_{\gamma}\right] \quad \text{(H.524)}$$

where one can define the trace $h = h^{\mu}{}_{\mu}$ and recovers the d'Alembert-operator $\Box = \partial_{\mu}\partial^{\mu}$.

Further contraction of the Ricci-tensor with $\eta^{\mu\nu}$ gives the Ricci-scalar,

$$R = \frac{1}{2}\left[\partial^{\alpha}\partial_{\alpha}\, h + \Box\, h_{\alpha}{}^{\alpha} - \partial^{\alpha}\partial^{\mu}\, h_{\alpha\mu} - \partial^{\alpha}\partial^{\mu}\, h_{\alpha\mu}\right] \quad \text{(H.525)}$$

with a particular compact form using the trace h and the d'Alembert-operator $\Box$,

$$R = \Box\, h - \partial^{\alpha}\partial^{\mu}\, h_{\alpha\mu} \quad \text{(H.526)}$$

With these approximations, one can write down the field equation $R_{\mu\nu} - R/2\, g_{\mu\nu} = -8\pi G/c^4\, T_{\mu\nu}$ (setting $\Lambda = 0$ as it is not relevant on small scales) in the weak field limit.

By redefining the amplitude $h_{\mu\nu}$ one can reach a significant simpification: The trace referse

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} \quad \text{(H.527)}$$

has the properties

$$\bar{h} = \eta^{\mu\nu}\, h_{\mu\nu} - \frac{h}{2}\eta^{\mu\nu}\eta_{\mu\nu} = h - \frac{h}{2}\cdot 4 = -h \quad \text{(H.528)}$$

as well as

$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} + \frac{h}{2}\eta_{\mu\nu} = h_{\mu\nu} \quad \text{(H.529)}$$

such that $h_{\mu\nu}$ is given by

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{h}{2}\eta_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} \quad \text{(H.530)}$$

because $\bar{h} = -h$. Then, the linearise gravitational field equation becomes

$$\Box\,\bar{h}_{\alpha\gamma} + \eta_{\alpha\gamma}\,\partial^{\mu}\partial^{\nu}\,\bar{h}_{\mu\nu} - \partial_{\alpha}\partial^{\mu}\,\bar{h}_{\mu\nu} - \partial_{\gamma}\partial^{\mu}\,\bar{h}_{\alpha\mu} = -\frac{16\pi G}{c^4}T_{\alpha\gamma} \tag{H.531}$$

After linearising the field equation, introducing a Minkowskian background and redefining the amplitudes there is still the freedom for picking a particular gauge, where the choice of the Lorenz-gauge would naturally come to mind. The gauge choice should be able to simplify the field equation further, discarding all terms apart from $\Box\,\bar{h}_{\alpha\gamma}$.

Now introducing the Lorenz-gauge and replacing $\bar{h}$ by $\bar{h}'$

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{h'}{2}\eta_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} - \frac{\eta_{\mu\nu}}{2}\left(h - 2\eta^{\alpha\beta}\,\partial_{\alpha}\,\xi_{\beta}\right) \tag{H.532}$$

such that one arrives at:

$$\bar{h}'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi - \partial_{\nu}\xi_{\mu} - \frac{h}{2}\eta_{\mu\nu} + h_{\mu\nu}\,\eta^{\alpha\beta}\,\partial_{\alpha}\xi_{\beta} \tag{H.533}$$

Applying ∂^{ν} to the equation then gives:

$$\partial^{\nu}\,\bar{h}'_{\mu\nu} = \partial^{\nu}\,\bar{h}_{\mu\nu} - \partial^{\nu}\partial_{\mu}\xi_{\nu} - \partial^{\nu}\partial_{\mu}\xi_{\nu} - \partial^{\nu}\partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\,\partial^{\nu}\eta^{\alpha\beta}\,\partial_{\alpha}\xi^{\beta} \tag{H.534}$$

Using the definitions $\partial^{\nu}\partial_{\nu} = \Box$, as well as $\eta_{\mu\nu}\partial^{\nu}\eta^{\alpha\beta}\,\partial_{\alpha}\,\xi_{\beta} = \partial_{\mu}\partial^{\beta}\,\xi_{\beta}$ one arrives finally at

$$\partial^{\nu}\bar{h}'_{\mu\nu} = \partial^{\nu}\bar{h}_{\mu\nu} - \Box\,\xi_{\mu} \tag{H.535}$$

such that, with the gauge choice $\Box\,\xi_{\mu} = \partial^{\nu}\,\bar{h}_{\mu\nu}$ implying $\partial^{\nu}\bar{h}'_{\mu\nu} = 0$, the linearised field equation in Lorenz-gauge reads

$$\Box\,\bar{h}'_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{H.536}$$

which is is perfect agreement with the expectations: $T_{\mu\nu}$ sources perturbations in $\bar{h}'_{\mu\nu}$ in a linear, Lorentz-covariant wave equation with propagation along the light cones: gravitational waves!

H.5 *vacuum solutions of the linearised field equation*

Vacuum solutions $T_{\mu\nu} = 0$ of the linearised field equation $\Box\,\bar{h}_{\mu\nu} = 0$ with the Lorenz gauge condition $\partial^{\nu}\,\bar{h}_{\mu\nu} = 0$ very naturally call for plane wave solutions, in complete analogy to the vacuum Maxwell-equation $\partial^{\mu}F_{\mu\nu} = 0$. By substituting $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ and assuming Lorenz gauge $\partial^{\mu}A_{\mu} = 0$ one obtains $\partial^{\mu}\partial_{\mu}A_{\nu} = \Box A_{\nu} = 0$, which is likewise solved by planed waves, where the gauge condition makes sure that the vector potential A_{μ} is oriented perpendicular to the wave vector k^{μ}, justifying the expression transverse gauge.

Plane waves of the form $\bar{h}_{\mu\nu} \sim \exp(\pm i\eta^{\alpha\beta}\,k_{\alpha}x_{\beta})$ have to have a light-like wave vector $\rightarrow \eta_{\mu\nu}\,k^{\mu}k^{\nu} = 0$ such that the propagation in the gravitational field takes place along the light cone without any dispersion at all. It should be emphasised that in the limit of linearised gravity that we are dealing with there the light cone is defined by the background alone, $g_{\mu\nu}k^{\mu}k^{\nu} = 0$ which becomes in the preferred coordinate system $\eta_{\mu\nu}k^{\mu}k^{\nu} = 0$ and that there is no effect of the gravitational field of the wave back onto the propagation of the wave.

H.6 *stationary sources and gravitomagnetism*

Stationary sources are peculiar as there is no time dependence is the source and hence none in the gravitational field. As a consequence, there is no corresponding retardation in the Green-function and the perturbation to the metric $\bar{h}_{\mu\nu}$ can be computed from the source $T_{\mu\nu}$:

$$\bar{h}_{\mu\nu}(x) = -\frac{4G}{c^4} \cdot \int d^3x' \, \frac{T_{\mu\nu}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.537}$$

There is a tremendous simplification in the energy momentum tensor if taken in the non-relativistic limit, where $p \ll \rho c2$:

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p \cdot g_{\mu\nu} \simeq \rho \, u_\mu u_\nu = \begin{pmatrix} \rho \, c^2 & \rho \, c u_i \\ \rho \, c u_j & \rho u_i u_j \end{pmatrix} \tag{H.538}$$

Solving for the metric perturbations then suggests a sourcing of Φ through ρc^2

$$\Phi(\boldsymbol{x}) = -G \cdot \int d^3x' \, \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.539}$$

suggesting that $\bar{h}_{tt} = \frac{4\Phi}{c^2}$, as well as of a vectorial contribution A_i

$$A_i(\boldsymbol{x}) = -\frac{4G}{c^2} \cdot \int d^3x' \, \frac{\rho(\boldsymbol{x}') \cdot u_i(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.540}$$

from ρu_i, appearing as $\bar{h}_{it} = \bar{h}_{ti} = \frac{A_i}{c}$. In order to construct the metric we need to revert back to $h_{\mu\nu}$,

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2} \, \eta_{\mu\nu} \tag{H.541}$$

and discard contributions to $\bar{h}_{ij}$, which is valid for small velocities $\beta \ll 1$. Then, the trace is simply given by the Newtonian potential, $\bar{h} = \bar{h}_{tt} \rightarrow h_{\mu\nu} = \pm\frac{2\Phi}{c^2}$ and the full line element reads

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \, dt^2 + 2A_i \, dtx^i - \left(1 - \frac{2\Phi}{c^2}\right) dx_i dx^i. \tag{H.542}$$

When computing the Christoffel-symbols from this metric, which would be needed for e.g. the geodesic equation

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} \, u^\mu u^\nu = 0 \tag{H.543}$$

describing the motion of a test particle, one realises that the scalar, Newtonian potential Φ appears in $\Gamma^i{}_{tt}$ while the vectorial potential A_i influences $\Gamma^i{}_{tj}$ and $\Gamma^i{}_{jk}$ terms, i.e. that one needs a nonzero velocity u^i to notice them, and that those terms will be proportional to the velocity at first and second power (and as inertial accelerations those would be exactly the Coriolis acceleration and the centrifugal acceleration). Velocity-dependent accelerations in relativistic motion are very typical, and in analogy to the Lorentz-force in electrodynamics these accelerations are called gravitomagnetic accelerations.

H.7 *wave equation and Lorenz-gauge condition*

Gravitational waves are a typical consequence of the hyperbolic gravitational field equation. After a suitable linearisation procedure and after writing the amplitudes with the trace reverse, one obtains the wave equation

$$\Box\, \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \tag{H.544}$$

in the Lorenz gauge, $\partial^\mu\, \bar{h}_{\mu\nu} = 0$. The waves necessarily follow null-geodesics which illustrates why in our consideration about the most general classical theory of gravity the parameter m was set to zero: Otherwise, the wave equation would have read $\Box\Phi = m^2\Phi$ such that for the wave vector $\eta_{\mu\nu}k^\mu k^\nu = m^2 > 0$ and would therefore lie inside the light cone. In addition, propagation of wave would not be dispersion-free.

The superposition principle applies to such a linear field equation and one can introduce plane waves as fundamental Fourier-modes:

$$h_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3}\, A_{\mu\nu}(\boldsymbol{k})\, \exp\left(\pm i\eta_{\alpha\beta}\, k^\alpha x^\beta\right) \tag{H.545}$$

with amplitudes $A_{\mu\nu}(\boldsymbol{k})$, and perhaps it's worth pointing out that in the context of a flat Minkowski-background with Cartesian coordinates the tuple x^μ is indeed a vector. While the $\Box$-operator generates a perfectly normal retardation,

$$\bar{h}_{\mu\nu}(x) = -\frac{4G}{c^4} \cdot \int d^3x'\, \frac{T_{\mu\nu}(\boldsymbol{x}', ct - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} \tag{H.546}$$

captured by the Green-function, it would be unnecessary to distinguish distances from different points of the source to the observer, $|\boldsymbol{x} - \boldsymbol{x}'| \sim r$ for all $\boldsymbol{x}'$, defining the compact source approximation:

$$\bar{h}_{\mu\nu}(x) = -\frac{4G}{c^4 \cdot r} \cdot \int d^3x'\, T_{\mu\nu}(\boldsymbol{x}', ct - r) \tag{H.547}$$

with a common retardation. It should be kept in mind that gravitational waves as vacuum solutions to the field equation only exhibit Weyl-curvature and that the Birkhoff-theorem forbids spherically symmetric gravitational waves, as spherically symmetric vacuum solutions need to be static.

H.8 *plane gravitational waves in traceless transverse gauge*

The wave equation fixes the wave vector k^μ to be lightlike, $\eta_{\mu\nu}k^\mu k^\nu = \omega^2/c^2 - k^2 = 0$, so for a propagation along the z-axis of a Cartesian coordinate frame one would write $k_\mu = (k, 0, 0, -k)^t$, so that the Lorenz-gauge condition $\partial^\mu\, \bar{h}_{\mu\nu} = 0$ makes sure that the amplitudes obey $A_{\mu\nu}k^\mu = 0$ so that they are confined to the (x, y)-plane of the coordinate system. Any further gauge transformation

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{H.548}$$

defined through the gauge function $\xi_\mu(x)$ does not interfere with the Lorenz-gauge if it obeys $\Box\, \xi_\mu = 0$, because due to the condition $\Box\, \xi_\mu = \partial^\alpha \bar{h}_{\alpha\mu}$ the gauging condition is maintained.

A specific choice for the gauge function would be $\xi_\mu = \epsilon_\mu \exp(\mathrm{i}\, k_\alpha x^\alpha)$ with constant ϵ_μ, which would obviously fulfil $\Box\, \xi_\mu = 0$ as a wave, and it would have the effect to change the gravitational wave amplitude to

$$\mathrm{A}'_{\mu\nu} = \mathrm{A}_{\mu\nu} - \mathrm{i}\epsilon_\mu k_\nu - \mathrm{i}\epsilon_\nu k_\mu + \mathrm{i}\epsilon_\alpha k^\alpha \cdot \eta_{\mu\nu} \tag{H.549}$$

Effectively, the new gauge introduces coordinates that oscillate along with the gravitational wave, and the best way to visualise this would be to draw the analogy to comoving coordinates. Specifically, the amplitudes in this coordinate frame with the null-vector k^μ become

$$\mathrm{A}'_{tt} = \mathrm{A}_{tt} - \mathrm{i}k\,(\epsilon_t + \epsilon_z) \quad \mathrm{A}'_{tx} = \mathrm{A}_{tx} - \mathrm{i}k\epsilon_x \tag{H.550}$$

$$\mathrm{A}'_{xx} = \mathrm{A}_{xx} - \mathrm{i}k\,(\epsilon_t - \epsilon_z) \quad \mathrm{A}'_{ty} = \mathrm{A}_{ty} - \mathrm{i}k\epsilon_y \tag{H.551}$$

$$\mathrm{A}'_{yy} = \mathrm{A}_{yy} - \mathrm{i}k\,(\epsilon_t - \epsilon_z) \quad \mathrm{A}'_{xy} = \mathrm{A}_{xy} \tag{H.552}$$

While transversality $\mathrm{A}_{\mu\nu}\, k^\nu = 0$ fixes the relation

$$\mathrm{A}_{\mu\nu}\, k^\nu = k \cdot \mathrm{A}_{\mu t} - k \cdot \mathrm{A}_{\mu z} = k\,(\mathrm{A}_{\mu t} - \mathrm{A}_{\mu z}) = 0 \tag{H.553}$$

to $\mathrm{A}_{\mu t} = \mathrm{A}_{\mu z}$. Then, the particular choice of the constants ϵ_μ : $\mathrm{A}'_{tt} = \mathrm{A}'_{tx} = \mathrm{A}'_{ty} = 0$, $\mathrm{A}'_{xx} = -\mathrm{A}'_{yy}$ implies

$$\to \mathrm{A}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{H.554}$$

which is referred to as the traceless transverse gauge, because of $\eta^{\mu\nu}\mathrm{A}_{\mu\nu} = 0$ and $k^\mu \mathrm{A}_{\mu\nu} = 0$. The shape of the central section of the matrix pertaining to the (x, y)-plane suggests the ansatz $a(t,z)\, \sigma^{(1)} + b(t,z)\, \sigma^{(3)} \sim \bar{h}_{\mu\nu}$ illustrating that there should be two polarisation modes, with oscillatory functions $a(t)$ and $b(t)$, such that the line element assumes the form

$$\mathrm{d}s^2 = \mathrm{c}^2\, \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 - a(t,z)\left[\mathrm{d}x^2 - \mathrm{d}y^2\right] - 2\, b(t,z)\, \mathrm{d}x\mathrm{d}y \tag{H.555}$$

i.e. effectively a Minkowski line element with periodic deformations in the plane transverse to the propagation direction. In comparison, the FLRW-line element for a flat Universe is given by

$$\mathrm{d}s^2 = \mathrm{c}^2\, \mathrm{d}t^2 - a^2(t) \cdot \left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] \tag{H.556}$$

which suggests that the two functions $a(t)$ and $b(t)$ should be thought of as scale factors, relating the comoving coordinates in the (x, y)-plane (which is actually the role of the traceless transverse gauge) to physical distances. Of course, the analogy does not go further than that as the two solutions could not be more different: FLRW-universes are systems of pure Ricci curvature and the effects of Λ are important, while gravitational waves are vacuum solutions with pure Weyl-curvature.

The motion of test particles is given by the geodesic equation $\mathrm{d}u^\alpha/\mathrm{d}\tau + \Gamma^\alpha_{\ \mu\nu}\, u^\mu u^\nu = 0$, and if the particle is initially at rest, $u^\mu = (c, 0)^t$ one would obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u^\alpha = \frac{\mathrm{d}}{\mathrm{d}t}u^\alpha = -\Gamma^\alpha_{\ \mu\nu}\, u^\mu u^\nu = -c^2\,\Gamma^\alpha_{\ tt} = -\frac{c}{2}\eta^{\alpha\beta}\left[\partial_t\, h_{\beta t} + \partial_t\, h_{t\beta} - \partial_\beta\, h_{tt}\right] = 0 \quad \text{(H.557)}$$

confirming that the test particles are indeed at rest in the traceless transverse (co-moving) coordinate frame. That of course does not mean that the physical distance between the particles does not change! Physical distances, as measured for instance at $\mathrm{d}t = 0$ or along the light cone $\mathrm{d}s = 0$ oscillate as given by $a(t)$ and $b(t)$.

H.9 *Huygens' principle and elementary waves*

There is a fundamental difference in the propagation of (spherical) waves in spacetimes with different dimensionalities. A plane wave obviously obeys the wave equation, for instance for a scalar field ϕ one gets

$$\eta^{\mu\nu}\,\partial_\mu\partial_\nu\,\phi = \Box\,\phi = \left[\partial_{ct}^2 - \sum_{i=1}^{n}\partial_i^2\right]\phi = 0 \quad \text{(H.558)}$$

with a light cone condition $\eta^{\mu\nu}k_\mu k_\nu = 0$, obtained by substitution of $\phi \propto \exp(\pm \mathrm{i}k_\mu x^\mu)$. If one now asks whether a spherical wave obeys a light cone condition, too, i.e. whether the radius r of a spherical wave front is given by $r = ct$, the answer would depend on the number of dimensions that spacetime has. This is in contrast to plane waves, because in fact eqn. H.558 always reduces to a wave equation in one temporal and one spatial dimension by orienting the coordinate system in the direction of k^i.

Spherical symmetry reduces the Laplace-operator to contain only derivatives along the r-direction, such that $\delta^{ij}\,\partial_i\partial_j\,\psi = \Delta_n\,\phi = \sum_{i=1}^{n}\partial_i^2\phi = 0$, with the definition $r^2 = \delta_{ij}\,x^i x^j = x_i x^i = \sum_{i=1}^{n} x_i^2$ typical for Euclidean space. A spherically symmetry blast wave would then increase its radius r as a function of time t, such that the system is effectively 2-dimensional. But even though it would be reasonable to assume that r and t fulfil a light cone condition $ct - r = 0$ we shall see that this is only the case in $1+1$ and $3+1$ dimensions!

From the derivatives

$$\partial_i r = \frac{x_i}{r} \quad \text{and} \quad \partial_i^2 r = \frac{1}{r} - \frac{x_i^2}{r^3} = \frac{r^2 - x_i^2}{r^3} \quad \text{(H.559)}$$

we can derive that

$$\sum_i(\partial_i r)^2 = \frac{1}{r^2}\cdot\sum_i x_i^2 = 1 \quad \text{and} \quad \sum_i \partial_i^2 r = \frac{1}{r}\sum_i 1 - \frac{1}{r^3}\sum_i x_i^2 = \frac{n-1}{r} \quad \text{(H.560)}$$

such that the double derivatives ∂_i^2 in the wave equation can be written as

$$\partial_i^2\,\phi = \partial_i\,(\partial_i\, r\partial_r\,\phi) = \partial_i^2 r\cdot\partial_r\phi + (\partial_i\, r)^2\cdot\partial_r^2\phi \quad \text{(H.561)}$$

Summing over i gives the Laplace-operator needed for the wave equation, then reformulated in radial derivatives,

$$\Delta\phi = \sum_i \partial_i^2 \phi = \sum_i \partial_i^2 \phi = \frac{n-1}{r}\partial_r \phi + \partial_r^2 \phi \tag{H.562}$$

so that the wave equation for a spherical wave reads

$$\Box \phi = \partial_{ct}^2 \phi - \partial_r^2 \phi - \frac{n-1}{r}\partial_r \phi = 0 \tag{H.563}$$

with an additional term $\partial_r\phi/r$ containing a first derivative. If it was not for that term, spherical waves in any number of dimensions would behave like plane waves, which is the case $n = 1$.

The asymptotic behaviour of the wave can be isolated by setting $\psi(r) \simeq r^k \cdot \phi(r)$ with a negative exponent k, as the amplitude is expected to decrease with increasing distance. Reformulating the wave equation in terms of ψ instead of ϕ gives

$$\partial_r \psi = r^k\,\partial_r\,\phi + k\cdot r^{k-1}\phi \quad \text{and} \quad \partial_r^2 \psi = r^k\,\partial_r^2 \phi + 2\cdot k\,r^{k-1}\partial_r\,\phi + k(k-1)\cdot r^{k-2}\phi \tag{H.564}$$

arriving by division with r^k at

$$\frac{1}{r^k}\partial_r^2\,\psi = \partial_r^2\,\phi + \frac{2k}{r}\cdot\partial_r\,\phi + \frac{k(k-1)}{r^2}\phi \tag{H.565}$$

If the energy flux is proportional to the squared amplitudes ϕ^2 and if it is conserved when integrated over shells of radius r which in turn have an area $\propto r^{n-1}$ in n spatial dimensions, the amplitudes need to scale as

$$\psi(r) = r^{\frac{n-1}{2}} \cdot \phi(r) \tag{H.566}$$

suggesting that $k = (n-1)/2$. Substitution of that particular scaling then

$$\frac{1}{r^{\frac{n-1}{2}}}\cdot\partial_r^2\,\psi = \partial_r^2\phi + \frac{n-1}{r}\partial_r\,\phi + \frac{(n-1)(n-3)}{4r}\phi \tag{H.567}$$

and finally

$$\partial_{ct}^2\psi = \partial_r^2\psi - \frac{(n-1)(n-3)}{4r^2}\psi \tag{H.568}$$

which is a truly surprising result: One recovers the archetypical wave equation in 1 and 3 spatial dimensions as the last term vanishes, but there will be in general additional effects from that term in propagation problems. Spherical waves in 3 + 1 dimensions behave in every aspect as plane waves as their radius obeys a light cone condition $ct - r = 0$ as as their propagation is therefore dispersionless. In spacetimes with other dimensionality one would see through numerical computation that there is no infinitesimally thin wave front, instead the entire bubble with radius $r = ct$ is filled with nonzero amplitudes, as not all partial waves propagate at the same speed. Formally, solutions to the spherical wave can be constructed with a power series ansatz, as eqn. H.567 is a differential equation of the Bessel-type.

I GRAVITY FROM A VARIATIONAL PRINCIPLE

I.1 *variational principles for particles and fields*

Variational principles play a huge role in theoretical physics, and only in the context of relativity becomes their true nature apparent: The Lagrange-function $\mathcal{L}$ is composed of invariants, and the Euler-Lagrange-equation carrying out the variation injects coordinates and generates a covariant equation of motion. There are fundamental properties of the Lagrange-function $\mathcal{L}$, for instance its convexity which makes sure that a global minimum for the variation exists and that the Legendre transform is well-defined, ultimately yielding the Hamilton-function $\mathcal{H}$ including possible conserved conjugate momenta.

While Hamilton's principle $\delta S = 0$ is straightforward to interpret for the motion of a particle as the arc length through spacetime, an analogous interpretation for fields is a bit more involved: After all, the field equation establishes a relation between the geometry of the field and the strength of the amplitudes and the source, so the variation is effectively searching among all field configurations for the single one that minimises the action. It is a curious property that vacuum solutions provide typically a lower bound on the action, for instance in electrodynamics: The Maxwell-action S is defined through the invariant Frobenius norm of $F_{\alpha\beta}$,

$$S = \frac{1}{4}\int d^4x\sqrt{-\det\eta}\,\eta^{\alpha\mu}\,\eta^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} \tag{I.569}$$

integrated over spacetime, where we already introduced the covolume $\sqrt{-\det\eta}$ to make d^4x invariant under coordinate transforms. For vacuum solutions such as plane waves $\eta^{\alpha\mu}\,\eta^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} \propto E_iE^i - B_iB^i = 0$ because for a wave the absolute values of E^i and B^i are equal. Incidentially, the (only) other quadratic invariant $\eta^{\alpha\mu}\,\eta^{\beta\nu}\,\tilde{F}_{\alpha\beta}F_{\mu\nu} \propto E_iB^i = 0$ as well, as the electric and magnetic fields are always perpendicular. Starting with squares of first derivatives of the potentials makes sure that one obtains a linear field equation which fulfils the superposition principle and excluding higher derivatives makes sure that the Ostrogradsky-theorem is respected and the Hamilton-function bounded from below.

As Lagrange-functions only ever appear as an integral in the action and as the Hamilton-principle makes a statement only about the action, any reformulation of the Lagrange-function by integration is permissible and should yield exactly the same equations of motion. For instance, a point particle would have an equivalent action if one writes

$$S = \int dt\,\mathcal{L} = \int dt\frac{1}{2}\delta_{ij}\,\dot{x}^i\dot{x}^j - \Phi(x) = -\int dt\frac{1}{2}\delta_{ij}\,\ddot{x}^i x^j + \Phi(x) \tag{I.570}$$

if the boundary term arising in the integration by parts is neglected. But of course this form of the action calls for a generalised Euler-Lagrange equation that is capable of dealing with second derivatives $\ddot{x}^i$ of the trajectory $x^i(t)$. In fact, the variation for $\mathcal{L}(x^i, \dot{x}^i, \ddot{x}^i)$ is given by

$$\delta S = \int dt\,\frac{\partial\mathcal{L}}{\partial x^i}\delta x^i + \frac{\partial\mathcal{L}}{\partial\dot{x}^i}\delta\dot{x}^i + \frac{\partial\mathcal{L}}{\partial\ddot{x}^i}\delta\ddot{x}^i = \int dt\left(\frac{\partial\mathcal{L}}{\partial x^i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot{x}^j}\right)\delta x^j = 0 \tag{I.571}$$

with a single integration for the second and a double integration for the third term.

In fact, this new Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\mathcal{L}}{\partial\ddot{x}^j} = 0 \tag{I.572}$$

works perfectly: $\mathcal{L} = \frac{1}{2}x\ddot{x} + \Phi(x)$ has the derivatives

$$\frac{\partial\mathcal{L}}{\partial x} = \frac{\ddot{x}}{2} + \partial\Phi, \quad \frac{\partial\mathcal{L}}{\partial\dot{x}} = 0, \quad \frac{\partial\mathcal{L}}{\partial\ddot{x}} = \frac{x}{2}, \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\mathcal{L}}{\partial\ddot{x}} = \frac{\ddot{x}}{2} \tag{I.573}$$

which get assembled in the Euler-Lagrange equation to $\ddot{x}^i + \partial^i\Phi = 0$.

Almost exactly the same argument holds for a scalar field on a Euclidean background: The Lagrange-density $\mathcal{L} = 1/2\ \delta^{ij}\partial_i\Phi\partial_j\Phi - \mathrm{V}(\Phi)$ can be integrated by parts to yield the equivalent form,

$$\mathrm{S} = \int \mathrm{d}^3x\,\mathcal{L} = \int \mathrm{d}^3x\,\frac{1}{2}\delta^{ij}\,\partial_i\Phi\,\partial_j\Phi = -\int \mathrm{d}^3x\,\frac{1}{2}\Phi\delta^{ij}\,\partial_i\partial_j\Phi\cdot = -\int \mathrm{d}^3x : \frac{1}{2}\Phi\Delta\Phi \tag{I.574}$$

where again a generalised Euler-Lagrange equation is required for the variation $\delta\mathrm{S} = 0$,

$$\delta\mathrm{S} = \int \mathrm{d}^3x\,\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial\partial_i\Phi}\delta\partial_i\Phi + \frac{\partial\mathcal{L}}{\partial\partial_i\partial_j\Phi}\delta\partial_i\partial_j\Phi \tag{I.575}$$

Single and double integration by parts while neglecting the boundary terms, where the variation is zero, yields

$$\delta\mathrm{S} = \int \mathrm{d}^3x\left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\Phi} + \partial_i\partial_j\frac{\partial\mathcal{L}}{\partial\partial_i\partial_j\Phi}\right)\delta\Phi = 0 \tag{I.576}$$

from which one can read off the suitable second-order Euler-Lagrange equation. Going through the example again recovers conventional Poisson-equation $\Delta\Phi = \mathrm{dV}/\mathrm{d}\Phi = 4\pi G\rho$ for $\mathrm{V}(\phi) = 4\pi G\rho\Phi$.

Things get a bit more interesting with the Maxwell-field: The variation of the field can not be, in general, set to zero on the surface of a spacetime volume, because for instance a plane wave as a perfectly valid solution to the field equation exists for arbitrarily early and late times. But there is the freedom to pick a gauge, and in fact the surface terms can be set to zero by demanding the Lorenz-gauge $\partial^\mu A_\mu = 0$ to be valid.

$$\mathrm{S} = \int \mathrm{d}^4x\,\sqrt{-\eta}\cdot\eta^{\alpha\mu}\eta^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} = \ldots = -\int \mathrm{d}^4x\,\sqrt{-\eta}\cdot 2\cdot\eta^{\beta\nu}A_\beta\Box\,A_\nu \tag{I.577}$$

with the d'Alembert-operator $\Box = \eta^{\alpha\mu}\,\partial_\alpha\partial_\mu$. The generalised Euler-Lagrange equation needed to deal with the second-order action is

$$\delta\mathrm{S} = \int \mathrm{d}^4x\,\sqrt{-\eta}\cdot\left(\frac{\partial\mathcal{L}}{\partial A_\alpha}\delta A_\alpha + \frac{\mathcal{L}}{\partial\partial_\mu A_\alpha}\delta\partial_\mu A_\alpha + \frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu A_\alpha}\delta\partial_\mu\partial_\nu A_\alpha\right) \tag{I.578}$$

with $\delta\partial_\mu A_\alpha = \partial_\mu\delta A_\alpha$ and $\delta\partial_\mu\partial_\nu A_\alpha = \partial_\mu\partial_\nu\delta A_\alpha$. Then, integration by parts suggests

$$S = \int d^4x\,\sqrt{-\eta}\left(\frac{\partial\mathcal{L}}{\partial A_\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu A_\alpha} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu A_\alpha}\right)\delta A_\alpha = 0 \tag{I.579}$$

where again Hamilton's principle determines the Euler-Lagrange equation. Substitution yields the perfectly normal vacuum field equation for the potential A_α

$$\partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\Phi} = \Box\,A_\alpha = \eta^{\mu\nu}\,\partial_\mu\partial_\nu\,A_\alpha = 0 \tag{I.580}$$

in Lorenz-gauge. In summary, there are possible reformulations of the Lagrange-densities involving the product of the fields and its second derivative (please notice the locality here!), which give exactly the same field equation after variation. Technically, there are subtleties related to the boundary terms of the integration, which can be set to zero in certain gauges, for instance by assuming Lorenz-gauge $g^{\mu\nu}\nabla_\mu A_\nu = 0$ on the boundary for the Maxwell-field A_μ.

I.2 *variational principles on manifolds*

Would it be possible to formulate a variational principle on a manifold? Clearly yes, but we would have to use the covariant derivative ∇_μ instead of the partial derivative ∂_μ as a general metric $g_{\mu\nu}$ as a globally Cartesian coordinate choice would not be possible. Let's try this with a scalar field first: Clearly, the action should be invariant under coordinate changes with a volume element $d^4x\sqrt{-g}$, and the Lagrange-function should depend on ϕ, the covariant derivative $\nabla_\mu\phi$ and the metric $g_{\mu\nu}$ that mediates the geometry of the manifold:

$$S = \int_V d^4x\,\sqrt{-g}\cdot\mathcal{L}\big(\phi, \nabla_\mu\,\phi, g_{\mu\nu}\big), \tag{I.581}$$

consisting of generally invariant scalars built from ϕ and $\nabla_\mu\phi$. Hamilton's principle $\delta S =$ would then imply for the variation that

$$\delta S = \int_V d^4x\,\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\Psi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\delta\,\nabla_\mu\phi\right) = 0 \tag{I.582}$$

We would continue with the usual $\delta\,\nabla_\mu\phi = \nabla_\mu\delta\phi$ but reach an impasse when it comes to the integration by parts, as there is the covariant ∇_μ instead of the partial ∂_μ: We need a generalisation of the Gauß-theorem for manifolds:

$$\int_V d^4x\,\sqrt{-g}\cdot\nabla_\mu\upsilon^\mu = \int_{\partial V} dA_\mu\,\sqrt{|\gamma|}\,\upsilon^\mu \tag{I.583}$$

with the induced metric γ on the boundary ∂V,

$$\sqrt{-g}\,\big|_{\partial V} \equiv \sqrt{|\gamma|} \tag{I.584}$$

The covariant divergence can be written as a conventional partial divergence with the covolume, such that

$$\int_V d^4x\,\sqrt{-g}\,\frac{1}{\sqrt{g}}\partial_\mu\big(\sqrt{-g}\cdot v^\mu\big) = \int_V d^4x\,\partial_\mu\big(\sqrt{-g}\cdot v^\mu\big) = \int_{\partial V} dA_\mu\,\sqrt{|\gamma|}\cdot v^\mu \tag{I.585}$$

With these tools, one can write:

$$\int_V d^4x\,\sqrt{-g}\,\nabla_\mu\Big(\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi\Big) = \int_{\partial V} dA_\mu\sqrt{|\gamma|}\cdot\Big(\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\delta\phi\Big) \tag{I.586}$$

Considering

$$\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi \equiv v^\mu \tag{I.587}$$

as the vector field v^μ, the product rule suggests that

$$= \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi + \int d^4x\,\sqrt{-g}\cdot\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\nabla_\mu\,\delta\phi \tag{I.588}$$

so that finally

$$\int_V d^4x\,\sqrt{-g}\,\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\nabla_\mu\,\delta\phi = -\int_V d^4x\,\sqrt{-g}\,\nabla_\mu\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\cdot\delta\phi \tag{I.589}$$

and the Euler-Lagrange equation on a manifold has exactly the same form as the conventional one, with a ∇_μ replacing the ∂_μ,

$$\nabla_\mu\frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \tag{I.590}$$

I.3 *gauge transformations on manifolds and source terms*

Clearly, coordinate transformations and a nontrivial geometry can be dealt with as discussed in the previous chapter, but what about gauge transformations? Writing

$$\mathcal{L} \to \mathcal{L} + \nabla_\mu\,Q^\mu(\phi) \tag{I.591}$$

and having the transformation generated by Q^μ would imply that S becomes

$$S \to S + \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\,Q^\mu \tag{I.592}$$

with the variation δS

$$\delta S \to \delta S + \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\,\delta Q^\mu = \delta S + \int_V d^4x\,\sqrt{-g}\,\nabla_\mu\Big(\frac{\partial Q^\mu}{\partial\phi}\delta\phi\Big) \tag{I.593}$$

Clearly, invariance is only given if

$$\int_V \mathrm{d}^4x\,\sqrt{-g}\,\nabla_\mu\left(\frac{\partial Q}{\partial\phi}\delta\phi\right) = \int_{\partial V} \mathrm{d}A_\mu\cdot\sqrt{|\gamma|}\cdot\frac{\partial Q^\mu}{\partial\phi}\delta\phi = 0 \qquad \text{(I.594)}$$

implying that the variation of the fields $\delta\phi = 0$ is valid on the boundary ∂V.

Let's look at Maxwell electrodynamics as an intuitive example. Acting on the Lagrange-density

$$\mathcal{L} = \frac{1}{4}g^{\alpha\mu}g^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} + +g^{\alpha\beta}\,A_\alpha\,\jmath_\beta \qquad \text{(I.595)}$$

with a gauge transformation $A_\alpha \rightarrow A_\alpha + \nabla_\alpha\chi$ with a gauge function χ does not change the field tensor $F_{\alpha\beta}$: Formally it transitions to

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \;\rightarrow\; F_{\alpha\beta} + \left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\chi = F_{\alpha\beta} \qquad \text{(I.596)}$$

but the additional term is zero as a consequence of the torsion-free condition ${\Gamma^\mu}_{\alpha\beta} = {\Gamma^\mu}_{\beta\alpha}$, making $\left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\chi = \left(\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha\right)\chi = 0$. That implies that the gauge-transformed Lagrange-density becomes:

$$\mathcal{L} = \frac{1}{4}g^{\alpha\mu}g^{\beta\nu}\,F_{\alpha\beta}F_{\mu\nu} + g^{\alpha\beta}A_\alpha\jmath_\beta + g^{\alpha\beta}\,\nabla_\alpha\,\chi\cdot\jmath_\beta \qquad \text{(I.597)}$$

with $g^{\alpha\beta}\,\nabla_\alpha\,\chi\cdot\jmath_\beta$ being an additional term. This term, however, is necessarily equivalent to

$$\int_V \mathrm{d}^4x\sqrt{-g}\,g^{\alpha\beta}\nabla_\alpha\chi\cdot\jmath_\beta = \int_V \mathrm{d}^4x\sqrt{-g}\,g^{\alpha\beta}\nabla_\alpha\left[\chi\cdot\jmath_\beta\right] - \int_V \mathrm{d}^4x\sqrt{-g}\,\chi\cdot g^{\alpha\beta}\nabla_\alpha\jmath_\beta = 0 \qquad \text{(I.598)}$$

where the first term vanishes as a boundary term and the second vanishes if charge is covariantly conserved, $g^{\alpha\beta}\,\nabla_\alpha\,\jmath_\beta = 0$.

The issue does not arise in the homogeneous Maxwell-equations. There, the covariant generalisation

$$g^{\alpha\mu}\,\nabla_\alpha\,\tilde{F}_{\mu\nu} = 0 \qquad \text{(I.599)}$$

of the Bianchi identity

$$\nabla_\alpha\,F_{\mu\nu} + \nabla_\mu\,F_{\nu\alpha} + \nabla_\nu\,F_{\alpha\mu} = 0 \qquad \text{(I.600)}$$

with the dual tensor $\tilde{F}_{\alpha\beta}$ is automatically gauge-independent, as $F_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$ do not change under gauge transformations. The relation between the two are

$$\tilde{F}_{\alpha\beta} = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\,F^{\mu\nu} \quad \text{and} \quad F_{\mu\nu} = +\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\,\tilde{F}^{\alpha\beta} \qquad \text{(I.601)}$$

so that both become auto-dual, $\tilde{\tilde{F}}_{\alpha\beta} = F_{\alpha\beta}$,

$$\tilde{\tilde{F}}_{\alpha\beta} = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\tilde{F}^{\mu\nu} = -\frac{1}{4}\epsilon_{\alpha\beta\mu\nu}\epsilon^{\mu\nu\gamma\delta}F_{\gamma\delta} = F_{\alpha\beta} \quad \text{with} \quad \epsilon_{\alpha\beta\mu\nu}\epsilon^{\mu\nu\gamma\delta} = -2!2!\cdot\delta_\alpha^\gamma\,\delta_\beta^\delta \qquad \text{(I.602)}$$

I.4 *invariant volume elements*

The integration measure for volumes needs to be independent of the coordinate choice. The transformation changes vectors according to

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial x'^\mu}\,\mathrm{d}x'^\nu \tag{I.603}$$

but clearly that coordinate change implies for the volume element:

$$\mathrm{d}^n x = \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\mathrm{d}^n x' \tag{I.604}$$

with the functional determinant as a prefactor. At the same time, the metric transforms like a rank-2 tensor,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\,\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta = g'_{\alpha\beta}\,\mathrm{d}x'^\alpha \mathrm{d}x'^\beta \tag{I.605}$$

i.e. inverse to the vector and with two powers of the Jacobian for the determinant of the metric (as the line element is invariant):

$$\det(g'_{\alpha\beta}) = \det(g_{\mu\nu})\cdot\left(\det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\right)^2 \tag{I.606}$$

implying the definition of an invariant volume element as

$$\mathrm{d}^n x\,\sqrt{-g} \to \mathrm{d}^n x'\cdot\det \mathrm{J}\cdot\frac{\sqrt{-g'}}{\sqrt{(\det \mathrm{J})^2}} = \mathrm{d}^n x'\sqrt{-g'} \tag{I.607}$$

with the functional determinant $\mathrm{J} = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)$.

It is important to realise that $\sqrt{-g}$ is a density, not a scalar, as $\mathrm{d}^n x\sqrt{-g}$ is scalar and therefore invariant under coordinate transformations. In particular,

$$\nabla_\mu\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\nabla_\mu g = \frac{1}{2\sqrt{-g}}\nabla_\mu\Big(\exp\operatorname{tr}\ln g_{\alpha\beta}\Big) = \\ \frac{-1}{2\sqrt{-g}}\,g\cdot\operatorname{tr} g^{-1}\cdot\nabla_\mu\,g = \frac{1}{2}\sqrt{-g}\cdot g^{\alpha\beta}\,\nabla_\mu\,g_{\alpha\beta} = 0 \tag{I.608}$$

as a consequence of metric compatibility of ∇_μ; but it would be wrong to write $\nabla_\mu\sqrt{-g} \neq \partial_\mu\sqrt{-g}$ because of the missing scalar property of $\sqrt{-g}$: The covolume is a density rather than a scalar, and the reduction of the covariant derivative $\nabla_\mu\phi = \partial_\mu\phi$ for scalar fields is not applicable for $\sqrt{-g}$.

I.5 *Einstein-Hilbert: gravity from a variational principle*

Up to this point we postulated the gravitational field equation and convinced ourselves that it had properties desirable in a field equation. A variational principle would require the construction of an action for the metric

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}\Big(g_{\alpha\beta}, \nabla_\mu\, g_{\alpha\beta},\ \nabla_\mu\nabla_\nu g_{\alpha\beta}\Big) \tag{I.609}$$

composed of invariants such that after variation a covariant field equation is obtained. The Lagrange-density can in principle depend on the metric $g_{\alpha\beta}$ as the dynamical field itself and its first and second derivatives. There, $\nabla_\mu\, g_{\alpha\beta}$ is impossible to use as it always vanishes due to metric compatibility, so $\partial_\mu\, g_{\alpha\beta}$ or $\Gamma^\mu_{\alpha\beta}$ would be better alternatives, but we have already argued that the gravitational field should rather be contained in the second than the first derivatives of the metric: According to the equivalence principle, first derivatives would automatically be zero in a freely falling frame.

As invariants containing second derivatives, the Ricci-scalar $R = g^{\alpha\mu}g^{\beta\nu}\, R_{\alpha\beta\mu\nu}$ or the Kretschmann-scalar $K = g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}\, R_{\alpha\beta\gamma\delta}R_{\mu\nu\rho\sigma}$ would be possible choices, although we would prefer R from the intuition on the contraction of freely falling clouds of point particles caused by Ricci-curvature. Perhaps a bit surprisingly, a straightforward constant Λ would be fine, too.

Postulating the Einstein-Hilbert-Lagrange density as being the simplest, local invariant second-order action

$$S = \int d^4x \sqrt{-g}\,\Big(R - 2\Lambda\Big) \tag{I.610}$$

one can in fact derive the gravitational field equation through variation of the metric, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. In the Ricci-scalar $R = g^{\mu\nu}R_{\mu\nu}$, however, there is the inverse metric $g^{\mu\nu}$ so actually one needs to vary with respect to that quantity, too. The two variations are related by

$$\delta\left(\delta^\mu_\nu\right) = \delta\left(g^{\mu\alpha}g_{\alpha\nu}\right) = \delta g^{\mu\alpha}\cdot g_{\alpha\nu} + g^{\mu\alpha}\cdot\delta g_{\alpha\nu} = 0 \tag{I.611}$$

where one can isolate $\delta^{\mu\alpha}$ by contraction with $g^{\nu\beta}$,

$$\delta g^{\mu\alpha}\, g_{\alpha\nu}\cdot g^{\nu\beta} = \delta g^{\mu\beta} = -g^{\nu\beta}g^{\mu\alpha}\,\delta g_{\alpha\nu} \tag{I.612}$$

with an additional minus-sign appearing.

Let's ignore the cosmological constant for a second, $\Lambda = 0$. Then, the variation δS of S becomes

$$\delta S = \int d^4x \left[\delta\sqrt{-g}\cdot R + \sqrt{-g}\cdot\delta g^{\mu\nu}\cdot R_{\mu\nu} + \sqrt{-g}\cdot g^{\mu\nu}\,\delta R_{\mu\nu}\right] \tag{I.613}$$

which requires a relation between $\delta\sqrt{-g}$ and $\delta g^{\mu\nu}$ as well as between $\delta R_{\mu\nu}$ and $\delta g^{\mu\nu}$, while the second term is already in good shape, being directly proportional to $\delta g^{\mu\nu}$. The variation of the covolume is done by

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\cdot\delta g = \frac{1}{2\sqrt{-g}}\cdot g\cdot g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\cdot g_{\mu\nu}\,\delta g^{\mu\nu} \tag{I.614}$$

keeping in mind that $\ln g = \ln\det g_{\mu\nu} = \mathrm{tr}\ln g_{\mu\nu}$, such that $g = \exp\mathrm{tr}\ln g_{\mu\nu}$. δ acts like a derivative, so that g is reproduced as the derivative of the exponential, the trace is linear and the derivative of a matrix valued logarithm is given by the matrix inverse, multiplied with the internal derivative. Switching from $\delta g_{\mu\nu}$ to $\delta g^{\mu\nu}$ then introduces yet another minus sign. Collecting all results so far gives the intermediate formula

$$\delta S = \int d^4x \sqrt{-g} \cdot \left(R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}\right) \delta g^{\mu\nu} + \int d^4x \sqrt{-g}\, \delta R_{\mu\nu}\, g^{\mu\nu} \to 0 \tag{I.615}$$

which is already very reminiscent of the field equation if Hamilton's principle $\delta S = 0$ is assumed, if only the last term was zero.

For continuing one needs the Palatini-identity, which relates the variation of the Ricci-tensor to covariant derivatives of the varied Christoffel-symbols, which I guess merits a few words of explanation. The Riemann-curvature $R_{\alpha\beta\mu\nu}$ is in general a function of $\Gamma^{\alpha}{}_{\mu\nu}$ and its derivatives $\partial_\beta \Gamma^{\alpha}{}_{\mu\nu}$, as suggested by parallel transport. In locally Cartesian coordinates $\Gamma^{\alpha}{}_{\mu\nu} = 0$ as in these coordinates partial derivatives of the metric vanish, but $\partial_\beta \Gamma^{\alpha}{}_{\mu\nu}$ are not necessarily zero. That implies that the Riemann-curvature only depends on the derivatives of the Christoffel-symbols but not on the squares. Secondly, the variation $\delta\Gamma^{\alpha}{}_{\mu\nu}$ of the Christoffel-symbols is a tensor, as the non-tensorial contributions drop out. And thirdly, $\nabla_\mu = \delta_\mu$ in locally Cartesian coordinates, as $\Gamma^{\alpha}{}_{\mu\nu} = 0$.

Putting everything together lets us write for the Riemann-tensor

$$\delta R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu\, \delta\Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu\, \delta\Gamma^{\alpha}{}_{\beta\mu} = \nabla_\mu\, \delta\Gamma^{\alpha}{}_{\beta\nu} - \nabla\, \delta\Gamma^{\alpha}{}_{\beta\mu} \tag{I.616}$$

and consequently for the Ricci-tensor

$$\delta R_{\beta\nu} = \nabla_\mu\, \delta\Gamma^{\mu}{}_{\beta\nu} - \nabla_\nu\, \delta\Gamma^{\mu}{}_{\beta\mu} \tag{I.617}$$

which is the sought after Palatini-identity. Inspecting the surplus term of the Einstein-Hilbert action

$$\int d^4x \sqrt{-g} \cdot \delta R_{\mu\nu} \cdot g^{\mu\nu} = \int d^4x \sqrt{-g}\, g^{\beta\nu} \left[\nabla_\mu\, \delta\Gamma^{\mu}{}_{\beta\nu} - \nabla_\nu\, \delta\Gamma^{\mu}{}_{\beta\mu}\right] \tag{I.618}$$

shows that both terms arising due to the Palatini-action are in fact covariant divergences, which would vanish when converted into surface integrals.

The cosmological constant Λ requires only the variation of the covolume in $S = \int d^4x \sqrt{-g} \cdot (2\Lambda)$, such that one gets:

$$\delta S = \int d^4x\, \delta\sqrt{-g}\, \Lambda = \int d^4x \sqrt{-g} \cdot \left(-\Lambda g_{\mu\nu}\right) \cdot \delta g^{\mu\nu} \tag{I.619}$$

Finally, one finds that the variation of the Einstein-Hilbert-Lagrange density

$$S = \int d^4x \sqrt{-g} \left(R - 2\Lambda\right) \tag{I.620}$$

in fact recovers the gravitational field equation (in vacuum)

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{I.621}$$

Let's have a quick look at the non-relativistic limit of the Einstein-Hilbert-Lagrange density. Analogous pairs of quantities are Φ and $g_{\mu\nu}$, then ∂^i and $\Gamma^{\alpha}{}_{\mu\nu}$, as well as $\partial^i\partial^j\Phi$ and $R_{\alpha\beta\mu\nu}$ and finally $\Delta\Phi$ and $R_{\mu\nu}$. The weakly perturbed line element on an otherwise flat Minkowski background

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) dx_i dx^i \tag{I.622}$$

can be used to extract the metric and to compute covolume through the determinant,

$$\det(g_{\mu\nu}) = -\left(1 + \frac{2\Phi}{c^2}\right)\left(1 - \frac{2\Phi}{c^2}\right)^3 = -\left[1 - \frac{6\Phi}{c^2} + \frac{2\Phi}{c^2} + \mathcal{O}\left(\left(\frac{\Phi}{c^2}\right)^2\right)\right] =\simeq -\left(1 - \frac{4\Phi}{c^2}\right) \tag{I.623}$$

such that the covolume becomes $\sqrt{-g} \simeq 1 - 2\Phi/c^2$ at lowest order. This means effectively, that in the classical, second-order Lagrange density for Newtonian gravity,

$$S = -\int d^3x\, \Phi\Delta\Phi = \int d^3x\, \delta_{ij}\partial^i\Phi\, \partial^j\Phi \tag{I.624}$$

the first factor of Φ could be thought of as a remainder of the covolume, while the second factor $\Delta\Phi$ appears as the Ricci-curvature. Integration by parts recovers the conventional form, which immediately poses the question if one could construct a gravitational action from squares of Christoffel symbols: This will be the Einstein-Palatini-action.

I.6 *Palatini-variation: metric $g_{\mu\nu}$ and connection $\Gamma^{\alpha}{}_{\mu\nu}$*

There is an alternative approach to deriving the field equation from a variational principle where the metric and the connection are interpreted as independent fields: Then, the field equation and the Levi-Civita connection are simultaneous results of the variational principle.

$$S = \int d^4x \sqrt{-g}\, R = \int d^4x\, \sqrt{-g} g^{\beta\nu}\, R^{\alpha}{}_{\beta\alpha\nu} =$$
$$\int d^4x\, \sqrt{-g} g^{\beta\nu} \left[\partial_\alpha \Gamma^{\alpha}{}_{\beta\alpha} - \partial_\nu \Gamma^{\alpha}{}_{\beta\alpha} + \Gamma^{\alpha}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\beta\alpha}\right] \tag{I.625}$$

where there is no a-priori assumption about the relationship between the metric $g_{\mu\nu}$ and connection $\Gamma^{\alpha}{}_{\mu\nu}$. The variation with the metric borrows from the derivation in the previous section and gives directly the vacuum-field equation

$$\delta S = \int d^4x\, \delta\left(\sqrt{-g}\, g^{\beta\nu}\right) R_{\beta\nu} = 0 \rightarrow R_{\beta\nu} = 0 \tag{I.626}$$

as the Ricci-tensor $R_{\beta\nu}$ was taken to depend only on $\Gamma^{\alpha}{}_{\mu\nu}$ and $\partial_\beta\Gamma^{\alpha}{}_{\mu\nu}$, not on $g_{\mu\nu}$. Then, the variation with respect to the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$ as the second independent field can be computed as follows. Firstly, on uses the Palatini-identity to get

$$\delta S = \int d^4x\, \sqrt{-g}\, g^{\beta\nu}\, \delta R_{\beta\nu} = \int d^4x\, \sqrt{-g}\, g^{\beta\nu} \cdot \left(\nabla_\mu\, \delta\Gamma^{\mu}{}_{\beta\nu} - \nabla_\nu\, \delta\Gamma^{\mu}{}_{\beta\mu}\right) \tag{I.627}$$

and rewrites it with the Leibnitz-rule: Please keep in mind that we did not yet make any assumption about e.g. metric compatibility, so terms of the type $\nabla_\mu g^{\mu\nu}$ are not automatically zero.

$$\delta S = \int d^4x\,\sqrt{-g}\cdot\nabla_\mu\left(g^{\beta\nu}\,\delta\Gamma^\mu{}_{\beta\nu}\right)-\nabla_\nu\left(g^{\beta\nu}\,\delta\Gamma^\mu{}_{\beta\mu}\right)=$$

$$-\int d^4x\,\sqrt{-g}\cdot\nabla_\mu\,g^{\beta\nu}\cdot\delta\Gamma^\mu{}_{\beta\nu}-\nabla_\nu\,g^{\beta\nu}\cdot\delta\Gamma^\mu{}_{\beta\mu} \quad \text{(I.628)}$$

The first two terms vanish as covariant divergences, as they can be rewritten as boundary integrals, leaving

$$\delta S = -\int d^4x\,\sqrt{-g}\cdot\left[\nabla_\mu\,g^{\beta\nu}-\delta^\beta_\mu\,\nabla_\alpha\,g^{\gamma\alpha}\right]\delta\Gamma^\mu{}_{\beta\nu} = 0 \quad \text{(I.629)}$$

Then, we realise that the Christoffel-symbol is symmetric in the lower two indices $\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu}$ if the connection was torsion free. The last equation has to be equal to zero as required by Hamilton's principle $\delta S = 0$, which then implies that the term in the square brackets, which does not have a perfect antisymmetry pertaining to the index pair μ, ν, has to vanish identically. From that one can conclude

$$\nabla_\mu\,g^{\beta\nu} = \frac{1}{2}\left[\delta^\beta_\mu\,\nabla_\alpha\,g^{\nu\alpha}+\delta^\beta_\nu\,\nabla_\alpha\,g^{\mu\alpha}\right] = 0 \quad \text{(I.630)}$$

after symmetrisation, and from that metric compatibility $\nabla_\mu g_{\beta\nu} = 0$, with the argument that

$$\nabla_\mu\left[g^{\alpha\beta}g_{\beta\gamma}\right] = \nabla_\mu\left(\delta^\alpha_\gamma\right) = 0 = \nabla_\mu g^{\alpha\beta}\cdot g_{\beta\gamma}+g^{\alpha\beta}\nabla_\mu g_{\beta\gamma} \quad \text{(I.631)}$$

implying that metric compatibility of the inverse metric is consistent with metric compatibility of the metric (please see Appendix X.1 for the detailed derivation).

These relations are sufficient to compute the Christoffel-symbol from the metric, as $\nabla_\mu\,g_{\beta\nu} = \partial_\mu\,g_{\beta\nu}-\Gamma^\alpha{}_{\mu\beta}\,g_{\alpha\nu}-\Gamma^\alpha{}_{\mu\nu}\,g_{\beta\alpha} = 0$ and the two cyclic permutations define already

$$\Gamma^\alpha{}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left[\partial_\mu\,g_{\beta\nu}+\partial_\nu\,g_{\mu\beta}-\partial_\beta\,g_{\mu\nu}\right]. \quad \text{(I.632)}$$

I.7 *coupling to matter and generation of the energy momentum tensor*

The field equation needs to be coupled to energy and momentum in the form of energy momentum-tensor $T_{\mu\nu}$, such that curvature is induced into spacetime. A combined action including geometry and the material fields could be

$$S = \int d^4x\left(\sqrt{-g}\left[R-2\Lambda\right]+\kappa\,L_m\right) \quad \text{(I.633)}$$

with an a-priori unknown coupling constant κ put as a prefactor to the Lagrange density L_m of the non-gravitational fields: Commonly, one calls this the matter-term, but actually it refers to any field that is defined on the spacetime. Variation would recover the field equation

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} - \Lambda\, g_{\mu\nu} \tag{I.634}$$

which would work out if

$$\delta S_m = \delta \int d^4x\, L = \int d^4x \frac{\delta L}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} = -\frac{1}{2} \int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} \tag{I.635}$$

by definition, with the energy momentum tensor $T_{\mu\nu}$ and the coupling constant $\kappa = 8\pi G/c^4$. Then, the symmetry of $T_{\mu\nu}$ is implied by $g^{\mu\nu}$, and the variation of S with respect to the (inverse) metric yields the correct field equation. Vice versa, this can only be consistent if

$$\frac{\delta L}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{2} T_{\mu\nu} \tag{I.636}$$

I.8 *dynamics of the energy-momentum tensor*

General relativity is the theory for the dynamics of spacetime for energy-momentum conserving fields, which is formulated in terms of the covariant divergence of the energy-momentum tensor $T_{\mu\nu}$,

$$g^{\alpha\mu}\, \nabla_\alpha\, T_{\mu\nu} = 0 \tag{I.637}$$

The variation in Hamilton's principle can be generated by an infinitesimal coordinate shift, which can have two important consequences: It should, applied to the matter-part of the action, reproduce covariant energy momentum conservation, as the working principle of the fields does not change across the manifold. Alternatively, it would as well generate a variation in the inverse metric, on which the Einstein-Hilbert-Lagrange density is built: Varying the gravitational part with respect to the inverse metric should yield the field equation, and varying the matter part the corresponding source of the gravitational field.

Infinitesimal coordinate shifts $x^\mu \rightarrow x'^\mu = x^\mu + \zeta^\mu(x)$ induce a change in the metric $g_{\mu\nu} \rightarrow g'_{\mu\nu}$ following

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta} \tag{I.638}$$

based on the Jacobians

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\left(x'^\alpha - \zeta^\alpha\right) = \frac{\partial x'^\alpha}{\partial x'^\mu} - \frac{\partial \zeta^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \partial_\mu\, \zeta^\alpha \tag{I.639}$$

Therefore, the metric changes according to

$$g'_{\mu\nu}(x') = \left(\delta^\alpha_\mu - \partial_\mu \zeta^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu \zeta^\beta\right) \cdot g_{\alpha\beta} = \delta^\beta_\nu g_{\alpha\beta} - \delta^\alpha_\mu \partial_\nu \zeta^\beta g_{\alpha\beta} - \delta^\beta_\nu \partial_\mu \zeta^\alpha g_{\alpha\beta} + \mathcal{O}(\zeta^2) \tag{I.640}$$

such that at order ζ^2 the changed metric is given by

$$g'_{\mu\nu}(x') = g_{\mu\nu} - \partial_\nu\, \zeta^\beta\, g_{\mu\beta} - \partial_\mu\, \zeta^\alpha \cdot g_{\alpha\nu} \tag{I.641}$$

The induced variation in $g_{\mu\nu}$ due to the coordinate change is given by

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) + g'_{\mu\nu}(x') - g'_{\mu\nu}(x') = \left[g'_{\mu\nu}(x') - g_{\mu\nu}(x)\right] - \left[g'_{\mu\nu}(x') - g'_{\mu\nu}(x)\right] \quad \text{(I.642)}$$

Subsituting eqn. I.641 and the Taylor expansion $g'_{\mu\nu}(x') - g'_{\mu\nu}(x) \simeq \partial_\alpha\, g'_{\mu\nu}\zeta^\alpha$

$$\delta g_{\mu\nu} = -g_{\mu\beta} \cdot \partial_\nu\, \zeta^\beta - g_{\alpha\nu}\, \partial_\mu\, \zeta^\alpha - \partial_\alpha\, g_{\mu\nu} \cdot \zeta^\alpha \quad \text{(I.643)}$$

if the approximation $\partial_\alpha\, g_{\mu\nu} = \partial_\alpha\, g'_{\mu\nu}$ is done

Replacing the partial derivatives ∂_μ with covariant ones ∇_μ according to

$$\nabla_\mu\, \zeta^\alpha = \partial_\mu \zeta^\alpha + \Gamma^\alpha{}_{\mu\tau}\, \zeta^\tau \quad \text{(I.644)}$$

as ζ^α is a vector yields

$$g_{\mu\beta}\partial_\nu\zeta^\beta + g_{\alpha\nu}\partial_\mu\zeta^\beta + \partial_\alpha g_{\mu\nu}\zeta^\alpha =$$
$$g_{\mu\beta}\nabla_\nu\zeta^\beta - g_{\mu\beta}\Gamma^\beta{}_{\nu\tau}\,\zeta^\tau + g_{\alpha\nu}\nabla_\mu\zeta^\alpha - g_{\alpha\nu}\Gamma^\beta{}_{\mu\tau}\,\zeta^\tau + \nabla_\alpha g_{\mu\nu}\zeta^\alpha + \left(\Gamma^\tau{}_{\alpha\mu}\; g_{\tau\nu} + \Gamma^\tau{}_{\alpha\nu}\; g_{\mu\tau}\right)\zeta^\alpha \quad \text{(I.645)}$$

where the metric compatibility condition $\nabla_\alpha g_{\mu\nu} = 0$ has been substituted. Two pairs of Christoffel-symbols drop out, leaving

$$\delta g_{\mu\nu} = -\left(g_{\alpha\nu}\nabla_\mu\zeta^\alpha + g_{\mu\beta}\nabla_\nu\zeta^\beta\right) = -\left(\nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu\right) \quad \text{(I.646)}$$

This result can be substituted into the variation δS_m of the part of the action S_m describing the material fields,

$$\delta S_m = -\frac{1}{2}\int d^4x\, \sqrt{-g} T^{\mu\nu}\delta g_{\mu\nu} = +\frac{1}{2}\int d^4x\, \sqrt{-g} T^{\mu\nu}\left(\nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu\right) \quad \text{(I.647)}$$

and using the Leibnitz-rule to orient the ∇_μ-differentiations to $T_{\mu\nu}$ rather than ζ_μ,

$$\delta S_m = -\frac{1}{2}\int d^4x\, \sqrt{-g}\left(\nabla_\mu\, T^{\mu\nu} \cdot \zeta_\nu + \nabla_\nu\, T^{\mu\nu} \cdot \zeta_\mu\right) = \int d^4x\, \sqrt{-g}\, \nabla_\mu\, T^{\mu\nu} \cdot \zeta_\nu \quad \text{(I.648)}$$

by exploiting the symmetry of the expression, and if the variation on the boundary vanishes, to be assumed when the Gauß-theorem is applied,

$$\int_V d^4x\, \sqrt{-g}\, \nabla_\mu \left[T^{\mu\nu}\, \zeta_\nu\right] = \int_{\partial V} dA_\mu \sqrt{|\gamma|}\, T^{\mu\nu}\, \zeta_\nu = 0 \quad \text{(I.649)}$$

Let's try out this relation for a straightforward scalar field ϕ with a self-interaction or a coupling $V(\phi)$, as the easiest example of a non-gravitational field serving as a model for the matter content of the theory. Variation of the action

$$S_\phi = \int d^4x\, \sqrt{-g}\left(\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\right) \quad \text{(I.650)}$$

with respect to $g^{\mu\nu}$ should recover the energy momentum tensor $T_{\mu\nu}$. In fact, there are two dependences on the metric, the covolume $\sqrt{-g}$ and the contraction $g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi$ in the kinetic term, such that the variation becomes

$$\delta S_\phi = \int d^4x\, \sqrt{-g}\Big[\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\Big] + \delta\sqrt{-g}\cdot\Big[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\Big]g^{\mu\nu} \tag{I.651}$$

using the relation

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}\; g\, g^{\mu\nu}\,\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\,\delta g^{\mu\nu} \tag{I.652}$$

for the variation of the covolume. Rewriting the variation yields

$$\delta S_\phi = \int d^4x\, \sqrt{-g}\Big[\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\,\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\right)\Big]\delta g^{\mu\nu} \tag{I.653}$$

Naturally, we obtain the energy momentum tensor

$$T_{\mu\nu} = \nabla_\mu\phi\,\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\,\nabla_\alpha\phi\,\nabla_\beta\phi - V(\phi)\right) \tag{I.654}$$

for a scalar field, by comparing eqn. I.653 with

$$\delta S_\phi = \int d^4x\, \sqrt{-g}\frac{1}{2}T_{\mu\nu}\,\delta g^{\mu\nu} \tag{I.655}$$

which is the correct form, that could otherwise be obtained by Legendre-transform or by taking the Lie-derivative of the Lagrange-function.

I.9 *symmetries on manifolds: Lie-derivatives and the Killing equation*

Spacetime as a manifold can have symmetries; whether they a particular choice of coordinates is compatible with them or not. Up to this point we have always relied on our intuition about choosing coordinates in which the symmetries became apparent in a very clear way, for instance the Schwarzschild coordinates for a spherically symmetric, static spacetime. But general covariance of relativity does not require that we find the best coordinate choice, instead, it should be possible to make a statement about symmetry without recursing to particular, properly adjusted coordinates; there should be a perfectly valid Schwarzschild solution for oscillating cylindrical coordinates, too. As all observables are associated with scalars, the coordinate choice does not matter for the prediction of measurable physical quantities.

Additionally, there should be conserved quantities along with any symmetry of a system as predicted by Noether's theorem. It is worth pointing out that certain statements are impossible or do not contribute substantially to statements on symmetry: concerning motion through manifolds, $g_{\mu\nu}u^\mu u^\nu = c^2$ or $g_{\mu\nu}k^\mu k^\nu = 0$ are expressing causality or define the choice of a sensible affine parameter rather than conservation, and symmetries of the metric are certainly not expressed by $\partial_\alpha g_{\mu\nu} = 0$ because of its unclear transformation properties, nor by $\nabla_\alpha g_{\mu\nu} = 0$, which is always true for a Levi-Civita connection.

Instead, we would require a new derivative, the Lie derivative $(\mathcal{L}_a g)_{\mu\nu} = 0$, which states that there is an isometry present: The metric does not change under shifts in the direction of a vector a^μ, as an expression of a spacetime symmetry. Ideally, we can link this new derivative to the already defined covariant derivative and possibly derive a relationship which allows us to find coordinates adopted to a spacetime with a given symmetry.

Imagine two distinct points $P(x^\mu)$ and $P'(x'^\mu)$ with coordinates x^μ and x'^μ, respectively. Then, the coordinates of the two points are related in general by

$$x'^\mu = x^\mu - \epsilon a^\mu \quad \text{and differentially, by} \quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu - \epsilon\,\partial_\nu\,a^\mu \tag{I.656}$$

where ϵ controls the infinitesimal shift into the direction a^μ. Any vector field υ^μ then transforms according to

$$\upsilon'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\,\upsilon^\nu(x) = \left(\delta^\mu_\nu - \epsilon\partial_\nu\,a^\mu\right)\upsilon^\nu(x) = \upsilon^\mu(x) - \epsilon\partial_\nu\,a^\mu\upsilon^\mu(x) \tag{I.657}$$

Clearly, $\upsilon'^\mu(x') - \upsilon^\mu(x)$ is not a vector because the two υ refer to physically different points on the manifold, so we could apply a Taylor-expansion

$$\upsilon'^\mu(x') = \upsilon'^\mu(x) + (x'-x)^\nu\,\partial_\nu\,\upsilon'^\mu(x) + \ldots = \upsilon'^\mu(x) - \epsilon a^\nu\,\partial_\nu\,\upsilon'^\mu(x) \tag{I.658}$$

such that

$$\upsilon^\mu(x) = \upsilon'^\mu(x') + \epsilon\partial_\nu\,a^\mu\upsilon^\nu(x) = \upsilon'^\mu(x) - \epsilon a^\nu\,\partial_\nu\,\upsilon'^\mu + \epsilon\,\partial_\nu\,a^\mu\cdot\upsilon^\nu(x) \tag{I.659}$$

and we can define the Lie-derivative $\left(\mathcal{L}_a\upsilon\right)^\mu$

$$\lim_{\epsilon\to 0}\,\frac{\upsilon^\mu(x) - \upsilon'^\mu(x)}{\epsilon} = -a^\nu\,\partial_\nu\,\upsilon'^\mu + \partial_\nu\,a^\mu\cdot\upsilon^\nu \equiv \left(\mathcal{L}_a\upsilon\right)^\mu \tag{I.660}$$

of the vector field υ^μ in the direction a^μ with all terms at order ϵ. If defined for linear forms, the Lie-derivative picks up a different sign in the Jacobian,

$$\left(\mathcal{L}_a\upsilon\right)_\mu = +a^\nu\,\partial_\nu\,\upsilon_\mu + \partial_\mu\,a^\nu\cdot\upsilon_\nu \tag{I.661}$$

and applied to a rank-2 tensor such as the metric one obtains

$$\left(\mathcal{L}_a g\right)_{\mu\nu} = g_{\mu\lambda}\cdot\partial_\nu\,a^\lambda + g_{\lambda\nu}\,\partial_\mu\,a^\lambda + a^\lambda\,\partial_\lambda\,g_{\mu\nu} \tag{I.662}$$

It is very important to realise that up to this point we did not use the concept of parallel transport nor the covariant derivative, but only partial derivatives. In fact, symmetries of vector or tensor fields on a manifold exist and and quantifiable with the Lie-derivative even when there is no differential structure and no parallel transport. But of course, one would like to define the Lie-derivative in a way that it becomes compatible with the covariant derivative, and that is in fact one motivation for Levi-Civita connections:

$$\nabla_\nu\,\upsilon^\mu\cdot a^\nu - \nabla_\nu\,a^\mu\cdot\upsilon^\nu = \partial_\nu\,\upsilon^\mu\cdot a^\nu - \partial_\nu\,a^\mu\cdot\upsilon^\nu - \left(\Gamma^\mu{}_{\kappa\lambda} - \Gamma^\mu{}_{\lambda k}\right)\cdot\upsilon^\kappa\cdot a^\lambda = \partial_\nu\,\upsilon^\mu\cdot a^\nu - \partial_\nu\,a^\mu\cdot\upsilon^\nu \tag{I.663}$$

if the connection is torsion free, $\Gamma^{\mu}{}_{k\lambda} = \Gamma^{\mu}{}_{\lambda k}$, and the covariant expression falls back onto the partial one. Applied to the metric this would mean that

$$\left(\mathcal{L}_a g\right)_{\mu\nu} = g_{\mu\lambda}\, \nabla_\lambda\, a^\lambda + g_{\lambda\nu}\, \nabla_\mu\, a^\lambda + a^\lambda \cdot \nabla_\lambda\, g_{\mu\nu} = \nabla_\nu\, a_\mu + \nabla_\mu\, a_\nu \tag{I.664}$$

with the last of the terms being canceled by metric compatibility $\nabla_\lambda\, g_{\mu\nu} = 0$, and using the index-lowering property of the metric. Again, we should be able to compute the Lie-derivative of the metric purely with partial derivatives instead of covariant ones. Indeed, replacing ∇ with ∂ and the Christoffel-symbols yields

$$\begin{aligned}\left(\mathcal{L}_a g\right)_{\mu\nu} &= \\ g_{\mu\lambda}\left[\partial_\nu\, a^\lambda + \Gamma^\lambda{}_{\nu\kappa}\, a^\kappa\right] &+ g_{\lambda\nu}\left[\partial_\mu a^\lambda + \Gamma^\lambda{}_{\mu\kappa}\, a^\kappa\right] + a^\lambda\left[\partial_\lambda\, g_{\mu\nu} - \Gamma^\kappa{}_{\lambda\mu}\, g_{\kappa\nu} - \Gamma^\kappa{}_{\lambda\nu}\, g_{\mu\kappa}\right] = \\ &g_{\mu\lambda}\, \partial_\nu\, a^\lambda + g_{\lambda\nu}\, \partial_\mu\, a^\lambda + a^\lambda \cdot \partial_\lambda\, g_{\mu\nu}\end{aligned} \tag{I.665}$$

because of the pairwise cancellation in the expression

$$g_{\mu\lambda}\Gamma^\lambda{}_{\nu\kappa}\, a^\kappa + g_{\lambda\nu}\Gamma^\lambda{}_{\mu\kappa}\, a^\kappa - \Gamma^\kappa{}_{\lambda\mu}\, g_{\kappa\nu} a^\lambda - \Gamma^\kappa{}_{\lambda\nu}\, g_{\mu\kappa} a^\lambda = 0 \tag{I.666}$$

.

If $\left(\mathcal{L}_a g\right)_{\mu\nu} = 0$ for a given shift field a^μ then the spacetime possesses a certain symmetry and $\nabla_\nu\, a_\mu + \nabla_\mu\, a_\nu = 0$. Then, a^μ is called a Killing-vector. There is a weird relationship between Killing vectors and the Riemann-curvature. For any vector we have the definition of curvature through the non-commutability of second covariant derivatives,

$$\left(\nabla_\kappa \nabla_\lambda - \nabla_\lambda \nabla_\kappa\right) a_\mu = \mathrm{R}^\tau{}_{\mu\kappa\lambda} a_\tau = \mathrm{R}_{\tau\mu\kappa\lambda} a^\tau \tag{I.667}$$

from which we can construct

$$\begin{aligned}\nabla_\kappa\left[\nabla_\nu\, a_\mu - \nabla_\mu\, a_\nu\right] + \nabla_\nu\left[\nabla_\mu\, a_\kappa - \nabla_\kappa\, a_\mu\right] + \nabla_\mu\left[\nabla_\kappa\, a_\nu - \nabla_\nu\, a_\kappa\right] &= \\ \left(\mathrm{R}_{\tau\mu\nu\kappa} + \mathrm{R}_{\tau\nu\kappa\mu} + \mathrm{R}_{\tau\kappa\mu\nu}\right] a^\tau &= 0\end{aligned} \tag{I.668}$$

which necessarily vanishes due to the algebraic Bianchi-identity. From the Killing-condition $\nabla_\nu\, a_\mu + \nabla_\mu\, a_\nu = 0$ we get $\nabla_\mu\, a_\nu = -\nabla_\nu\, a_\mu$, so we can change the sign in every second term,

$$\begin{aligned}\nabla_\kappa\left(\nabla_\nu a_\mu + \nabla_\nu a_\mu\right) + \nabla_\nu\left(\nabla_\mu a_\kappa + \nabla_\mu a_\kappa\right) + \nabla_\mu\left(\nabla_\kappa a_\nu + \nabla_\kappa a_\nu\right) &= \\ 2\left[\nabla_\kappa \nabla_\nu a_\mu + \nabla_\nu \nabla_\mu a_\kappa + \nabla_\mu \nabla_\kappa a_\nu\right] &= 0\end{aligned} \tag{I.669}$$

Inspecting the result $\nabla_\kappa \nabla_\nu a_\mu + \nabla_\nu \nabla_\mu a_\kappa + \nabla_\mu \nabla_\kappa a_\nu = 0$ in more detail we can carry out this treatment: Let's keep the first term unchanged, but switch the indices $\mu \leftrightarrow \kappa$ in the second term. Because of the Killing-condition, this can be done as $\nabla_\mu a_\kappa + \nabla_\kappa a_\mu = 0$, so one picks up a minus-sign. The analogous index switch can be performed on the last term. $\kappa \leftrightarrow \mu$ is possible because $\nabla_\kappa a_\nu + \nabla_\nu a_\kappa = 0$, again introducing a minus sign:

$$\nabla_\kappa \nabla_\nu a_\mu + \nabla_\nu \nabla_\mu a_\kappa + \nabla_\mu \nabla_\kappa a_\nu = \nabla_\kappa \nabla_\nu a_\mu - \nabla_\nu \nabla_\kappa a_\mu - \nabla_\mu \nabla_\nu a_\kappa \tag{I.670}$$

The first two terms are just double covariant derivatives with interchanged order applied to the vector a_μ which yields the Riemann-curvature: Making this identification yields the Killing-equation

$$\nabla_\mu \nabla_\nu a_\kappa = \left(\nabla_\nu \nabla_\kappa - \nabla_\kappa \nabla_\nu\right) a_\mu = R^\tau{}_{\mu\kappa\nu} a_\tau \tag{I.671}$$

The Killing-equation is a tool of determining the Killing-vectors a^μ for a spacetime with a given metric $g_{\mu\nu}$: Think of it as an eigenvalue equation, which yields the shift-vectors for any spacetime where the covariant derivatives and the Riemann-curvature are given in an arbitrary coordinate choice, and effectively isolate the spacetime symmetries in the form of the set of a_μ. If the connection is of the Levi-Civita type, both the covariant derivative ∇ as well as the Riemann-curvature are completely computable from $g_{\mu\nu}$, so that all ingredients of the Killing equation for a given metric are present.

Euclidean space, for instance, has two types of symmetries: shifts and rotations. By using intuition and introducing global Cartesian coordinates one simplifies everything tremendously as $g_{\mu\nu} = \delta_{\mu\nu}$, $\Gamma^\alpha{}_{\mu\nu} = 0$ such that $\nabla_\mu = \partial_\mu$ and of course $R^\tau{}_{\kappa\mu\nu} = 0$. Then, the Killing-equation reduces to $\nabla_\mu \nabla_\nu a_\kappa = 0 = \partial_\mu \partial_\nu a_\kappa$ and one can search for solutions to $\partial_\mu \partial_\nu a_\kappa = 0$, which are obviously given by $a_\kappa = q_{\nu\kappa} x^\kappa + p_\kappa$ with 6 constants $q_{\nu\kappa}$ (due to the antisymmetry $q_{\nu\kappa} = -q_{\kappa\nu}$, from the Lie-derivative) and 3 constants p_κ, corresponding to the rotations and shifts, respectively.

There is a tight connection between Killing-vectors $\nabla_\nu a_\mu + \nabla_\mu a_\nu = 0$ expressing an isometry of spacetime and geodesics, which are defined through their autoparallelity condition $u^\nu \nabla_\nu u^\mu = 0$. If the scalar product $a_\mu u^\mu$ is shifted by $u^\lambda \nabla_\lambda$ into the direction of u^λ, we obtain

$$u^\lambda \nabla_\lambda \left[a_\mu \cdot u^\mu\right] = u^\lambda \left[\nabla_\lambda a_\mu \cdot u^\mu + a_\mu \nabla_\lambda u^\mu\right] = \nabla_\lambda a_\mu \cdot u^\mu u^\lambda + a_\mu \cdot u^\lambda \nabla_\lambda u^\mu = 0 \tag{I.672}$$

as $\nabla_\lambda a_\mu \cdot u^\mu u^\lambda = 0$ because of the antisymmetry $\nabla_\nu a_\mu = -\nabla_\mu a_\nu$ and $u^\lambda \nabla_\lambda u^\mu = 0$ because of geodesic motion. Hence, the projection of the tangent u^μ onto the Killing vector field a^μ is conserved along the geodesic.

X MATHEMATICAL SUPPLEMENT

X.1 *metric compatibility of the inverse metric*

Metric compatibility $\nabla_\alpha g_{\mu\nu} = 0$ of the metric $g_{\mu\nu}$ itself implies metric compatibility $\nabla_\alpha g^{\mu\nu} = 0$ of the inverse metric $g^{\mu\nu}$. This can be seen by starting from the definition of the inverse metric, $g^{\mu\tau} g_{\tau\nu} = \delta^\mu_\nu$ and have a covariant derivative act on this relation, keeping in mind that the covariant differentiation ∇_α obeys the Leibnitz-rule:

$$\nabla_\alpha g^{\mu\tau} g_{\tau\nu} = \nabla_\alpha g^{\mu\tau} \cdot g_{\tau\nu} + g^{\mu\tau} \nabla_\alpha g_{\tau\nu} = \nabla_\alpha \delta^\mu_\nu \tag{X.673}$$

The covariant derivative of the Kronecker-symbol is determined from the fact that it is a tensor with a co- and a contravariant index, i.e.

$$\nabla_\alpha \delta^\mu_\nu = \partial_\alpha \delta^\mu_\nu + \Gamma^\mu{}_{\alpha\beta} \delta^\beta_\nu - \Gamma^\beta{}_{\alpha\nu} \delta^\mu_\beta = \partial_\alpha \delta^\mu_\nu + \Gamma^\mu{}_{\alpha\nu} - \Gamma^\mu{}_{\alpha\nu} = \partial_\alpha \delta^\mu_\nu \tag{X.674}$$

renaming the indices in the second step. The Kronecker-δ is peculiar as a tensor, because it assumes the same values of 0 and 1 in every coordinate choice: One needs a Jacobian for μ and an inverse Jacobian for ν,

$$\delta^{\mu}_{\nu} \to \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\delta^{\alpha}_{\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x'^{\nu}} = \frac{\partial x'^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}_{\nu} \tag{X.675}$$

so that the transformation does not have any influence on the tensor. Therefore, $\partial_{\alpha}\delta^{\mu}_{\nu} = 0$, and one gets

$$\nabla_{\alpha} g^{\mu\tau} \cdot g_{\tau\nu} + g^{\mu\tau}\nabla_{\alpha} g_{\tau\nu} = 0 \tag{X.676}$$

Metric compatibility of the metric sets the second term to zero, so that one is left with

$$\nabla_{\alpha} g^{\mu\tau} \cdot g_{\tau\nu} = 0 \tag{X.677}$$

from where one can isolate the metric compatibility condition for the inverse metric by contraction with $g^{\nu\beta}$

$$\nabla_{\alpha} g^{\mu\tau} \cdot g_{\tau\nu} g^{\nu\beta} = \nabla_{\alpha} g^{\mu\tau} \cdot \delta^{\beta}_{\tau} = \nabla_{\alpha} g^{\mu\beta} = 0 \tag{X.678}$$

ACKNOWLEDGEMENTS

I'd like to acknowledge the great help in typesetting the script provided by Rebecca Kuntz. My students Eileen Sophie Giesel, Robert Felix Reischke and Tim Tugendhat have supported the lecture expertly as head tutors, and I appreciate many discussions with Hannes Keppler and Jonah Cedric Strauss. I would like to thank Karl-Heinz Lotze, Tilman Plehn, Frederic Paul Schuller and Eduard Thommes for their support in relativity and beyond.