Quantum Field Theory I

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Literature

This is a writeup of my Master programme course on Quantum Field Theory I. The primary source for this course has been

- Peskin, Schröder: An introduction to Quantum Field Theory, ABP 1995,
- Itzykson, Zuber: Quantum Field Theory, Dover 1980,
- Kugo: Eichtheorie, Springer 1997,

which I urgently recommend for more details and for the many topics which time constraints have forced me to abbreviate or even to omit. Among the many other excellent textbooks on Quantum Field Theory I particularly recommend

- Weinberg: Quantum Field Theory I + II, Cambridge 1995,
- Srednicki: Quantum Field Theory, Cambridge 2007,
- Banks: Modern Quantum Field Theory, Cambridge 2008

as further reading. All three of them oftentimes take an approach different to the one of this course. Excellent lecture notes available online include

- A. Hebecker: Quantum Field Theory,
- D. Tong: Quantum Field Theory.

Special thanks to Robert Reischke[1] for his fantastic work in typing these notes.

[1] For corrections and improvement suggestions please send a mail to reischke@stud.uni-heidelberg.de.

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Chapter 1

The free scalar field

1.1 Why Quantum Field Theory?

In (non-relativistic) Quantum Mechanics, the dynamics of a particle is described by the time-evolution of its associated wave-function $\psi(t,\vec{x})$ with respect to the non-relativistic Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = H\psi(t,\vec{x}), \tag{1.1}$$

with the Hamilitonian given by $H = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{x})$. In order to achieve a Lorentz invariant framework, a naive approach would start by replacing this non-relativistic form of the Hamiltonian by a relativistic expression such as

$$H = \sqrt{c^2\hat{\vec{p}}^2 + m^2c^4} \tag{1.2}$$

or, even better, by modifying the Schrödinger equation altogether such as to make it symmetric in $\frac{\partial}{\partial t}$ and the spatial derivative $\vec{\nabla}$. However, the central insight underlying the formulation of Quantum Field Theory is that this is not sufficient. Rather, combining the principles of Lorentz invariance and Quantum Theory requires abandoning the single-particle approach of Quantum Mechanics.

- In any relativistic Quantum Theory, particle number need not be conserved, since the relativistic dispersion relation $E^2 = c^2\vec{p}^2 + m^2c^4$ implies that energy can be converted into particles and vice versa. This requires a **multi-particle framework**.

- **Unitarity and causality** cannot be combined in a single-particle approach: In Quantum Mechanics, the probability amplitude for a particle to propagate from position $\vec{x}$ to $\vec{y}$ is

  $$G(\vec{x},\vec{y}) = \langle\vec{y}|\, e^{-\frac{i}{\hbar}Ht}\,|\vec{x}\rangle\,. \tag{1.3}$$

  One can show that e.g. for the free non-relativistic Hamiltonian $H = \frac{\hat{p}^2}{2m}$ this is non-zero even if $x^\mu = (x^0,\vec{x})$ and $y^\mu = (y^0,\vec{y})$ are at a spacelike distance. The problem persists if we replace H by a relativistic expression such as (1.2).

Quantum Field Theory (QFT) solves both these problems by a radical change of perspective:

- The fundamental entities are not the particles, but the field, an abstract object that penetrates spacetime.
- Particles are the excitations of the field.

Before developing the notion of an abstract field let us try to gain some intuition in terms of a mechanical model of a field. To this end we consider a mechanical string of length L and tension T along the x-axis and excite this string in the transverse direction. Let $\phi(x,t)$ denote the transverse excitation of the string. In this simple picture $\phi(x,t)$ is our model for the field. This system arises as the continuum limit of N mass points of mass m coupled by a mechanical spring to each other. Let the distance of the mass points from each other projected to the x-axis be Δ and introduce the transverse coordinates $q_r(t)$, $r = 1,\ldots,N$ of the mass points. In the limit $\Delta \to 0$ with L fixed, the profile $q_r(t)$ asymptotes to the field $\phi(x,t)$. In this sense the field variable x is the continuous label for infinitely many degrees of freedom.

We can now linearise the force between the mass points due to the spring. As a result of a simple exercise in classical mechanics the energy at leading order is found to be

$$E = \sum_{r=0}^{N} \left(\frac{1}{2} m \left(\frac{\mathrm{d}q_r(t)}{\mathrm{d}t} \right)^2 + k(q_r^2 - q_r q_{r-1}) \right) + O(q^3), \qquad k = \frac{T}{L}. \tag{1.4}$$

In the continuum limit this becomes

$$E = \int_0^L \left[\frac{1}{2}\rho \left(\frac{\partial \phi(x,t)}{\partial t} \right)^2 + \frac{1}{2}\rho c^2 \left(\frac{\partial \phi(x,t)}{\partial x} \right)^2 \right] \mathrm{d}x \tag{1.5}$$

in terms of the mass density ρ of the string and a suitably defined characteristic velocity c. Note that the second term indeed includes the nearest neighbour interaction because

$$\left(\frac{\partial \phi(x,t)}{\partial x} \right)^2 \simeq \left(\lim_{\delta x \to 0} \frac{\phi(x+\delta x,t) - \phi(x,t))}{\delta x} \right)^2 \tag{1.6}$$

contains the off-diagonal terms $\phi(x+\delta x,t)\phi(x,t)$.

The nearest-neighbour interaction implies that the equation of motion for the mass points q_i obey *coupled* linear differential equations. This feature persists in the continuum limit. To solve the dynamics it is essential that we are able to *diagonalise the interaction in terms* of the **Fourier modes**,

$$\begin{aligned} \phi(x,t) &= \sum_{k=1}^{\infty} A_k(t) \sin\left(\frac{k\pi x}{L} \right), \\ E &= \frac{L}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2}\rho \dot{A}_k^2 + \frac{1}{2}\rho \omega_k^2 A^2 \right), \end{aligned} \tag{1.7}$$

where $\omega_k = k\pi c/L$. We are now dealing with a collection of infinitely many, decoupled harmonic oscillators $A_k(t)$.

In a final step, we quantise this collection of harmonic oscillators. According to Quantum Mechanics, each mode $A_k(t)$ can take energy values

$$E_k = \hbar\omega_k(n_k + 1/2) \quad n_k = 0, 1, 2, ..., \infty. \tag{1.8}$$

The total energy is given by summing over the energy associated with all the modes, $E = \sum E_k$. A state of definite energy E corresponds to mode numbers $(n_1, n_2, ..., n_\infty)$, where we think of n_r as an excitation of the string or of the field ϕ, i.e. as a quantum. In condensed matter physics, these quantised excitations in terms of harmonic modes are called **quasi-particles**, e.g. phonons for mechanical vibrations of a solid. Note that the above decoupling of the degrees of freedom rested on the quadratic form of the potential. Including higher terms will destroy this and induce interactions between modes.

The idea of Quantum Field Theory is to adapt this logic to particle physics and to **describe a particle as the quantum of oscillation of an abstract field** - just like in solid state physics we think of a quasi-particle as the vibrational excitation of a solid. The only difference is that the fields are now more abstract objects defined all over spacetime as opposed to concrete mechanical fields of the type above.
As a familiar example for a field we can think of the Maxwell field $A_\mu(x, t)$ in classical electrodynamics. A photon is the quantum excitation of this. It has spin 1. Similarly we assign one field to each particle species, e.g. an electron is the elementary excitation of the electron field (Spin $1/2$). We will interpret the sum over harmonic oscillator energies as an integral over possible energies for given momentum,

$$E = \sum_{k=1}^{\infty} \hbar\omega_k(n_k + 1/2) \to \int \mathrm{d}p\, \hbar\omega_p(n_p + 1/2). \tag{1.9}$$

A single particle with momentum p corresponds to $n_p = 1$ while all others vanish, but this is just a special example of a more multi-particle state with several $n_{p_i} \neq 0$. In particular, in agreement with the requirements of a multi-particle framework, at fixed E transitions between various multi-particle states are in principle possible. Such transitions are induced by interactions corresponding to the higher order terms in the Hamiltonian that we have discarded so far. As a triumph this formalism also solves the problem of causality, as we will see.

1.2 Classical scalar field: Lagrangian formulation

We now formalise the outlined transition from a classical system with a finite number of degrees of freedom $q_i(t)$ to a classical field theory in terms of a scalar field $\phi(t, \vec{x}) \equiv \phi(x^\mu)$. In classical mechanics we start from an action

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(q_i(t), \dot{q}_i(t)) \text{ with } L = \frac{1}{2}\sum_i (\dot{q}_i(t))^2 - V(q_1, , q_N), \tag{1.10}$$

where we have included the mass m in the definition of $q_i(t)$. In a first step replace

$$q_i \rightarrow \phi(x^\mu) \equiv \phi(x), \tag{1.11}$$

$$\dot{q}_i(t) \rightarrow \frac{\partial\phi(x)}{\partial t}, \tag{1.12}$$

thereby substituting the label $i = 1, ...N$ by a continous coordinate $\vec{x} \equiv x^i$ with $i = 1, 2, 3$. For the moment we consider a real scalar field i.e. $\phi(x) = \phi^*(x)$ which takes values in $\mathbb{R}$, i.e.

$$\phi : x^\mu \rightarrow \phi(x^\mu) \in \mathbb{R}. \tag{1.13}$$

We will see that such a field describes spin-zero particles. Examples of scalar particles in nature are the Higgs boson or the inflaton, which cosmologists believe to be responsible for the exponential expansion of the universe during in inflation.

To set up the Lagrange function we first note that in a relativistic theory the partial time derivative can only appear as part of

$$\partial_\mu\phi(x) \equiv \frac{\partial}{\partial x^\mu}\phi(x). \tag{1.14}$$

Thus the Lagrange function can be written as

$$L = \int \mathrm{d}^3x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x)), \tag{1.15}$$

where $\mathcal{L}$ is the Lagrange density. The action therefore is

$$S = \int \mathrm{d}^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x)). \tag{1.16}$$

While, especially in condensed matter physics, also non-relativistic field theories are relevant, we focus on relativistic theories in this course.

Note furthermore that throughout this course we use conventions where

$$\boxed{\hbar = c = 1.} \tag{1.17}$$

Then $\mathcal{L}$ has the dimension mass^4, i.e. $[\mathcal{L}] = 4$, since $[S] = 0$ and $[\mathrm{d}^4x] = -4$.

The next goal is to find the Lagrangian: In a relativistic setting $\mathcal{L}$ can contain powers of ϕ and[1] $\partial_\mu\phi\partial^\mu\phi \equiv \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, which is the simplest scalar which can be built from $\partial_\mu\phi$. The action in this case is

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + O(\phi^n(\partial\phi)^m)\right], \tag{1.18}$$

where

$$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 \tag{1.19}$$

[1] Note that the only remaining option $\partial^\mu\partial_\mu\phi$ is a total derivative and will therefore not alter the equations of motion under the usual assumptions on the boundary terms.

and the last type of terms consists of higher derivative terms with $m \geq 2$ or mixed terms with $n \geq 1$. Notice that

- the signature for the metric is, in our conventions, $(+,-,-,-)$, such that the sign in the action is indeed chosen correctly such that the kinetic term appears with a positive prefactor;
- ϕ has dimension 1 (mass^1).

The potential $V(\phi(x))$ is in general a power series of the form

$$V(\phi(x)) = a + b\phi(x) + c\phi^2(x) + d\phi^3(x) + \dots . \tag{1.20}$$

We assume that the potential has a global minimum at $\phi(x) = \phi_0(x)$ such that

$$\frac{\partial V(\phi)}{\partial \phi}|_{\phi=\phi_0} = 0, \qquad V(\phi_0) = V_0 \tag{1.21}$$

By a field redefinition we ensure that the minimum is at $\phi_0(x) \equiv 0$ and expand $V(\phi(x))$ around this minimum as

$$V(\phi(x)) = V_0 + \frac{1}{2}m^2\phi^2(x) + O(\phi^3(x)). \tag{1.22}$$

Here we used that the linear terms vanish at the extremum and the assumption that we are expanding around a minimum implies $m^2 > 0$. The constant V_0 is the classical contribution to the ground state or vacuum energy. Since in a theory without gravity absolute energies are not measurable, we set $V_0 = 0$ for the time being, but keep in mind that in principle V_0 is arbitrary. We will have considerably more to say about V_0 in the quantum theory in section (1.8).
Therefore the action becomes

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + ...\right]. \tag{1.23}$$

We will find that m^2, the prefactor of the quadratic term, is related to the mass of the particles and that the omitted higher powers of ϕ as well as the terms $O(\phi^n(\partial\phi)^m)$ will give rise to interactions between these particles.
As an aside note that a negative value of m^2 signals that the extremum around which we are expanding the potential is a maximum rather than a minimum. Therefore $m^2 < 0$ signals a tachyonic instability: quantum fluctuations will destabilise the vacuum and cause the system to roll down its potential until it has settled in its true vacuum.
We will start by ignoring interaction terms and studying the action of the free real scalar field theory

$$\boxed{S = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right], \ \ \phi = \phi^*.} \tag{1.24}$$

The equations of motion are given by the Euler-Lagrange equations. As in classical mechanics we derive them by varying S with respect to ϕ and $\partial_\mu\phi$ subject to $\delta\phi|_{\text{boundary}} = \delta\partial_\mu\phi|_{\text{boundary}} = 0$. This

yields

$$\begin{aligned} 0 \stackrel{!}{=} \delta S &= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} \delta\partial_\mu \phi(x) \right] \\ &= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} \delta\phi(x) + \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} \partial_\mu \delta\phi(x) \right]. \end{aligned} \tag{1.25}$$

Integrating by parts gives

$$\int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))} \right] \delta\phi(x) + \text{boundary terms}. \tag{1.26}$$

Since the boundary terms vanish by assumption, therefore the integrand has to vanish for all variations $\delta\phi(x)$. This yields the Euler-Lagrange equations

$$\boxed{\frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} = \partial_\mu \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_\mu \phi(x))}.} \tag{1.27}$$

By inserting (1.24) into (1.27) we find the equations of motion for the free scalar field

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \tag{1.28}$$

i.e. the **Klein-Gordon equation**

$$\boxed{(\partial^2 + m^2)\phi(x) = 0.} \tag{1.29}$$

Note that (1.29) is a relativistic wave equation and that it is solved by

$$e^{\pm ipx} \text{ with } p \equiv p^\mu = (p^0, \vec{p}) \tag{1.30}$$

subject to the dispersion relation $-p^2 + m^2 = 0$, i.e. $p^0 = \pm\sqrt{\vec{p}^2 + m^2}$. We now set

$$E_{\vec{p}} := \sqrt{\vec{p}^2 + m^2} \equiv E_p \tag{1.31}$$

and write the general solution of (1.29) in the form

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(f(\vec{p}) e^{-ipx} + g(\vec{p}) e^{ipx} \right), \tag{1.32}$$

where $p := (E_p, \vec{p})$ and $f^*(\vec{p}) = g(\vec{p})$ for real ϕ and $px = p \cdot x = p^\mu x_\mu$.

1.3 Noether's Theorem

A key role in Quantum Field Theory is played by symmetries. We consider a field theory with Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$. A symmetry of the theory is then defined to be a field transformation by which $\mathcal{L}$ changes at most by a total derivative such that the action stays invariant. This ensures that the equations of motion are also invariant. Symmetries and conservation laws are related by **Noether's Theorem**[2]:

[2] Emmy Noether, 1882-1935.

Every continuous symmetry in the above sense gives rise to a Noether current $j^\mu(x)$ such that

$$\partial_\mu j^\mu(x) = 0 \tag{1.33}$$

upon use of the equations of motion (≡ "on-shell").

This can be proven as follows:

For a continuous symmetry we can write infinitesimally:

$$\phi \to \phi + \epsilon\delta\phi + O(\epsilon^2) \text{ with } \delta\phi = X(\phi, \partial_\mu\phi). \tag{1.34}$$

Off-shell (i.e. without use of the equations of motion) we know that

$$\mathcal{L} \to \mathcal{L} + \epsilon\delta\mathcal{L} + O(\epsilon^2) \quad \text{with} \quad \delta\mathcal{L} = \partial_\mu F^\mu \tag{1.35}$$

for some F^μ. Now, under an arbitrary transformation $\phi \to \phi + \epsilon\delta\phi$, which is not necessarily a symmetry, $\delta\mathcal{L}$ is given by

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\ &= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] + \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi. \end{aligned} \tag{1.36}$$

If $\delta\phi = X$ is a symmetry, then $\delta\mathcal{L} = \partial_\mu F^\mu$. Setting

$$\boxed{j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X - F^\mu} \tag{1.37}$$

we therefore have

$$\partial_\mu j^\mu = -\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)X \tag{1.38}$$

off-shell. Note that the terms in brackets are just the Euler-Lagrange equation. Thus, if we use the equations of motion, i.e. on-shell, $\partial_\mu j^\mu = 0$. □

This immediately yields the following Lemma:

Every continuous symmetry whose associated Noether current satisfies $j^i(t, \vec{x}) \to 0$ sufficiently fast for $|\vec{x}| \to \infty$ gives rise to a conserved charge Q with

$$\boxed{\dot{Q} = 0.} \tag{1.39}$$

Indeed if we take

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3x\, j^0(t, \vec{x}), \tag{1.40}$$

then the total time derivative of Q is given by

$$\begin{aligned}\dot{Q} &= \int_{\mathbb{R}^3} \mathrm{d}^3x \frac{\partial}{\partial t} j^0 \\ &= -\int_{\mathbb{R}^3} \mathrm{d}^3x\, \partial_i j^i(t,\vec{x}) = 0\end{aligned} \tag{1.41}$$

by assumption of sufficiently fast fall-off of $j^i(t,\vec{x})$. We used that $\partial_\mu j^\mu = 0$ in the first step. □

The technical assumption $j^i(t,\vec{x}) \to 0$ for $|\vec{x}| \to \infty$ is really an assumption of 'sufficiently fast fall-off' of the fields at spatial infinity, which is typically satisfied. Note that in a finite volume $V = \text{const.}$, the quantity $Q_V = \int_V \mathrm{d}V j^0(t,\vec{x})$ satisfies local charge conservation,

$$\dot{Q}_V = -\int_V \mathrm{d}V\, \nabla\vec{j} = -\int_{\partial V} \vec{j}\cdot \mathrm{d}\vec{s}. \tag{1.42}$$

We now apply Noether's theorem to deduce the canonical energy-momentum tensor: Under a global spacetime transformation $x^\mu \to x^\mu + \epsilon^\mu$ a scalar field $\phi(x^\mu)$ transforms like

$$\phi(x^\mu) \to \phi(x^\mu - \epsilon^\mu) = \phi(x^\mu) - \epsilon^\nu \underbrace{\partial_\nu \phi(x^\mu)}_{\equiv X_\nu(\phi)} + \mathcal{O}(\epsilon^2). \tag{1.43}$$

Because $\mathcal{L}$ is a local function of x it transforms as

$$\begin{aligned}\mathcal{L} \to \mathcal{L} - \epsilon^\nu \partial_\nu \mathcal{L} &= \mathcal{L} - \eta^\mu{}_\nu \epsilon^\nu \partial_\mu \mathcal{L} \\ &= \mathcal{L} - \epsilon^\nu \partial_\mu \eta^\mu{}_\nu \mathcal{L}.\end{aligned} \tag{1.44}$$

For each component ν we therefore have a conserved current $(j^\mu)_\nu$ given by

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\partial_\nu \phi}_{\equiv X_\nu} - \underbrace{\eta^\mu{}_\nu \mathcal{L}}_{\equiv (F^\mu)_\nu}. \tag{1.45}$$

With both indices up, we arrive at the **canonical energy-momentum tensor**

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} \text{ with } \partial_\mu T^{\mu\nu} = 0 \text{ on-shell.}} \tag{1.46}$$

The conserved charges are the energy $E = \int d^3x\, T^{00}$ associated with time translation invariance and the spatial momentum $P^i = \int d^3x\, T^{0i}$ associated with spatial translation invariance. We can combine them into the conserved 4-momentum

$$P^\nu = \int \mathrm{d}^3x\, T^{0\nu} \tag{1.47}$$

with the property $\dot{P}^\nu = 0$.
Two comments are in order:

- In general, $T^{\mu\nu}$ may not be symmetric - especially in theories with spin. In such cases it can be useful to modify the energy-momentum tensor without affecting its conservedness or the associated conserved charges. Indeed we state as a fact that the **Belinfante-Rosenfeld tensor**

$$\Theta^{\mu\nu}_{BR} := T^{\mu\nu} + \partial_\rho S^{\rho\mu\nu} \tag{1.48}$$

  can be defined in terms of a suitable $S^{\rho\mu\nu} = -S^{\mu\rho\nu}$ such that $\Theta^{\mu\nu}_{BR}$ is symmetric and obeys $\partial_\mu \Theta^{\mu\nu}_{BR} = 0$.

- In General Relativity (GR), there exists yet another definition of the energy-momentum tensor: With the metric $\eta_{\mu\nu}$ replaced by $g_{\mu\nu}$ and

$$S = \int \mathrm{d}^4x \sqrt{-g}\mathcal{L}^{\text{matter}}(g_{\mu\nu}, \phi, \partial\phi), \tag{1.49}$$

  where $g \equiv \det g$, one defines the **Hilbert energy-momentum tensor**:

$$(\Theta_H)^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\mathcal{L}^{\text{matter}})}{\partial g_{\mu\nu}}, \tag{1.50}$$

  which is obviously symmetric and it the object that appears in the Einstein equations

$$R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi G (\Theta_H)_{\mu\nu}. \tag{1.51}$$

  In fact one can choose the Belinfante-Rosenfeld tensor such that it is equal to the Hilbert energy-momentum tensor.

1.4 Quantisation in the Schrödinger Picture

Before quantising field theory let us briefly recap the transition from classical to quantum mechanics. We first switch from the Lagrange formulation to the canonical formalism of the classical theory. In classical mechanics the canonical momentum conjugate to $q_i(t)$ is

$$p_i(t) = \frac{\partial L}{\partial \dot{q}_i(t)}. \tag{1.52}$$

The Hamiltonian is the Legendre transformation of the Lagrange function L

$$H = \sum_i p_i(t)\dot{q}_i(t) - L. \tag{1.53}$$

To quantise in the Schrödinger picture we drop the time dependence of q_i and p_i and promote them to self-adjoint operators without any time dependence such that the fundamental commutation relation

$$[q_i, p_j] = i\delta_{ij} \tag{1.54}$$

holds. Then all time dependence lies in the states.

This procedure is mimicked in a field theory by first defining clasically

$$\boxed{\Pi(t,\vec{x}) := \frac{\partial \mathcal{L}}{\partial \dot{\phi}(t,\vec{x})}} \tag{1.55}$$

to be the conjugate momentum density. The Hamiltonian is

$$\boxed{H = \int \mathrm{d}^3x\, \mathcal{H} = \int \mathrm{d}^3x [\Pi(t,\vec{x})\dot{\phi}(t,\vec{x}) - \mathcal{L}],} \tag{1.56}$$

where $\mathcal{H}$ is the Hamiltonian density. For the scalar field action (1.24) one finds

$$\Pi(t,\vec{x}) = \dot{\phi}(t,\vec{x}) \tag{1.57}$$

and therefore

$$\begin{aligned} H &= \int \mathrm{d}^3x \left[\dot{\phi}^2(t,\vec{x}) - \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}m^2\phi^2(t,\vec{x}) \right] \\ &= \int \mathrm{d}^3x \Big[\underbrace{\frac{1}{2}\dot{\phi}^2(t,\vec{x})}_{=\frac{1}{2}\Pi^2(t,\vec{x})} + \frac{1}{2}(\nabla\phi(t,\vec{x}))^2 + \frac{1}{2}m^2\phi^2(t,\vec{x}) \Big]. \end{aligned} \tag{1.58}$$

Note that as in classical mechanics one can define a Poisson bracket which induces a natural sympletic structure on phase space. In this formalism the Noether charges Q are the generators of their underlying symmetries with the respect to the Poisson bracket (see Assignment 1 for details).

We now quantise in the Schrödinger picture. Therefore we drop the time-dependence of ϕ and Π and promote them to Schrödinger-Picture operators $\phi^{(s)}(\vec{x})$ and $\Pi^{(s)}(\vec{x})$. For real scalar fields we get self-adjoint operators $\phi^{(s)}(\vec{x}) = (\phi^{(s)}(\vec{x}))^\dagger$ with the canonical commutation relations (dropping $^{(s)}$ from now on)

$$\boxed{[\phi(\vec{x}),\Pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,, \quad [\phi(\vec{x}),\phi(\vec{y})] = 0 = [\Pi(\vec{x}),\Pi(\vec{y})].} \tag{1.59}$$

1.5 Mode expansion

Our Hamiltonian (1.58) resembles the Hamiltonian describing a collection of harmonic oscillators, one at each point $\vec{x}$, but the term

$$(\nabla\phi)^2 \approx \left(\frac{\phi(\vec{x}+\delta\vec{x}) - \phi(\vec{x})}{|\delta\vec{x}|} \right)^2 \tag{1.60}$$

couples the degrees of freedom at $\vec{x}$ and $\vec{x}+\delta\vec{x}$. To arrive at a description in which the harmonic oscillators are decoupled, we must diagonalise the potential. Now, a basis of eigenfunctions with

respect to ∇ is $e^{i\vec{p}\cdot\vec{x}}$. Thus the interaction will be diagonal in momentum space. With this motivation we Fourier-transform the fields as

$$\begin{aligned}\phi(\vec{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\tilde{\phi}(\vec{p})e^{i\vec{p}\cdot\vec{x}},\\ \Pi(\vec{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\tilde{\Pi}(\vec{p})e^{i\vec{p}\cdot\vec{x}},\end{aligned} \tag{1.61}$$

where $\tilde{\phi}^\dagger(\vec{p}) = \tilde{\phi}(-\vec{p})$ ensures that $\phi(\vec{x})$ is self-adjoint. To compute H in Fourier space we must insert these expressions into (1.58). First note that

$$\begin{aligned}\frac{1}{2}\int \mathrm{d}^3x(\nabla\phi(\vec{x}))^2 &= \frac{1}{2}\int \mathrm{d}^3x\left(\int \frac{\mathrm{d}^3p}{(2\pi)^3}\nabla e^{i\vec{p}\cdot\vec{x}}\tilde{\phi}(\vec{p})\right)^2\\ &= \frac{1}{2}\int \mathrm{d}^3x\int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6}(-\vec{p}\cdot\vec{q})e^{i(\vec{p}+\vec{q})\cdot\vec{x}}\tilde{\phi}(\vec{p})\tilde{\phi}(\vec{q}).\end{aligned} \tag{1.62}$$

Thanks to the important equality

$$\boxed{\int \mathrm{d}^3x\, e^{i(\vec{p}+\vec{q})\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q}),} \tag{1.63}$$

the latter equation yields

$$\frac{1}{2}\int \mathrm{d}^3x(\nabla\phi(\vec{x}))^2 = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\vec{p}^2\underbrace{\tilde{\phi}(\vec{p})\tilde{\phi}(-\vec{p})}_{\equiv|\tilde{\phi}(\vec{p})|^2}, \tag{1.64}$$

and therefore with (1.58) altogether

$$\boxed{H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{1}{2}|\tilde{\Pi}(\vec{p})|^2 + \frac{1}{2}\omega_p^2|\tilde{\phi}(\vec{p})|^2\right],} \tag{1.65}$$

where $\omega_p = \sqrt{\vec{p}^2+m^2}$. This is a collection of *decoupled harmonic oscillators* of frequency ω_p - and indeed the Hamiltonian is diagonal in momentum space.

The next step is to solve these oscillators in close analogy with the quantum mechanical treatment of a harmonic oscillator of frequency ω with Hamiltonian

$$H = \frac{1}{2}\Pi^2 + \omega^2q^2. \tag{1.66}$$

The quantum mechanical definition of ladder operators $a, a^\dagger$ such that

$$q = \frac{1}{\sqrt{2\omega}}(a+a^\dagger)\ ,\ \ \Pi = -\sqrt{\frac{\omega}{2}}i(a-a^\dagger) \tag{1.67}$$

and with commutation relation $[a,a^\dagger] = 1$ can be generalised to field theory as follows: By taking into account that

$$\tilde{\phi}(\vec{p}) = \tilde{\phi}^\dagger(-\vec{p})\ ,\ \ \tilde{\Pi}(\vec{p}) = \tilde{\Pi}^\dagger(-\vec{p}), \tag{1.68}$$

we define the operators

$$\begin{aligned} a(\vec{p}) &= \frac{1}{2}\left[\sqrt{2\omega_p}\,\tilde{\phi}(\vec{p}) + i\sqrt{\frac{2}{\omega_p}}\,\tilde{\Pi}(\vec{p})\right], \\ a^\dagger(\vec{p}) &= \frac{1}{2}\left[\sqrt{2\omega_p}\,\tilde{\phi}(-\vec{p}) - i\sqrt{\frac{2}{\omega_p}}\,\tilde{\Pi}(-\vec{p})\right]. \end{aligned} \tag{1.69}$$

Solving for $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(\vec{p})$ and plugging into (1.61) yields

$$\boxed{\begin{aligned} \phi(\vec{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right), \\ \Pi(\vec{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} - a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right). \end{aligned}} \tag{1.70}$$

From (1.61) and (1.59) we furthermore deduce the commutation relations

$$[\tilde{\phi}(\vec{p}),\tilde{\Pi}(\vec{q})] = \int \mathrm{d}^3x\mathrm{d}^3y e^{-i\vec{p}\cdot\vec{x}}e^{-i\vec{q}\cdot\vec{y}}\underbrace{[\phi(\vec{x}),\Pi(\vec{y})]}_{=i\delta^{(3)}(\vec{x}-\vec{y})} = (2\pi)^3 i\delta^{(3)}(\vec{p}+\vec{q}), \tag{1.71}$$

where again (1.63) was used, and

$$[\tilde{\phi}(\vec{p}),\tilde{\phi}(\vec{q})] = 0 = [\tilde{\Pi}(\vec{p}),\tilde{\Pi}(\vec{q})]. \tag{1.72}$$

The ladder operators therefore obey the commutation relation

$$\boxed{\begin{aligned} &[a(\vec{p}),a^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}), \\ &[a^\dagger(\vec{p}),a^\dagger(\vec{q})] = 0 = [a(\vec{p}),a(\vec{q})]. \end{aligned}} \tag{1.73}$$

The Hamiltonian (1.65) in mode expansion is

$$\begin{aligned} H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\Bigg[&\frac{1}{2}\left(i\sqrt{\frac{\omega_p}{2}}\right)^2\left(a(\vec{p}) - a^\dagger(-\vec{p})\right)\left(a(-\vec{p}) - a^\dagger(\vec{p})\right) \\ &+ \frac{\omega_p^2}{2}\frac{1}{2\omega_p}\left(a(\vec{p}) + a^\dagger(-\vec{p})\right)\left(a(-\vec{p}) + a^\dagger(\vec{p})\right)\Bigg]. \end{aligned} \tag{1.74}$$

Only cross-terms survive due to the commutation relations:

$$\begin{aligned} H &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_p}{4}\left[a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(-\vec{p})a(-\vec{p})\right]\cdot 2 \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_p}{2}\left[a(\vec{p})a^\dagger(\vec{p}) + a^\dagger(-\vec{p})a(-\vec{p})\right] \\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_p}{2}\left[a^\dagger(\vec{p})a(\vec{p}) + (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}) + a^\dagger(-\vec{p})a(-\vec{p})\right]. \end{aligned} \tag{1.75}$$

Therefore H in its final form is given by

$$\boxed{H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_p \, a^\dagger(\vec{p}) a(\vec{p}) + \Delta_H,} \tag{1.76}$$

where we renamed $-\vec{p} \to \vec{p}$ in the second term. The additional constant Δ_H is

$$\Delta_H = \frac{1}{2} \int \mathrm{d}^3 p \, \omega_p \, \delta^{(3)}(0), \tag{1.77}$$

which is clearly divergent. An interpretation will be given momentarily. By explicit computation one finds that H obeys the commutation relations

$$\boxed{\begin{aligned} [H, a(\vec{p})] &= -\omega_p \, a(\vec{p}), \\ [H, a^\dagger(\vec{p})] &= \omega_p \, a^\dagger(\vec{p}). \end{aligned}} \tag{1.78}$$

Similarly one computes the spatial momentum operator

$$P^i = \int \mathrm{d}^3 x \, \dot{\phi}(\vec{x}) \partial^i \phi(\vec{x}), \tag{1.79}$$

to be

$$P^i = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^i a^\dagger(\vec{p}) a(\vec{p}) + \Delta_{p^i}, \tag{1.80}$$

with

$$\Delta_{p^i} = \frac{1}{2} \int \mathrm{d}^3 p \, p^i \, \delta^{(3)}(0) \equiv 0. \tag{1.81}$$

We combine H and P into the 4-momentum operator

$$\boxed{P^\mu = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^\mu a^\dagger(\vec{p}) a(\vec{p}) + \Delta_{p^\mu},} \tag{1.82}$$

with $p^\mu = (p^0, \vec{p}) = (\omega_p, \vec{p})$. It obeys the commutation relations

$$\boxed{\begin{aligned} [P^\mu, a^\dagger(\vec{p})] &= p^\mu a^\dagger(\vec{p}), \\ [P^\mu, a(\vec{p})] &= -p^\mu a(\vec{p}). \end{aligned}} \tag{1.83}$$

1.6 The Fock space

We now find the Hilbert space on which the 4-moment operator P^μ acts. The logic is analogous to the considerations leading to the representation theory of the harmonic oscillator in Quantum Mechanics:

- Since P^μ is self-adjoint it has eigenstates with real eigenvalues. Let $|k^\mu\rangle$ be such an eigenstate with

$$P^\mu |k^\mu\rangle = k^\mu |k^\mu\rangle . \tag{1.84}$$

Then as a result of (1.83)

$$\begin{aligned} P^\mu a^\dagger(\vec{q})\,|k^\mu\rangle &= a^\dagger(\vec{q})P^\mu\,|k^\mu\rangle + q^\mu a^\dagger(\vec{q})\,|k^\mu\rangle \\ &= (k^\mu + q^\mu)a^\dagger(\vec{q})\,|k^\mu\rangle \end{aligned} \tag{1.85}$$

and similarly

$$P^\mu a(\vec{q})\,|k^\mu\rangle = (k^\mu - q^\mu)a(\vec{q})\,|k^\mu\rangle\,. \tag{1.86}$$

This means that $a(\vec{q})$ and $a^\dagger(\vec{q})$ are indeed ladder operators which respectively subtract and add 4-momentum q^μ to or from $|k^\mu\rangle$.

- Next we observe that the Hamiltonian $H = P^0$ given by (1.76) is non-negative, i.e. $\langle\psi|H|\psi\rangle \geq 0\ \forall$ states $|\psi\rangle$ because

$$\langle\psi|H|\psi\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p\langle\psi|a^\dagger(\vec{p})a(\vec{p})|\psi\rangle + \Delta_H\langle\psi|\psi\rangle \geq 0. \tag{1.87}$$

Thus there exists a state $|0\rangle$ such that

$$a(\vec{q})\,|0\rangle = 0\ \ \forall\,\vec{q}. \tag{1.88}$$

Otherwise successive action of $a(\vec{q})$ would lead to negative eigenvalues of H. $|0\rangle$ is called the **vacuum** of the theory. It has 4-momentum

$$P^\mu\,|0\rangle = \Delta_{p^\mu}\,|0\rangle = \begin{cases} \Delta_H, & \mu = 0 \\ 0\ , & \mu = i \end{cases}\ . \tag{1.89}$$

- We interpret the divergent constant Δ_H given in (1.77) as the **vacuum energy**. A more thorough discussion of the significance of the divergence will be given later. For now we should note that in a theory without gravity absolute energy has no meaning. We can hus discard the additive constant Δ_{p^μ} by defining

$$\tilde{P}^\mu := P^\mu - \Delta_{p^\mu} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}p^\mu a^\dagger(\vec{p})a(\vec{p}), \tag{1.90}$$

with $\tilde{P}^\mu\,|0\rangle = 0$. From now on we only work with $\tilde{P}^\mu$ and drop the tilde.

- The state $a^\dagger(\vec{p})\,|0\rangle$ then has 4-momentum p^μ,

$$P^\mu a^\dagger(\vec{p})\,|0\rangle = p^\mu a^\dagger(\vec{p})\,|0\rangle\,, \tag{1.91}$$

with $p^\mu = (E_p, \vec{p})$ and $E_p = \sqrt{\vec{p}^2 + m^2}$. Since this is the relativistic dispersion relation for a single particle with mass m we interpret $a^\dagger(\vec{p})\,|0\rangle$ as a 1-particle state with energy E_p and momentum $\vec{p}$.

- More generally an N-particle state with energy $E = E_{p_1} + ... + E_{p_N}$ and momentum $\vec{p} = \vec{p}_1 + ...\vec{p}_N$ is given by
$$a^\dagger(\vec{p}_1)a^\dagger(\vec{p_2})...a^\dagger(\vec{p}_N)\,|0\rangle\,. \tag{1.92}$$

So much about the formalism. To get a better feeling for the objects we have introduced, let us recap what we have done: We have started with the assertion that spacetime - in our case $\mathbb{R}^{1,3}$ - is filled with the real scalar field $\phi(\vec{x})$, which we have taken to be a free field with Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$. This field is interpreted as a field operator, i.e. in the Schrödinger picture at every space point $\vec{x}$ the object $\phi(\vec{x})$ represents a self-adjoint operator that acts on a Hilbert space. This Hilbert space possesses a state of lowest energy, the vacuum $|0\rangle$. The vacuum corresponds to the absence of any excitations of the field ϕ (at least on-shell). If one pumps energy E_p and momentum $\vec{p}$ into some region of spacetime such that the relativistic dispersion relation $E_p = \sqrt{\vec{p}^2 + m^2}$ holds, a particle $a^\dagger(\vec{p})\,|0\rangle$ is created as an excitation of $\phi(\vec{x})$. In particular, for the free theory the parameter m in $\mathcal{L}$ is interpreted as the mass of such a particle. Since the underlying field $\phi(x)$ is a scalar field, the associated particle is called scalar particle. This realizes the shift of paradigm advertised at the very beginning of this course that the fundamental entity in Quantum Field Theory is not the particle, but rather the field:

> *The field $\phi(\vec{x})$ is the property of spacetime that in the presence of energy and momentum $(E_p, \vec{p})$ a particle of energy $(E_p, \vec{p})$ can be created.*

In particular, this naturally gives rise to a multi-particle theory. The particles are just the excitations of the field and transitions with varying particle number can occur as long as the kinematics allows it. A first new, non-trivial conclusion we can draw from the formalism developed so far is the following special case of the **spin-statistics theorem**:

> *Scalar particles obey Bose statistics.*

All we have to show that the N-particle wavefunction is symmetric under permutations. This follows immediately from the commutation relations, specifically the second line in (1.73):

$$\begin{aligned}|\vec{p}_1, ..., \vec{p}_i, ..., \vec{p}_j, ..., \vec{p}_n\rangle &\simeq a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_i)...a^\dagger(\vec{p}_j)...a^\dagger(\vec{p}_N)\,|0\rangle \\ &= a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_j)...a^\dagger(\vec{p}_i)...a^\dagger(\vec{p}_N)\,|0\rangle \\ &\simeq |\vec{p}_1, ..., \vec{p}_j, ..., \vec{p}_i, ..., \vec{p}_n\rangle,\end{aligned} \tag{1.93}$$

where we have not fixed the normalization of the Fock state yet.

1.7 Some important technicalities

1.7.1 Normalisation

For reasons that will become clear momentarily, we choose to normalise the 1-particle momentum eigenstates as $|\vec{p}\rangle := \sqrt{2E_p}a^\dagger(\vec{p})\,|0\rangle$ and, more generally,

$$|\vec{p}_1, ..., \vec{p}_n\rangle := \sqrt{2E_{p_1}\cdot ...\cdot 2E_{p_N}}a^\dagger(\vec{p}_1)...a^\dagger(\vec{p}_N)\,|0\rangle\,. \tag{1.94}$$

Then

$$\langle \vec{q}|\vec{p}\rangle = \sqrt{2E_p}\sqrt{2E_q}\,\langle 0|\, a(\vec{q})a^\dagger(\vec{p})\,|0\rangle\,. \tag{1.95}$$

To compute the inner product of two such states we use an important trick: Move all a's to the right and all $a^\dagger$ to the left with the help of

$$a(\vec{q})a^\dagger(\vec{p}) = a^\dagger(\vec{q})a(\vec{p}) + (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}). \tag{1.96}$$

This gives rise to terms of the form $a(\vec{q})\,|0\rangle = 0 = \langle 0|\, a^\dagger(\vec{p})$. Therefore

$$\boxed{\langle \vec{q}|\vec{p}\rangle = (2\pi)^3 2E_p\delta^{(3)}(\vec{p}-\vec{q}).} \tag{1.97}$$

Note that, as in Quantum Mechanics, momentum eigenstates are not strictly normalisable due to the appearance of the delta-distribution, but we can form normalisable states as wavepackets

$$|f\rangle = \int \mathrm{d}^3p f(\vec{p})\,|\vec{p}\rangle\,. \tag{1.98}$$

1.7.2 The identity

With the above normalisation the identity operator on the 1-particle Hilbert space is

$$\boxed{\mathbb{1}_{1-\text{particle}} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\,|\vec{p}\rangle\,\langle\vec{p}|\,.} \tag{1.99}$$

One should notice that $\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}$ is a Lorentz-invariant measure. This, in turn, is part of the motivation for the normalisation of the 1-particle momentum eigentstates. To see this we rewrite the measure as

$$\begin{aligned}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p} &= \int \frac{\mathrm{d}^3p}{(2\pi)^4}\frac{2\pi}{2E_p}\\ &= \int \frac{\mathrm{d}^4p}{(2\pi)^4}2\pi\delta(p^2-m^2)\Theta(p^0),\end{aligned} \tag{1.100}$$

where we used that $\delta(ax) = \frac{1}{a}\delta(x)$ and that

$$\delta(p^2-m^2) = \delta\big((p^0-E_p)(p^0+E_p)\big) \tag{1.101}$$

in the last step. (1.100) is manifestly Lorentz-invariant: First, $\mathrm{d}^4p \to \det(\Lambda)\mathrm{d}^4p$ under a Lorentz-transformation, and since the determinat of a Lorentz-transformation is 1, it follows that d^4p is Lorentz-invariant. Moreover the sign of p^0 is unchanged under a Lorentz transformation.

1.7.3 Position-space representation

In Quantum Mechanics, the position eigenstate is related by a Fourier transformation to the momentum eigenstates, $|x\rangle = \int \frac{\mathrm{d}p}{2\pi}e^{-ipx}\,|p\rangle$ with $\langle x|p\rangle = e^{ipx}$. Due to our normalisation

$$|\vec{p}\rangle = \sqrt{2E_p}a^\dagger(\vec{p})\,|0\rangle\,, \tag{1.102}$$

the correct expression for $|\vec{x}\rangle$ in QFT is

$$\boxed{|\vec{x}\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle} \tag{1.103}$$

because then

$$\langle \vec{x}|\vec{p}\rangle = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2E_q} e^{i\vec{q}\cdot\vec{x}} \langle \vec{q}|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}}, \tag{1.104}$$

where we used (1.97). Note that

$$|\vec{x}\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(e^{-i\vec{p}\cdot\vec{x}} a^\dagger(\vec{p}) + e^{i\vec{p}\cdot\vec{x}} a(\vec{p})\right)|0\rangle = \phi(\vec{x})\,|0\rangle\,. \tag{1.105}$$

In other words, the field operator $\phi(x)$ acting on the vacuum $|0\rangle$ creates a 1-particle position eigenstate.

1.8 On the vacuum energy

We had seen that originally

$$H = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_p a^\dagger(\vec{p}) a(\vec{p}) + \Delta_H, \qquad \Delta_H = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_p\, (2\pi)^3 \delta(0), \tag{1.106}$$

where Δ_H is the vacuum energy E_0 such that $H|0\rangle = \Delta_H |0\rangle \equiv E_0 |0\rangle$. E_0 is the first example of a divergent quantity in QFT. In fact, it realises the two characteristic sources of a possible divergence in QFT:

- The divergent factor $(2\pi)^3 \delta^{(3)}(0)$ is interpreted as follows: We know that

$$\int_{\mathbb{R}^3} \mathrm{d}^3 x\, e^{i\vec{p}\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{p}), \tag{1.107}$$

  so formally the volume of $\mathbb{R}^3$ is given by

$$V_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \mathrm{d}^3 x = (2\pi)^3 \delta^{(3)}(0). \tag{1.108}$$

  The divergence of $\delta^{(3)}(0)$ is rooted in the fact that the volume of $\mathbb{R}^3$ is infinite, and the corresponding divergent factor in E_0 arises because we are computing an energy in an infinite volume. This divergent factor thus results from the long-distance (i.e. small energy) behaviour of the theory and is an example of an **infra-red (IR) divergence**. Generally in QFT, IR divergences signal that we are either making a mistake or ask an unphysical question. In our case, the mistake is to consider the theory in a strictly infinite volume, which is of course unphysical. One can regularise the IR divergence by instead considering the theory in a given, but finite volume. What is free of the IR divergence is in particular the vacuum energy density

$$\epsilon_0 = \frac{E_0}{V_{\mathbb{R}^3}} = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \omega_p. \tag{1.109}$$

- Nonetheless, even ϵ_0 remains divergent because

$$\epsilon_0 = \frac{E_0}{V_{\mathbb{R}^3}} = \int \frac{\mathrm{d}^3 p}{2(2\pi)^3} \sqrt{\vec{p}^2 + m^2} = \frac{1}{2} \frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}p p^2 \sqrt{p^2 + m^2}, \tag{1.110}$$

  which goes to infinity due to the integration over all momenta up to $p \to \infty$. This is an **ultraviolet (UV) divergence**. The underlying reason for this (and all other UV divergences in QFT) is the breakdown of the theory at high energies (equivalently at short distances) - or at least a breakdown of our treatment of the theory.

To understand this last point it is beneficial to revisit the mechanical model of the field $\phi(t, x)$ as an excitation of a mechanical string as introduced in section 1.1. Recall that the field $\phi(t, x)$ describes the transverse position of the string in the continuum limit of vanishing distance Δ between the individual mass points at position $q_i(t)$ which were thought of as connected by an elastic spring. However, in reality the string is made of atoms of finite, typical size R. A continuous string profile $\phi(t, x)$ is therefore not an adequate description at distances $\Delta \leq R$ or equivalently at energies $E \geq \Lambda \simeq 1/R$ resolving such small distances. Rather, if we want to describe processes at energies $E \geq \Lambda$, the continuous field theory $\phi(t, x)$ is to be replaced by the more fundamental, microscopic theory of atoms in a lattice. In this sense the field $\phi(t, x)$ gives merely an effective description of the string valid at energies $E \leq \Lambda$. Extrapolation of the theory beyond such energies is doubtful and can give rise to infinities - the UV divergences.

In a modern approach to QFT, this reasoning is believed to hold also for the more abstract relativistic fields we are considering in this course. According to this logic, **QFT is really an effective theory that eventually must be replaced at high energies by a more fundamental theory**. A necessary condition for such a fundamental theory to describe the microscopic degrees of freedom correctly is that it must be free of pathologies of all sort and in particular be UV finite.[3] At the very least, gravitational degrees of freedom become important in the UV region and are expected to change the qualitative behaviour of the theory at energies around the Planck scale $M_P \simeq 10^{19} GeV$.

Despite these limitations, in a 'good' QFT the UV divergences can be removed - *for all practical purposes* - by the powerful machinery of **regularisation and renormalisation**. We will study this procedure in great detail later in the course, but let us take this opportunity to very briefly sketch the logic for the example of the vacuum energy density:

- The first key observation is that in the classical Lagrangian we can have a constant term V_0 of dimension mass^4,

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 - V_0 \,, \quad V_0 = V|_{\min}, \tag{1.111}$$

  corresponding to the value of the classical potential at the minimum around which we expand. We had set $V_0 \equiv 0$ in our analysis because we had argued that such an overall energy offset is not measurable in a theory without gravity. However, as we have seen the vacuum energy

[3] To date, the only known fundamental theory that meets this requirement including gravity is string theory.

density really consists of two pieces - the classical offset V_0 and the quantum piece Δ_H. Thus, let us keep V_0 for the moment and derive the Hamiltonian

$$H = \int \mathrm{d}^3x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}m^2\,\phi^2 + \frac{1}{2}(\nabla\phi)^2\right) + \int \mathrm{d}^3x\, V_0. \tag{1.112}$$

- Next, we regularise the theory: Introduce a cutoff-scale Λ and for the time being only allow for energies $E \leq \Lambda$. If we quantise the theory with such a cutoff at play, the overall vacuum energy is now

$$H\,|0\rangle = V_{\mathbb{R}^3}(\epsilon_0(\Lambda) + V_0)\,|0\rangle \tag{1.113}$$

with

$$\epsilon_0(\Lambda) = \frac{1}{(2\pi)^2}\int_0^\Lambda \mathrm{d}p\; p^2\;\sqrt{p^2+m^2}. \tag{1.114}$$

Note that the momentum integral only runs up to the cutoff Λ in the regularised theory.

- Since V_0 is just a parameter, we can set

$$V_0 = V_0(\Lambda) = -\epsilon_0(\Lambda) + \chi, \qquad \chi \text{ finite as } \Lambda \to \infty. \tag{1.115}$$

With this choice

$$H\,|0\rangle = V_{\mathbb{R}^3}\,\chi\,|0\rangle \tag{1.116}$$

independently of Λ!This way we absorb the divergence into a cutoff-dependent counterterm $V_0(\Lambda)$ in the action such that the total vacuum energy density is finite. This step is called renormalisation. Crucially, note that the finite piece χ is completely arbitrary a priori. It must be determined experimentally by measuring a certain observable - in this case the vacuum energy (which is a meaningful observable only in the presence of gravity - see below).

- Finally, we can remove the cutoff by taking $\Lambda \to \infty$.

To summarize, we *define* the quantum theory as the result of quantising the classical Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\,\phi^2 - V_0(\Lambda)$ with $V_0(\Lambda) = -\epsilon_0(\Lambda) + \chi$ and taking the limit $\Lambda \to \infty$ at the very end. Since Λ appears only in the classical - or bare - Lagrangian, but in no observable, physical quantity at any of the intermediate stages, we can safely remove it at the end. In this sense, the theory is *practically* defined up to all energies.

This way to deal with UV divergences comes at a prize: We lose the prediction of one observable per type of UV divergence as a result of the inherent arbitrariness of the renormalisation step. In fact, above we have *chosen* $V_0(\Lambda)$ such that $H\,|0\rangle = V_{\mathbb{R}^3}\,\chi\,|0\rangle$. It is only in the absence of gravity that the vacuum energy is unobservable and thus χ is irrelevant. More generally, the value of the vacuum energy must now be taken from experiment - χ is an input parameter of the theory rather than a prediction. While we have succeeded in removing the divergence by renormalising the original Lagrangian,

the actual value of the physical observable associated with the divergence - here the vacuum energy density - must be taken as an input parameter from experiment or from other considerations. In a renormalisable QFT it is sufficient to do this for a finite number of terms in the Lagrangian so that once the associated observables are specified, predictive power is maintained for the computation of all subsequent observables.

The Cosmological Constant in gravity

In gravity, the vacuum energy density is observable because it gravitates and we have to carry the vacuum energy with the field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu} + (\epsilon_0 + V_0)g_{\mu\nu}. \tag{1.117}$$

This form of the Einstein field equations leads to accelerated expansion of the Universe. Indeed, observations indicate that the universe expands in an accelerated fashion, and the simplest - albeit not the only possible - explanation would be to identify the underlying 'Dark energy' with the vacuum energy. Observationally, this would then point to $\epsilon_0 + V_0 = \chi = (10^{-3}\text{eV})^4$. Of course in our QFT approach it is impossible to *explain* such a value of the vacuum energy because as a side-effect of renormalisation we gave up on predicting it. In this respect, Quantum Field Theory remains an effective description. In a fundamental, UV finite theory this would be different: There the net vacuum energy would be the difference of a finite piece V_0 and a finite quantum contribution ϵ_0, both of which would be computable from first principles (if the theory is truly fundamental). Thus the two quantities should almost cancel each other. There is a problem, though: On dimensional grounds one expects both V_0 and ϵ_0 to be of the order of the fundamental scale in the theory, which in a theory of quantum gravity is the Planck scale $M_P = 10^{19}$ GeV. The expected value for the vacuum energy density is thus M_P^4, which differs by the observed value by about 122 orders of magnitude difference. Thus, the difference of V_0 and ϵ_0 must be by 122 orders of magnitude smaller than both individual numbers. Such a behaviour is considered immense fine-tuning and thus unnatural. The famous Cosmological Constant Problem is therefore the puzzle of why the observed value is so small.[4]

1.9 The complex scalar field

We now extend the formalism developed so far to the theory of a complex scalar field, which classically no longer satisfies $\phi(x) = \phi^*(x)$. A convenient way to describe a complex scalar field of mass m is to note that its real and imaginary part can be viewed as independent real scalar fields ϕ_1 and ϕ_2 of mass m, i.e. we can write

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right). \tag{1.118}$$

The real fields ϕ_1 and ϕ_2 are canonically normalised if we take as Lagrangian for the complex field

$$\mathcal{L} = \partial_\mu\phi^\dagger(x)\partial^\mu\phi(x) - m^2\phi^\dagger(x)\phi(x). \tag{1.119}$$

[4] In fact, the problem is even more severe as becomes apparent in the Wilsonian approach to be discussed in QFT2. For more information see e.g. the review arXiv:1309.4133 by Cliff Burgess.

It is then a simple matter to repeat the programme of quantisation, e.g. by quantizing ϕ_1 and ϕ_2 as before and rewriting everything in terms of the complex field $\phi(x)$. At the end of this rewriting we can forget about ϕ_1 and ϕ_2 and simply describe the theory in terms of the complex field $\phi(x)$. The details of this exercise will be provided in the tutorials so that we can be brief here and merely summarise the main formulae.

- The fields $\phi(x)$ and $\phi^\dagger(x)$ describe independent degrees of freedom with respective conjugate momenta

$$\boxed{\begin{aligned}\Pi(t,\vec{x}) =& \frac{\partial \mathcal{L}}{\partial \dot{\phi}(t,\vec{x})} = \dot{\phi}^\dagger(t,\vec{x}),\\ \Pi^\dagger(t,\vec{x}) =& \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger(t,\vec{x})} = \dot{\phi}(t,\vec{x}).\end{aligned}} \tag{1.120}$$

  The Hamiltonian H is

$$\begin{aligned}H &= \int \mathrm{d}^3x \left(\Pi^\dagger(\vec{x})\dot{\phi}^\dagger(\vec{x}) + \Pi(\vec{x})\dot{\phi}(\vec{x}) - \mathcal{L}\right)\\ &= \int \mathrm{d}^3x \left(\Pi^\dagger(\vec{x})\Pi(\vec{x}) + \nabla\phi^\dagger(\vec{x})\nabla\phi(\vec{x}) + m^2\phi^\dagger(\vec{x})\phi(\vec{x})\right).\end{aligned} \tag{1.121}$$

- These fields are promoted to Schrödinger picture operators with non-vanishing commutators

$$\boxed{[\phi(\vec{x}),\Pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) = \left[\phi^\dagger(\vec{x}),\Pi^\dagger(\vec{y})\right]} \tag{1.122}$$

  and all other commutators vanishing.

- The mode expansion is conveniently written as

$$\begin{aligned}\phi(\vec{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec{p})e^{i\vec{p}\cdot\vec{x}} + b^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right)\\ \phi^\dagger(\vec{x}) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(b(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right),\end{aligned} \tag{1.123}$$

  where the mode operators $a(\vec{p})$ and $b(\vec{p})$ are independent and $a^\dagger(\vec{p})$ and $b^\dagger(\vec{p})$ describe the respective conjugate operators. A quick way to arrive at this form of the expansion is to plug the mode expansion of the real fields ϕ_1 and ϕ_2 into (1.118). This identifies

$$\begin{aligned}a =& \frac{1}{\sqrt{2}}(a_1 + ia_2),\\ b^\dagger =& \frac{1}{\sqrt{2}}(a_1^\dagger + ia_2^\dagger).\end{aligned} \tag{1.124}$$

  In particular this implies that

$$[a(\vec{p}),a^\dagger(\vec{q})] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) = [b(\vec{p}),b^\dagger(\vec{q})], \tag{1.125}$$

  while all other commutators vanish.

- The mode expansion of the 4-momentum operator is

$$P^\mu = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} p^\mu \left(a^\dagger(\vec{p}) a(\vec{p}) + b^\dagger(\vec{p}) b(\vec{p}) \right). \tag{1.126}$$

  Crucially, one can establish that it has 2 types of momentum eigenstates

$$a^\dagger(\vec{p})\,|0\rangle \text{ and } b^\dagger(\vec{p})\,|0\rangle\,, \tag{1.127}$$

  both of energy $E_p = \sqrt{\vec{p}^2 + m^2}$. I.e. both states have mass m, but they differ in their $U(1)$ charge as we will see now:

- The Lagrange density is invariant under the global continuous $U(1)$ symmetry

$$\phi(x) \to e^{i\alpha} \phi(x), \tag{1.128}$$

  where α is a constant in $\mathbb{R}$. Recall that the unitary group $U(N)$ is the group of complex $N \times N$ matrices A satisfying $A^\dagger = A^{-1}$. The dimension of this group is N^2. In particular $e^{i\alpha} \in U(1)$. According to Noether's theorem there exists a conserved current (see Ass. 3)

$$\boxed{j^\mu = -i(\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi)} \tag{1.129}$$

  and charge

$$\boxed{Q = \int \mathrm{d}^3 x j^0 = -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(a^\dagger(\vec{p}) a(\vec{p}) - b^\dagger(\vec{p}) b(\vec{p}). \right)} \tag{1.130}$$

  This Noether charge acts on a particle with momentum $\vec{p}$ as follows:

$$\begin{aligned} Q a^\dagger(\vec{p})\,|0\rangle &= -\,a^\dagger(\vec{p})\,|0\rangle \;:\; \text{charge } -1, \\ Q b^\dagger(\vec{p})\,|0\rangle &= +\,b^\dagger(\vec{p})\,|0\rangle \;:\; \text{charge } +1. \end{aligned} \tag{1.131}$$

  We interpret $a^\dagger(\vec{p})\,|0\rangle$ as a particle of mass m and charge -1 and $b^\dagger(\vec{p})\,|0\rangle$ as a particle with the same mass, but positive charge, i.e. as its anti-particle. For the real field the particle is its own anti-particle. Note that the term 'charge ± 1' so far refers simply to the eigenvalue of the Noether charge operator Q associated with the global $U(1)$ symmetry of the theory. That this abstract charge really coincides with what we usually call charge in physics - i.e. that it describes the coupling to a Maxwell type field - will be confirmed later when we study Quantum Electrodynamics.

1.10 Quantisation in the Heisenberg picture

So far all field operators have been defined in the Schrödinger picture, in which the time-dependence is carried entirely by the states on which these operators act. From Quantum Mechanics we recall

that alternatively quantum operators can be described in Heisenberg picture (HP), where the time-dependence is carried by the operators $A^{(H)}(t)$ and not the states. The HP operator $A^{(H)}(t)$ is defined as

$$A^{(H)}(t) = e^{iH^{(S)}(t-t_0)}A^{(S)}e^{-iH^{(S)}(t-t_0)}, \tag{1.132}$$

where $A^{(S)}$ is the corresponding Schrödinger picture operator and $H^{(S)}$ is the Schrödinger picture Hamilton operator. At the time t_0 the Heisenberg operator and the Schrödinger operator coincide, i.e. $A^{(H)}(t_0) = A^{(S)}$. We will set $t_0 \equiv 0$ from now on. The definition (1.132) has the following implications:

- $H^{(H)}(t) = H^{(S)}\ \forall t$.
- The time evolution of the Heisenberg picture operators is governed by the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{(H)}(t) = i[H, A^{(H)}(t)]. \tag{1.133}$$

- The Schrödinger picture commutators translate into **equal time** commutators, e.g.

$$[q_i^{(H)}(t), p_j^{(H)}(t)] = i\,\delta_{ij}. \tag{1.134}$$

In field theory we similarly define the Heisenberg fields via

$$\begin{aligned}
\phi^{(H)}(t,\vec{x}) \equiv&\phi(x) = e^{iH^{(S)}t}\phi^{(S)}(\vec{x})e^{-iH^{(S)}t},\\
\Pi^{(H)}(t,\vec{x}) \equiv&\Pi(x) = e^{iH^{(S)}t}\Pi^{(S)}(\vec{x})e^{-iH^{(S)}t},\\
\mathcal{H}^{(H)}(t,\vec{x}) \equiv&\mathcal{H}(x) = e^{iH^{(S)}t}\mathcal{H}^{(S)}(\vec{x})e^{-iH^{(S)}t}.
\end{aligned} \tag{1.135}$$

These obey **equal-time** canonical commutation relations

$$\boxed{\begin{aligned}
[\phi(t,\vec{x}),\Pi(t,\vec{y})] &= i\delta^{(3)}(\vec{x}-\vec{y})\\
[\phi(t,\vec{x}),\phi(t,\vec{y})] &= 0 = [\Pi(t,\vec{x}),\Pi(t,\vec{y})].
\end{aligned}} \tag{1.136}$$

The Heisenberg equation of motion for $\phi(x)$ reads

$$\frac{\partial}{\partial t}\phi(t,\vec{x}) = i[H,\phi(t,\vec{x})] = i\int \mathrm{d}^3y[\mathcal{H}(t,\vec{y}),\phi(t,\vec{x})], \tag{1.137}$$

with

$$\mathcal{H}(t,\vec{y}) = \frac{1}{2}\Pi^2(t,\vec{y}) + \frac{1}{2}(\nabla\phi(t,\vec{y}))^2 + \frac{1}{2}m^2\phi^2(t,\vec{y}) \tag{1.138}$$

for a real scalar field. In the latter equation we used that H is time-independent, i.e. we can evaluate the commutator at arbitrary times and thus choose equal time with $\phi(t,\vec{x})$ so that we can exploit the equal-time commutation relations.
In evaluating (1.137) we observe that the only non-zero term comes from

$$\frac{1}{2}[\Pi^2(t,\vec{y}),\phi(t,\vec{x})] = (-i)\Pi(t,\vec{y})\delta^{(3)}(\vec{x}-\vec{y}), \tag{1.139}$$

where used the standard relation $[A,BC] = [A,B]C + B[A,C]$ together with (1.136). This gives

$$\frac{\partial}{\partial t}\phi(t,\vec{x}) = \Pi(t,\vec{x}) \tag{1.140}$$

as expected. One can similarly show that

$$\begin{aligned}\frac{\partial}{\partial t}\Pi(t,\vec{x}) &= i[H,\Pi(t,\vec{y})] \\ &= \nabla^2\phi(t,\vec{x}) - m^2\phi(t,\vec{x}).\end{aligned} \tag{1.141}$$

Therefore altogether we can establish that the Klein-Gordon equation

$$\boxed{(\partial^2 + m^2)\phi(x) = 0} \tag{1.142}$$

holds as an operator equation at the quantum level.
The covariantisation of (1.137) is

$$\boxed{\partial^\mu\phi(x) = i[P^\mu,\phi(x)]} \tag{1.143}$$

with $P^i = \int \mathrm{d}^3y\,\Pi(y)\,\partial^i\phi(y)$. Indeed this equation can be explicitly confirmed by evaluating the commutator $[P^i,\phi(x)]$. As a consequence we will check, on sheet 3, that

$$\boxed{\phi(x^\mu + a^\mu) = e^{ia^\mu P_\mu}\phi(x)e^{-ia^\mu P_\mu}.} \tag{1.144}$$

This, in fact, is simply the transformation property of the quantum field $\phi(x)$ under translation.
Let us now compute the mode expansion for the Heisenberg field. From the mode expansion for the Schrödinger picture operator we find

$$\phi^{(H)}(t,\vec{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(e^{iHt}a(\vec{p})e^{-iHt}e^{i\vec{p}\cdot\vec{x}} + e^{iHt}a^\dagger(\vec{p})e^{-iHt}e^{-i\vec{p}\cdot\vec{x}}\right). \tag{1.145}$$

To simplify this we would like to commute e^{iHt} through $a(\vec{p})$ and $a^\dagger(\vec{p})$. Since $[H,a(\vec{p})] = -a(\vec{p})E_p$, we can infer that

$$Ha(\vec{p}) = a(\vec{p})(H - E_p) \tag{1.146}$$

and by induction that

$$H^n a(\vec{p}) = a(\vec{p})(H - E_p)^n. \tag{1.147}$$

Thus

$$e^{iHt}a(\vec{p}) = a(\vec{p})e^{i(H-E_p)t}, \tag{1.148}$$

which gives the mode expansion in the Heisenberg picture

$$\boxed{\phi^{(H)}(t,\vec{x}) \equiv \phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right),} \tag{1.149}$$

with

$$p\cdot x = p^0x_0 - \vec{p}\cdot\vec{x}, \qquad (p^0,\vec{p}) = (E_p,\vec{p}). \tag{1.150}$$

In other words, the coefficient of $e^{-ip\cdot x}$ in the mode expansion of the Heisenberg field corresponds to the annihilator and the coefficient of $e^{ip\cdot x}$ to the creator. Note that indeed this mode expansion solves the operator equation of motion (1.142).
For later purposes we also give the inverted expression

$$\boxed{\begin{aligned} a(\vec{q}) &= \frac{i}{\sqrt{2E_q}} \int \mathrm{d}^3x\, e^{iq\cdot x} \overset{\leftrightarrow}{\partial_0} \phi(x), \\ a^\dagger(\vec{q}) &= \frac{-i}{\sqrt{2E_q}} \int \mathrm{d}^3x\, e^{-iq\cdot x} \overset{\leftrightarrow}{\partial_0} \phi(x), \end{aligned}} \tag{1.151}$$

where

- $u(x) \overset{\leftrightarrow}{\partial_0} v(x) := u(x)\partial_0 v(x) - (\partial_0 u(x))v(x)$,
- the integrals $\int \mathrm{d}^3x$ are evaluated at arbitrary times $t = x^0$.

You will check these expressions in the tutorial.

1.11 Causality and Propagators

We are finally in a position to come back to the question of causality in a relativistic quantum theory, which in section 1.1 served as one of our two prime motivations to study Quantum Field Theory. We will investigate the problem from two related points of view - via commutators and propagators.

1.11.1 Commutators

For causality to hold we need two measurements at spacelike distance not to affect each other. This is guaranteed if any two local observables $O_1(x)$ and $O_2(y)$ at spacelike separation commute, i.e.

$$[O_1(x), O_2(y)] \overset{!}{=} 0 \qquad \text{for} \quad (x-y)^2 < 0. \tag{1.152}$$

By a local observable we mean an observable in the sense of quantum mechanics (i.e. a hermitian operator) that is defined locally at a spacetime point x, i.e. it depends only on x and at most on a local neighborhood of x. Any such local observable is represented by a local (hermitian) operator, by which we mean any local (hermitian) expression of the fundamental operators $\phi(x)$ and $\partial_\mu\phi(x)$ such as products or powers series (e.g. exponentials) of the operators. Indeed such operators are local in the sense that they depend only on a local neighborhood of the spacetime point x.
To check for (1.152) we thus need to compute the commutator

$$\Delta(x-y) := [\phi(x), \phi(y)] \tag{1.153}$$

not just at equal times $x^0 = y^0$, but for general times. In Fourier modes we have

$$\begin{aligned} \Delta(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \\ \times \left[a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}, a(\vec{q})e^{-iq\cdot y} + a^\dagger(\vec{q})e^{iq\cdot y}\right]. \end{aligned} \tag{1.154}$$

Using the commutation relations for the modes yields

$$\boxed{\Delta(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{-ip\cdot(y-x)}\right).} \tag{1.155}$$

Now assume $(x-y)^2 < 0$. Since $\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}$ is Lorentz invariant (see the discussion in section 1.7.2) we can apply a Lorentz transformation such that $(x^0 - y^0) = 0$. Indeed this can be always be achieved if two points are at spacelike distance. This gives

$$\Delta(x-y)\Big|_{(x-y)^2<0} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right), \tag{1.156}$$

which is evidently zero because we can change the integration variable from $\vec{p}$ to $-\vec{p}$ in the second term. Consequently, also commutators of the form $[\partial_{x^\mu}\phi(x), \phi(y)] = \partial_{x^\mu}[\phi(x), \phi(y)]$ vanish for x and y at spacelike distance. Therefore for all local operators we have

$$[O_i(x), O_j(y)] = 0 \quad \text{if } (x-y)^2 < 0. \tag{1.157}$$

This establishes that causality is maintained at the operational level.

Remark:
In the above we have been able to *derive* (1.157) because we are working with a free theory, for which the free mode expansion implies $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$. In an interacting theory, such a derivation may not be possible since in this case a free mode expansion is no longer available. More generally, therefore, one must postulate $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$ in form of an axiom of QFT.

A note on Quantum Entanglement

As we have seen, locality in Quantum Field Theory refers to the fact (or requirement) that local operators commute at spacelike distances. By contrast, the states on which these operators act, i.e. the elements of the Hilbert space, do exhibit non-local behaviour just as in non-relativistic Quantum Mechanics. Indeed the fact that in QFT local operators commute is not at odds with the existence of entangled states, known from Quantum Mechanics. Consider e.g. an entangled 2-spin state $|\frac{1}{2}, -\frac{1}{2}\rangle - |-\frac{1}{2}, \frac{1}{2}\rangle$. Even though the state is entangled, the spin operator S_1 for measurement of the spin of particle 1 at x and S_2 for measurement of the spin of particle 2 at y commute. In particular the expectation value of S_2 is not changed by measuring S_1. Therefore by measuring the expectation value of S_2 we cannot determine if S_1 was measured and vice versa and the two measurements 'do not affect each other' in this sense. More generally, the states of the Quantum Field Theory are non-local objects. We will understand this better when discussing the Schrödinger Representation of states in the context of path integral quantization in QFT 2. The states are represented as functionals of the field configuration and are necessarily dependent on non-local information.

1.11.2 Propagators

Consider now the probability amplitude for a particle emitted at y to propagate to x (where it can be measured). This is given by the propagator

$$\boxed{D(x-y) := \langle 0|\, \phi(x)\phi(y)\, |0\rangle\,,} \tag{1.158}$$

because $|x\rangle = \phi(x)\,|0\rangle$ is a position eigenstate. In particular, unlike in the non-relativistic expression (1.3), the time-evolution operator need not be inserted by hand because $|x\rangle$ contains information about the time variable x^0. By a mode expansion we find

$$\begin{aligned} D(x-y) = \iint \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\,\langle 0|\left(a(\vec{q})e^{-iq\cdot x}+a^\dagger(\vec{q})e^{iq\cdot x}\right) \\ \times\left(a(\vec{p})e^{-ip\cdot y}+a^\dagger(\vec{p})e^{ip\cdot y}\right)|0\rangle\,. \end{aligned} \tag{1.159}$$

Using $a(\vec{p})\,|0\rangle = 0 = \langle 0|\, a^\dagger(\vec{q})$ we find that the only contribution is due to the term

$$e^{-iq\cdot x}e^{ip\cdot x}\,\langle 0|\, a(\vec{q})a^\dagger(\vec{p})\,|0\rangle = e^{-iq\cdot x}e^{ip\cdot x}(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p}). \tag{1.160}$$

This yields

$$\boxed{D(x-y) = \langle 0|\, \phi(x)\phi(y)\, |0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}.} \tag{1.161}$$

We see that $D(x-y)$ is non-zero even for x and y at spacelike distance. On the other hand, in view of (1.156) this cannot affect causality. So how can this be consistent? What saves the day is the observation that

$$[\phi(x),\phi(y)] = D(x-y) - D(y-x). \tag{1.162}$$

We can therefore interpret the commutator as describing 2 physical processes, whose quantum probability amplitude apparently cancel each other for $(x-y)^2 < 0$:

- $D(x-y)$ is the quantum amplitude for a particle to travel from $y \to x$,
- $D(y-x)$ is the quantum amplitude for a particle to travel from $y \to x$.

Indeed if x and y are at spacelike distance from each other, there is no Lorentz invariant notion of whether (x^0-y^0) is bigger or smaller than zero. Therefore both processes can occur. In the expression for $[\phi(x),\phi(y)]$ these two processes cancel each other in the sense of a destructive quantum mechanical interference.
Even more interestingly we can consider a complex scalar field, which contains the mode operators according to

$$\begin{aligned} \phi(x) &\sim a(\vec{p}),\ b^\dagger(\vec{p}), \\ \phi^\dagger(x) &\sim a^\dagger(\vec{p}),\ b(\vec{p}). \end{aligned} \tag{1.163}$$

For a complex scalar field $[\phi(x), \phi(y)] = 0$ for all x, y. More interesting is the commutator

$$[\phi(x), \phi^\dagger(y)] = \langle 0| \phi(x)\phi^\dagger(y) |0\rangle - \langle 0| \phi^\dagger(y)\phi(x) |0\rangle , \tag{1.164}$$

which vanishes for $(x-y)^2 < 0$ while being in general non-zero otherwise. The first term corresponds to a particle which travels from y to x, while the second term corresponds to an anti-particle travelling from x to y. Again both processes cancel each other in the expression for the commutator. The important conclusion is that the field formalism saves causality in the QM sense even though the QM probability for a propagation $x \to y$ itself is non-zero if $(x-y)^2 < 0$. This is why a single-particle approach must fail. The field commutators, on the other hand, know about processes of all possible particles and anti-particles. Thus it is the intrinsic nature of Quantum Field Theory as a multi-particle framework which is responsible for causality.

1.11.3 The Feynman-propagator

We will see in the next chapter that an object of crucial importance in QFT is the Feynman-propagator

$$D_F(x-y) := \langle 0| T\phi(x)\phi(y) |0\rangle , \tag{1.165}$$

where

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & \text{if } x^0 \geq y^0, \\ \phi(y)\phi(x) & \text{if } y^0 > x^0 \end{cases} \tag{1.166}$$

is the time-ordered product. The Feynman-propagator can be written as

$$\begin{aligned} D_F(x-y) &= \Theta(x^0-y^0) \underbrace{\langle 0| \phi(x)\phi(y) |0\rangle}_{D(x-y)} + \Theta(y^0-x^0) \underbrace{\langle 0| \phi(y)\phi(x) |0\rangle}_{D(y-x)} \\ &= \Theta(x^0-y^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0=+E_p} \\ &\quad + \Theta(y^0-x^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_p} e^{ip\cdot(x-y)} \Big|_{p^0=+E_p} . \end{aligned} \tag{1.167}$$

Relabeling $\vec{p}$ by $-\vec{p}$ in the second term gives

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left[\Theta(x^0-y^0) \frac{1}{2E_p} e^{-iE_p(x^0-y^0)} + \Theta(y^0-x^0) \frac{1}{2E_p} e^{iE_p(x^0-y^0)} \right]. \tag{1.168}$$

The term in brackets can be rewritten as a complex contour integral. As a preparation recall **Cauchy's integral formula**:

> *Let $g : U \to \mathbb{C}$ be a holomorphic function defined in an open subset U of $\mathbb{C}$ and consider a closed disk D with $C = \partial D \subset U$. Then for z_0 in the interior of D we have*
>
> $$g(z_0) = \frac{1}{2\pi i} \oint_C \frac{g(z)}{z - z_0} dz. \tag{1.169}$$

In this spirit the first term in $D_F(x-y)$ can be written as

$$\begin{aligned}&\Theta(x^0-y^0)\frac{1}{2E_p}e^{-iE_p(x^0-y^0)}\\&\quad=-\Theta(x^0-y^0)\frac{1}{2\pi i}\oint_{C_1}\mathrm{d}p^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_p)(p^0+E_p)},\end{aligned}\tag{1.170}$$

where

- the poles are avoided in ϵ-surroundings as drawn in the picture.
- we close the contour in the lower half-plane such as to pick up the residue at $p^0=+E_p$ and
- the integral is clockwise, which explains the overall minus sign.

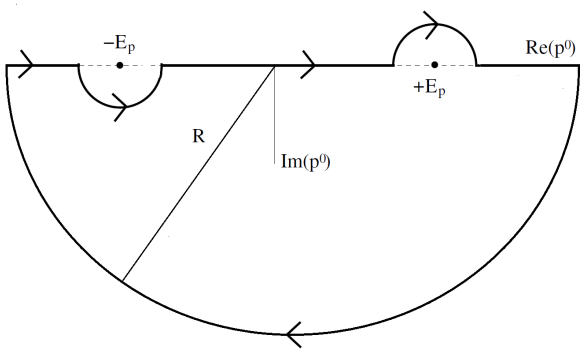


Figure 1.1: Contour C_1.

It is important to note that this result holds for any contour that encloses the poles as drawn. In particular, since $x^0-y^0>0$ the integral along the lower half-plane asymptotically vanishes if we choose to deform the contour to infinity, corresponding to $R\to\infty$. For the second term in (1.168) we can similarly write

$$\begin{aligned}&\Theta(y^0-x^0)\frac{1}{2E_p}e^{iE_p(x^0-y^0)}\\&\quad=-\Theta(y^0-x^0)\frac{1}{2\pi i}\oint_{C_2}\mathrm{d}p^0\frac{e^{-ip^0(x^0-y^0)}}{(p^0-E_p)(p^0+E_p)}.\end{aligned}\tag{1.171}$$

This time C_2 runs counter-clockwise, but picks up the pole at $p_0=-E_p$, which again yields an overall minus.

Now, as stressed above both expressions hold for any $R>E_p$, but if $R\to\infty$, then the integral in the lower-/upper halfplane each vanishes due to the appearance of $\Theta(x^0-y^0)$ and $\Theta(y^0-x^0)$, respectively. It is at this place that the time ordering becomes crucial.

We can therefore evaluate both integrals for $R\to\infty$, add them with the help of $1=\Theta(x^0-y^0)+\Theta(y^0-x^0)$ and arrive at

$$\boxed{D_F(x-y)=\oint_C\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)},}\tag{1.172}$$

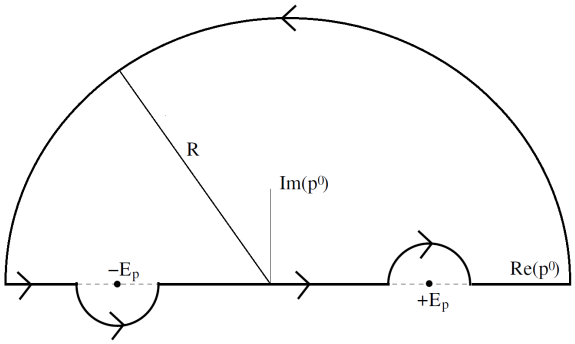


Figure 1.2: Contour C_2.

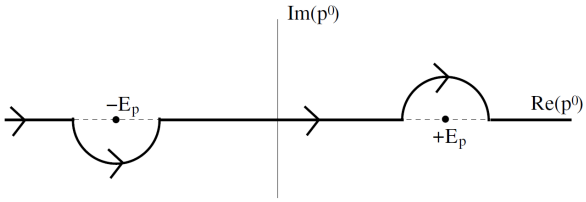


Figure 1.3: Contour C of the Feynman propagator.

where $(p^0 - E_p)(p^0 + E_p) = p^2 - m^2$ was used. In this expression the contour integral in p^0 must be taken along the path C shown above.

Since this is a complex contour integral, all that matters is the relative position of the contour to the poles. Therefore we can equivalently pick the contour C on top of the real axis but shift the poles e.g. by an amount $\pm i\tilde{\epsilon}/E_p$ in the limit $\tilde{\epsilon} \to 0$.

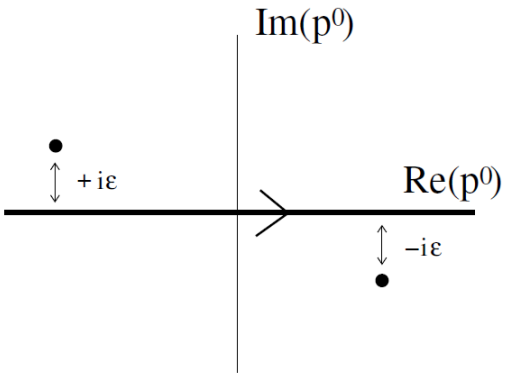


Figure 1.4: Shifted poles.

This modifies the denominators as

$$p^0 = -E_p + i\tilde{\epsilon}/E_p \quad \text{and} \quad p^0 = E_p - i\tilde{\epsilon}_p/E_p. \tag{1.173}$$

This must be combined with taking the limit $\tilde{\epsilon} \to 0$ after performing the integral. With this understood and using furthermore $(p^0 - (E_p - i\tilde{\epsilon}/E_p))(p^0 + (E_p - i\tilde{\epsilon}/E_p)) = \left(p^0\right)^2 - E_p^2 + 2i\tilde{\epsilon} + \epsilon^2/E_p^2 = p^2 - m^2 + i\epsilon$ with $\epsilon = 2i\tilde{\epsilon} - i\tilde{\epsilon}/E_p$, the Feynman propagator can be written as the integral

$$\boxed{D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}} \tag{1.174}$$

with the p^0 integration along the real axis and with the limit $\epsilon \to 0$ after performing the integral.

1.11.4 Propagators as Green's functions

Direct computation reveals that $D_F(x-y)$ is a Green's function for the Klein-Gordon equation

$$(\partial_x^2 + m^2) D_F(x-y) = -i\delta^{(4)}(x-y). \tag{1.175}$$

The general solution to the equation

$$(\partial^2 + m^2)\Delta(x) = -i\delta^{(4)}(x) \tag{1.176}$$

is found by Fourier transforming both sides as

$$\Delta(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{\Delta}(p) e^{-ip\cdot x}, \;\; \delta^{(4)}(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip\cdot x} \tag{1.177}$$

and noting that the Klein-Gordon equation becomes an algebraic equation for the Fourier transforms,

$$(-p^2 + m^2)\tilde{\Delta}(p) = -i \;\Rightarrow \tilde{\Delta}(p) = \frac{i}{p^2 - m^2}. \tag{1.178}$$

Special care must now be applied in performing the Fourier backtransformation for this solution due to the appearance of the two poles at $p^0 = \pm\sqrt{E_{\vec{p}}^2}$. Any consistent prescription to avoid divergences in performing the contour integral in p^0 leads to a solution to the original equation (1.175).
In fact, there are 2×2 different ways to evaluate the contour integral in p^0 by avoiding the two poles in the upper and lower half-plane. As will be shown in the tutorials, if we avoid both poles along a contour in the upper half-plane, the solution is the retarded Green's function

$$\begin{aligned} D_R(x-y) &= \Theta(x^0 - y^0)[D(x-y) - D(y-x)] \\ &\equiv \Theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \end{aligned}, \tag{1.179}$$

while avoiding both poles in the lower half-plane yields the the advanced Green's function

$$D_A(x-y) = \Theta(y^0 - x^0)[D(x-y) - D(y-x)]. \tag{1.180}$$

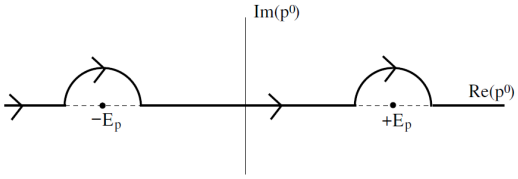


Figure 1.5: The retarded contour.

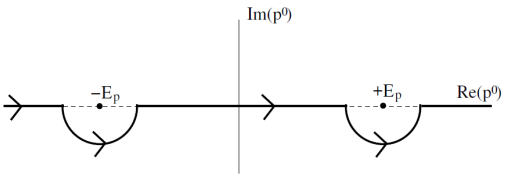


Figure 1.6: The advanced contour.

Note that D_R and D_A appear also in classical field theory in the context of constructing solutions to the inhomogenous Klein-Gordon equation, where they propagate the inhomogeneity forward (D_R) and backward (D_A) in time. In particular the classical version of causality is the statement that D_R and D_A vanish if $(x-y)^2 < 0$, which we proved from a different perspective before.

By contrast $D_F(x-y)$ is the solution corresponding to the contour described previously in this section. It does not appear in classical field theory. The reason is that $D_F(x-y)$ propagates 'positive frequency modes' e^{-ipx} forward and 'negative frequency modes' e^{ipx} backward in time (see (1.168)). Remember that $D_F(x-y)$ is non-vanishing, even for x and y at spacelike distance. Finally there is a fourth prescription which has no particularly important interpretation in classical or quantum field theory.

Chapter 2

Interacting scalar theory

2.1 Introduction

So far we have considered a free scalar theory

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - V_0(\phi) \tag{2.1}$$

with $V_0(\phi) = \frac{1}{2} m_0^2 \phi^2$.

- The theory is exactly solvable: The Hilbert space is completely known as the Fock space of multi-particle states created from the vacuum $|0\rangle$.
- There are no interactions between the particles.

Interactions are described in QFT by potentials $V(\phi)$ beyond quadratic order. We can think of $V(\phi)$ as a formal power series in ϕ,

$$V(\phi) = \underbrace{\frac{1}{2} m_0^2 \phi^2}_{V_0} + \underbrace{\frac{1}{3!} g \phi^3 + \frac{1}{4!} \lambda \phi^4 + ...}_{V_{\text{int}}}, \tag{2.2}$$

and decompose the full Langrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \;\; \mathcal{L}_{\text{int}} = -V_{\text{int}}. \tag{2.3}$$

This yields the decomposition of the full Hamiltonian as $H = H_0 + H_{\text{int}}$. Introducing interaction terms leads to a number of important changes in the theory:

- The Hilbert space is different from the Hamiltonian of the free theory.
  - This is true already for the vacuum, i.e. the full Hamiltonian H has a ground state $|\Omega\rangle$ different from the ground state $|0\rangle$ of the free Hamiltonian H_0:

$$\begin{aligned} |0\rangle &\;\leftrightarrow\; \text{vacuum of } H_0 : \; H_0 |0\rangle = E_0 |0\rangle , \\ |\Omega\rangle &\;\leftrightarrow\; \text{vacuum of } H : \; H |\Omega\rangle = E_\Omega |\Omega\rangle \end{aligned} \tag{2.4}$$

    with $|\Omega\rangle \neq |0\rangle$.

  - The mass of the momentum eigenstates of H does no longer equal the parameter m_0 that appears in $\mathcal{L}_0$.
  - Bound states may exist in the spectrum.

- The states interact.

The dream of Quantum Field Theorists is to find the exact solution of a non-free QFT, i.e. find the exact spectrum and compute all interactions exactly. This has only been possible so far for very special theories with a lot of symmetry, e.g. certain 2-dimensional QFTs with conformal invariance (Conformal Field Theory), or certain 4-dimensional QFTs with enough supersymmetry.
However, for small coupling parameters such as g and λ in $V(\phi)$ we can view the higher terms as small perturbations and apply perturbation theory. Note that depending on the mass dimension of the couplings it must be specified in what sense these parameters must be small, but e.g. for the dimensionless parameter λ this would mean that $\lambda \ll 1$ for perturbation theory to be applicable.
Before dealing with interactions in such a perturbative approach, however, we will be able to establish a number of non-trivial important results on the structure of the spectrum and interactions in a non-perturbative fashion.

2.2 Källén-Lehmann spectral representation

We take a first look at the spectrum of an interacting real scalar field theory in a manner valid for all types of interactions and without relying on perturbation theory. As a consequence of Lorentz invariance the Hamiltonian and the 3-momentum operator must of course still commute, $[H, \vec{P}] = 0$[1], and can thus be diagonalised simultaneously. By $|\lambda_{\vec{p}}\rangle$ we denote such an eigenstate of H and $\vec{P}$ in the full theory such that

$$\begin{aligned} H\,|\lambda_{\vec{p}}\rangle &= E_p(\lambda)\,|\lambda_{\vec{p}}\rangle\,, \\ \vec{P}\,|\lambda_{\vec{p}}\rangle &= \vec{p}\,|\lambda_{\vec{p}}\rangle\,. \end{aligned} \tag{2.5}$$

Each $|\lambda_{\vec{p}}\rangle$ is related via a Lorentz boost with the corresponding state at rest, called $|\lambda_0\rangle$. We can have the following types of $|\lambda_{\vec{p}}\rangle$:

- 1-particle states $|1_{\vec{p}}\rangle$ with $E_p = \sqrt{\vec{p}^2 + m^2}$ and rest mass m. Remember that m is no more identical to m_0 in $\mathcal{L}_0$.

- Bound states with no analogue in the free theory.

- 2- and N-particle states formed out of 1-particle and the bound states. In this case we take $\vec{p}$ to be the centre-of-mass momentum of the multi-particle state.

[1] This is simply the statement that the momentum operator is conserved - cf. (1.137).

All these states are created from the vacuum $|\Omega\rangle$. The crucial difference to the free theory is, though, that $\phi(x)$ cannot simply be written as a superposition of its Fourier amplitudes $a(\vec{p})$ and $a^\dagger(\vec{p})$ because it does not obey the free equation of motion, i.e.

$$(\partial^2 + m^2)\phi \neq 0. \tag{2.6}$$

Rather,

$$(\partial^2 + m^2)\phi = j \tag{2.7}$$

for a suitable current j. Thus, acting with ϕ on $|\Omega\rangle$ does not simply create a 1-particle state as in the free theory.

We will make frequent use of the completeness relation of this Hilbert space,

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} |\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}| . \tag{2.8}$$

Here the formal sum over λ includes the sum over the 1-particle state, over all types of bound states as well as over all multi-particle states, while the integral $\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{E_p(\lambda)}$ refers to the centre-of-mass momentum of a state of species λ. In particular, since specifying a multi-particle state requires specifying the relative momenta of the individual states (in addition to the centre-of-mass momentum $\vec{p}$), the sum over λ is really a sum over a continuum of states.

Our first goal is to compute the interacting Feynman-propagator

$$\langle\Omega| T\phi(x)\phi(y) |\Omega\rangle \tag{2.9}$$

and to establish a physical interpretation for it. Even though we cannot rely on the mode expansion of the field any more, we will be able to make a great deal of progress with the help of two tricks.

- First we insert $\mathbb{1}$ between $\phi(x)$ and $\phi(y)$[2]. We first ignore time ordering. Then, without loss of generality we can assume that

  $$\langle\Omega|\phi(x)|\Omega\rangle = 0, \tag{2.10}$$

  because since

  $$\phi(x) = e^{ix^\mu P_\mu}\phi(0)e^{-ix^\mu P_\mu} \tag{2.11}$$

  (cf. (1.144)) and $P^\mu|\Omega\rangle = 0$ we have

  $$\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle \ \forall x. \tag{2.12}$$

  So if $c \equiv \langle\Omega|\phi(0)|\Omega\rangle \neq 0$ we simply redefine $\phi \to \phi - c$ to achieve (2.10). Therefore

  $$\langle\Omega|\phi(x)\,\mathbb{1}\,\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} \langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle\langle\lambda_{\vec{p}}|\phi(y)|\Omega\rangle . \tag{2.13}$$

[2] Whenever we do not know what to do, and Fourier transformation is not the answer, we insert a $\mathbb{1}$.

Now

$$\langle\Omega|\,\phi(x)\,|\lambda_{\vec{p}}\rangle = \langle\Omega|\,e^{iP\cdot x}\phi(0)e^{-iP\cdot x}\,|\lambda_{\vec{p}}\rangle\,, \tag{2.14}$$

and with $\langle\Omega|\,e^{iP\cdot x} = \langle\Omega|$ and $e^{-iP\cdot x}\,|\lambda_{\vec{p}}\rangle = |\lambda_{\vec{p}}\rangle\,e^{-ip\cdot x}$ we find

$$\langle\Omega|\,\phi(x)\,|\lambda_{\vec{p}}\rangle = \langle\Omega|\,\phi(0)\,|\lambda_{\vec{p}}\rangle\,e^{-ip\cdot x}. \tag{2.15}$$

- The next trick is to relate $|\lambda_{\vec{p}}\rangle$ to $|\lambda_0\rangle$ by a Lorentz boost. To this end we investigate the **transformation behaviour of a scalar field** under a Lorentz transformation

$$x \mapsto x' = \Lambda x. \tag{2.16}$$

In the spirit of Quantum Mechanics the action of the Lorentz group is represented on the Hilbert space in terms of a unitary operator $U(\Lambda)$ such that all states transform like

$$|\alpha\rangle \mapsto |\alpha'\rangle = U(\Lambda)\,|\alpha\rangle\,. \tag{2.17}$$

What is new to us is that also the transformation of the field is determined in terms of $U(\Lambda)$. More precisely, the scalar field transforms as

$$\phi(x) \mapsto \phi'(x') = \phi(x(x')) \tag{2.18}$$

with $\phi'(x') = U^{-1}(\Lambda)\phi(x')U(\Lambda)$, i.e.

$$\boxed{U^{-1}(\Lambda)\phi(x')U(\Lambda) = \phi(x).} \tag{2.19}$$

To see this we start with the familiar transformation of a classical scalar field under a Lorentz transformation given by

$$\phi \mapsto \phi'(x') = \phi(x). \tag{2.20}$$

We now need to find the analogue of this equation for operator-valued fields. The analogue of the classical value of $\phi(x)$ is the matrix element $\langle\alpha|\,\phi(x)\,|\beta\rangle$ evaluated on a basis of the Hilbert space. The transformed field $\phi'(x)$ then corresponds to transformed matrix elements $\langle\alpha'|\,\phi(x)\,|\beta'\rangle$. Thus the classical relation (2.20) translates into

$$\underbrace{\langle\alpha'|\,\phi(x')\,|\beta'\rangle}_{\langle\alpha|U^{-1}\phi(x')U|\beta\rangle} = \langle\alpha|\,\phi(x)\,|\beta\rangle\,, \tag{2.21}$$

for all states $|\alpha\rangle$ and $|\beta\rangle$.

Now, let U denote a Lorentz boost such that

$$|\lambda_{\vec{p}}\rangle = U^{-1}\,|\lambda_0\rangle\,. \tag{2.22}$$

We can then further manipulate (2.15) by writing

$$\langle\Omega|\,\phi(x)\,|\lambda_{\vec{p}}\rangle = \underbrace{\langle\Omega|\,U^{-1}}_{\langle\Omega|}\,\underbrace{U\phi(0)U^{-1}}_{\phi(0)}\,\underbrace{U\,|\lambda_{\vec{p}}\rangle}_{|\lambda_0\rangle}\,e^{-ip\cdot x}. \tag{2.23}$$

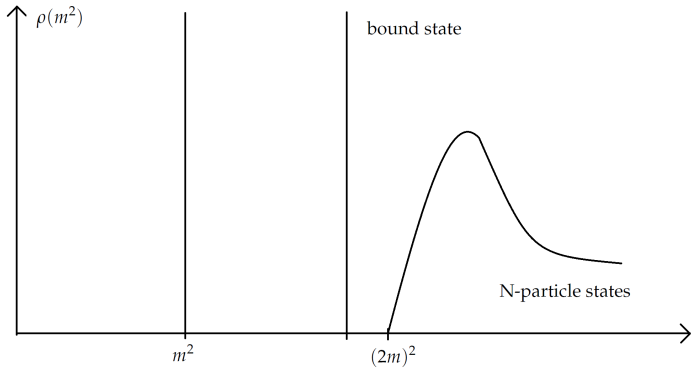


Figure 2.1: Spectral function.

Thus the Feynman propagator without time ordering can be expressed as

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \sum_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} e^{-ip\cdot(x-y)} \Big|\langle\Omega|\,\phi(0)\,|\lambda_0\rangle\Big|^2, \tag{2.24}$$

which is to be compared with the free scalar result (1.161). Including time ordering, we can peform the same manipulations for the integral as in the free theory and therefore conclude

$$\boxed{\langle\Omega|\,T\phi(x)\phi(y)\,|\Omega\rangle = \sum_\lambda \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m_\lambda^2+i\epsilon} e^{-ip\cdot(x-y)} \Big|\langle\Omega|\,\phi(0)\,|\lambda_0\rangle\Big|^2.} \tag{2.25}$$

One can define

$$D_F(x-y;M^2) := \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-M^2+i\epsilon} e^{-ip\cdot(x-y)} \tag{2.26}$$

to write

$$\langle\Omega|\,T\phi(x)\phi(y)\,|\Omega\rangle = \int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\rho(M^2)D_F(x-y;M^2) \tag{2.27}$$

in terms of the spectral function

$$\rho(M^2) = \sum_\lambda 2\pi\,\delta(M^2-m_\lambda^2)\Big|\langle\Omega|\,\phi(0)\,|\lambda_0\rangle\Big|^2, \tag{2.28}$$

which has a typical form like in Figure 2.1. It is crucial to appreciate that the 1-particle states leads to an isolated δ-function peak around $M^2 = m^2$. Therefore below $M^2 \cong (2m)^2$ or $M^2 \cong m^2_{\text{bound}}$ the spectral function takes the form

$$\rho(M^2) = 2\pi\,\delta(M^2-m^2)\,Z. \tag{2.29}$$

Here we have defined the field-strength or **wavefunction renormalisation**

$$Z = \Big|\langle\Omega|\,\phi(0)\,|1_0\rangle\Big|^2, \tag{2.30}$$

where $|1_0\rangle$ is a 1-particle state at rest.

Consider now the Fourier-transformation

$$\begin{aligned}\int \mathrm{d}^4x\, e^{ip\cdot(x-y)}\ \langle\Omega|\, T\phi(x)\phi(y)\,|\Omega\rangle &= \int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\rho(M^2)\frac{i}{p^2-M^2+i\epsilon} \\ &= \frac{iZ}{p^2-m^2+i\epsilon}+\int_{m^2_{\text{bound}}}^\infty \frac{\mathrm{d}M^2}{2\pi}\rho(M^2)\frac{i}{p^2-M^2+i\epsilon}.\end{aligned} \tag{2.31}$$

This identifies the 1-particle state as the first analytic pole at m^2. This is an important observation:

> *The mass-square m^2 of the particle is the location of the lowest-lying pole of the Fourier transformed propagator.*

I.e. computation of the propagator gives us, among other things, a way to read off the mass of the particle. Note furthermore that

- bound states appear at higher isolated poles and
- N-particle states give rise to a branch cut beginning at $p^2 = 4m^2$.

The field-strength renormalisation Z was 1 in free theory because $\phi(0)$ just creates the free particle from vacuum. In an interacting theory

$$\left|\langle\Omega|\,\phi(0)\,|1_0\rangle\right| = \sqrt{Z} < 1 \tag{2.32}$$

because ϕ creates not only 1-particle states and thus the overlap with the 1-particle states is smaller. In fact, by exploiting the properties of the spectral function one can prove formally that

> $Z = 1$ *if and only if the theory is free.*

We give a guided tour through this proof in the tutorials.

2.3 S-matrix and asymptotic in/out-states

We now consider scattering of incoming states $|i\rangle$ to outgoing states $|f\rangle$ with the aim of computing the QM transition amplitude, i.e. the probability amplitude for scattering of $|i\rangle$ to $|f\rangle$. The process is formulated in terms of the theory of asymptotic in- and out-states.

- In the asymptotic past, $t \to -\infty$, the in-states $|i, \text{in}\rangle$ are described as distinct wave-packets corresponding to well-separated single particle states. Being far apart for $t \to -\infty$, they travel freely as individual states. This is a consequence of locality of the interactions, which we assume in the sequel.
- As these states approach each other, they start to "feel each other", interact and scatter into the final states $|f\rangle$.

- For $t \to \infty$ these final states are again asymptotically free and well-separated 1-particle states.

The concept of free asymptotic in/out states is formalised by the so-called in- and out-fields ϕ_{in} and ϕ_{out} with the following properties:

- The in-state $|i, \text{in}\rangle$ is created from the asymptotic vaccum $|\text{vac}, \text{in}\rangle$ by action of ϕ_{in} as $t \to -\infty$. We will see that

$$|\text{vac}, \text{in}\rangle = |\Omega\rangle\,, \tag{2.33}$$

  the vacuum of the interacting theory.

- $|i, \text{in}\rangle$ has $E = \sqrt{p^2 + m^2}$ with m the value of the 1-particle pole in the Feynman propagator of the **full** interacting theory. In particular $m \neq m_0$. Therefore, ϕ_{in} is a free field obeying the free Klein-Gordon-equation, but with the full mass $m \neq m_0$,

$$(\partial^2 + m^2)\phi_{\text{in}} = 0. \tag{2.34}$$

  It is thus possible to expand ϕ_{in} in terms of $a_{\text{in}}(\vec{p})$ and $a^\dagger_{\text{in}}(\vec{p})$ so that

$$\phi_{\text{in}}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\text{in}}(\vec{p}) e^{-ip\cdot x} + a^\dagger_{\text{in}}(\vec{p}) e^{ip\cdot x}\right), \tag{2.35}$$

  where $p^0 = \sqrt{\vec{p}^2 + m^2}$.

- ϕ_{in} satisfies the following relation to the interacting field ϕ: Asymptotically for $t \to -\infty$ the above logic suggests identifying $\phi_{\text{in}}(x)$ with $\phi(x)$, at least in weak sense that their their matrix elements with a basis of the Hilbert space must agree in a suitable manner. We therefore make the ansatz

$$\phi \to C\,\phi_{\text{in}} \tag{2.36}$$

  in the weak sense only, i.e.

$$\langle\alpha|\,\phi\,|\beta\rangle \to C\,\langle\alpha|\,\phi_{\text{in}}\,|\beta\rangle \tag{2.37}$$

  for all $|\alpha\rangle$ and $|\beta\rangle$ as $t \to -\infty$. With this input one can show (see Examples Sheet 5 for the proof) that

$$\langle 1_{\vec{p}}|\,\phi(0)\,|\Omega\rangle = C\,\langle 1_{\vec{p}}|\,\phi_{\text{in}}(0)\,|\Omega\rangle\,. \tag{2.38}$$

  Since by construction $\langle 1_{\vec{p}}|\,\phi_{\text{in}}(0)\,|\Omega\rangle = 1$ this identifies $C = \sqrt{Z}$ with Z the wavefunction renormalisation of the full theory as defined in (2.30). Thus

$$\langle\alpha|\,\phi\,|\beta\rangle \to \sqrt{Z}\,\langle\alpha|\,\phi_{\text{in}}\,|\beta\rangle \tag{2.39}$$

  as $t \to -\infty$. Note that this is really true only in this weak sense. What does not hold in an interacting field theory is that operator products (i.e. powers of $\phi(x)$) approach corresponding products of $\phi_{\text{in}}(x)$.[3]

[3] If this were to hold, then the field theory would be free: Indeed the assumption that $\phi(x)\dot{\phi}(y) \to Z\,\phi_{in}(x)\dot{\phi}_{in}(y)$ and similarly for $\dot{\phi}(y)\phi(x)$ implies that $Z = 1$ by exploiting the commutation relations for the fields.

- ϕ_{out} has of course analogous properties as $t \to +\infty$.

Our considerations can be summarised as follows: $\phi_{\text{in/out}}$ are free fields with single particle states of energy $E_p = \sqrt{p^2 + m^2}$, with the mass m of the full theory. We can think of switching off all interactions of the theory as $t \to \mp\infty$ except for self-interactions of the field. This leads to mass $m \neq m_0$ and $Z \neq 1$. We will be able to understand what is meant by these self-interactions very soon when discussing the resummed propagator, and it will become clear then that indeed it is imperative to take into account $m \neq m_0$ and $Z \neq 1$ for $\phi_{\text{in/out}}$.

The Hilbert spaces of asymptotic in- and out-states are isomorphic Fock spaces. Thus there exists an operator S which maps the out-states onto the in-states, i.e.

$$|i, \text{in}\rangle = S\,|i, \text{out}\rangle\,. \tag{2.40}$$

On Examples Sheet 5 we prove the following properties of S:

- S is unitary, i.e $S^\dagger = S^{-1}$,
- $\phi_{\text{in}}(x) = S\phi_{\text{out}}(x)S^{-1}$,
- $|\text{vac,in}\rangle = |\text{vac,out}\rangle = |\Omega\rangle$ and $S\,|\Omega\rangle = |\Omega\rangle$.

Our aim is to compute the **transition amplitude**

$$\langle f, \text{out}|\, i, \text{in}\rangle = \underbrace{\langle f, \text{in}|\, S\,|i, \text{in}\rangle}_{S\text{-matrix element}}, \tag{2.41}$$

so that $|\langle f, \text{in}|\, S\,|i, \text{in}\rangle|^2$ it the probability for scattering from the initial states to the final states.

2.4 The LSZ reduction formula

Let us now compute the **S-matrix element**

$$\langle p_1, ..., p_n, \text{out}|\, q_1, ..., q_r, \text{in}\rangle \tag{2.42}$$

for scattering of asymptotic in- and out-states of definite momenta. One can think of this as the building block to describe scattering of asymptotically localised wave-packets $|f_{\text{in}}\rangle = \int \mathrm{d}^3 p\, f(\vec{p})\,|p_{\text{in}}\rangle$.

- We first use the definition of $|q_i, \text{in}\rangle$ in terms of the creation operator of the in-field

$$\phi_{\text{in}}(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_{\text{in}}(\vec{k}) e^{-ik\cdot x} + a^\dagger_{\text{in}}(\vec{k}) e^{ik\cdot x}\right) \tag{2.43}$$

  given by

$$|q_i, \text{in}\rangle = \sqrt{2E_{q_i}}\, a^\dagger_{\text{in}}(q_i)\,|\Omega\rangle\,. \tag{2.44}$$

We will use that

$$\boxed{\begin{aligned} a_{\rm in}(\vec{q}) &= \frac{i}{\sqrt{2E_q}} \int \mathrm{d}^3x\, e^{iq\cdot x} \overset{\leftrightarrow}{\partial_0} \phi_{\rm in}(x)\big|_{x^0=t} \ , \\ a^\dagger_{\rm in}(\vec{q}) &= \frac{-i}{\sqrt{2E_q}} \int \mathrm{d}^3x\, e^{-iq\cdot x} \overset{\leftrightarrow}{\partial_0} \phi_{\rm in}(x)\big|_{x^0=t} \ , \end{aligned}} \tag{2.45}$$

(cf. (1.151)), where the integral can be evaluated at arbitrary time t. This allows us to trade $|q_i\rangle$ by $\phi_{\rm in}(x)$ as follows:

$$\begin{aligned} \langle p_1,...,p_n,\mathrm{out}|q_1,...,q_r,\mathrm{in}\rangle &= \sqrt{2E_{q_1}}\,\langle p_1,...,p_n,\mathrm{out}|\, a^\dagger_{\rm in}(\vec{q_1})\,|q_2,...,q_r,\mathrm{in}\rangle \\ &= \frac{1}{i}\int \mathrm{d}^3x\, e^{-iq_1\cdot x} \overset{\leftrightarrow}{\partial_0}\, \langle p_1,...,p_n,\mathrm{out}|\,\phi_{\rm in}(t,\vec{x})\,|q_2,...,q_r,\mathrm{in}\rangle\,\big|_{x^0=t}. \end{aligned} \tag{2.46}$$

- Since t is arbitrary, we can take $t\to-\infty$ because in that limit we can make use of the relation

$$\lim_{t\to-\infty}\langle 1_{\vec{p}}|\,\phi_{\rm in}(t,\vec{x})\,|\Omega\rangle = \lim_{t\to-\infty} Z^{-1/2}\,\langle 1_{\vec{p}}|\,\phi(t,\vec{x})\,|\Omega\rangle\,, \tag{2.47}$$

or more generally

$$\lim_{t\to-\infty}\langle p_1,...,p_n,\mathrm{out}|\,\phi_{\rm in}(t,\vec{x})\,|q_2,...,q_r,\mathrm{in}\rangle = \lim_{t\to-\infty} Z^{-1/2}\,\langle p_1,...,p_n,\mathrm{out}|\,\phi(x)\,|q_2,...,q_r,\mathrm{in}\rangle\,. \tag{2.48}$$

This leads to

$$\begin{aligned} &\langle p_1,...,p_n,\mathrm{out}|\,q_1,...,q_r,\mathrm{in}\rangle \\ &= \lim_{t\to-\infty} Z^{-1/2}\,\underbrace{\frac{1}{i}\int \mathrm{d}^3x\, e^{-iq_1\cdot x} \overset{\leftrightarrow}{\partial_0}\,\langle p_1,...,p_n,\mathrm{out}|\,\phi(x)\,|q_2,...,q_r,\mathrm{in}\rangle}_{\equiv\int \mathrm{d}^3x\, f(t,\vec{x})}\,. \end{aligned} \tag{2.49}$$

- Our next aim is to let $\phi(x)$ act from the right on the out-states in order to annihilate one of the states. To this end we need to relate the above matrix element with $t\to-\infty$ to a matrix element in the limit $t\to\infty$, where we can re-express $\phi(x)$ by $\phi_{\rm out}(x)$. We can do so by exploiting that for all functions $f(t,\vec{x})$

$$(\lim_{t\to\infty} - \lim_{t\to-\infty})\int \mathrm{d}^3x\, f(t,\vec{x}) = \underbrace{\lim_{\substack{t_f\to\infty\\ t_i\to-\infty}} \int_{t_i}^{t_f} \mathrm{d}t \frac{\partial}{\partial t}\int \mathrm{d}^3x\, f(t,\vec{x})}_{\equiv\int \mathrm{d}^4x\,\partial_0 f(x)}, \tag{2.50}$$

i.e.

$$\lim_{t\to-\infty}\int \mathrm{d}^3x\, f(t,\vec{x}) = \lim_{t\to+\infty}\int \mathrm{d}^3x\, f(t,\vec{x}) - \int \mathrm{d}^4x\,\partial_0 f(x). \tag{2.51}$$

Therefore we can write the overlap of the in- and out-states as

$$\langle p_1,...,p_n,\mathrm{out}|\,q_1,...,q_r,\mathrm{in}\rangle = \ A - B \tag{2.52}$$

with

$$B = \int \mathrm{d}^4x\, Z^{-1/2}\, \partial_0 \left[e^{-iq_1\cdot x}\, \frac{1}{i} \overset{\leftrightarrow}{\partial_0} \langle p_1, ..., p_n, \mathrm{out}|\, \phi(x)\, |q_2, ..., q_r, \mathrm{in}\rangle \right] \tag{2.53}$$

and

$$\begin{aligned} A &= \lim_{t\to\infty} \int \mathrm{d}^3x\, \frac{1}{i}\, e^{-iq_1\cdot x} \overset{\leftrightarrow}{\partial_0} \underbrace{Z^{-1/2}\, \langle p_1, ..., p_n, \mathrm{out}|\, \phi(x)\, |q_2, ..., q_r, \mathrm{in}\rangle}_{\text{for } t\to\infty\text{: } \langle p_1,...,p_n,\mathrm{out}|\phi_{\mathrm{out}}(x)|q_2,...,q_r,\mathrm{in}\rangle} \\ &= \langle p_1, ..., p_n, \mathrm{out}|\, a^\dagger_{\mathrm{out}}(\vec{q}_1)\, |q_2, ..., q_r, \mathrm{in}\rangle\, \sqrt{2E_{q_1}}. \end{aligned} \tag{2.54}$$

Altogether

$$\begin{aligned} \langle p_1, ..., p_n, \mathrm{out}|\, q_1, ..., q_r, \mathrm{in}\rangle &= \langle p_1, ..., p_n, \mathrm{out}|\, a^\dagger_{\mathrm{out}}(\vec{q}_1)\, |q_2, ..., q_r, \mathrm{in}\rangle\, \sqrt{2E_{q_1}} \\ &\quad + i \int \mathrm{d}^4x\, Z^{-1/2}\, \partial_0 \left[e^{-iq_1\cdot x} \overset{\leftrightarrow}{\partial_0} \langle p_1, ..., p_n, \mathrm{out}|\, \phi(x)\, |q_2, ..., q_r, \mathrm{in}\rangle \right]. \end{aligned} \tag{2.55}$$

- The first term gives

$$\begin{aligned} &\langle p_1, ..., p_n, \mathrm{out}|\, a^\dagger_{\mathrm{out}}(\vec{q}_1)\, |q_2, ..., q_r, \mathrm{in}\rangle\, \sqrt{2E_{q_1}} = \\ &= \sum_{k=1}^{n} 2E_{p_k} (2\pi)^3 \delta^{(3)}(\vec{p}_k - \vec{q}_1)\, \langle p_1, ..., \hat{p}_k, ..., p_n, \mathrm{out}|\, q_2, ..., q_r, \mathrm{in}\rangle, \end{aligned} \tag{2.56}$$

where $\hat{p}_k$ has to be taken out. This describes a process where one of the in- and outgoing states are identical and do not participate in scattering. Such an amplitude corresponds to a disconnected diagram and its computation reduces to computing an S-matrix element involving only $(r-1)$ in- and $(n-1)$ out-states (since one in- and out-state factor out). The second term in (2.55) gives

$$\begin{aligned} &i \int \mathrm{d}^4x\, Z^{-1/2} \partial_0 \left[e^{-iq_1\cdot x} \partial_0 \langle ... \rangle - \left(\partial_0 e^{-iq_1\cdot x} \right) \langle ... \rangle \right] \\ &= i \int \mathrm{d}^4x\, Z^{-1/2} \left[e^{-iq_1\cdot x} \partial_0^2 \langle ... \rangle - \left(\partial_0^2 e^{-iq_1\cdot x} \right) \langle ... \rangle \right], \end{aligned} \tag{2.57}$$

because the cross-terms cancel each other. Now consider that

$$-\partial_0^2 e^{-iq_1\cdot x} = \left(q_1^0\right)^2 e^{-iq_1\cdot x} = \left(q_1^2 + \vec{q}_1^2\right) e^{-iq_1\cdot x} = \left(m^2 - \nabla^2\right) e^{-iq_1\cdot x}. \tag{2.58}$$

With respect to the spatial variable we can integrate two times by parts,

$$\int \mathrm{d}^4x \left(m^2 - \nabla^2\right) e^{-iq_1\cdot x} \langle ... \rangle = \int \mathrm{d}^4x\, e^{-iq_1\cdot x} \left(m^2 - \nabla^2\right) \langle ... \rangle, \tag{2.59}$$

since boundary terms at spatial infinity vanish. To justify this recall that the momentum eigenstates are to be thought of as convoluted with a wavefunction profile as in $|f_{\mathrm{in}}\rangle = \int \mathrm{d}^3p\, f(\vec{p})\, |p_{\mathrm{in}}\rangle$ so that the full states are really localised in space. Altogether this gives

$$\begin{aligned} &\langle p_1, ..., p_n, \mathrm{out}|\, q_1, ..., q_r, \mathrm{in}\rangle = \\ &\sum_{k=1}^{n} 2E_{p_k} (2\pi)^3 \delta^{(3)}(\vec{p}_k - \vec{q}_1)\, \langle p_1, ..., \hat{p}_k, ..., p_n, \mathrm{out}|\, q_2, ..., q_r, \mathrm{in}\rangle \\ &+ i\, Z^{-1/2} \int \mathrm{d}^4x_1\, e^{-iq_1\cdot x_1} \left(\Box_1 + m^2\right) \langle p_1, ..., p_n, \mathrm{out}|\, \phi(x_1)\, |q_2, ..., q_r, \mathrm{in}\rangle\,. \end{aligned} \tag{2.60}$$

- Now we repeat this for all remaining states. First consider replacing $\langle p_1|$ by ϕ_{out} as follows:

$$\begin{aligned}
&\langle p_1, ..., p_n, \text{out}|\, \phi(x_1)|q_2, ..., q_r, \text{in}\rangle \\
&= \sqrt{2E_{p_1}}\, \langle p_2, ..., p_n, \text{out}|\, a_{\text{out}}(p_1)\phi(x_1)|q_2, ..., q_r, \text{in}\rangle \\
&= \lim_{y_1^0\to\infty} i Z^{-1/2} \int \mathrm{d}^3 y_1\, e^{ip_1\cdot y_1}\, \overset{\leftrightarrow}{\partial}_{y_1^0} \langle p_2, ..., p_n, \text{out}|\, \phi(y_1)\phi(x_1)\, |q_2, ..., q_r, \text{in}\rangle\,.
\end{aligned} \tag{2.61}$$

We would like to repeat the previous logic and transform this into a sum of two terms, one of which being the disconnected term $\langle p_2, ..., p_n, \text{out}|\, \phi(x_1)a_{\text{in}}(\vec{p}_1)\, |q_2, ..., q_r, \text{in}\rangle$. However, we need to be careful with the ordering of operators as we cannot simply commute $a_{\text{in}}(\vec{p}_1)$ through $\phi(x_1)$. This is where the the time-ordering symbol T comes in. Namely, observe that for finite values of x_1^0

$$\begin{aligned}
&\lim_{\substack{t_f\to\infty \\ t_i\to-\infty}} \int_{t_i}^{t_f} \mathrm{d}y_1^0 \left[\frac{\partial}{\partial y_1^0}\, i Z^{-1/2} \int \mathrm{d}^3 y_1\, e^{ip_1\cdot y_1}\, \overset{\leftrightarrow}{\partial}_{y_1^0} \langle p_2, ..., p_n, \text{out}|\, T\phi(y_1)\phi(x_1)\, |q_2, ..., q_r, \text{in}\rangle\right] \\
&= \lim_{y_1^0\to\infty} \left[i Z^{-1/2} \int \mathrm{d}^3 y_1\, e^{ip_1\cdot y_1}\, \overset{\leftrightarrow}{\partial}_{y_1^0} \langle p_2, ..., p_n, \text{out}|\, \phi(y_1)\phi(x_1)\, |q_2, ..., q_r, \text{in}\rangle\right] \\
&\underbrace{- \lim_{y_1^0\to-\infty} \left[i Z^{-1/2} \int \mathrm{d}^3 y_1\, e^{ip_1\cdot y_1} \overset{\leftrightarrow}{\partial}_{y_1^0} \langle p_2, ..., p_n, \text{out}|\, \phi(x_1)\phi(y_1)\, |q_2, ..., q_r, \text{in}\rangle\right]}_{\equiv\langle p_2,...,\text{out}|\phi(x_1)a_{\text{in}}(\vec{p}_1)|q_2,...,\text{in}\rangle\sqrt{2E_{p_1}}\ \to\ \text{disconnected term}}.
\end{aligned} \tag{2.62}$$

Note the different ordering of $\phi(y_1)\phi(x_1)$ in both terms. Indeed since the limit $x_1^0 \to \pm\infty$ appears outside of the correlator in eq. (2.60), the time-ordering symbol precisely yields these orderings as $y_1^0 \to \pm\infty$. The term on the lefthand-side of (2.62) is, as before,

$$i Z^{-1/2} \int \mathrm{d}^4 y_1\, e^{ip_1\cdot y_1} \left(\Box_{y_1} + m^2\right) \langle p_2, ..., p_n, \text{out}|\, T\phi(y_1)\phi(x_1)\, |q_2, ..., q_r, \text{in}\rangle\,. \tag{2.63}$$

- This can be repeated for all in- and out states to get the **Lehmann-Symanzik-Zimmermann reduction formula**

$$\boxed{\begin{aligned}
&\langle p_1, ..., p_n, \text{out}|\, q_1, ..., q_r, \text{in}\rangle \equiv \langle p_1, ..., p_n, \text{in}|\, S\, |q_1, ..., q_r, \text{in}\rangle \\
&= (\Sigma \text{ disconnected terms}) + \\
&+ \left(i Z^{-1/2}\right)^{n+r} \int \mathrm{d}^4 y_1 ... \mathrm{d}^4 y_n \int \mathrm{d}^4 x_1 ... \mathrm{d}^4 x_r \\
&\times e^{i\left(\sum_{k=1}^n p_k\cdot y_k - \sum_{l=1}^r q_l\cdot x_l\right)} \\
&\times \left(\Box_{y_1} + m^2\right) ... \left(\Box_{x_1} + m^2\right) ... \langle\Omega|\, T\phi(y_1) ... \phi(y_n)\phi(x_1) ... \phi(x_r)\, |\Omega\rangle\,.
\end{aligned}} \tag{2.64}$$

This formula reduces the computation of the S-matrix to the computation of **time-ordered correlation functions** of the full interacting theory.

- In terms of the Fourier transformed quantities it reads as follows. First note that

$$\left(\Box_y + m^2\right)\phi(y) = \int \frac{\mathrm{d}^4\tilde{p}}{(2\pi)^4}\left(-\tilde{p}^2 + m^2\right)e^{-i\tilde{p}\cdot y}\tilde{\phi}(\tilde{p}). \tag{2.65}$$

We can plug this into (2.64), perform the integrals $\int d^4y_k\, e^{i(p_k-\tilde{p}_k)y_k} = (2\pi)^4\delta^{(4)}(p_k-\tilde{p}_k)$ (and similarly for x_l, where we define accordingly $\left(\Box_y + m^2\right)\phi(x) = \int \frac{\mathrm{d}^4\tilde{q}}{(2\pi)^4}\left(-\tilde{q}^2 + m^2\right)e^{i\tilde{q}\cdot x}\tilde{\phi}(\tilde{q})$) and arrive at

$$\begin{aligned}\langle p_1,...p_n|\, S\, |q_1,...,q_r\rangle\big|_{\text{connected}} = \left(iZ^{-1/2}\right)^{n+r}\prod_{k=1}^{n}\left(-p_k^2+m^2\right)\prod_{l=1}^{r}\left(-q_l^2+m^2\right)\\ \times\langle\Omega|\, T\tilde{\phi}(p_1)...\tilde{\phi}(p_n)\tilde{\phi}(q_1)...\tilde{\phi}(q_r)\,|\Omega\rangle\,.\end{aligned} \tag{2.66}$$

Note that the $p_1,\ldots,p_n$ and $q_1,\ldots q_r$ which appear on both sides of this equation are on-shell since these correspond to the physical 4-momenta of the out- and incoming 1-particles states. Therefore $p_k^2 - m^2 = 0 = q_l^2 - m^2$. In order for $\langle p_1,...p_n|\, S\, |q_1,...,q_r\rangle\big|_{\text{connected}}$ to be non-zero, the correlation function appearing on the right must therefore have a suitable pole structure such as to cancel precisely the kinematic factors $\prod_{k=1}^n\left(-p_k^2+m^2\right)\prod_{l=1}^r\left(-q_l^2+m^2\right)$.

In fact, as will be confirmed by explicit computation, $\langle\Omega|\, T\tilde{\phi}(p_1)...\tilde{\phi}(p_n)\tilde{\phi}(q_1)...\tilde{\phi}(q_r)\,|\Omega\rangle$ will in general be a sum of terms with different poles in the momenta. Only the term with the pole structure given precisely by $\prod_{k=1}^n\frac{1}{p_k^2-m^2}\prod_{l=1}^n\frac{1}{q_l^2-m^2}$ contributes to the connected S-matrix element $\langle p_1,...p_n|\, S\, |q_1,...,q_r\rangle\big|_{\text{connected}}$. The terms with fewer poles will contribute at best to disconnected scattering processes. On the other hand, since the S-matrix, being a QM probability amplitude, is non-singular, the correlation functions cannot have more poles than what is cancelled by the kinematic factors on the right, and this prediction of the LSZ-formula will indeed be confirmed in explicit computations.

Thus

$$\boxed{\begin{aligned}&\prod_{k=1}^{n}\int \mathrm{d}^4y_k\, e^{ip_k\cdot y_k}\prod_{l=1}^{r}\int \mathrm{d}^4x_l\, e^{-iq_l\cdot x_l}\,\langle\Omega|\, T\prod_k\phi(y_k)\prod_l\phi(x_l)\,|\Omega\rangle\\ &\quad=\left(\prod_{k=1}^{n}\frac{i\sqrt{Z}}{p_k^2-m^2}\right)\left(\prod_{l=1}^{r}\frac{i\sqrt{Z}}{q_l^2-m^2}\right)\langle p_1,...p_n|\, S\, |q_1,...,q_r\rangle\Big|_{\text{connected}}.\end{aligned}} \tag{2.67}$$

We have arrived at a very precise prescription to compute the connected piece of the S-matrix element $\langle p_1,...p_n|\, S\, |q_1,...,q_r\rangle\Big|_{\text{connected}}$:

- Compute the Fourier transformation of the corresponding time-ordered $(n+r)$-correlation function and take all momenta p_k and q_l on-shell, i.e. $p_k^0 = \sqrt{\vec{p}_k^2+m^2}$ and $q_l^0 = \sqrt{\vec{p}_l^2+m^2}$.
- The result will be a sum of terms each a function of the momenta, which are distinguished by the structure of their poles.
- The connected S-matrix element (times $\left(i\sqrt{Z}\right)^{n+r}$)) is the residue with respect to $\prod_{k=1}^n\frac{1}{p_k^2-m^2}\prod_{l=1}^r\frac{1}{q_l^2-m^2}$.

2.5 Correlators in the interaction picture

To further evaluate the LSZ formula we need to compute the (n+r)-correlation function

$$\langle\Omega|\, T\phi(y_1)...\phi(y_n)\phi(x_1)...\phi(x_r)\,|\Omega\rangle \tag{2.68}$$

of the full interacting theory with Hamiltonian

$$H = H_0 + H_{\text{int}}. \tag{2.69}$$

There are two ways how to do this, either by performing a path-integral computation, or by computing in the Interaction Picture. The path-integral formalism is reserved for the course QFT II. In the sequel we consider the latter approach. Our strategy is to reduce the computation of the full correlator to a calculation in terms of

- free-field creation/annihilation operators and
- the free-field vacuum $|0\rangle$.

This is achieved in the Interaction Picture (≡ Dirac picture). Let $\phi(t,\vec{x})$ denote the Heisenberg Picture field of the full interacting theory and fix some reference time t_0. Then we define the **Interaction Picture operators**

$$\boxed{\begin{aligned}\Phi_I(t,\vec{x}) &= e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)}\\ \Pi_I(t,\vec{x}) &= e^{iH_0(t-t_0)}\Pi(t_0,\vec{x})e^{-iH_0(t-t_0)}.\end{aligned}} \tag{2.70}$$

The motivation behind this definition is that $\Phi_I(t,\vec{x})$ satisfies the free Klein-Gordon-equation

$$\boxed{(\partial^2 + m_0^2)\Phi_I(t,\vec{x}) = 0} \tag{2.71}$$

with mass m_0 as in H_0. One can see this as follows: The defintion (2.70) implies that

$$\begin{aligned}\partial_t\Phi_I(t,\vec{x}) &= i[H_0,\Phi_I(t,\vec{x})]\\ &= e^{iH_0(t-t_0)}\underbrace{i[H_0,\phi(t_0,\vec{x})]}_{\Pi(t_0,\vec{x})}e^{-iH_0(t-t_0)}.\end{aligned} \tag{2.72}$$

Here we are using that $\Pi(t_0,\vec{x}) = i[H,\phi(t_0,\vec{x})] = i[H_0,\phi(t_0,\vec{x})]$ because H and H_0 differ only by powers in $\phi(t,\vec{x})$.[4]
Therefore

$$\partial_t\Phi_I(t,\vec{x}) = \Pi_I(t,\vec{x}) \tag{2.73}$$

and likewise

$$\partial_t^2\Phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\underbrace{i[H_0,\Pi(t_0,\vec{x})]}_{(\nabla^2-m_0^2)\phi(t_0,\vec{x})}e^{-iH_0(t-t_0)} \tag{2.74}$$

$$= (\nabla^2 - m_0^2)\Phi_I(t,\vec{x}). \tag{2.75}$$

[4] If we allow also for time derivative terms in the interactions, we should be writing here and in the sequel $i[H_0,\phi(t_0,\vec{x})] = \Pi_0(t_0,\vec{x})$ with $\Pi_0(t_0,\vec{x}) \neq \Pi(t,\vec{x})$. It can be checked that this does not alter the conclusions.

Here it is crucial to appreciate that m_0 appears because the commutator involves only H_0 and the computation of $i[H_0, \Pi(t_0, \vec{x})]$ proceeds as in the free theory.
Equ. (2.71) implies that $\Phi_I(t, \vec{x})$ enjoys a free mode expansion of the form

$$\Phi_I(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_I(\vec{p}) e^{-ip\cdot x} + a_I^\dagger(\vec{p}) e^{ip\cdot x} \right), \qquad p^0 = E_{\vec{p}} = \vec{p}^2 + m_0^2. \tag{2.76}$$

Furthermore it is easy to see that the interaction picture fields and, as a result of (2.73), also the modes satisfy the free-field commutation relations

$$[\Phi_I(t, \vec{x}), \Pi_I(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}), \qquad [a_I(\vec{p}), a_I^\dagger(\vec{q})] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}). \tag{2.77}$$

One then verifies that

$$[H_0, a_I(\vec{p})] = -E_p a_I(\vec{p}), \qquad [H_0, a_I^\dagger(\vec{p})] = +E_p a_I^\dagger(\vec{p}) \tag{2.78}$$

as in the free theory. This can be seen e.g. by noting that $H_0 = e^{iH_0(t-t_0)} H_0 e^{-iH_0(t-t_0)} \equiv (H_0)_I$ and therefore we can replace in H_0 all fields $\phi(x)$ by $\phi_I(x)$ so that all free field results carry over. Consequently by the same arguments as in the free theory there must exist a vacuum state annihilated by all $a_I(\vec{p})$ and by H_0. This identifies this state as the unique vacuum of the free theory, $H_0 |0\rangle = 0$, and therefore

$$\boxed{a_I(\vec{p}) |0\rangle = 0.} \tag{2.79}$$

At $t \neq t_0$, the Heisenberg Picture $\phi(t, \vec{x})$ and the Interaction Picture $\Phi_I(t, \vec{x})$ relate as

$$\begin{aligned} \phi(t, \vec{x}) &= e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \Phi_I(t, \vec{x}) e^{iH_0(t-t_0)} e^{-iH(t-t_0)}, \end{aligned} \tag{2.80}$$

which yields

$$\boxed{\phi(t, \vec{x}) = U^\dagger(t, t_0) \Phi_I(t, \vec{x}) U(t, t_0),} \tag{2.81}$$

where

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \tag{2.82}$$

is the **time-evolution operator**. Note that, since H and H_0 do not commute, this is not just $e^{-iH_{\text{int}}(t-t_0)}$.
The logic is now to replace the Heisenberg Picture operators $\phi(x)$ in the correlator (2.68) by the Interaction Picture operators $\Phi_I(x)$ because these obey a free-mode expansion. To further evaluate the resulting expression, we first derive a useful expression for $U(t, t_0)$ and second establish a relation between the vacuum $|\Omega\rangle$ of the interacting theory as appearing in (2.68) and the free vacuum $|0\rangle$ on which the interaction picture modes act.

2.5.1 Time evolution

To compute $U(t,t_0)$ we note that the time-evolution operator satisfies

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0) \tag{2.83}$$

with $H_I(t) = e^{iH_0(t-t_0)}H_{\rm int}e^{-iH_0(t-t_0)}$ because

$$\begin{aligned} i\partial_t e^{iH_0(t-t_0)}e^{-iH(t-t_0)} &= i\left[iH_0 e^{iH_0(t-t_0)}e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}(-iH)e^{-iH(t-t_0)}\right] \\ &= e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\rm int}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= H_I(t)U(t,t_0). \end{aligned} \tag{2.84}$$

As in Quantum Mechanics we solve the differential equation (2.83)by rewriting it as an integral equation,

$$U(t,t_0) = \frac{1}{i}\int_{t_0}^{t} H_I(t')U(t',t_0)\mathrm{d}t' + U(t_0,t_0). \tag{2.85}$$

The latter can be solved iteratively with initial value $U^{(0)}(t,t_0) = \mathbb{1}$,

$$\begin{aligned} U^{(1)}(t,t_0) &= \mathbb{1} + \frac{1}{i}\int_{t_0}^{t}\mathrm{d}t_1\, H_I(t_1)\underbrace{U^{(0)}(t_1,t_0)}_{=\mathbb{1}}, \\ U^{(2)}(t,t_0) &= \mathbb{1} + \frac{1}{i}\int_{t_0}^{t}\mathrm{d}t_2\, H_I(t_2)U^{(1)}(t_2,t_0), \end{aligned} \tag{2.86}$$

and so on at each iteration. The exact solution is given by the n-th iteration for $n\to\infty$,

$$\lim_{n\to\infty} U^{(n)}(t,t_0) \equiv U(t,t_0). \tag{2.87}$$

Indeed it is easy to check that this expression, which is just

$$\boxed{U(t,t_0) = \mathbb{1} + \sum_{n=1}^{\infty}\left(\frac{1}{i}\right)^n \int_{t_0}^{t}\mathrm{d}t_1 \int_{t_0}^{t_1}\mathrm{d}t_2 ... \int_{t_0}^{t_{n-1}}\mathrm{d}t_n H_I(t_1)H_I(t_2)...H_I(t_n),} \tag{2.88}$$

solves equ. (2.83). Note that the $H_I(t_i)$ under the integral are time-ordered.
The solution can be simplified further. Consider e.g. the second term and observe that it can be rewritten as

$$\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2 H_I(t_1)H_I(t_2) = \frac{1}{2}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t}\mathrm{d}t_2 T\left(H_I(t_1)H_I(t_2)\right). \tag{2.89}$$

Indeed the right-hand side is

$$\underbrace{\frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)}_{\text{a) } t_1>t_2} + \underbrace{\frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 H_I(t_2)H_I(t_1)}_{\text{b) } t_2>t_1} \tag{2.90}$$

and intgral b) is in fact the same as integral a). To see this rotate the square in the $t_1 - t_2$ plane over which we integrating by 90°, which gives

$$\text{b) } = \frac{1}{2}\int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 H_I(t_2)H_I(t_1) \equiv \text{a) } . \tag{2.91}$$

With this reasoning the time-evolution operator is

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 ... \int_{t_0}^{t} dt_n \, T\, H_I(t_1)H_I(t_n). \tag{2.92}$$

For the latter infinite series one introduces the notation

$$\boxed{U(t,t_0) = T e^{-i\int_{t_0}^{t} dt' H_I(t')}} . \tag{2.93}$$

From the series expression one can verify that the time-evolution operator has the following properties:

- $U^\dagger(t_1,t_2) = U^{-1}(t_1,t_2) = U(t_2,t_1)$,
- $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$ for $t_1 \geq t_2 \geq t_3$.

2.5.2 From the interacting to the free vacuum

Having understood how to relate the full Heisenberg fields to the free interaction picture fields, we now try to set up a relation between the free vacuum $|0\rangle$ and the interacting vacuum $|\Omega\rangle$. Let $|n\rangle$ be an eigenstate of the full Hamiltonian, i.e.

$$H|n\rangle = E_n|n\rangle \quad \text{with} \quad H = H_0 + H_{\text{int}}. \tag{2.94}$$

Then the time-evolution of the free vacuum $|0\rangle$ is[5]

$$\begin{aligned} e^{-iHT}|0\rangle &= e^{-iHT}\sum_n |n\rangle\langle n|0\rangle \\ &= \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle \\ &= e^{-iE_\Omega T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{|n\rangle\neq|\Omega\rangle} e^{-iE_nT}|n\rangle\langle n|0\rangle. \end{aligned} \tag{2.95}$$

[5] You must not confuse time ordering T and time T over the next pages.

If $H_0|0\rangle = 0$, then $H|\Omega\rangle = E_\Omega|\Omega\rangle$ with $E_\Omega \neq 0$, because we now compare the vacuum energy of 2 theories or, put differently, the counter-term V_0 in the Lagrangian has already been used to set $E_0 = 0$ so that we are stuck with E_Ω, whatever it is. Be that as it may, we have $E_n > E_\Omega \;\forall\; |n\rangle \neq |\Omega\rangle$. So if we formally take the limit $T \to \infty(1-i\epsilon)$, then e^{-iE_nT} is stronger suppressed and only the vacuum $|\Omega\rangle$ survives in[6]

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|0\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iE_\Omega T}\langle\Omega|0\rangle|\Omega\rangle\,. \tag{2.96}$$

Solving this for $|\Omega\rangle$ yields

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_\Omega T}\langle\Omega|0\rangle}.} \tag{2.97}$$

Note that we are assuming here that $\langle\Omega|0\rangle \neq 0$, which is guaranteed at least for small perturbations H_{int} in H.

To bring this to a form involving the time-evolution operator we shift $T \to T + t_0 = t_0 - (-T))$ and write for $e^{-iH(T+t_0)}|0\rangle$

$$e^{-iH(t_0-(-T))}\underbrace{e^{-iH_0(-T-t_0)}|0\rangle}_{=|0\rangle \text{ since } H_0|0\rangle = 0} = U(t_0,-T)|0\rangle\,, \tag{2.98}$$

because

$$\begin{aligned} U(t_0,-T) = U(-T,t_0)^{-1} &= \left[e^{iH_0(-T-t_0)}e^{-iH(-T-t_0)}\right]^{-1} \\ &= e^{-iH(T+t_0)}e^{-iH_0(-T-t_0)}. \end{aligned} \tag{2.99}$$

So the vacuum is

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle\,.} \tag{2.100}$$

Likewise, starting from $\lim_{T\to\infty(1-i\epsilon)}\langle 0|e^{-iHT}$ one gets

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,t_0)\left(e^{-iE_\Omega(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}. \tag{2.101}$$

Finally we can compute $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$. Suppose first that $x^0 \geq y^0 \geq t_0$: Then $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ becomes

$$\begin{aligned} \langle\Omega|\phi(x)\phi(y)|\Omega\rangle = &\lim_{T\to\infty(1-i\epsilon)} (e^{-iE_\Omega(T-t_0)}\langle 0|\Omega\rangle)^{-1} \\ &\times \langle 0|U(T,t_0)\underbrace{U(x^0,t_0)^\dagger \Phi_I(x)U(x^0,t_0)}_{\phi(x)} \\ &\times \underbrace{U(y^0,t_0)^\dagger\Phi_I(x)U(y^0,t_0)}_{\phi(y)}U(t_0,-T)|0\rangle\left(e^{-iE_\Omega(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} \\ = &\lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_\Omega 2T}\right)^{-1} \\ &\times \langle 0|U(T,x_0)\Phi_I(x)U(x^0,y^0)\Phi_I(y)U(y^0,-T)|0\rangle\,. \end{aligned} \tag{2.102}$$

[6] Alternatively, the argument can be phrased as follows: Renormalise your theory such that $E_\Omega = 0$, but $E_0 \neq 0$. Then it is clear that only the term involving $|\Omega\rangle$ survives unsupressed. The following equations must then be adjusted, but the final result is the same.

Note that the contraction $U(x^0,t_0)U(t_0,y^0) = U(x^0,y^0)$ only works because of time-ordering. We can eliminate the constant prefactor by noting that

$$\mathbb{1} = \langle\Omega|\,\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(|\langle 0|\,\Omega\rangle|^2 e^{-iE_\Omega(2T)}\right)^{-1} \langle 0|\,U(T,t_0)U(t_0,-T)\,|0\rangle\,. \tag{2.103}$$

For $x^0 \geq y^0 \geq t_0$ we arrive at

$$\langle\Omega|\,\phi(x)\phi(y)\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,U(T,x^0)\Phi_I(x)U(x^0,y^0)\Phi_I(y)U(y^0,-T)\,|0\rangle}{\langle 0|\,U(T,-T)\,|0\rangle}. \tag{2.104}$$

The nominator is

$$\langle 0|\,T\Phi_I(x)\Phi_I(y)\,\underbrace{U(T,x^0)U(x^0,y^0)U(y^0,-T)}_{U(T,-T)}\,|0\rangle\,, \tag{2.105}$$

where the time-ordering symbol takes care of order. Similar conclusions are obtained for $x^0 \leq y^0$. Therefore altogether

$$\boxed{\langle\Omega|\,T\phi(x)\phi(y)\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,T\left(\Phi_I(x)\Phi_I(y)e^{-i\int_{-T}^{T}\mathrm{d}tH_I(t)}\right)|0\rangle}{\langle 0|\,Te^{-i\int_{-T}^{T}\mathrm{d}tH_I(t)}\,|0\rangle}.} \tag{2.106}$$

This same reasoning goes through for higher n-point correlators. Our master formula for computing correlation function becomes

$$\boxed{\langle\Omega|\,T\prod_i \phi(x_i)\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,T\prod_i \Phi_I(x_i)e^{-i\int_{-T}^{T}\mathrm{d}tH_I(t)}\,|0\rangle}{\langle 0|\,Te^{-i\int_{-T}^{T}\mathrm{d}tH_I(t)}\,|0\rangle}.} \tag{2.107}$$

This formula can be applied to concrete interactions, i.e.

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}},\ \text{e.g.}\ \ \mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4(x), \tag{2.108}$$

where λ has mass-dimension 0. The interaction picture Hamiltonian is then

$$H_I = \int \mathrm{d}^3x \frac{\lambda}{4!}\Phi_I(t,\vec{x})^4 \tag{2.109}$$

and we can expand the time-evolution perturbatively order by order in λ if $\lambda \ll 1$ by expanding the exponential

$$\lim_{T\to\infty(1-i\epsilon)} e^{-i\int_{-T}^{T}\mathrm{d}tH_I(t)} = \lim_{T\to\infty(1-i\epsilon)} e^{-i\int \mathrm{d}^4x\frac{\lambda}{4!}\Phi_I^4(x)}. \tag{2.110}$$

As we will see the $(1-i\epsilon)$ prescription for the boundaries of the integral will pose no problems (cf. discussion after (2.133)).

There are basically two remaining problems:

- Perform a systematic evaluation of $\langle 0|\,T\prod_i \Phi_I(x_i)\,|0\rangle$ and
- deal with the denominator.

We will solve these problems by exploiting the action of the creation and annihilation operators on the vacuum.

2.6 Wick's theorem

To compute an expression of the form $\langle 0|\, T\prod_i \Phi_I(x_i)\,|0\rangle$ we decompose Φ_I into free modes,

$$\Phi_I = \underbrace{\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_I(\vec{p})e^{-ip\cdot x}}_{=:\,\Phi_I^+(x)} + \underbrace{\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a_I^\dagger(\vec{p})e^{ip\cdot x}}_{=:\,\Phi_I^-(x)}. \tag{2.111}$$

Now we want to commute all $\Phi_I^-(x)$ to the left of all $\Phi_I^+(x)$ to use

$$\Phi_I^+(x)\,|0\rangle = 0 = \langle 0|\,\Phi_I^-(x). \tag{2.112}$$

To do so we definie **normal-ordering**:

> *An operator O is normal-ordered if all creation/annihilation operators appear on the left/right. For such O we write* $:O:$

Consider for example the operator:

$$:a^\dagger(\vec{p}_1)a(\vec{p}_2)a^\dagger(\vec{p}_3): = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_3)a(\vec{p}_2) = a^\dagger(\vec{p}_3)a^\dagger(\vec{p}_1)a(\vec{p}_2). \tag{2.113}$$

It is obvious that

$$\boxed{\langle 0|:O:|0\rangle = 0} \tag{2.114}$$

for every non-trivial operator $O \neq c\mathbb{1}$ for $c \in \mathbb{C}$. We begin with $\langle 0|\,T\Phi_I(x)\Phi_I(y)\,|0\rangle$ and drop the subscript "I" from now on. There are two cases to consider, either $x^0 \geq y^0$ or $y^0 \geq x^0$. If $x^0 \geq y^0$, then $T\Phi(x)\Phi(y) = \Phi(x)\Phi(y)$ and

$$\Phi(x)\Phi(y) = \Phi^-(x)\Phi^-(y) + \Phi^-(x)\Phi^+(y) + \Phi^+(x)\Phi^+(y) + \Phi^+(x)\Phi^-(y) \tag{2.115}$$

The first three terms are already normal-ordered and the last term can be put in normal-ordered form by commuting the fields through each other,

$$\Phi(x)\Phi(y) = :\Phi(x)\Phi(y): + [\Phi^+(x), \Phi^-(y)]. \tag{2.116}$$

Similar expressions follow for $x^0 \leq y^0$, so altogether

$$\begin{aligned} T(\Phi(x)\Phi(y)) = :\Phi(x)\Phi(y): &+ \Theta(x^0-y^0)[\Phi^+(x), \Phi^-(y)] \\ &+ \Theta(y^0-x^0)[\Phi^+(y), \Phi^-(x)]. \end{aligned} \tag{2.117}$$

We notice that the last two terms are a $\mathbb{C}$-number c and with

$$\langle 0|\,T\Phi(x)\Phi(y)\,|0\rangle = \underbrace{\langle 0|:\Phi(x)\Phi(y):|0\rangle}_{\equiv 0} + \langle 0|\,c\,|0\rangle = c \equiv D_F^{(0)}(x-y) \tag{2.118}$$

we find that

$$\boxed{T(\Phi(x)\Phi(y)) = :\Phi(x)\Phi(y): + D_F^{(0)}(x-y).} \tag{2.119}$$

We use the notation $D_F^{(0)}(x-y)$ for the free Feynman propagator in the sequel to emphasize that this object is defined in terms of the free mass parameter m_0 appearing in the Lagrangian. If we define the **contraction**

$$\overbrace{\Phi(x)\Phi(y)} = D_F^{(0)}(x-y) = D_F^{(0)}(y-x), \tag{2.120}$$

we can write this as

$$\boxed{T(\Phi(x)\Phi(y)) = :\Phi(x)\Phi(y) + \overbrace{\Phi(x)\Phi(y)}: \quad .} \tag{2.121}$$

Note that by definition $: c := c$ for a $\mathbb{C}$-number, which explains the notation $: \overbrace{\Phi(x)\Phi(y)} :$.[7] This generalises to higher products. E.g. one can show by direct computation that

$$\begin{aligned} T(\Phi(x_1)\Phi(x_2)\Phi(x_3)) &\equiv T(\Phi_1\Phi_2\Phi_3) \\ &= :\Phi_1\Phi_2\Phi_3: + :\Phi_1\overbrace{\Phi_2\Phi_3}: + :\overbrace{\Phi_1\Phi_2}\Phi_3: + :\overbrace{\Phi_1\Phi_2\Phi_3}: \\ &= :\Phi_1: D_F^{(0)}(x_2-x_3) + :\Phi_3: D_F^{(0)}(x_1-x_2) + :\Phi_2: D_F^{(0)}(x_1-x_3). \end{aligned} \tag{2.122}$$

Wick's theorem generalises this for N fields,

$$\begin{aligned} T(\Phi_1...\Phi_N) = &:\Phi_1...\Phi_N: \\ &+ \sum_{1\le i<j\le N} :\Phi_1...\overbrace{\Phi_i...\Phi_j}...\Phi_N: \\ &+ \sum_{1\le i<k<j<l\le N} \Big(:\Phi_1...\overbrace{\Phi_i...\Phi_k}...\overbrace{\Phi_j...\Phi_l}...\Phi_N: \\ &\qquad\qquad + \text{ 2 more contractions} \Big) \\ &+ ... \end{aligned} \tag{2.123}$$

Indeed one can prove this by induction and summarize Wick's theorem as

$$T(\Phi_1...\Phi_N) = :\Phi_1...\Phi_N: + :\textit{all contractions of distinct pairs}: \quad . \tag{2.124}$$

The proof is reserved to Examples Sheet 6.
Wick's theorem together with (2.114) has two important consequences:

- $\langle 0|\, T\Phi_1...\Phi_{2N+1}\,|0\rangle = 0$, because there is always an odd number of normal-ordered fields remaining and
- $\langle 0|\, T\Phi_1...\Phi_{2N}\,|0\rangle = D_F^{(0)}(x_1-x_2)D_F^{(0)}(x_3-x_4)...D_F^{(0)}(x_{2N-1}-x_{2N}) + \sum(\text{all other contractions})$.

[7] See also the remark right after equ. (2.114).

2.7 Feynman diagrams

There exists a practical graphical representation of the systematics of contractions in terms of **Feynman diagrams**. Consider for instance

$$\begin{aligned}\langle 0|\, T(\Phi_1\Phi_2\Phi_3\Phi_4)\,|0\rangle &= D_F^{(0)}(x_1-x_2)D_F^{(0)}(x_3-x_4)\\ &\quad + D_F^{(0)}(x_1-x_3)D_F^{(0)}(x_2-x_4) + D_F^{(0)}(x_1-x_4)D_F^{(0)}(x_2-x_3).\end{aligned} \tag{2.125}$$

The result can be translated into a Feynman diagram as follows:

- First, draw 1 point for all x_i in $\Phi(x_i)$ and connect these by lines in all possible ways. This gives, in the present case, three distinct diagrams displayed in Figure 2.2. Note that these are distinct because we distinguish between x_1, x_2, x_3, x_4.

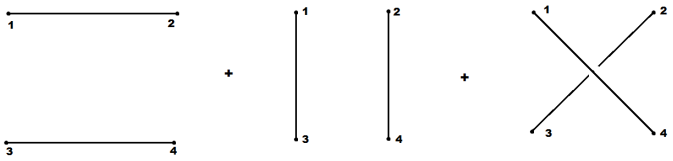


Figure 2.2: Possible connections between all x_i.

- Then, to each line between i and j we attach a factor of $D_F^{(0)}(x_i - x_j)$.
- We then multiply all $D_F^{(0)}(x_i - x_j)$ and sum over all distinct diagrams.

In interacting theories also contractions involving fields at the same spacetime point occur. For example consider the propagator in Φ^4-theory up to first order in λ:

$$\begin{aligned}&\langle 0|T\Phi(x)\Phi(y)e^{-i\frac{\lambda}{4!}\int \mathrm{d}^4z\Phi^4(z)}\,|0\rangle =\\ &\quad \langle 0|\, T\Phi(x)\Phi(y)\,|0\rangle + \langle 0|\, T\Phi(x)\Phi(x)\int \mathrm{d}^4z\left(-\frac{i\lambda}{4!}\right)\Phi^4(z)\,|0\rangle + O(\lambda^2).\end{aligned} \tag{2.126}$$

The $O(\lambda^0)$-term corresponds to a straight line between x and y, i.e. the propagation from x to y, which is assigned a factor $D_F^{(0)}(x-y)$. The term to first order in λ is

$$\frac{-i\lambda}{4!}\Bigg[\int \mathrm{d}^4z\,\overbrace{\phi(x)\phi(y)}\,\overbrace{\Phi(z)\,\overbrace{\Phi(z)\Phi(z)}\,\Phi(z)} \tag{2.127}$$

$$+\int \mathrm{d}^4z\overbrace{\Phi(x)\Phi(y)\Phi(z)}\,\Phi(z)\,\overbrace{\Phi(z)\Phi(z)}\Bigg]. \tag{2.128}$$

This can be written as

$$\begin{aligned}&\int \mathrm{d}^4 z D_F^{(0)}(x-y)\binom{4}{2}\frac{1}{2}D_F^{(0)}(z-z)D_F^{(0)}(z-z)\\&+\int \mathrm{d}^4 z\, D_F^{(0)}(x-z)D_F^{(0)}(y-z)D_F^{(0)}(z-z)\cdot 4\cdot 3.\end{aligned} \tag{2.129}$$

The graphical representation of these two different contributions is given in Figure 2.3. Note the appearance of the point z, which is integrated over. The first term is proportional $-i\lambda\frac{1}{8}$, while the second term has a factor of $-i\lambda\frac{1}{2}$.

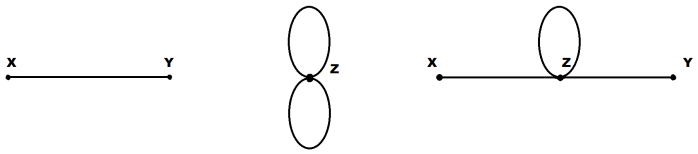


Figure 2.3: First and second term of $O(\lambda^1)$.

To give an interpretation of these combinatorial factors of $1/8$ and $1/2$, we observe that they coincide with 1 over the order (≡ number of elements) of the symmetry group of the diagrams. These are called the **symmetry factors**. The symmetry factors are given by the number of ways one can exchange components of the diagram without changing the diagram, where by components we mean either the two ends of a line starting and ending on the same point, entire lines between points or internal points (vertices - see below).
For the two diagrams these are in turn:

- The single loop at z is symmetric under the exchange of the out- and in-going end of the loop, but the direction must be consistent (i.e. one out- and one ingoing arrow). Therefore the symmetry group is $G = \mathbb{Z}_2$ and its order is $|G| = 2$.
- The double loop is separately symmetric under exchange of the out- and in-going end of both loops, which gives a $\mathbb{Z}_2 \times \mathbb{Z}_2$ as well as under exchange of the two loops as a whole. Therefore the symmetry group is $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and its order is $|G| = 8$.

One can show in general that if a Feynman diagram with symmetry group G always carries a combinatorial factor $\frac{1}{|G|}$. This is because if some symmetry remains the factor of $\frac{1}{n!}$ in the interaction term $-\frac{\lambda}{n!}\phi^n$ is only partially cancelled by counting the various contractions that yield the same diagram. In the above examples the symmetry group was due to the ambiguity in assigning arrows to the lines at a vertex. In addition, if k vertices are identical, the symmetry group includes the group of permutations of these k vertices of order $|S_k| = k!$.

Let us introduce some jargon:

- The points x, y associated to $\Phi(x)\Phi(y)$ in the correlator, which are not integrated over, are called **external points**.

- All other points - here z - over which is integrated over are **internal points**.
- For a Φ^n interaction, at an internal point n lines meet. Such points are called **vertices**.

2.7.1 Position space Feynman-rules

The translation of Wick's theorem into Feynman-graphs as exemplified above can be proven to hold generally (the only non-trivial aspect being the symmetry factors). We summarize this in terms of the **position space Feynman-rules** for the computation of

$$\langle 0|\, T \prod_{i=1}^{m} \Phi(x_i) e^{-i\frac{\lambda}{n!}\int d^4z\, \phi^n(z)} \,|0\rangle \tag{2.130}$$

at order λ^k, where we are assuming that all x_i are distinct:

- Draw one external point for all x_i and k internal points z_j.
- Connect the points by lines such that
  - to each external point x_i 1 line is attached,
  - to each internal point z_j n lines are attached.
- To each line between points y_i and y_j (both external and internal) we associate a free propagator

  $$D_F^{(0)}(y_i - y_j) = D_F^{(0)}(y_j - y_i), \tag{2.131}$$

  where we stress again that the superscript reminds us to take the Feynman propagator of the free theory.
- To each vertex associate a factor $-i\lambda \int \mathrm{d}^4 z_j$.
- To each external point associate a 1.
- Multiply all factors, Feynman propagators etc. and divide by the symmetry factor of the diagram.
- Then sum up all distinct such Feynman diagrams.

2.7.2 Momentum space Feynman-rules

More practical are the **momentum space Feynman-rules**: To each free propagator

$$D_F^{(0)}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 - m_0^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{2.132}$$

associate an arrow and momentum p in that direction. This means an arrow from x to y corresponds to the propagator $D_F^{(0)}(x-y)$. Obviously the direction is arbitrary because $D_F^{(0)}(x-y) = D_F^{(0)}(y-x)$,

but let us make one such choice for each line. We can then perform the integral $\int d^4z$ at each vertex explicitly. E.g. for the Φ^4-theory 4 lines meet at a vertex, and if, say, the momenta p_1 and p_2 point into that vertex and p_3, p_4 point out of it, this integral yields

$$\int \mathrm{d}^4z\, e^{ip_1\cdot z} e^{ip_2\cdot z} e^{-ip_3\cdot z} e^{-ip_4\cdot z} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4). \tag{2.133}$$

This implements momentum conservation at each vertex. The delta-function will effectively eliminate one integral $\int \mathrm{d}^4 p_i$ per vertex.[8]

Note at this point that the formal prescription that we integrate $\lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T}$ in (2.107) translates into a corresponding prescription for $\int \mathrm{d}^4z$. We should therefore view this integral as a complex contour integral. From the fact that such integrals can be deformed in the complex plane as long as no poles are hit it follows that the $(1-i\epsilon)$ prescription does not affect the result.

The momentum-space Feynman rules[9] for the computation of the n-point function

$$\langle 0|\, T \prod_{i=1}^{m} \Phi(x_i) e^{-i\frac{\lambda}{n!}\int d^4z\, \phi^n(z)} |0\rangle \tag{2.134}$$

at order λ^k are:

- Draw one external point for all x_i and k internal points z_j.
- Connect the points by lines such that
  - to each external point x_i 1 line is attached,
  - to each internal point z_j n lines are attached.
- To each line between points y_i and y_j (both external and internal) we associate a free propagator

  $$D_F^{(0)}(y_i - y_j) \tag{2.135}$$

  with one choice of direction and to each such $D_F^{(0)}(y_i - y_j)$ we associate directed momentum p from y_i to y_j and a factor

  $$\frac{i}{p^2 - m_0^2 + i\epsilon}. \tag{2.136}$$

- For each vertex we multiply a factor of $(2\pi)^4\delta^{(4)}(\sum_{\text{ingoing}} p_i - \sum_{\text{outgoing}} p_k) \times (-i\lambda)$.
- For each external point we multiply a factor of $e^{-ip\cdot x}$ for momentum pointing out of the external point, or $e^{ip\cdot x}$ for momentum pointing into the point.

[8] The elimination of the z_i-integration works this way only if the vertex is connected to at least one other point, either internal or external. This is, for example, not the case for the second diagram in Fig. 2.3. More generally, one overall factor of $\int d^4z_i = \mathrm{Vol}_{\mathbb{R}^{1,3}} = (2\pi)^4\delta^{(4)}(0)$ remains for diagrams not connected to any of the external points. We will see in the next subsection how to deal with such 'disconnected diagrams'.

[9] Careful: By momentum space Feynman rules we do not mean that we compute the Fourier transform of the correlator, but rather that we give an equivalent set of rules for the computation of the correlator where the integral over the vertex positions has been performed explicitly.

- Integrate over each momentum $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$ and divide by the symmetry factor.
- Then sum up all distinct such Feynman diagrams.

For example, consider again the Φ^4 theory to order λ, i.e.

$$\langle 0|\, T \prod_{k=1}^{4} \Phi_k e^{-i\frac{\lambda}{4!}\int \mathrm{d}^4 x \Phi^4} \,|0\rangle \,. \tag{2.137}$$

λ^0 corresponds to the diagrams in Figure 2.2. For λ^1 we can have

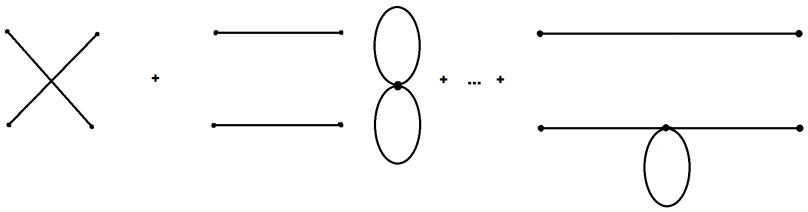


Figure 2.4: Diagrams for λ^1.

Consider the first diagram at order λ, make a consistent choice of momenta and assign $e^{ik\cdot x}$ for in-going arrows at external points, $e^{-ik\cdot x}$ for out-going arrows and $-i\lambda\delta^{(4)}(k_1+k_2-k_3-k_4)$ at the vertex due to momentum consservation. We end up with

$$\begin{aligned}
&\langle 0|\, T \prod_{k=1}^{4}\Phi(x_k)\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4 z\, \Phi^4(z)\right)|0\rangle\Big|_{\text{Diagram}\times} \\
&\quad = \prod_{j=1}^{4}\int \frac{\mathrm{d}^4 k_j}{(2\pi)^4} e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_3\cdot x_3} e^{ik_4\cdot x_4} \\
&\qquad \times\ (-i\lambda)(2\pi)^4\delta^{(4)}(k_1+k_2-k_3-k_4)\prod_{j=1}^{4}\frac{i}{k_j^2-m_0^2+i\epsilon}.
\end{aligned} \tag{2.138}$$

As we will see soon, in the context of the LSZ formalism we actually need not the correlation function, but rather its Fourier transform. E.g. we will need expressions of the form

$$\begin{aligned}
&\int \mathrm{d}^4 x_1\, e^{ip_1\cdot x_1}\int \mathrm{d}^4 x_2\, e^{ip_2\cdot x_2}\int \mathrm{d}^4 x_3\, e^{-ip_3\cdot x_3}\int \mathrm{d}^4 x_4\, e^{-ip_4\cdot x_4}\,\langle 0|\ldots|0\rangle\Big|_{\text{Diagram}\times} \\
= \quad & (2\pi)^4\delta^{(4)}(p_1+p_2-q_3-q_4)(-i\lambda)\times \\
& \frac{i}{p_1^2-m_0^2+i\epsilon}\,\frac{i}{p_2^2-m_0^2+i\epsilon}\,\frac{i}{p_3^2-m_0+i\epsilon}\,\frac{i}{p_4^2-m_0+i\epsilon}.
\end{aligned} \tag{2.139}$$

2.8 Disconnected diagrams

A typical diagram contains disconnected pieces, i.e. subdiagrams which are not connected to any of the external points. For example consider the second diagram in Figure 2.4: The double loop in this

case is not connected to any external point and therefore called 'disconnected piece' (or disconnected diagram). A disconnected piece contains only internal points. By contrast, the part of the diagram which is connected to at least one external point is called 'partially connected diagram'.
According to the Feynman rules disconnected pieces appear as overall factors of the Feynman diagram. If we sum up all Feynman diagrams that contribute to

$$\langle 0|\, T\Phi(x_1)...\Phi(x_k)e^{-i\frac{\lambda}{N!}\int \Phi^N \mathrm{d}^4x}\,|0\rangle \tag{2.140}$$

the result factorises into the sum of all 'partially connected diagrams' in the above sense multiplied by the sum of all disconnected diagrams. If you don't believe this, just draw a few pages of Feynman diagrams relevant for, say, the 4-point function in ϕ^4-theory and convince yourself that the infinite sum over all relevant diagrams can be organized this way.
Let $\{V_j\}$ denote the set of all individual disconnected pieces (i.e. V_i is a Feynman diagram containing only internal points which itself is connected as a diagram). Then the sum over all diagrams not connected to any of the external points can be organized as a sum over all V_i, where each V_i appears n_i-times with $n_i = 0, 1, ..., \infty$. If some V_i appears n_i-times we must divide its contribution by $n_i!$ to account for the symmetry factor from interchanging identical V_i. For instance, the sum over all disconnected diagrams contains

$1 +$  $+\frac{1}{2!}$  $+\frac{1}{3!}$  $+ ... +$ arbitrary products and number of loops.

Denoting by V_i also the value of the corresponding diagram, we have that the sum over all disconnected diagrams is

$$\prod_i \sum_{n_i=0}^{\infty} (V_i)^{n_i}\frac{1}{n_i!} = \prod_i e^{V_i} = e^{\sum_i V_i}. \tag{2.141}$$

This is called exponentiation of disconnected pieces. The final result for (2.140) is therefore of the form

$$\sum(\text{ at least partially connected pieces}) \times e^{\sum_i V_i}. \tag{2.142}$$

Now consider the full correlator

$$\langle\Omega|\, T\prod_i \phi_i\,|\Omega\rangle = \frac{\langle 0|\, T\prod_i \phi_i e^{-i\int H_I \mathrm{d}t}\,|0\rangle}{\langle 0|\, Te^{-i\int H_I \mathrm{d}t}\,|0\rangle}. \tag{2.143}$$

The denominator contains no external points and thus yields precisely

$$\langle 0|\, Te^{-i\int H_I \mathrm{d}t}\,|0\rangle = e^{\sum_i V_i} \equiv \textbf{ the partition function}. \tag{2.144}$$

This cancels the factor from the nominator and thus

$$\boxed{\langle\Omega|\, T\prod_i^n \phi_i\,|\Omega\rangle = \sum(\text{all partially connected diagrams with } n \text{ external points}).} \tag{2.145}$$

Note that the partition function plays an important role also in Statistical Mechanics. In computing the partition function in that context you will encounter many times the same reasoning that organizes the sum over all its contributions in a form similar to the one above. In fact, when discussing the path integral approach to Quantum Field Theory in QFT II we will see that this is no coincidence.

2.8.1 Vacuum bubbles

The disconnected diagrams contributing to the partition function are called vacuum bubbles and they have a remarkable physical interpretation: From eq. (2.103) we recall that

$$e^{\sum_i V_i} = \langle 0| T e^{-i\int \mathrm{d}t H_I} |0\rangle = \lim_{T\to\infty(1-i\epsilon)} \left|\langle \Omega| 0\rangle\right|^2 e^{-iE_\Omega 2T} \tag{2.146}$$

so

$$-iE_\Omega 2T\big|_{T\to\infty(1-i\epsilon)} = \sum V_i - \underbrace{\log\left(|\langle\Omega|0\rangle|^2\right)}_{\text{finite number}} \tag{2.147}$$

and therefore

$$E_\Omega = \lim_{T\to\infty(1-i\epsilon)} \frac{i\sum V_i}{2T} \tag{2.148}$$

because the finite term plays no role in the limit we are taking. E_Ω is the vacuum energy of the interacting theory in the scheme where E_0 - the vacuum energy of the free theory - has been set to 0 by renormalisation of the original Lagrangian after adding a term $-\frac{E_0}{\mathrm{Vol}_{\mathbb{R}^3}}$ to $\mathcal{L}$. In terms of the corresponding energy density this means

$$\frac{E_\Omega}{\mathrm{Vol}_{\mathbb{R}^3}} = \lim_{T\to\infty(1-i\epsilon)} \frac{i\sum_j V_j}{2T\mathrm{Vol}_{\mathbb{R}^3}} = i\frac{\sum_j V_j}{(2\pi)^4\delta^{(4)}(0)}. \tag{2.149}$$

Note that the factor of $(2\pi)^4\delta^{(4)}(0)$ in the denominator cancels a corresponding factor appearing in the computation of each of the V_i from an uncancelled overall $\int d^4z$ - see footnote 8. This factor represents the familiar IR divergence of the vacuum energy from integration over spacetime. As we will see later when actually computing loops of this type, $\frac{E_\Omega}{\mathrm{Vol}_{\mathbb{R}^3}}$ is also UV divergent because of divergent momentum integration for the vacuum loops. This is no surprise as already $\frac{E_0}{\mathrm{Vol}_{\mathbb{R}^3}}$ was divergent. We organise these divergences order by order in perturbation theory and can renormalise them away by counter-terms, i.e. by adding to the classical Lagrangian terms of form[10]

$$\mathcal{L} \to \mathcal{L} - \lambda^1 V_0^{(1)} - \lambda^2 V_0^{(2)} - ..., \tag{2.150}$$

just like we did for E_0 (i.e. at order λ^0) in the free theory.

2.9 1-particle-irreducible diagrams

As a first application of our formula for the computation of n-point functions in an interacting theory we consider perturbative corrections to the propagator in Φ^4-theory,

$$D_F(x-y) = \langle\Omega| T\Phi(x)\Phi(y)|\Omega\rangle = \sum_i O(\lambda^i). \tag{2.151}$$

Some of the first few Feynman diagrams are shown in Figure 2.5.

[10] In fact, due to renormalisation of the other operators order by order in λ this is a bit of an oversimplification, but we will come to this in more detail later in the course.

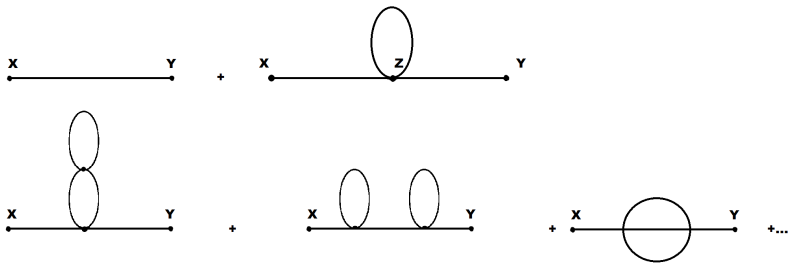


Figure 2.5: Perturbative corrections.

To organise the sum over Feynman diagrams it is useful to define the concept of 1− **particle-irreducible** (1PI) Feynman diagrams. These are diagrams out of which one cannot produce 2 separate non-trivial diagrams (diagrams containing more than just one line) by cutting a single line. E.g. in Figure 2.5 only the 4th diagram is not 1PI (one can cut it in 2 by cutting the line between the two loops). It is standard to introduce the notation

$$\text{(1PI)} = \sum(\text{all non-trivial 1PI diagrams}), \tag{2.152}$$

where it is understood that we do not attach external points to both ends (i.e. no factors $e^{-ip\cdot x}$ or $i/(p^2-m_0^2+i\epsilon)$ from the left or right of (1PI).
We now define $-iM^2(p^2)$ to be the value of (1PI), where p^2 denotes the in- and outgoing momentum. Clearly computing $-iM^2(p^2)$ can be a hard task, and the result will be a complicated function of p^2. In particular this quantity may be divergent (again due to integration over internal momenta). We will compute $-iM^2(p^2)$ explicitly for an electron in Quantum Electrodynamics, in which case it is called **self-energy**, later in the course and find ways to deal with its divergence systematically. Irrespective of the outcome for $-iM^2(p^2)$, the sum over all diagrams contributing to $\langle\Omega|\,T\Phi(x)\Phi(y)\,|\Omega\rangle$ can be organised as a geometric series in 1 PI diagrams. Let us compute the Fourier transform $D_F(p^2)$ defined such that

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} D_F(p^2). \tag{2.153}$$

$D_F(p^2)$ can be expanded as

$$D_F(p^2) = \frac{i}{p^2-m_0^2+i\epsilon} + \frac{i}{p^2-m_0^2+i\epsilon}\,\text{(1PI)}\,\frac{i}{p^2-m_0^2+i\epsilon} + \frac{i}{p^2-m_0^2+i\epsilon}\,\text{(1PI)}\,\frac{i}{p^2-m_0^2+i\epsilon}\,\text{(1PI)}\,\frac{i}{p^2-m_0^2+i\epsilon} + \ldots \tag{2.154}$$

The fact that we are not drawing points at the two ends of diagrams symbolizes that we are omitting the factors of e^{ipx} or e^{-ipy} as these are, by definition, not part of $D_F(p^2)$. The Fourier transformed

propagator is then

$$\begin{aligned} D_F(p^2) &= \frac{i}{p^2-m_0^2+i\epsilon} + \frac{i}{p^2-m_0^2+i\epsilon}\left(-iM^2(p^2)\right)\frac{i}{p^2-m_0^2+i\epsilon} + ... \\ &= \frac{i}{p^2-m_0^2+i\epsilon}\left[1 + \underbrace{\frac{M^2(p^2)}{p^2-m_0^2+i\epsilon}}_{\equiv q} + \left(\frac{M^2(p^2)}{p^2-m_0^2+i\epsilon}\right)^2 + ...\right] \\ &= \frac{i}{p^2-m_0^2+i\epsilon}\quad\frac{1}{1-\frac{M^2(p^2)}{p^2-m_0^2+i\epsilon}}. \end{aligned} \tag{2.155}$$

Here we assumed that $q < 1$ so that the geometric series $\sum_{n=0}^{\infty} q^n$ converges to $\frac{1}{1-q}$. Once again, we will deal with the explicit form of $-iM^2(p^2)$ later in QED and justify this assumption. This procedure is called **Dyson resummation** and yields the **resummed propagator**

$$\boxed{D_F(p^2) = \frac{i}{p^2-(m_0^2+M^2(p^2))+i\epsilon} \equiv}\quad \tag{2.156}$$

Now, $-iM^2(p^2)$ is a complicated function of p^2 that we compute order by order in λ. We can then extract the first analytic pole in $D_F(p^2)$ and call it m^2. Then

$$D_F(p^2) = \frac{iZ}{p^2-m^2+i\epsilon} + \text{terms regular at } m^2 \tag{2.157}$$

for some Z. I.e. Z is by definition the residue (in the above sense) of $D_F(p^2)$ at its first analytical pole m^2. This way we can compute

- m^2 - the physical 1-particle-mass ≡ pole mass,
- Z - the wavefunction renormalisation,

perturbatively to given order in λ. This is a beautiful result because we can now understand quantitatively why the mass of the 1-particle momentum eigenstates in an interacting theory differs from m_0: The reason are the self-interactions of the field, which are resummed as above to shift the pole of the full propagator from m_0 to m. In particular this picture justifies our assertion made in the context of asymptotic in- and out-states that the asymptotic states behave as free particles, but with fully 'renormalised mass' $m \neq m_0$: By sending the particles infinitely far apart from each other we effectively 'switch off' the interactions between the different particles, but we cannot switch off the interactions of the asymptotic particles with themselves (or rather of the field with itself). These are precisely the 1PI contributions to $D_F(p^2)$ and thus the in-and out-states do have the fully resummed mass $m^2 \neq m_0^2$.

2.10 Scattering amplitudes

Recall the LSZ-reduction formula for r in- and n out-states

$$\begin{aligned}&\prod_{k=1}^{n}\int \mathrm{d}^4y_k\, e^{ip_k\cdot y_k}\prod_{l=1}^{r}\int \mathrm{d}^4x_l\, e^{-iq_l\cdot x_l}\,\langle\Omega|\,T\Phi(y_1)...\Phi(x_1)...\,|\Omega\rangle\\ &=\prod_k \frac{iZ^{1/2}}{p_k^2-m^2+i\epsilon}\prod_l \frac{iZ^{1/2}}{q_l^2-m^2+i\epsilon}\,\langle p_1...|\,S\,|q_1...\rangle\big|_{\text{connected}},\end{aligned} \tag{2.158}$$

where all q_l and p_k are **on-shell**. Now, recall that $\langle p_1...|\,S\,|q_1...\rangle\big|_{\text{connected}}$ itself cannot be proportional to $p_k^2-m^2$ or $q_l^2-m^2$ because otherwise it is zero on-shell and thus no scattering occurs; likewise $\langle p_1...|\,S\,|q_1...\rangle\big|_{\text{connected}}$ cannot contain any factors of $(p_k^2-m^2)^{-1}$ or $(q_l^2-m^2)^{-1}$ because then it would be divergent on-shell, in contradiction with its definition as a quantum mechanical amplitude. As a result only those Feynman diagrams are relevant with exactly $(n+r)$ poles at m^2 in the above sense. It is not hard to see that these are precisely the ones that contribute to the fully connected correlation function. A fully connected correlation function has the structure displayed in Figure 2.6.

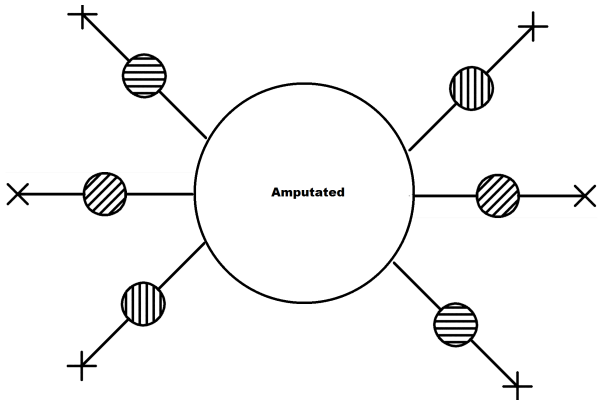
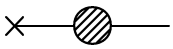
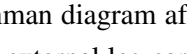
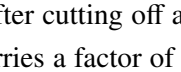


Figure 2.6: Fully connected correlation function with r in and n out states.

Here  denotes the fully resummed propagator from lines connected to external points. The fully resummed propagator would appear if we computed to all orders in perturbation theory. More realistically, to a given finite order in perturbation theory we should think of the propagator as computed to suitable order.

The big blop in the middle denotes the **amputated correlator**. By amputated correlator we mean the Feynman diagram after cutting off all external legs carrying  or  Since each external leg carries a factor of $\frac{iZ}{p^2-m^2+i\epsilon}$ near m^2 if p^2 is on-shell, all $(n+r)$ external legs yield together yield the right singularity structure.

By contrast, partially connected diagrams such as the one in Figure 2.7 carry fewer factors of $\frac{i}{p^2-m^2+i\epsilon}$ and thus do not contribute to $\langle p_1...|\,S\,|q_1...\rangle\big|_{\text{connected}}$. Therefore the **final result for the computation**

of scattering amplitudes is

$$\langle p_1,...,p_n|\,S\,|q_1,...,q_r\rangle\,\big|_{\text{connected}} = \left(\sqrt{Z}\right)^{n+r}\times \tag{2.159}$$
$$\left(\prod_k \int \mathrm{d}^4 y_k e^{ip_k\cdot y_k}\prod_l \int \mathrm{d}^4 x_l e^{-iq_l\cdot x_l}\,\langle\Omega|\,T\Phi(y_1)...\Phi(x_1)...\,|\Omega\rangle\,|_{\text{fully connected}}\right)|_{\text{Amputated}}$$

Note that the wavefunction renormalisation factor Z itself is of the form $1+O(\lambda)$ in perturbation theory, so to leading order in the coupling constant λ the Z-factors play no role as only $O(\lambda)$ diagrams can be fully connected.

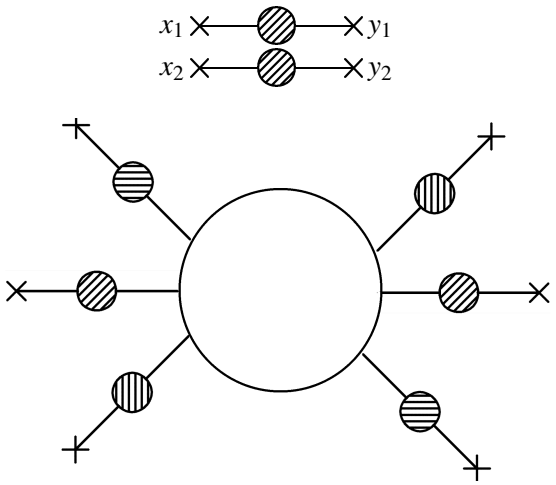


Figure 2.7: Partially connected diagramm, i.e. $(n-2)$ and $(r-2)$.

As an example we consider again the Φ^4-theory to leading order. The only **fully connected** diagram at $O(\lambda)$ is the first diagram in Figure 2.4. We make a consistent choice of ingoing versus outgoing momenta and recall the result

$$\begin{aligned}&\langle\Omega|T\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\,|\Omega\rangle\,\big|_{\times} = \\ &= \prod_{j=1}^{4}\int \frac{\mathrm{d}^4 k_j}{(2\pi)^4} e^{-ik_1\cdot x_1}e^{-ik_2\cdot x_2}e^{ik_3\cdot y_1}e^{ik_4\cdot y_2} \\ &\quad\times(-i\lambda)(2\pi)^4\delta^{(4)}(k_1+k_2-k_3-k_4)\prod_{l=1}^{4}\frac{i}{k_l^2-m_0^2+i\epsilon}.\end{aligned} \tag{2.160}$$

To compute $\langle p_1,p_2|\,S\,|q_1,q_2\rangle\,|_{\text{connected}}$ we take first the Fourier transform as

$$\begin{aligned}&\int \mathrm{d}^4 x_1 e^{-iq_1\cdot x_1}\int \mathrm{d}^4 x_2 e^{-iq_2\cdot x_2}\int \mathrm{d}^4 y_1 e^{ip_1\cdot y_1}\int \mathrm{d}^4 y_2 e^{ip_2\cdot y_2} \\ &\quad\times\langle\Omega|\,T\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\,|\Omega\rangle\,\big|_{\times} \\ &= (-i\lambda)(2\pi)^4\delta^{(4)}(q_1+q_2-p_1-p_2)\prod_{j=1}^{2}\frac{i}{q_j^2-m_0^2+i\epsilon}\frac{i}{p_j^2-m_0^2+i\epsilon}\end{aligned} \tag{2.161}$$

and then we amputate by discarding all propagators from external lines. The result is

$$\langle p_1, p_2| S\, |q_1, q_2\rangle\big|_{\text{connected, } O(\lambda)} = (-i\lambda)(2\pi)^4\delta^{(4)}(q_1 + q_2 - p_1 - p_2), \tag{2.162}$$

where we have discarded the factor $\sqrt{Z}^4$ because, as discussed, it does not contribute to the leading order result.

2.10.1 Feynman-rules for the S-matrix

This procedure can be summarised in general in terms of the **Feynman-rules for the computation of** $\langle p_1, ..., p_n| S\, |q_1, ..., q_r\rangle\big|_{\text{connected}}$ as follows:

- Draw the relevant fully connected Feynman diagrams with $(n + r)$ external points to given order in λ.
- Assign ingoing momenta q_l and outgoing momenta p_k and label momenta of internal lines with k_j.
- Each vertex carries $(-i\lambda)(2\pi)^4\delta^{(4)}$ ($\sum$ ingoing momenta – outgoing momenta).
- Each internal line carries $\frac{i}{k_j^2 - m_0^2 + i\epsilon}$.
- Integrate over all internal momenta $\prod_j \int \frac{\mathrm{d}^4 k_j}{(2\pi)^4}$ and divide by the symmetry factor.
- Sum up all diagrams and multiply by $\left(\sqrt{Z}\right)^{n+r}$ to given order in λ.

Deeper interpretation of the Feynman rules

The connection between correlation functions and S-matrix elements gives Feynman diagrams an intuitive physical meaning: In Figure 2.8 two particles come in and interact at Z_1 to form two so-called "virtual" particles, which in turn join again at Z_2 and form two outgoing states. The obvious

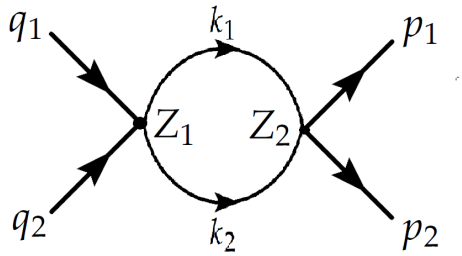


Figure 2.8: Feynman graph with two virtual particles.

interpretation of the Feynman diagram elements is therefore:

- A line —— corresponds to the **worldline** of a particle.

- $\longrightarrow e^{-ip\cdot x}$ is the wavefunction for a momentum-eigenstate.
- A vertex ● Z is a localised interaction at a spacetime point Z.
- Summing up diagrams and integrating over $\int \mathrm{d}^4 z$ amounts to coherently summing up the QM probability amplitudes for all possible processes - called "channels" - with the same macroscopic result. In particular, this will lead to QM interference between the different channels.

Intermediate particles, e.g. those running in the loop as k_1 and k_2 in Figure 2.8, are called virtual because they are generally off-shell: This means that these particles do in general not satisfy the relation $k_i^2 - m_0^2 = 0$. E.g. for the loop-diagram above we integrate over all internal momenta (where the delta-functions at the vertices implies that some of these integrals become trivial). In the present diagram momentum conservation at the first vertex implies that

$$q_1 + q_2 = k_1 + k_2 \;\Leftrightarrow\; k_2 = q_1 + q_2 - k_1, \tag{2.163}$$

where k_1 is free and thus integrated over. For most values of k_1 we have $k_1^2 - m_0^2 \neq 0$. We can think of this as a quantum-mechanical violation of the energy-momentum relation because $E_j^2 \neq \vec{k_j}^2 + m_0^2$ for virtual particles. In Quantum Mechanics this is allowed for sufficiently short times - in fact this precisely what is meant by the energy-time uncertainty relation arising in quantum mechanical perturbation theory. Note, however, that 4-momentum conservation is guaranteed at each vertex due to the factors

$$(2\pi)^4 \Big(\sum \text{ingoing momenta} - \text{outgoing momenta} \Big). \tag{2.164}$$

LSZ versus Interaction Picture Perturbation Theory

On assignment 7 will work out a detailed a comparison of the LSZ approach to scattering as completed in this chapter and a more naive approach based on quantum mechanical Interaction Picture perturbation theory.

2.11 Cross-sections

S-matrix elements are in general of the form

$$\langle f|\, S\, |i\rangle = \underbrace{\delta_{fi}}_{\text{no scattering}} + \underbrace{i(2\pi)^4 \delta^{(4)}(p_f - p_i)}_{\text{momentum conservation}} \cdot \underbrace{\mathcal{M}_{fi}}_{=:\text{scattering amplitude}} \tag{2.165}$$

In particular, momentum conservation is a consequence of the insertion of delta-functions at each vertex in the Feynman diagrams. The first summand δ_{fi} accounts for the situation that the initial and final states are identical, in which case no scattering occurs. In the sequel we will usually exclude this possibility and focus on non-trivial scattering events.
The QM probability for scattering of a given initial state $\{|i\rangle\}$ into a range of final states $\{|f\rangle\}$ is

$$\mathcal{P}_{|i\rangle \to \{|f\rangle\}} = \sum_{|f\rangle \in \{|f\rangle\}} \big| \langle f|\, S\, |i\rangle \big|^2. \tag{2.166}$$

Assume now that $|i\rangle \notin \{|f\rangle\}$ (so that in particular the term δ_{fi} is irrelevant), then

$$\mathcal{P}_{|i\rangle\to\{|f\rangle\}} = \sum_{|f\rangle\in\{|f\rangle\}} \underbrace{\left[(2\pi)^4\delta^{(4)}(p_f-p_i)\right]^2}_{=(2\pi)^4\delta^{(4)}(p_f-p_i)\,\underbrace{(2\pi)^4\,\delta^{(4)}(0)}_{=V_{\mathbb{R}^{1,3}}}} \left|\mathcal{M}_{fi}\right|^2. \tag{2.167}$$

We define the **transition rate** as the probability normalised per spacetime volume,

$$\omega_{fi} = \frac{\mathcal{P}_{|i\rangle\to\{|f\rangle\}}}{\text{unit time} \times \text{unit volume}} \tag{2.168}$$

and therefore

$$\omega_{fi} = \sum_{|f\rangle\in\{|f\rangle\}} (2\pi)^4\delta^{(4)}(p_f-p_i)\left|\mathcal{M}_{fi}\right|^2. \tag{2.169}$$

For scattering into N identical particles this is

$$\boxed{\omega_{fi} = \frac{1}{N!}\prod_{n=1}^{N}\int \frac{\mathrm{d}^3k_n}{(2\pi)^3}\frac{1}{2E_n}(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_n k_n\right)\left|\mathcal{M}_{fi}\right|^2,} \tag{2.170}$$

where the symmetry factor $\frac{1}{N!}$ accounts for the indistinguishability of the N identical particles.
Now, a typical scattering experiment is of the form

$$\boxed{\text{Incoming beam of density } \rho_B \text{ and length } l_B} \longrightarrow \boxed{\text{Target } A \text{ of density } \rho_A \text{ and length } l_A},$$

where $\rho_{A,B}$ denote the respective number densities (i.e. number of particles per volume). The number (#) of scattered particles is proportional[11] to $l_A\, l_B \int \mathrm{d}^2x\,\rho_a(x)\,\rho_B(x)$. The factor of proportionality has dimension $[\text{Area}]$ and is called **cross-section** σ,

$$\boxed{\#\text{ of events } =:\ \sigma\, l_A\, l_B \int \mathrm{d}^2x\,\rho_A(x)\,\rho_B(x).} \tag{2.171}$$

If ρ_A and ρ_B are constant and both beams overlap over the area $\mathcal{A}$, then

$$\sigma = \frac{\#\text{ of scattered particles}}{l_A\, l_B\, \rho_A\, \rho_B\, \mathcal{A}}. \tag{2.172}$$

This suggests the following intuitively clear interpretation of the meaning of σ: The cross-section σ is the effective area of the beam B that participates in the scattering.

Consider now a $2 \to N$ scattering process such that all out-going states $|k_j\rangle$ are momentum eigenstates. The initial states are the momentum eigenstates $|p_A\rangle$ and $|p_B\rangle$. Let us go to the rest frame of

[11] Here we are assuming that all particles of the beam hitting the target get to interact with the target particles with equal probability. For simplicity we are also assuming that the particle density varies only in the directions x^1, x^2 transverse to the beam.

the target A called **lab frame**. Suppose for simpilcity that the beam B hits the target A over its entire area $\mathcal{A}$ and that the densities of beam and target do not vary much. Then

$$\frac{\#\text{ of events}}{\text{Volume} \times \text{Time}} = \frac{\sigma \rho_A \rho_B l_A l_B \mathcal{A}}{\text{Volume} \times \text{Time}} = \sigma \rho_A \rho_B \left|\vec{v}_B^L\right|, \tag{2.173}$$

where the relevant interaction volume is is just Volume $= \mathcal{A} l_A$ and the relevant interaction time is what it takes for the beam to pass over a given slice in the target orthogonal to the beam direction, i.e. $\frac{l_B}{\text{Time}} = \left|\vec{v}_B^{(L)}\right|$. Sometimes it is also useful to think in terms of the beam **flux** $\mathcal{F}$ defined as

$$\rho_B \left|\vec{v}_B^L\right| =: \mathcal{F}. \tag{2.174}$$

Since $E = \gamma m$ and $\vec{p} = m\gamma\vec{v}$ we can use

$$\left|\vec{v}_B^L\right| = \frac{\left|\vec{p}_B^L\right|}{E_B} \tag{2.175}$$

to obtain

$$\boxed{\frac{\#\text{ events}}{\text{Volume} \times \text{Time}} = \sigma \rho_A \rho_B \frac{\left|\vec{p}_B^L\right|}{E_B} = \sigma \rho_A \mathcal{F}.} \tag{2.176}$$

What we actually compute in evaluating the S-matrix elements are not absolute numbers of events, but rather QM probabilites. In order make the transition between both these concepts we replace the particle number densities ρ_A and ρ_B by the quantum mechanical densities of the corresponding single particles states, i.e. the probabilities of finding a particle A or B per given volume. If $\langle p_A|p_A\rangle = 1$ this probability would be $1/V_{\mathbb{R}^3} = \langle p_A|p_A\rangle/V_{\mathbb{R}^3}$. With our normalisation of momentum states

$$\langle p_A|p_A\rangle = (2\pi)^3 2E_A \delta^{(3)}(\vec{p}_A - \vec{p}_A) = 2E_A V_{\mathbb{R}_3}, \tag{2.177}$$

we must replace

$$\rho_A \to \frac{\langle p_A|p_A\rangle}{V_{\mathbb{R}^3}} = 2E_A \quad \text{and} \quad \rho_b \to 2E_B. \tag{2.178}$$

Since we are in the rest frame of A we have $2E_A = 2m$. Therefore in the lab frame ($\vec{v}_A^{(L)} = 0$)

$$\boxed{\omega_{fi} = \frac{\text{QM probability}}{\text{Volume} \times \text{Time}} = 4\sigma m p_B^{(L)}.} \tag{2.179}$$

Note that

$$4mp_B^{(L)} = 4\, E_A\, E_B|\vec{v}_B^{(L)}| = 4E_A\, E_B|\vec{v}_A^{(L)} - \vec{v}_B^{(L)}| \tag{2.180}$$

in the lab frame. Finally, one can show that $4\, E_A\, E_B|\vec{v}_A^{(L)} - \vec{v}_B^{(L)}|$ is invariant under Lorentz boosts in direction $A \to B$. Therefore $\omega_{fi} = 4\, E_A\, E_B|\vec{v}_A - \vec{v}_B|\,\sigma$ is the correct general expression for the transition rate valid in any frame [12]. To conclude, the **differential cross-section** is

$$\boxed{\mathrm{d}\sigma = \frac{(2\pi)^4}{4E_A\, E_B\, |\vec{v}_A - \vec{v}_B|}\, \mathrm{d}\Pi_N\, \delta^{(4)}\left(p_a + p_B - \sum_i k_i\right)\left|\mathcal{M}_{fi}\right|^2,} \tag{2.181}$$

[12] For a more formal, but also considerably more complicated proof see Peskin-Schröder p.102 – 108.

where

$$\mathrm{d}\Pi_N \equiv \frac{1}{N!}\prod_{n=1}^{N}\int \frac{\mathrm{d}^3k_n}{(2\pi)^3}\frac{1}{2E_n}. \tag{2.182}$$

For example consider $2-2$ scattering as shown in Figure 2.9. On Assignment 7 we show that if all

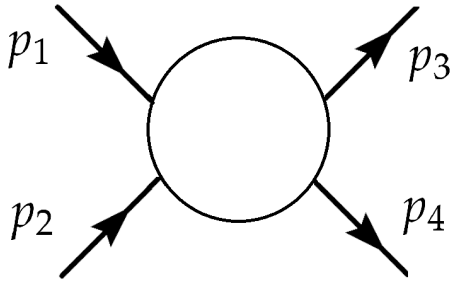


Figure 2.9: $2-2$ scattering.

four particles have the same mass, the differential cross-section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_3} = \frac{1}{2!}\frac{1}{64\pi^2}\frac{1}{s}|\mathcal{M}|^2, \tag{2.183}$$

where the relativistically invariant **Mandelstam variable**

$$s = (p_1+p_2)^2 = (p_3+p_4)^2 \tag{2.184}$$

appears. Note that in Φ^4-theory $|\mathcal{M}|^2 = \lambda^2$ to first order in λ, i.e. for the maximally localised, pointlike interaction given by the first Feynman diagram in Figure 2.4. Therefore

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim \frac{1}{s} \tag{2.185}$$

to first order in λ. This is a famous result, which in fact holds more generally:

> *The differential cross-section for hard scattering off a pointlike target (i.e. a target with no substructure of length $l \geq 1/\sqrt{s}$) falls off as $1/s$.*

This characteristic behaviour as observed in deep inelastic scattering experiments with hardons was a crucial clue to the the parton structure of hadrons as you will surely recall from your particle physics course.

Chapter 3

Quantising spin $\frac{1}{2}$-fields

3.1 The Lorentz algebra $so(1,3)$

Relativistic fields as representations of the Lorentz algebra

Relativistic fields are classified by their behaviour under Lorentz transformations

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_{\ \nu} x^\nu \,, \quad \Lambda^\mu_{\ \nu} \in SO(1,3). \tag{3.1}$$

Quite generally a field $\Phi^a(x)$ transforms as a representation[1] of the Lorentz group $SO(1,3)$ - or, more precisely, as we will see momentarily, of its double cover $\mathrm{Spin}(1,3)$. This means that the classical field $\Phi^a(x)$ is a map

$$\Phi^a : \mathbb{R}^{1,3} \to V, \tag{3.2}$$

$$x \mapsto \Phi^a(x), \qquad a = 1, \ldots, \dim(V) \tag{3.3}$$

with V a vector space such that the specific transformation behaviour of the field $\Phi^a(x)$ under (3.1) is given by

$$\Phi^a(x) \mapsto R(\Lambda)^a_{\ b} \Phi^b(\Lambda^{-1} x') = R(\Lambda)^a_{\ b} \Phi^b(x). \tag{3.4}$$

Here for every element $\Lambda \in SO(1,3)$ the object $R(\Lambda)$ is an automorphism (an invertible linear map) acting on V in a manner compatible with the group action of $SO(1,3)$. Specifying the representation in which the field transforms amounts to assigning such an automorphism R to every element $\Lambda \in SO(1,3)$, where compatibility with the group action of $SO(1,3)$ means that[2]

$$\begin{aligned} R(\Lambda_2)^a_{\ b} R(\Lambda_1)^b_{\ c} &= R(\Lambda_2 \Lambda_1)^a_{\ c} \\ R(\Lambda^{-1})^a_{\ b} &= \left(R(\Lambda)^{-1}\right)^a_{\ b}. \end{aligned} \tag{3.5}$$

[1] The concept of Lie groups, Lie algebras and their representations has been discussed in detail in the course on Quantum Mechanics. A good book in the present context is: Urbantke, Sexl: Relativity, Groups, Particles. As a quick reminder of the main points see e.g. http://www.thphys.uni-heidelberg.de/$\sim$weigand/Skript-QM2011/skript.pdf.

[2] In short, a representation R of a group G is a group homomorphism from G to $\mathrm{Aut}(V)$ for some vector space V.

The vector space V is also called representation space. Note that the index a refers to the components of the fields with respect to a basis of the vector space V - called the representation space -, while μ, ν are spacetime indices. These two are in general completely different objects.

Consider the following examples:

- For a real (or complex) scalar field $\Phi(x)$ the representation space V is just $\mathbb{R}$ (or $\mathbb{C}$) and

$$R(\Lambda) = \mathbb{1} \ \ \forall \Lambda. \tag{3.6}$$

  The scalar field is said to transform in the trivial or **scalar representation** and describes particles with spin 0.

- A vector field $A^\mu(x)$ (e.g. the gauge potential of electro-magnetism) transforms in the **vector representation**: The representation space V is identified with spacetime $\mathbb{R}^{1,3}$ itself (or rather its tangent space) and

$$R(\Lambda)^\mu_{\ \nu} = \Lambda^\mu_{\ \nu} \ \ \forall \Lambda. \tag{3.7}$$

  Since here $V = \mathbb{R}^{1,3}$, we have in this case that $a, b = \mu, \nu$. A field in the vector representation describes particles with spin 1 (as we will see later in this course when quantising such vector fields).

An important concept in representation theory is that of an **irreducible representation** (irrep): An Irreducible representation is one whose representation space does not split into a direct sum of two vector spaces in a manner respected by the group action, i.e. it is not possible to find a basis of the representation space in which

$$R(\Lambda) = \left(\begin{array}{c|c} \star & 0 \\ \hline 0 & \star \end{array}\right) \ \ \forall \Lambda. \tag{3.8}$$

Irreducible representations form the building blocks of which larger representations can be formed by considering direct sums of representation spaces.

Spin $1/2$ particles are described by fields in the **spinor representation**. To find this representation our starting point will not be the irreducible representations of the Lorentz group $SO(1,3)$, but of

The Lorentz algebra $so(1,3)$

This is the algebra of infinitesimal Lorentz transformations connected to the identity. This is analogous to the precedure applied in quantum mechanics to find spin $\frac{1}{2}$ representations of the algebra of spatial rotations $so(3) \simeq su(2)$.

A Lorentz transformation can be written infinitesimally as

$$\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}. \tag{3.9}$$

Since the metric must be invariant under a Lorentz transformation in the sense that

$$\Lambda^{\mu}_{\ \alpha}\,\Lambda^{\nu}_{\ \beta}\,\eta^{\alpha\beta}=\eta^{\mu\nu}, \tag{3.10}$$

the infinitesimal transformations satisfy $\omega^{\mu\nu}=-\omega^{\nu\mu}$. We introduce a basis of antisymmetric 4×4 matrices into which we would like to expand $\omega^{\mu\nu}$. This basis contains 6 elements. Let the basis be

$$\left(M^{A}\right)^{\mu\nu} \quad \text{with } A=1,...,6. \tag{3.11}$$

Then

$$\omega^{\mu\nu}=-\frac{i}{2}\Omega_{A}\left(M^{A}\right)^{\mu\nu} \tag{3.12}$$

for some Ω_A with summation over A understood. We introduce the double-index notation $A=\rho\sigma$ with $\rho,\sigma=0,1,2,3$ such that

$$(M^{\rho\sigma})^{\mu\nu}=-(M^{\sigma\rho})^{\mu\nu} \tag{3.13}$$

is antisymmetric in $\sigma\rho$. Therefore

$$\omega^{\mu\nu}=-\frac{i}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^{\mu\nu}. \tag{3.14}$$

So an infinitesimal Lorentz transformation is

$$\Lambda^{\mu}_{\ \nu}=\delta^{\mu}_{\ \nu}-\frac{i}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^{\mu}_{\ \nu}. \tag{3.15}$$

A finite transformation is given by applying the infnitesimal version N times for $N\to\infty$ in the sense

$$\begin{aligned}\Lambda^{\mu}_{\ \nu}(M^{\rho\sigma}) &= \lim_{N\to\infty}\left(\delta^{\mu}_{\ \nu}-\frac{1}{N}\frac{i}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^{\mu}_{\ \nu}\right)^{N}\\ &=\left[e^{-\frac{i}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})}\right]^{\mu}_{\ \nu}.\end{aligned} \tag{3.16}$$

This is what is meant by saying that the $M^{\rho\sigma}$ are the generators of the Lie group $SO(1,3)$, or equivalently form a basis of the Lie algebra $so(1,3)$.
The structure of the Lie algebra $so(1,3)$ is encoded in the commutation relations of its basis elements. To find these one expands the defining relation of the underlying Lie algebra $SO(1,3)$

$$\Lambda^{\mu}_{\ \alpha}\,\Lambda^{\nu}_{\ \beta}\,\eta^{\alpha\beta}=\eta^{\mu\nu} \tag{3.17}$$

to second order in Ω to conclude that

$$\boxed{\left[M^{\rho\sigma},M^{\tau\nu}\right]=-i\left(\eta^{\rho\nu}M^{\sigma\tau}+\eta^{\sigma\tau}M^{\rho\nu}-\eta^{\rho\tau}M^{\sigma\nu}-\eta^{\sigma\nu}M^{\rho\tau}\right).} \tag{3.18}$$

This defines the structure constants of the Lie-algebra $so(1,3)$. Recall that in general a basis T^A of a Lie algebra satisfies

$$[T^{A},T^{B}]=\sum_{C} i f^{AB}{}_{C}T^{C}, \tag{3.19}$$

where $f^{AB}{}_{C}$ are the **structure constants**. They obey

- ${f^{AB}}_{C} = -{f^{BA}}_{C}$
- and the Jacobi identity

$$ {f^{AB}}_{D} {f^{CD}}_{E} + {f^{CA}}_{D} {f^{BD}}_{E} + {f^{BC}}_{D} {f^{AD}}_{E} = 0. \tag{3.20} $$

We view the commutator (3.18) as the defining relation for abstract objects $M^{\rho\sigma}$. An ***n*-dimensional representation of the Lie algebra** $so(1,3)$ is an assignment that associates to each $M^{\rho\sigma}$ an invertible map

$$ (R^{\rho\sigma})^{a}_{\ b} = R(M^{\rho\sigma})^{a}_{\ b} \tag{3.21} $$

acting on an n-dimensional vector space V such that the same relation (3.18) holds for the representation matrices $(R^{\rho\sigma})^{a}_{\ b}$. The indices $a, b = 1, \ldots, n$ refer to a basis e_a of V in the sense that $v \in V$ is expanded as $v = v^a e_a$. Note again that in general a and b are unrelated to μ and ν.
Consider as a special case the vector representation by choosing $V = \mathbb{R}^{1,3}$ (viewed as the tangent space to spacetime). In this case we do identify $a, b \equiv \mu, \nu$ and set

$$ (\mathcal{J}^{\rho\sigma})^{\mu\nu} := i(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\nu}\eta^{\sigma\mu}). \tag{3.22} $$

One can check that this indeed satisfies the $so(1,3)$ relations (3.18) and that

$$ \omega^{\mu\nu} = -\frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^{\mu\nu}. \tag{3.23} $$

Therefore

$$ \boxed{\Lambda^{\mu}_{\ \nu} = \left[e^{-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}}\right]^{\mu}_{\ \nu}.} \tag{3.24} $$

We can use this result to deduce the matrix representation of spatial rotations by an angle α around an axis $\vec{n}$. The corresponding $\omega_{\mu\nu}$ can be written as

$$ \omega_{ij} = \alpha\epsilon_{ijk}\, n^k. \tag{3.25} $$

E.g. for a rotation around the x^1-axis we have

$$ \vec{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \omega_{\rho\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix}. \tag{3.26} $$

The corresponding infinitesimal rotations are then of the form

$$ \Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\alpha \\ 0 & 0 & \alpha & 1 \end{pmatrix}, \tag{3.27} $$

whose finite version is

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \tag{3.28}$$

3.2 The Dirac spinor representation

To find the spinor representation of $so(1,3)$ we start from the Clifford algebra $\mathrm{Cliff}(1,3)$ defined as the algebra spanned by $n \times n$-matrices $(\gamma^{\mu})^{A}_{\ B}$, $\mu = 0,1,2,3$ and $A,B = 1,\ldots,n$ such that the anti-commutator is

$$\boxed{\{\gamma^{\mu},\gamma^{\nu}\} := \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}\,\mathbb{1}.} \tag{3.29}$$

Reinstating for clarity the indices A and B the defining relation is

$$(\gamma^{\mu})^{A}_{\ B}(\gamma^{\nu})^{B}_{\ C} + (\gamma^{\nu})^{A}_{\ B}(\gamma^{\mu})^{B}_{\ C} = 2\eta^{\mu\nu}\,\mathbb{1}^{A}_{\ C}. \tag{3.30}$$

This implies

$$\gamma^{\mu}\gamma^{\nu} = \begin{cases} \eta^{\mu\nu} & \text{if } \mu = \nu \\ -\gamma^{\nu}\gamma^{\mu} & \text{if } \mu \neq \nu \end{cases}. \tag{3.31}$$

Therefore

$$\boxed{(\gamma^{0})^{2} = \mathbb{1}, \quad (\gamma^{i})^{2} = -\mathbb{1}.} \tag{3.32}$$

The central point is that given $(\gamma^{\mu})^{A}_{\ B}$ as above the objects

$$\boxed{(S^{\rho\sigma})^{A}_{\ B} := \frac{i}{4}[\gamma^{\rho},\gamma^{\sigma}]^{A}_{\ B}} \tag{3.33}$$

form a representation of $so(1,3)$, i.e.

$$\left[S^{\rho\sigma}, S^{\tau k}\right] = -i\left(\eta^{\rho k}S^{\sigma\tau} + \eta^{\sigma\tau}S^{\rho k} - \eta^{\rho\tau}S^{\sigma k} - \eta^{\sigma k}S^{\rho\tau}\right), \tag{3.34}$$

therefore (3.18) holds. Thus we have constructed a representation of $so(1,3)$ from the Clifford algebra:

Every representation of Cliff*(1,3) induces a representation of so(1,3).*

One can prove this by direct computation. To this end note that

$$S^{\mu\nu} = \left\{ \begin{array}{ll} 0 & \text{if } \mu = \nu \\ \frac{i}{2}\gamma^\mu\gamma^\nu & \text{if } \mu \neq \nu \end{array} \right\} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}). \tag{3.35}$$

The claim then follows with the help of the anti-commutation relations (3.29) after some algebra as worked out in the tutorial.

Now we want to find an explicit representation of the Clifford algebra. Since it is useful to know the result for an arbitrary number of spacetime dimensions let us give the general result valid for $\text{Cliff}(1,d-1)$. The problem is therefore to find $n \times n$-matrices

$$(\gamma^\mu)^A{}_B \text{ with } \mu = 0,1,2,...,d-1 \text{ and } A,B = 1,...,n \tag{3.36}$$

subject to (3.29). This will then also give a representation of the Lorentz algebra $so(1,d-1)$. In the tutorials we will prove the following famous theorem:

The irreducible representations of $\text{Cliff}(1,d-1)$ *are of dimension*

$$\begin{array}{ll} n = & 2^{\frac{d}{2}} \text{ if } d \text{ is even} \\ n = & 2^{\frac{1}{2}(d-1)} \text{ if } d \text{ is odd} \end{array} . \tag{3.37}$$

Let us now specialise to $d = 4$ as is relevant for QFT in four spacetime dimensions. In this case $n = 4$. One choice of γ^μ called **chiral or Dirac representation** is

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3.38}$$

where σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.39}$$

with

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}. \tag{3.40}$$

Given an invertible matrix U, every other choice $U\gamma^\mu U^{-1}$ is also a representation of $\text{Cliff}(1,3)$. The complex vector space on which $(\gamma^\mu)^A{}_B$ acts is called the space of **Dirac spinors** ψ^A, $A = 1,...,n$ with $n = 4$ for $d = 4$.

Now, from the above it is clear that the ψ^A also form a representation of $so(1,3)$ because the γ^μ induce a representation of $so(1,3)$. More precisely, a Dirac spinor ψ^A transforms under a Lorentz transformation

$$\Lambda^\mu{}_\nu = \left[e^{-\frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}} \right]^\mu{}_\nu \tag{3.41}$$

as

$$\psi^A \mapsto \left[e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}} \right]^A{}_B \psi^B. \tag{3.42}$$

A **Dirac spinor field** $\psi^A(x)$ transforms as

$$x \mapsto x' = \Lambda x, \tag{3.43}$$

$$\psi^A(x) \mapsto [S(\Lambda)]^A{}_B \psi^B(\Lambda^{-1}x') = [S(\Lambda)]^A{}_B \psi^B(x) \tag{3.44}$$

with

$$[S(\Lambda)]^A{}_B = \left[e^{-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}}\right]^A{}_B. \tag{3.45}$$

The indices $A = 1, ..., n$, with $n = 4$ for Cliff$(1,3)$ are called **spinor indices**. Even though in 4 dimensions A happens to run from $1, \ldots, 4$, we must not cunfuse them at any time with spacetime indices μ, ν.

Now that we found a new type of representation of the Lorentz algebra $so(1,3)$ we would like to give it a physical interpretation. The important claim is that

A Dirac spinor field $\psi^A(x)$ behaves like a spin $\frac{1}{2}$-field.

To prove this we consider a spatial rotation around $\vec{n}$ with

$$\omega_{ij} = \alpha\, \epsilon_{ijk}\, n^k \tag{3.46}$$

and

$$S[\Lambda] = \left[e^{-\frac{i}{2}\omega_{ij}S^{ij}}\right]^A{}_B. \tag{3.47}$$

Now,

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \begin{pmatrix} -\frac{i}{2}\sigma^i\sigma^j & 0 \\ 0 & -\frac{i}{2}\sigma^i\sigma^j \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sigma^k & 0 \\ 0 & \frac{1}{2}\sigma^k \end{pmatrix} \tag{3.48}$$

with (ijk) cyclic permutations of 123. The latter equations follows from the properties of the Pauli matrices. So

$$S[\Lambda] = \begin{pmatrix} e^{-i\frac{\alpha}{2}\vec{n}\vec{\sigma}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}\vec{n}\vec{\sigma}} \end{pmatrix}. \tag{3.49}$$

What is crucial is the factor of $\frac{1}{2}$, which indeed indicates a transformation as a spin $\frac{1}{2}$ under the subalgebra $so(3) \subset so(1,3)$ of spatial rotations. In particular consider a rotation by $\alpha = 2\pi$ around $\vec{n}^T = (0,0,1)$,

$$S[\Lambda] = \exp\begin{pmatrix} -i\pi\sigma_3 & 0 \\ 0 & -i\pi\sigma_3 \end{pmatrix} = -\mathbb{1}. \tag{3.50}$$

Therefore

$$\psi^A \mapsto -\psi^A \tag{3.51}$$

under a rotation by 2π. □

While we have found a representation of the Lorentz-algebra $so(1,3)$, this does not give a representation of the Lorentz-group $SO(1,3)$, but rather of its **double cover** $\mathrm{Spin}(1,3)$. The latter is isomorphic to $SL(2,\mathbb{C})$, the group of complex 2×2 matrices with determinant 1 (not to confuse with $SU(2)$, the group of complex unitary such matrices). In the tutorial we will investigate this isomorphism and the corresponding description of spinors.

A Dirac spinor forms a representation of $\mathrm{Spin}(1,3) \simeq SL(2,\mathbb{C})$*, not of* $SO(1,3)$.

The reason is that $S[\Lambda = \mathbb{1}] \neq \mathbb{1}$, which is incompatible with the group law of $SO(1,3)$. In $\mathrm{Spin}(1,3)$ by contrast, a rotation around 2π does not correspond to the $\mathbb{1}$ because it is defined as the double cover of $SO(1,3)$.[3] This is the relativistic version of the statement familiar from Quantum Mechanics that the $j = \frac{1}{2}$ spinor representation of the algebra of spatial rotations $so(3)$ does not furnish a representation of the Lie group $SO(3)$ but only of its double cover $SU(2)$.

3.3 The Dirac action

We now want to construct Lorentz scalars and vectors out of $\psi^A(x)$ in order to construct a covariant action. To do so we define the **conjugate spinor** $\psi^\dagger := (\psi^*)^T$, i.e.

$$\psi^\dagger = \left((\psi^1)^*, (\psi^2)^*, (\psi^3)^*, (\psi^4)^*\right). \tag{3.52}$$

We now would like to know how $\psi^\dagger(x)\psi(x)$ behaves under a Lorentz transformation Λ, under which

$$\begin{aligned} x &\mapsto x' = \Lambda x, && (3.53)\\ \psi(x) &\mapsto S(\Lambda)\,\psi(\Lambda^{-1}x'), && (3.54)\\ \psi^\dagger(x) &\mapsto \psi^\dagger(\Lambda^{-1}x')\,S^\dagger(\Lambda). && (3.55) \end{aligned}$$

In order for $\psi^\dagger(x)\psi(x)$ to transform as a Lorentz scalar, we would need $S(\Lambda)^\dagger = S^{-1}(\Lambda)$. However,

$$S^\dagger(\Lambda) = e^{\left[-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right]^\dagger} = e^{\frac{i}{2}\omega_{\rho\sigma}(S^{\rho\sigma})^\dagger} \tag{3.56}$$

and

$$(S^{\rho\sigma})^\dagger = \left(\frac{i}{4}[\gamma^\rho,\gamma^\sigma]\right)^\dagger = -\frac{i}{4}\left[\gamma^{\sigma\dagger},\gamma^{\rho\dagger}\right] = \frac{i}{4}\left[\gamma^{\rho\dagger},\gamma^{\sigma\dagger}\right]. \tag{3.57}$$

It is not possible to pick all γ^μ hermitian at the same time since

$$\begin{aligned} (\gamma^0)^2 = 1 &\;\rightarrow\; \text{real eigenvalues},\\ (\gamma^i)^2 = -1 &\;\rightarrow\; \text{imaginary eigenvalues}. \end{aligned} \tag{3.58}$$

Therefore, $(S^{\rho\sigma})^\dagger \neq (S^{\rho\sigma})$ and thus $S(\Lambda)^\dagger \neq S^{-1}(\Lambda)$.

[3] For further reading see e.g. Urbantke, Sexl: *Relativity, Groups, Particles*.

Rather, it is possible to pick γ^μ such that

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i, \tag{3.59}$$

which is indeed satisfied by the Dirac representation above. In this case

$$\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0,} \tag{3.60}$$

because $\gamma^0\gamma^0 = \mathbb{1}$ and $\gamma^0\gamma^i = -\gamma^i\gamma^0$. Therefore

$$\gamma^0(S^{\rho\sigma})^\dagger\gamma^0 = \frac{i}{4}\gamma^0\left[\gamma^{\rho\dagger}, \gamma^{\sigma\dagger}\right]\gamma^0 = \frac{i}{4}\left[\gamma^\rho, \gamma^\sigma\right] = S^{\rho\sigma} \tag{3.61}$$

and so

$$\boxed{\gamma^0 S\left[\Lambda\right]^\dagger\gamma^0 = S\left[\Lambda\right]^{-1}.} \tag{3.62}$$

Inspired by this result we define the **Dirac conjugate spinor**

$$\bar{\psi} := \psi^\dagger\gamma^0. \tag{3.63}$$

It transforms as

$$\bar{\psi}(x) \mapsto \psi^\dagger(\Lambda^{-1}x')S^\dagger(\Lambda)\gamma^0 = \psi^\dagger(\Lambda^{-1}x')\gamma^0 S^{-1}(\Lambda) = \bar{\psi}(\Lambda^{-1}x')S^{-1}(\Lambda). \tag{3.64}$$

Thus the spinor bilinear $\bar{\psi}(x)\psi(x)$ transforms as

$$\bar{\psi}(x)\psi(x) \mapsto \bar{\psi}(\Lambda^{-1}x')S^{-1}(\Lambda)S\left(\Lambda\right)\psi((\Lambda^{-1}x')) = \bar{\psi}(\Lambda^{-1}x')\psi(\Lambda^{-1}x'). \tag{3.65}$$

So $\bar{\psi}(x)\psi(x)$ is a scalar quantity. Furthermore one can show that $\bar{\psi}(x)\gamma^\mu\psi(x)$ transforms as a vector because

$$\bar{\psi}(x)\gamma^\mu\psi(x) \mapsto \bar{\psi}(\Lambda^{-1}x')S^{-1}(\Lambda)\gamma^\mu S\left(\Lambda\right)\psi(\Lambda^{-1}x') \tag{3.66}$$

and

$$\boxed{S^{-1}(\Lambda)\gamma^\mu S\left(\Lambda\right) = \Lambda^\mu{}_\nu\gamma^\nu,} \tag{3.67}$$

as will be proven in the tutorials. This logic can be repeated for tensors of rank n.

We can now build an action for $\psi(x)$, where we follow the principle of simplicity. The simplest Lorentz scalars built from $\psi(x)$ which include non-trivial dynamics (i.e. at least one derivative has to appear) are

$$\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x), \qquad \bar{\psi}(x)\psi(x). \tag{3.68}$$

So we take as the action for the free classical Dirac spinor field

$$\boxed{S = \int \mathrm{d}^4x\ \bar{\psi}(x)\left[i\gamma^\mu\partial_\mu - m\right]\psi(x),} \tag{3.69}$$

where

- the factor i is required for the action to be real
- and $|m|$ will be the mass of the Dirac spinor particle.

This is the simplest action we can build. Interestingly - unlike in for the scalar field - it is possible to construct a Lorentz invariant action with only one derivative. Furthermore note that $\psi(x)$ has mass dimension $[\text{mass}]^{3/2}$.

Since $\psi(x)$ is complex we treat $\psi(x)$ and $\psi^\dagger(x)$ as independent when deriving the equations of motion. Varying with respect to $\psi^\dagger(x)$ yields the **Dirac equation**

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi(x) = 0,} \tag{3.70}$$

which is linear in the derivatives. In order to see that this indeed describes a field with mass $|m|$ we note that

$$\begin{aligned} 0 &= (i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m)\psi \\ &= (-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi \\ &= -\Big(\underbrace{\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}}_{=\eta^{\mu\nu}}\partial_\mu\partial_\nu + m^2\Big)\psi \end{aligned} \tag{3.71}$$

and thus obtain the Klein-Gordon equation (1.29) for the spinor field $\psi(x)$. Loosely speaking:

$$\text{Dirac equation} = \sqrt{\text{Klein-Gordon equation.}} \tag{3.72}$$

This is a consequence of the particular manner how we constructed a representation of the Lorentz algebra from the Clifford algebra. Equ. (3.71) also justifies the relative factor of i in the Dirac action, for which we can take m to be real.

3.4 Chirality and Weyl spinors

The Dirac spinor representation of $\text{Cliff}(1,3)$ is not irreducible as a representation of $\text{Spin}(1,3)$. Indeed for our special choice of Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{3.73}$$

ψ^A transforms under a spatial rotation as

$$\psi \mapsto \begin{pmatrix} e^{-\frac{i}{2}\alpha\vec{n}\cdot\vec{\sigma}} & 0 \\ 0 & e^{-\frac{i}{2}\alpha\vec{n}\cdot\vec{\sigma}} \end{pmatrix}\psi \tag{3.74}$$

and under a Lorentz boost with $\omega_{0j} = \chi_j$ as

$$\psi \mapsto \begin{pmatrix} e^{-\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} & 0 \\ 0 & e^{+\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} \end{pmatrix}\psi. \tag{3.75}$$

Therefore the subspaces spanned by

$$\psi_-^T = (\psi^1, \psi^2, 0, 0) \quad \text{and} \quad \psi_+^T = (0, 0, \psi^3, \psi^4) \tag{3.76}$$

transform separately. Irrespective of the concrete representation the reducibility of the Dirac spinor repsentation as a representation of Spin(1,3) can be seen as follows: Define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.77}$$

with the properties

$$(\gamma^5)^2 = \mathbb{1}, \quad \{\gamma^5, \gamma^\mu\} = 0, \quad [S^{\rho\sigma}, \gamma^5] = 0 \tag{3.78}$$

following from the properties of the Clifford-algebra. Now consider the orthogonal projection operators

$$\mathbb{P}_\pm := \frac{1}{2}(\mathbb{1} \pm \gamma^5) \tag{3.79}$$

with the properties

$$(\mathbb{P}_\pm)^2 = \mathbb{P}_\pm, \quad \mathbb{P}_+\mathbb{P}_- = 0 = \mathbb{P}_-\mathbb{P}_+, \quad \mathbb{1} = \mathbb{P}_+ + \mathbb{P}_-. \tag{3.80}$$

Defining

$$\psi_\pm := \mathbb{P}_\pm\psi \tag{3.81}$$

yields

$$\mathbb{P}_\pm\psi_\mp = 0. \tag{3.82}$$

Now, since $[S^{\rho\sigma}, \gamma^5] = 0$ we have $\mathbb{P}_\mp S[\Lambda]\psi_\pm = 0$, i.e. the $\pm$ subspaces transform separately under $\mathrm{Spin}(1,3)$. The $\psi_\pm$ are called positive/negative-chirality spinors.
In the special representation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{3.83}$$

we find

$$\psi_- = \begin{pmatrix} u_- \\ 0 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} 0 \\ u_+ \end{pmatrix}. \tag{3.84}$$

The 2-component objects $u_\pm$ are called **Weyl-spinors**. It is the Weyl spinors that form irreducible representations of Spin(1,3).

It is instructive to rewrite the Dirac action in this language. The Dirac action decomposes as follows: Since $\gamma^0\gamma^5 = -\gamma^5\gamma^0$ we have

$$\gamma^0\mathbb{P}_\pm = \mathbb{P}_\mp\gamma^0 \tag{3.85}$$

and therefore

$$\overline{\psi_\pm} = (\mathbb{P}_\pm\psi)^\dagger\gamma^0 = \psi^\dagger\mathbb{P}_\pm\gamma^0 = \bar{\psi}\mathbb{P}_\mp. \tag{3.86}$$

This means

$$\begin{aligned}\overline{\psi_\pm}\psi_\pm &= \bar\psi \mathbb{P}_\mp \mathbb{P}_\pm \psi = 0,\\ \overline{\psi_\pm}\psi_\mp &= \bar\psi \mathbb{P}_\mp \psi \neq 0\end{aligned} \tag{3.87}$$

and similary (using in addition that $\gamma^\mu \mathbb{P}_\pm = \mathbb{P}_\mp \gamma^\mu$)

$$\begin{aligned}\overline{\psi_\pm}\gamma^\mu\psi_\pm &= \bar\psi \gamma^\mu \mathbb{P}_\pm \psi \neq 0\\ \overline{\psi_\mp}\gamma^\mu\psi_\pm &= 0.\end{aligned} \tag{3.88}$$

The Dirac action can then be written as

$$\begin{aligned}S &= \int \mathrm{d}^4x\, \bar\psi(i\gamma^\mu\partial_\mu - m)\psi\\ &= \int \mathrm{d}^4x \Big[\overline{\psi_+} i\gamma^\mu\partial_\mu\psi_+ + \overline{\psi_-} i\gamma^\mu\partial_\mu\psi_- - m(\overline{\psi_+}\psi_- + \overline{\psi_-}\psi_+)\Big],\end{aligned} \tag{3.89}$$

or in Weyl-spinor notation as

$$S = \int \mathrm{d}^4x \Big[u_+^\dagger\, i\sigma^\mu\,\partial_\mu\, u_+ + u_-^\dagger\, i\bar\sigma^\mu\,\partial_\mu\, u_- - m(u_+^\dagger u_- + u_-^\dagger u_+)\Big]. \tag{3.90}$$

Here

$$\sigma^\mu \equiv (\mathbb{1}_2, \sigma^i) \;\text{ and }\; \bar\sigma^\mu = (\mathbb{1}_2, -\sigma^i). \tag{3.91}$$

From the decomposed Dirac equation we can draw the following important conclusions about the underlying physics:

- If $m = 0$, u_+ and u_- decouple and describe independent degrees of freedom subject to the **Weyl equations**

  $$\boxed{i\sigma^\mu\partial_\mu u_+(x) = 0, \quad i\bar\sigma^\mu\partial_\mu u_-(x) = 0.} \tag{3.92}$$

  Both u_+ and u_- transform in the $s = \frac{1}{2}$ representation of $\mathrm{SU}(2) \subset \mathrm{Spin}(1,3)$, in the sense that under spatial rotations

  $$x \;\mapsto\; R\,x, \tag{3.93}$$

  $$u_\pm(x) \;\mapsto\; e^{-\frac{i}{2}\alpha\,\vec n\cdot\vec\sigma}\, u_\pm(R^{-1}x). \tag{3.94}$$

  We define the **helicity**

  $$\boxed{h = \epsilon_{ijk}\hat p^i S^{jk} = \frac{1}{2}\hat{\vec p}\cdot\vec\sigma,} \tag{3.95}$$

  which is the projection of the spin onto the momentum direction $\hat{\vec p}$. To solve the Weyl equations of momentum p^μ one makes the ansatz (see the next chapter for details)

  $$u_\pm(x) = u_\pm(p)e^{-ip\cdot x} \tag{3.96}$$

  and finds

  $$\boxed{h\,u_\pm(p) = \pm\frac{1}{2}u_\pm(p),} \tag{3.97}$$

  where u_+ **corresponds to right-handed and** u_- **to left-handed spinors**.

- If $m \neq 0$, u_+ and u_- do not decouple. In this sense the full 4-component Dirac spinor ψ is needed to describe massive spin-$\frac{1}{2}$ fields. Note for fields of mass $m \neq 0$ it is impossible to define a Lorentz invariant notion of helicity because massive particles travel with a velocity $v < c$ and it is always possible to find a Lorentz frame in which the particle moves into the opposite direction. This causes a change in the helicity.

3.5 Classical plane-wave solutions

To solve the classical equation of motion (3.70) we make the ansatz

$$\psi(x) = u(\vec{p})e^{-ip\cdot x} \tag{3.98}$$

with $p = (E_p, \vec{p})$ and $E_p = \sqrt{\vec{p}^2 + m^2}$. Here we have used that the dispersion relation $p^2 - m^2 = 0$ is satisfied because $\psi(x)$ obeys the Klein-Gordon equation (3.71). Plugging this ansatz into the Dirac equation yields

$$(p^\mu \gamma_\mu - m)u(\vec{p}) = 0. \tag{3.99}$$

For our choice of γ^μ this is

$$\begin{pmatrix} -m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & -m \end{pmatrix} u(\vec{p}) = 0. \tag{3.100}$$

We make the ansatz

$$u(\vec{p}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3.101}$$

which leads to the relations

$$\begin{aligned} p_\mu \sigma^\mu u_2 &= m u_1, \\ p_\mu \bar{\sigma}^\mu u_1 &= m u_2. \end{aligned} \tag{3.102}$$

These are consistent because

$$(p_\mu \sigma^\mu)(p_\mu \bar{\sigma}^\mu) = (p^0)^2 - p_i p_j \underbrace{\sigma^i \sigma^j}_{\delta_{ij}} = m^2. \tag{3.103}$$

The general solution can be parametrised by introducing some 2-component Weyl spinor ξ' and writing

$$u(\vec{p}) = \begin{pmatrix} \frac{1}{m} p \cdot \sigma \, \xi' \\ \xi' \end{pmatrix}. \tag{3.104}$$

We make a conventional choice

$$\xi' = \sqrt{p \cdot \bar{\sigma}}\, \xi \tag{3.105}$$

with $\xi^\dagger \xi = 1$. Here $\sqrt{M}$ denotes the matrix whose eigenvalues are the square root of those of M. With this choice we find the so-called **positive frequency solution**

$$\boxed{\psi(x) = u(\vec{p})e^{-ip\cdot x}, \qquad u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}.} \tag{3.106}$$

Likewise

$$\psi(x) = v(\vec{p})e^{ip\cdot x} \tag{3.107}$$

is a solution if

$$(\gamma \cdot p + m)v(\vec{p}) = 0. \tag{3.108}$$

This yields the **negative frequency solution**

$$\boxed{\psi(x) = v(\vec{p})e^{ip\cdot x}, \qquad v(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ -\sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix}.} \tag{3.109}$$

Let us introduce a basis of the space of 2-spinors by ξ_s with

$$\xi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.110}$$

and therefore

$$\xi_s^\dagger \xi_{s'} = \delta_{ss'}. \tag{3.111}$$

Note that ξ_s, viewed as a Weyl spinor, transforms in the $s = \frac{1}{2}$ representation of SU(2). In particular

$$\frac{1}{2}\sigma_3 \xi_{\pm\frac{1}{2}} = \pm\frac{1}{2}\xi_{\pm\frac{1}{2}}. \tag{3.112}$$

Correspondingly u_s and v_s with $s = \pm\frac{1}{2}$ describe spinors with spin $\pm\frac{1}{2}$ in direction x_3. In the tutorial we will convince ourselves of the important identities

$$\boxed{\begin{aligned} \bar{u}_s(\vec{p})u_{s'}(\vec{p}) &= 2m\,\delta_{ss'}\,, \quad u_s^\dagger(\vec{p})u_{s'}(\vec{p}) = 2p^0\,\delta_{ss'}, \\ \bar{v}_s(\vec{p})v_{s'}(\vec{p}) &= -2m\,\delta_{ss'}\,, \quad v_s^\dagger(\vec{p})v_{s'}(\vec{p}) = 2p^0\,\delta_{ss'}, \\ \bar{u}_s(\vec{p})v_{s'}(\vec{p}) &= 0 \end{aligned}} \tag{3.113}$$

and

$$\boxed{\begin{aligned} \sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) &= \gamma\cdot p + m, \\ \sum_s v_s(\vec{p})\bar{v}_s(\vec{p}) &= \gamma\cdot p - m. \end{aligned}} \tag{3.114}$$

Let us consider the following example: Suppose we have chosen coordinates such that $p^\mu = (E, 0, 0, p^3)$. According to the above, the Dirac spinor solution $u_{\frac{1}{2}}(\vec{p})$ with spin $\frac{1}{2}$ along x_3 is given by

$$u_{\frac{1}{2}}(\vec{p}) = \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi_{\frac{1}{2}} \\ \sqrt{\bar{\sigma}\cdot p}\,\xi_{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{E-p^3}\,\xi_{\frac{1}{2}} \\ \sqrt{E+p^3}\,\xi_{\frac{1}{2}} \end{pmatrix}. \tag{3.115}$$

If $m = 0$, then $E = p^3$ and

$$u_{\frac{1}{2}}(\vec{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \equiv u_+(\vec{p}) \tag{3.116}$$

since $\xi_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}$. The solution correspnding to $\xi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix}$ is

$$u_{-\frac{1}{2}}(\vec{p}) = \begin{pmatrix}\sqrt{E+p^3}\,\xi_{-\frac{1}{2}}\\ \sqrt{E-p^3}\,\xi_{-\frac{1}{2}}\end{pmatrix} \overset{m=0}{\longrightarrow} \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \equiv u_-(\vec{p}). \tag{3.117}$$

The helicity of the solutions is

$$\begin{aligned} h\,u_+(\vec{p}) &= \frac{1}{2}u_+(\vec{p}) \Rightarrow \quad \text{right-handed,}\\ h\,u_-(\vec{p}) &= -\frac{1}{2}u_-(\vec{p}) \Rightarrow \quad \text{left-handed.}\end{aligned} \tag{3.118}$$

3.6 Quantisation of the Dirac field

So far our analysis of spinors has been classical. To define the quantum theory of spinor fields we follow the same procedure as in the scalar case, but we will encounter a problem if we just impose the same commutation relation as in the scalar case. Its resolution will prove the fermionic nature of spin $\frac{1}{2}$ particles.

3.6.1 Using the commutator

Starting from the classical Lagrangian

$$S = \int \mathrm{d}^4x\,\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{3.119}$$

we find the conjugate momentum density

$$\Pi_A = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^A} = i\psi_A^\dagger. \tag{3.120}$$

We promote $\psi(x)$ and $\Pi(x)$ to Schrödinger picture quantum operators and expand these as

$$\begin{aligned}\psi(\vec{x}) &= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_s(\vec{p})u_s(\vec{p})e^{i\vec{p}\cdot\vec{x}} + b_s^\dagger(\vec{p})v_s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right],\\ \psi^\dagger(\vec{x}) &= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b_s(\vec{p})v_s^\dagger(\vec{p})e^{i\vec{p}\cdot\vec{x}} + a_s^\dagger(\vec{p})u_s^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right],\end{aligned} \tag{3.121}$$

where u_s and v_s are the spinor-valued solutions to the classical equation (3.70) corresponding to the ansatz (3.98). The quantum operators a_s and b_s are independent because the field ψ is complex. This expansion guarantees that the corresponding Heisenberg fields with $e^{\pm i\vec{p}\cdot\vec{x}} \mapsto e^{\mp ip\cdot x}$ satisfy the Dirac equation as an operator equation.

We are now careful with operator orderings and proceed without specifying the commutation relations. The Hamiltonian density is

$$\mathcal{H} = \Pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^j\partial_j + m)\psi. \tag{3.122}$$

Now,

$$\begin{aligned}(-i\gamma^j\partial_j + m)\psi = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\Big[&(-\gamma^j p_j + m)u_s(\vec{p})a_s(\vec{p})e^{i\vec{p}\cdot\vec{x}} \\ &+ (\gamma^j p_j + m)v_s(\vec{p})b_s^\dagger(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\Big]\end{aligned} \tag{3.123}$$

because

$$\partial_j e^{i\vec{p}\cdot\vec{x}} = \partial_j e^{-ip_k x^k} = -ip_j e^{i\vec{p}\cdot\vec{x}}. \tag{3.124}$$

Now use that v_s and u_s solve the classical equation of motion, i.e.

$$\begin{aligned}(\gamma^\mu p_\mu - m)u_s(\vec{p}) = 0 \;&\Rightarrow\; (-\gamma^j p_j + m)u_s(\vec{p}) = \gamma^0 p_0\, u_s(\vec{p}), \\ (\gamma^\mu p_\mu + m)v_s(\vec{p}) = 0 \;&\Rightarrow\; (\gamma^j p_j + m)v_s(\vec{p}) = -\gamma^0 p_0\, v_s(\vec{p}),\end{aligned} \tag{3.125}$$

and therefore

$$(-i\gamma^j\partial_j + m)\psi = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\gamma^0\Big[a_s(\vec{p})u_s(\vec{p})e^{i\vec{p}\cdot\vec{x}} - b_s^\dagger(\vec{p})v_s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\Big]. \tag{3.126}$$

To compute the Hamiltonian

$$H = \int \mathrm{d}^3x\, \psi^\dagger\gamma^0(-i\gamma^j\partial_j + m)\psi \tag{3.127}$$

we furthermore exploit the spinor identities

$$\begin{aligned}v_r^\dagger(\vec{p})u_s(\vec{p}) = 0\,, \quad u_r^\dagger(\vec{p})v_s(\vec{p}) = 0, \\ u_r^\dagger(p)u_s(p) = 2p^0\delta_{rs} = v_r^\dagger(p)v_s(p),\end{aligned} \tag{3.128}$$

which eventually give

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_p \sum_s \Big[a_s^\dagger(\vec{p})a_s(\vec{p}) - b_s(\vec{p})b_s^\dagger(\vec{p})\Big]. \tag{3.129}$$

So far no re-ordering of operators has been performed. Now, the naive guess would be to impose, at the next step, canonical commutation relations as for the scalar field,

$$\begin{aligned}\Big[\psi^A(\vec{x}), \Pi_B(\vec{x}')\Big] &\stackrel{?}{=} i\,\delta_B^A\,\delta^{(3)}(\vec{x}-\vec{x}'), \\ \Big[\psi^A(\vec{x}), \psi_B^\dagger(\vec{x}')\Big] &\stackrel{?}{=} \delta_B^A\,\delta^{(3)}(\vec{x}-\vec{x}'), \\ \Big[\psi^A(\vec{x}), \psi^B(\vec{x}')\Big] &\stackrel{?}{=} 0.\end{aligned} \tag{3.130}$$

We will now show why **this is wrong**. The commutation relations would imply for the modes

$$\begin{aligned}\left[a_r(\vec{p}), a_s^\dagger(\vec{q})\right] &= (2\pi)^3, \delta_{rs}, \delta^{(3)}(\vec{p}-\vec{q}),\\ \left[b_r(\vec{p}), b_s^\dagger(\vec{q})\right] &= -(2\pi)^3, \delta_{rs}, \delta^{(3)}(\vec{p}-\vec{q}),\\ \left[a_r(\vec{p}), b_s^{(\dagger)}(\vec{q})\right] &= 0.\end{aligned} \tag{3.131}$$

Indeed we have

$$\begin{aligned}[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \sum_{r,s}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{4E_pE_q}}\Big(&\left[a_r(\vec{p}), a_s^\dagger(\vec{q})\right]u_r(\vec{p})u_s(\vec{q})^\dagger e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\\ &+\left[b_r^\dagger(\vec{p}), b_s(\vec{q})\right]v_r(\vec{p})v_s^\dagger(\vec{q})e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\Big).\end{aligned} \tag{3.132}$$

Then, using $\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = \gamma\cdot p + m$ and $\sum_s v_s(\vec{p})\bar{v}_s(\vec{p}) = \gamma\cdot p - m$, the commutator relations (3.130) of the field ψ would follow from the commutator relations (3.131) of the modes. Note that

$$[a_r(\vec{p}), a_s^\dagger(\vec{q})] \quad\text{and}\quad [b_r^\dagger(\vec{p}), b_s(\vec{q})] \tag{3.133}$$

appear on same footing in the commutator of the field. This ordering is the reason for the crucial relative minus sign in (3.131). If this were correct, then we could reorder the Hamiltonian to find

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3}E_p\sum_s\left[a_s^\dagger(\vec{p})a_s(\vec{p}) - b_s^\dagger(\vec{p})b_s(\vec{p}) + (2\pi)^3\delta(0)\right]. \tag{3.134}$$

To give the mode operators a physical interpretation we observe that

$$\begin{aligned}[H, a_s(\vec{p})] &= -E_p\, a_s(\vec{p}),\\ [H, a_s^\dagger(\vec{p})] &= E_p\, a_s^\dagger(\vec{p})\end{aligned} \tag{3.135}$$

and also

$$\begin{aligned}[H, b_s(\vec{p})] &= -E_p\, b_s(\vec{p}),\\ [H, b_s^\dagger(\vec{p})] &= E_p\, b_s^\dagger(\vec{p})\end{aligned} \tag{3.136}$$

because the minus sign in the Hamiltonian cancels with the sign we pick up in the commutator. So as in the scalar theory we would interpret

$$a_s(\vec{p}), b_s(\vec{p}) \quad\leftrightarrow\quad \text{annihilation operators} \tag{3.137}$$

$$a_s^\dagger(\vec{p}), b_s^\dagger(\vec{p}) \quad\leftrightarrow\quad \text{creation operators} \tag{3.138}$$

and define $|0\rangle$ such that $a_s(\vec{p})\,|0\rangle = b_s(\vec{p})\,|0\rangle$. The creation operators give positive energy modes by construction.

While at first sight everything looks fine, in actuality this whole construction is in deep conflict with unitarity. The reason is that due to the minus sign in the commutator

$$\left[b_r(\vec{p}), b_s^\dagger(\vec{q})\right] = -(2\pi)^3\,\delta_{rs}\,\delta^{(3)}(\vec{p}-\vec{q}) \tag{3.139}$$

the b-mode excitations are negative norm states. To see this note that

$$\langle 0|\,[b_r(\vec{p}), b_s^\dagger(\vec{q})]\,|0\rangle = -(2\pi)^3\,\delta_{rs}\delta^{(3)}(\vec{p}-\vec{q})\;\langle 0|\,0\rangle. \tag{3.140}$$

Evaluated for $r = s$ and $\vec{p} = \vec{q}$ we conclude

$$0 > -(2\pi)^3\delta^{(3)}(0) = \langle 0|\,[b_r(\vec{p}), b_r^\dagger(\vec{p})]\,|0\rangle = \langle 0|\,b_r(\vec{p})b_r^\dagger(\vec{p})\,|0\rangle = \|b_r^\dagger\,|0\rangle\,\|^2. \tag{3.141}$$

Therefore no positive norm Hilbert space interpretation is possible!
One might hope to avoid this problem by switching the interpretation of creation versus annihilation operators for the b-modes. But then the energy spectrum becomes arbitrarily negative by exciting more and more such states from the vacuum because of the minus sign in $H \sim -b^\dagger b$!
We therefore conclude that this procedure results in

- either loss of unitarity due to appearance of negative-norm states
- or unboundness from below of H, i.e. instability of the vacuum.

The origin of this problem lies in the fact that the signs in the spinor theory conspire such that to establish a commutation relation of the schematic form $[\psi, \psi^\dagger] \sim 1$ we must impose

$$[a, a^\dagger] \sim 1 \;\text{ and }\; [b^\dagger, b] \sim 1 \;\rightarrow\; [b, b^\dagger] \sim -1. \tag{3.142}$$

If instead we impose a relation symmetric in ψ and $\psi^\dagger$ this minus sign for the b-mode relation would not occur. The task is therefore to promote the classical Poisson-bracket relations not to operator commutation relations, but to an analogous 'bracket' which is symmetric in both entries. The simplest such bracket is the anti-commutator. It turns out that this procedure is successful.

3.6.2 Using the anti-commutator

The correct procedure for quantisation of spin-$\frac{1}{2}$ fields is to impose the canonical **anti-commutation** relations

$$\boxed{\begin{aligned}&\{\psi^A(\vec{x}), \psi_B^\dagger(\vec{x}')\} = \delta^A_{\ B}\,\delta^{(3)}(\vec{x}-\vec{x}'),\\ &\{\psi^A(\vec{x}), \psi^B(\vec{x}')\} = 0 = \{\psi_A^\dagger(\vec{x}), \psi_B^\dagger(\vec{x}')\},\end{aligned}} \tag{3.143}$$

where $\{A, B\} = AB + BA = \{B, A\}$. This induces the mode relations

$$\boxed{\begin{aligned}\left\{a_r(\vec{p}), a_s^\dagger(\vec{q})\right\} &= (2\pi)^3\,\delta_{rs}\,\delta^{(3)}(\vec{p}-\vec{q}),\\ \left\{b_r(\vec{p}), b_s^\dagger(\vec{q})\right\} &= (2\pi)^3\,\delta_{rs}\,\delta^{(3)}(\vec{p}-\vec{q}),\\ \left\{a_r(\vec{p}), b_s^{(\dagger)}(\vec{q})\right\} &= 0.\end{aligned}} \tag{3.144}$$

Starting from $H \sim \sum a^\dagger a - bb^\dagger$ we now find

$$\boxed{H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_p \sum_s \left[a_s^\dagger(\vec{p})a_s(\vec{p}) + b_s^\dagger(\vec{p})b_s(\vec{p}) - (2\pi)^3\delta(0)\right].} \tag{3.145}$$

The divergent vacuum energy has opposite sign compared to a scalar theory. In theories with scalars and spin-$\frac{1}{2}$fields cancellations in the vacuum energy are indeed possible.[4]

Since for this Hamiltonian the anti-commutation relations still imply the commutation relations (3.135) and (3.136), the vacuum is again defined by $a_s(\vec{p})\,|0\rangle = 0 = b_s(\vec{p})\,|0\rangle$. From this vacuum we define the Fock space of a- and b-mode excitations. Let us start with the a-modes, which we will call the particle sector.

1-particle states

The state $|\vec{p}, s\rangle := \sqrt{2E_p}a_s^\dagger(\vec{p})\,|0\rangle$ is a 1-particle state with momentum $\vec{p}$, energy $E_p = \sqrt{\vec{p}^2 + m^2}$ and spin s in the x_3-direction, normalized such that

$$\langle \vec{p}, s|\, \vec{q}, r\rangle = 2E_p(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\delta^{rs}. \tag{3.146}$$

***N*-particle states**

The state

$$|p_1, s_1; ...; p_N, s_N\rangle = \prod_{i=1}^{N} \sqrt{2E_{p_i}}a_{s_1}^\dagger(\vec{p}_1)...a_{s_N}^\dagger(\vec{p}_N)\,|0\rangle \tag{3.147}$$

is an N-particle momentum eigenstate. This allows us, in complete analogy to the scalar field, to state the following theorem

> *The wavefunction of N-particle states of spin $\frac{1}{2}$ particles is anti-symmetric under particle exchange.*

Indeed, if we exchange two particles, we pick up a minus sign due to the anti-commutation relations:

$$a_{s_i}^\dagger(\vec{p}_i)a_{s_j}^\dagger(\vec{p}_j) = -a_{s_j}^\dagger(\vec{p}_j)a_{s_i}^\dagger(\vec{p}_i). \tag{3.148}$$

This leads to the following **Corollary**

> *Spin $\frac{1}{2}$ particles obey Fermi-statistics, i.e. they are fermions.*

In particular they obey the **Pauli exclusion principle**:

> *No two fermionic states of exactly the same quantum numbers are possible.*

[4] More generally, scalars and spinors contribute with opposite signs in loops and theories with supersymmetry, i.e. with an equal number of bosonic and fermionic degrees of freedom, therefore have a chance to exhibit better UV properties.

The **Pauli exclusion principle** is again a result of the anti-commutation relation because

$$a_s^\dagger(\vec{p})a_s^\dagger(\vec{p})\,|0\rangle = 0. \tag{3.149}$$

This exemplifies the much more general **Spin-Statistics-Theorem**[5]:

Lorentz invariance, positivity of energy, unitarity and causality imply that:

- *Particles of half-integer spin are fermions and*
- *particles of integer spin are bosons.*

Excitations $b_s^\dagger(\vec{p})\,|0\rangle$ describe the corresponding anti-particles. Indeed the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{3.150}$$

enjoys a global $U(1)$ symmetry because it is invariant under

$$\psi \mapsto e^{-i\alpha}\psi, \qquad \bar{\psi} \mapsto \bar{\psi}e^{i\alpha}, \qquad \alpha \in \mathbb{R}. \tag{3.151}$$

The associated conserved Noether current will be found in the tutorial to take the form

$$\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi,} \tag{3.152}$$

with Noether charge

$$Q = \int \mathrm{d}^3x\, j^0 = \int \mathrm{d}^3x\, \psi^\dagger\psi = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\sum_s \Big(a_s^\dagger(\vec{p})a_s(\vec{p}) - b_s^\dagger(\vec{p})b_s(\vec{p})\Big), \tag{3.153}$$

after dropping a normal ordering constant. The charge acts on the 1-particle state as follows:

$$\begin{aligned} Q\,a_s^\dagger(\vec{p})\,|0\rangle &= +\,a_s^\dagger(\vec{p})\,|0\rangle\,, \text{ thus defining a fermion,} \\ Q\,b_s^\dagger(\vec{p})\,|0\rangle &= -\,b_s^\dagger(\vec{p})\,|0\rangle\,, \text{ thus defining an anti-fermion.} \end{aligned} \tag{3.154}$$

Finally, a careful analysis of the angular momentum operator via Noether's theorem reveals that

$$J_{x_3}\,a_s^\dagger(\vec{p}=0)\,|0\rangle = s\,a_s^\dagger(\vec{p}=0)\,|0\rangle\,, \tag{3.155}$$

but

$$J_{x_3}\,b_s^\dagger(\vec{p}=0)\,|0\rangle = -s\,b_s^\dagger(\vec{p}=0)\,|0\rangle\,, \tag{3.156}$$

with $s = \pm\frac{1}{2}$, For details of the derivation see Peskin-Schröder, page 61. This shows that $b_s^\dagger(\vec{p}=0)\,|0\rangle$ has spin (in x_3-direction) $-s$, while $a_s^\dagger(\vec{p}=0)\,|0\rangle$ has spin $+s$.

[5] For a general proof see Weinberg (1, 5.7).

3.7 Propagators

As for the scalar fields, we now move to the Heisenberg picture by considering the time-dependent free fields (with free mass denoted by m_0 to avoid confusion)

$$\begin{aligned}\psi(x) &= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_s(\vec{p})u_s(\vec{p})e^{-ip\cdot x} + b_s^\dagger(\vec{p})v_s(\vec{p})e^{ip\cdot x}\right],\\ \psi^\dagger(x) &= \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b_s(\vec{p})v_s^\dagger(\vec{p})e^{-ip\cdot x} + a_s^\dagger(\vec{p})u_s^\dagger(\vec{p})e^{ip\cdot x}\right],\end{aligned} \tag{3.157}$$

which, as noted already, satisfy the free Dirac equation as an operator equation.

To examine causality of the theory we define the **anti-commutator**

$$S^A{}_B(x-y) := \{\psi^A(x), \bar{\psi}_B(y)\} \tag{3.158}$$

and compute

$$\begin{aligned}S(x-y) = \sum_{s,r}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{E_p}}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{E_q}}\\ \times\Big[\underbrace{\{a_s(\vec{p}), a_r^\dagger(\vec{q})\}}_{\delta_{sr}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})}u_s(\vec{p})\bar{u}_r(\vec{q})e^{-ip\cdot x}e^{iq\cdot y}\\ + \{b_s^\dagger(\vec{p}), b_r(\vec{q})\}v_s(\vec{p})\bar{v}_r(\vec{q})e^{ip\cdot x}e^{-iq\cdot y}\Big].\end{aligned} \tag{3.159}$$

The identities

$$\sum_s u_s(\vec{p})\bar{u}_s(\vec{p}) = \gamma\cdot p + m_0 \;\text{ and }\; \sum_s v_s(\vec{p})\bar{v}_s(\vec{p}) = \gamma p - m_0 \tag{3.160}$$

imply that

$$S(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}\left[(\gamma\cdot p + m_0)e^{-ip\cdot(x-y)} + (\gamma\cdot p - m_0)e^{-ip\cdot(y-x)}\right] \tag{3.161}$$

or in a more compact form

$$\boxed{S(x-y) = (i\gamma^\mu\partial_{x^\mu} + m)\left[D^{(0)}(x-y) - D^{(0)}(y-x)\right].} \tag{3.162}$$

Here

$$D^{(0)}(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)} \tag{3.163}$$

is the propagator from the scalar theory with mass m_0. In particular this implies that $S(x-y) = 0$ for $(x-y)^2 < 0$. This in turn guarantees that $[O_1(x), O_2(y)] = 0$ for $(x-y)^2 < 0$ for $O_i(x)$ any local expression of fermion bilinears $\bar{\psi}\psi$. Since all physical observables are bosonic this establishes causality of the Dirac theory.

The **time-ordering symbol** in the fermionic theory is defined as

$$T(\psi(x)\bar{\psi}(y)) = \begin{cases} \psi(x)\bar{\psi}(y) & \text{if } x^0 \geq y^0, \\ -\bar{\psi}(y)\psi(x) & \text{if } y^0 > x^0. \end{cases} \tag{3.164}$$

Note the crucial minus sign. It is required because if $(x-y)^2 < 0$, we have

$$\psi(x)\bar{\psi}(y) = -\bar{\psi}(y)\psi(x), \tag{3.165}$$

because $S(x-y) = 0$ for $(x-y)^2 < 0$. Now, for $(x-y)^2 < 0$ the question of whether $x^0 \geq y^0$ or $x^0 < y^0$ depends on the Lorentz frame we have chosen. To arrive at a Lorentz frame independent definition of the time-ordering symbol T the expression for $T(\psi(x)\bar{\psi}(y))$ for $x^0 \geq y^0$ and $y^0 \geq x^0$ must agree.

The **Feynman propagator** is

$$\boxed{S_F(x-y) = \langle 0|\, T\psi(x)\bar{\psi}(y)\,|0\rangle\,,} \tag{3.166}$$

while

$$\langle 0|\, T\psi(x)\psi(y)\,|0\rangle = 0 = \langle 0|\, T\bar{\psi}(x)\bar{\psi}(y)\,|0\rangle\,. \tag{3.167}$$

Now by the usual tricks one evaluates

$$\begin{aligned} S_F(x-y) &= \Theta(x^0-y^0)\,\langle 0|\,\psi(x)\bar{\psi}(y)\,|0\rangle - \Theta(y^0-x^0)\,\langle 0|\,\bar{\psi}(y)\psi(x)\,|0\rangle \\ &= \Theta(x^0-y^0)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}(\gamma\cdot p + m_0)e^{-ip\cdot(x-y)} \\ &\quad - \Theta(y^0-x^0)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p}(\gamma\cdot p - m_0)e^{+ip\cdot(x-y)} \\ &= (i\gamma\cdot\partial_x + m)\,\underbrace{D_F^{(0)}(x-y)}_{\text{free scalar theory}}\,, \end{aligned} \tag{3.168}$$

with

$$D_F^{(0)}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m_0^2+i\epsilon}e^{-ip\cdot(x-y)}. \tag{3.169}$$

This gives

$$\boxed{S_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(\gamma\cdot p + m_0)}{p^2-m_0^2+i\epsilon}e^{-ip\cdot(x-y)}.} \tag{3.170}$$

Due to $(\gamma\cdot p + m_0)(\gamma\cdot p - m_0) = p^2 - m_0^2$ it is convenient to write

$$\frac{(\gamma\cdot p + m_0)}{p^2-m_0^2} = (\gamma\cdot p - m_0)^{-1}. \tag{3.171}$$

The Feynman propagator $S_F(x-y)$ is a Green's function in that it represents one of the 4 possible solutions to

$$(i\gamma^\mu\partial_\mu - m)G(x-y) = i\delta^{(4)}(x-y). \tag{3.172}$$

The interpretation and closure procedure in the complex plane for these are as in the scalar theory.

3.8 Wick's theorem and Feynman diagrams

The time ordering of several fields picks up a minus sign whenever 2 fermionic fields are exchanged, e.g.

$$T(\psi_1\psi_2\psi_3) = \begin{cases} (-1)\psi_1\psi_3\psi_2 & \text{if } x_1^0 > x_3^0 > x_2^0, \\ (-1)^2\psi_3\psi_1\psi_2 & \text{if } x_3^0 > x_1^0 > x_2^0, \\ (-1)^3\psi_3\psi_2\psi_1 & \text{if } x_3^0 > x_2^0 > x_1^0, \\ & \cdots \end{cases} \tag{3.173}$$

We define **normal-ordered products** as expressions with all creation operators to the left of all annihilation operators, where, unlike in the scalar theory, each exchange of two operators induces a minus sign, e.g.

$$: b_s(\vec{p})a_r^\dagger(\vec{q})b_v^\dagger(\vec{k}) : = (-1)^2 a_r^\dagger(\vec{q})b_v^\dagger(\vec{k})b_s(\vec{p}) \tag{3.174}$$

and so on. Then

$$T(\psi(x)\bar{\psi}(y)) = : \psi(x)\bar{\psi}(y) : + \overbrace{\psi(x)\bar{\psi}(y)}, \tag{3.175}$$

with

$$\overbrace{\psi(x)\bar{\psi}(y)} = \langle 0|\, T\psi(x)\bar{\psi}(y)\,|0\rangle = S_F(x-y) \tag{3.176}$$

and

$$\overbrace{\psi(x)\psi(y)} = 0 = \overbrace{\bar{\psi}(x)\bar{\psi}(y)}. \tag{3.177}$$

Direct computation confirms that Wick's theorem goes through, with the understanding that we include the minus signs from operator exchanges. For instance

$$\begin{aligned} T(\psi_1\bar{\psi}_2\bar{\psi}_3) = & : \psi_1\bar{\psi}_2\bar{\psi}_3 : + : \overbrace{\psi_1\bar{\psi}_2}\,\psi_3 : \\ & + \underbrace{: \overbrace{\psi_1\bar{\psi}_2\bar{\psi}_3} :}_{-:\overbrace{\psi_1\bar{\psi}_3}\,\bar{\psi}_2: \,=-S_f(x_1-x_3):\bar{\psi}_2:} . \end{aligned} \tag{3.178}$$

With this in mind Wick's theorem becomes

$$\boxed{T(\bar{\psi}_1\bar{\psi}_2\psi_3...) = : \bar{\psi}_1\bar{\psi}_2\psi_3... + \text{ all contractions with signs } :} \tag{3.179}$$

In particular

$$\langle 0|\, T \prod_i \psi(x_i) \prod_j \bar{\psi}(\bar{x}_j)\,|0\rangle \neq 0 \tag{3.180}$$

only for equal numbers of ψ and $\bar{\psi}$ fields. Physically this just reflects charge conservation.

To compute a 2n-point function of this type, we draw the corresponding Feynman diagrams, but now

- label the points x_i associated with $\psi(x_i)$ and $\bar{x}_j$ associated with $\overline{\psi}(\bar{x}_j)$ separately,
- only connect x_i with $\bar{x}_j$ and

- associate each directed line from $\bar{x}_j$ to x_i with a propagator $S_F(x_i - \bar{x}_j)$.

Be sure to always draw the arrow from $\bar{x}_j$ to x_i in order to account for the correct sign in $S_F(x_i - \bar{x}_j)$. Apart from an overall sign (which is typically unimportant because we will eventually take the square of the amplitude), the relative signs between the diagrams (which are important due to interference) equal the number of crossing lines. For instance, for

$$\langle 0|\, T\psi(x_1)\bar{\psi}(\bar{x}_2)\psi(x_3)\bar{\psi}(\bar{x}_4)\,|0\rangle \tag{3.181}$$

this prescription gives (see Figure 3.1)

$$S_F(x_1 - \bar{x}_2)S(x_3 - \bar{x}_4) + (-1)^1 S_F(x_1 - \bar{x}_4)S(x_3 - \bar{x}_2). \tag{3.182}$$

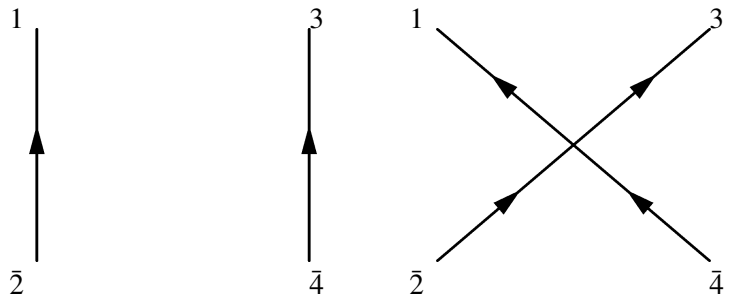


Figure 3.1: Possible Feynman diagrams

3.9 LSZ and Feynman rules

We will examine interacting spin $\frac{1}{2}$ fields in great detail in the context of Quantum Electrodynamics. Another example of an interesting interacting theory is **Yukawa theory**, which couples a spin $\frac{1}{2}$ field to a real boson via a cubic coupling. Its form is given in the tutorial.

In this section we only briefly summarise the logic behind the computation of scattering amplitudes with spin $\frac{1}{2}$. As in the scalar theory, in the presence of interactions we define asymptotic in- and out-fields satisfying the free Dirac equation with mass $m \neq m_0$, where m_0 is the mass in the free Dirac action. We then express the creation and annihilation modes by the in- and out-fields, e.g. for the

in-fields

$$\begin{aligned}
a_{\text{in},s}(\vec{q}) &= \frac{1}{\sqrt{2E_q}} \int \mathrm{d}^3x\, \bar{u}_s(\vec{q}) e^{iq\cdot x}\gamma^0\psi_{\text{in}}(x),\\
a^\dagger_{\text{in},s}(\vec{q}) &= \frac{1}{\sqrt{2E_q}} \int \mathrm{d}^3x\, \bar{\psi}_{\text{in}}(x)\gamma^0 e^{-iq\cdot x} u_s(\vec{q}),\\
b_{\text{in},s}(\vec{q}) &= \frac{1}{\sqrt{2E_q}} \int \mathrm{d}^3x\, \bar{\psi}_{\text{in}}(x)\gamma^0 e^{iq\cdot x} v_s(\vec{q}),\\
b^\dagger_{\text{in},s}(\vec{q}) &= \frac{1}{\sqrt{2E_q}} \int \mathrm{d}^3x\, \bar{v}_s(\vec{q}) e^{-iq\cdot x}\gamma^0\psi_{\text{in}}(x).
\end{aligned} \tag{3.183}$$

Using these one can perform exactly the same LSZ reduction procedure as in the scalar field case to extract the S-matrix. In this process we make heavy use of the equations

$$(\gamma\cdot p - m)u(\vec{p}) = 0, \qquad (\gamma\cdot p + m)v(\vec{p}) = 0, \tag{3.184}$$

where now m is the fully renormalized physical mass. Consider incoming fermions $|q,s,+\rangle$ and anti-fermions $|q's',-\rangle$ and outgoing fermions $\langle p,r,+|$ and anti-fermions $\langle q',r',-|$. The final result for the S-matrix element is

$$\begin{aligned}
&\langle ...(p,r,+)...(p'r',-)....|\, S\, |...(q,s,+)...(q',s',-)...\rangle\big|_{\text{connected}}\\
=&\left((-iZ)^{-\frac{1}{2}}\right)^n\left((iZ)^{-\frac{1}{2}}\right)^{n'}\int \mathrm{d}^4x... \int \mathrm{d}^4x'... \int \mathrm{d}^4y... \int \mathrm{d}^4y'\\
&\times e^{-i(q\cdot x+q'\cdot x'-p\cdot y-p'\cdot y'+...)}\bar{u}_r(\vec{p})(i\gamma\cdot\partial_y - m)...\bar{v}_{s'}(\vec{q}')(i\gamma\cdot\partial_{x'}-m)...\\
&\times\langle\Omega|\, T...\bar{\psi}(y')...\psi(y)...\bar{\psi}(x)...\psi(x')...\,|\Omega\rangle\,(-i\gamma\cdot\overleftarrow{\partial}_x - m)u_s(\vec{q})...(-i\gamma\cdot\overleftarrow{\partial}_y - m)v_{r'}(\vec{p}').
\end{aligned} \tag{3.185}$$

Thus to compute the S-matrix we compute the Fourier transform of the amputated fully connected associated Feynman diagram, where for each external particle we include

- $u_s(\vec{q})$ for an incoming particle of spin s,
- $\bar{v}_{s'}(\vec{q}')$ for an incoming anti-particle of spin $-s'$,
- $\bar{u}_r(\vec{p})$ for an outgoing particle of spin r,
- $v_{r'}(\vec{p}')$ for an outgoing anti-particle of spin $-r'$.

Alternatively, the appearance of these spinor polarisations can also be deduced by the Interaction picture procedure discussed in the tutorials. We will exemplify this for Yukawa theory in the tutorial and later in the course for Quantum Electrodynamics.

Chapter 4

Quantising spin 1-fields

4.1 Classical Maxwell-theory

Let us first recall the main aspects of classical Maxwell theory.

- The classical Maxwell equations are

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho\,, & \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t}, \\ \vec{\nabla}\times\vec{B} &= \vec{j}+\frac{\partial\vec{E}}{\partial t}, & \vec{\nabla}\cdot\vec{B} &= 0, \end{aligned} \tag{4.1}$$

  where the sources are subject to local charge conservation, i.e.

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0. \tag{4.2}$$

- By virtue of the inhomogeneous Maxwell equations and Helmholtz's theorem the fields $\vec{E}$ and $\vec{B}$ can locally be expressed as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\,, \qquad \vec{B} = \vec{\nabla}\times\vec{A}. \tag{4.3}$$

- This description in terms of the scalar potential ϕ and the vector potential $\vec{A}$ is redundant because $\vec{E}$ and $\vec{B}$ are invariant under a gauge transformation, i.e. a transformation

$$\begin{aligned} \phi(x) &\to \phi(x) + \frac{\partial}{\partial t}\alpha(x), \\ \vec{A}(x) &\to \vec{A}(x) - \vec{\nabla}\alpha(x). \end{aligned} \tag{4.4}$$

- To establish a Lorentz invariant formulation we introduce the 4-vector gauge potential A^μ and the 4-current j^μ as

$$A^\mu = \begin{pmatrix}\phi\\ \vec{A}\end{pmatrix}, \qquad j^\mu = \begin{pmatrix}\rho\\ \vec{j}\end{pmatrix}. \tag{4.5}$$

The fields $\vec{E}$ and $\vec{B}$ are really components of the field strength tensor $F^{\mu\nu}$,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{4.6}$$

which in matrix notation reads

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{4.7}$$

- Using the field strength tensor the inhomogenous Maxwell equations can be written as

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{4.8}$$

The homogenous Maxwell equations are automatically satisfied because they were used to express $\vec{E}$ and $\vec{B}$ in terms of the potentials. Indeed they correspond to the Bianchi identity

$$\partial_{[\mu} F_{\nu\rho]} = 0, \tag{4.9}$$

where $[\,]$ denotes all cyclic permutations. Note that $\partial_\mu j^\mu = 0$ follows as a consistency condition because $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ since we contract the symmetric tensor $\partial_\nu \partial_\mu$ with the anti-symmetric tensor $F^{\mu\nu}$.

- We stress again that $F^{\mu\nu}$ and thus $\vec{E}$ and $\vec{B}$ are invariant under a **local gauge transformation**

$$\boxed{A^\mu(x) \to A^\mu(x) + \partial^\mu \alpha(x).} \tag{4.10}$$

Configurations related by gauge transformations are physically equivalent. Gauge symmetries merely denote a redundancy in the description of the system. To determine the true local physical degrees of freedom we must be sure to divide out by this redundancy. This will be the main difficulty in quantising the system.

- The Maxwell equations follow as the equation of motion of A^μ from the action

$$\boxed{S = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu(x)\, j^\mu(x) \right).} \tag{4.11}$$

Note that S is gauge invariant if and only if $\partial_\mu j^\mu = 0$. The equation of motion can be rewritten as

$$\Box A^\mu - \partial^\mu (\partial_\nu A^\nu) = j^\mu. \tag{4.12}$$

- Due to gauge invariance we can always pick A^μ such that

$$\partial_\mu A^\mu = 0. \tag{4.13}$$

This partially fixes the gauge in **Lorenz gauge**, but we are still free to perform a **residual gauge transformation**

$$A^\mu \to A^\mu + \partial^\mu \phi \quad \text{with} \quad \Box\phi = 0 \tag{4.14}$$

without violating the Lorenz gauge condition. In Lorenz gauge the equation of motion is

$$\Box A^\mu = j^\mu. \tag{4.15}$$

- Lorenz gauge can be implemented by adding a Lagrange multiplier term in the action:

$$S = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial \cdot A)^2 - A_\mu\, j^\mu \right). \tag{4.16}$$

Therefore we now have two equations of motion, namely for A^μ

$$\Box A^\mu - (1-\lambda)\partial^\mu(\partial \cdot A) = j^\mu \tag{4.17}$$

and for λ

$$\partial \cdot A = 0, \tag{4.18}$$

where by equation of motion for λ we mean that the variation of S with respect to λ is proportional to $\partial \cdot A$.

4.2 Canonical quantisation of the free field

We now set the current $j^\mu = 0$ and consider the free gauge potential. The free non-gauge fixed Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{4.19}$$

It is not suitable for quantisation because the momentum density canonically conjugate to A^μ is

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F_{\mu 0}, \tag{4.20}$$

and since

$$\Pi_0 = 0, \tag{4.21}$$

(A^μ, Π_μ) are no good canonical variables. Instead quantisation starts from the gauge fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial \cdot A)^2, \tag{4.22}$$

with

$$\Pi_\mu = F_{\mu 0} - \lambda \eta_{\mu 0}(\partial \cdot A). \tag{4.23}$$

We could proceed for a general Lagrange multiplier λ, but for simplicity we set

$$\boxed{\lambda = 1 \quad \text{corresponding to Feynman gauge.}} \tag{4.24}$$

Explicitly the Lagrangian in Feynman gauge is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \quad \text{together with} \quad \partial \cdot A = 0. \tag{4.25}$$

Since the Lagrangian multiplier λ has been integrated out by setting $\lambda = 1$, its equation of motion $\partial \cdot A = 0$ must now be imposed by hand as a **constraint**.

Note two important facts about the gauge fixed Lagrangian:

- The Langrangian $\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$ is simply the Lagrangian of 3 free massless scalars A^i, i=1,2,3, but for $\mu = 0$ the sign of the kinetic terms is wrong. This will be important in the sequel.
- The extra constraint $\partial \cdot A = 0$ is a consequence of the underlying gauge symmetry of the system and will ensure that a consistent quantization is possible despite the wrong sign for $\mu = 0$.

The equation of motion for A^μ which follows from (4.25) is

$$\Box A^\mu = 0 \quad \text{together with} \quad \partial \cdot A = 0. \tag{4.26}$$

With (4.25) as a starting point, the canonical momentum density is

$$\Pi_\mu = -\dot{A}_\mu. \tag{4.27}$$

We quantise the system by promoting A^μ and Π_ν to Heisenberg picture fields with canonical equal-time commutators

$$[A^\mu(t,\vec{x}), \Pi_\nu(t,\vec{y})] = i\delta^\mu_{\ \nu}\delta^{(3)}(\vec{x}-\vec{y}) \tag{4.28}$$

and therefore

$$\boxed{\begin{aligned}[A^\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{y})] &= -i\eta^{\mu\nu}\delta^{(3)}(\vec{x}-\vec{y}),\\ [A^\mu(t,\vec{x}), A^\nu(t,\vec{y})] &= 0 = [\dot{A}^\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{y})].\end{aligned}} \tag{4.29}$$

As observed above there is an odd minus sign for $\mu = \nu = 0$. Despite this issue we proceed and consider the mode expansion

$$A^\mu(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{\lambda=0}^{3}\epsilon^\mu(\vec{p},\lambda)\left[a_\lambda(\vec{p})e^{-ip\cdot x} + a^\dagger_\lambda(\vec{p})e^{ip\cdot x}\right], \tag{4.30}$$

which is compatible with the equation of motion $\Box A^\mu = 0$ as an operator equation provided

$$p^2 = 0 \;\Rightarrow\; E_p = |\vec{p}|. \tag{4.31}$$

Furthermore the vectors $\epsilon^\mu(\vec{p},\lambda),\ \lambda = 0,1,2,3$ are the 4 linearly independent real polarisation vectors whose definition depends on the value of the lightlike vector p^μ (satisfying $p^2 = 0$). Our conventions to define these are as follows: Let n^μ denote the time axis such that $n^2 = 1$.

- $\epsilon^\mu(\vec{p},0) \equiv n^\mu$ is the timelike or scalar polarisation vector.
- $\epsilon^\mu(\vec{p},i)$ for $i=1,2$ are called transverse polarisation vectors. They are defined via $\epsilon(\vec{p},i)\cdot n = 0 = \epsilon(\vec{p},i)\cdot p$ and
$$\epsilon(\vec{p},i)\cdot\epsilon(\vec{p},j) = -\delta_{ij}. \tag{4.32}$$
- $\epsilon^\mu(\vec{p},3)$ is called longitudinal polarisation and is defined via $\epsilon(\vec{p},3)\cdot n = 0 = \epsilon(\vec{p},3)\cdot\epsilon(\vec{p},i)$ for $i=1,2$ and $\epsilon(\vec{p},3)^2 = -1$. Since $p^2=0$ it can therefore be expressed as
$$\epsilon(\vec{p},3) = \frac{p - n(p\cdot n)}{p+n}. \tag{4.33}$$

So altogether
$$\boxed{\epsilon(\vec{p},\lambda)\cdot\epsilon(\vec{p},\lambda') = \eta_{\lambda,\lambda'}.} \tag{4.34}$$

We stress that this basis of polarisation vectors depends on the concrete momentum vector p with $p^2=0$. Consider e.g. a momentum vector $p^\mu = (1,0,0,1)^T$, then
$$\epsilon(\vec{p},0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \epsilon^\mu(\vec{p},1) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \epsilon^\mu(\vec{p},2) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \epsilon^\mu(\vec{p},3) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{4.35}$$

The canonical commutation relations imply for the modes
$$\boxed{[a_\lambda(\vec{p}), a^\dagger_{\lambda'}(\vec{p}')] = -\eta_{\lambda\lambda'}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')} \tag{4.36}$$

and
$$[a_\lambda(\vec{p}), a_{\lambda'}(\vec{p}')] = 0 = [a^\dagger_\lambda(\vec{p}), a^\dagger_{\lambda'}(\vec{p}')]. \tag{4.37}$$

Note again the minus sign for timelike modes $\lambda = \lambda' = 0$.
The Hamiltonian is
$$H = \int \mathrm{d}^3x\,(-\dot{A}^\mu\dot{A}_\mu - \mathcal{L}) = \frac{1}{2}\int \mathrm{d}^3x\left[-\dot{A}^\mu\dot{A}_\mu + \partial_i A_\mu \partial^i A^\mu\right] \tag{4.38}$$

and in modes
$$\begin{aligned} H =& -\int \frac{\mathrm{d}^3p}{(2\pi)^3}|\vec{p}|\sum_\lambda a^\dagger_\lambda(\vec{p})a_\lambda(\vec{p})\epsilon^\mu(\vec{p},\lambda)\epsilon^\nu(\vec{p},\lambda)\eta_{\mu\nu} \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}|\vec{p}|\left[\sum_i a^\dagger_i(\vec{p})a_i(\vec{p}) - a^\dagger_0(\vec{p})a_0(\vec{p})\right] \end{aligned} \tag{4.39}$$

after dropping the vacuum energy. This leads to the following commutation relations (valid $\forall\lambda$)
$$\begin{aligned} [H, a^\dagger_\lambda(\vec{p})] =& +|\vec{p}|\,a^\dagger_\lambda(\vec{p}), \\ [H, a_\lambda(\vec{p})] =& -|\vec{p}|\,a_\lambda(\vec{p}). \end{aligned} \tag{4.40}$$

We define again the vacuum $|0\rangle$ such that $a_\lambda(\vec{p})\,|0\rangle = 0$ and the 1-particle states

$$\boxed{|\vec{p},\lambda\rangle := \sqrt{2E_p}a^\dagger_\lambda(\vec{p})\,|0\rangle} \tag{4.41}$$

as the states of momentum $\vec{p}$ and polarisation λ. The corresponding particles are called photons.

Two crucial problems remain though:

- The previous analysis seems to suggest that the theory gives rise to 4 independent degrees of freedom per momentum eigenstate, but from classical electrodynamics we only expect 2 transverse degrees of freedom.
- Timelike polarisation states have negative norm,

$$\langle\vec{p},0|\,\vec{q},0\rangle \propto \langle 0|\,[a(\vec{p},0),a^\dagger(\vec{q},0)]\,|0\rangle = -(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}). \tag{4.42}$$

  Such **negative norm states** are called **ghosts** and spoil unitarity.

4.3 Gupta-Bleuler quantisation

The two above problems arose because the quantisation procedure so far is incomplete: The point is that the constraint $\partial\cdot A = 0$ has not been implemented yet. It is impossible to implement this constraint as an operator equation for Heisenberg fields because then we would conclude

$$0 \overset{!}{=} [\partial_\mu A^\mu(t,\vec{x}),A^\nu(t,\vec{y})] = \left[\dot{A}^0(t,\vec{x}),A^\nu(t,\vec{y})\right] = i\eta^{0\nu}\delta^{(3)}(\vec{x}-\vec{y}). \tag{4.43}$$

The idea of the Gupta-Bleuler formalism is to implement $\partial\cdot A = 0$ not at the level of operators, but directly on the Hilbert space, i.e. as a defining constraint on the so-called physical states. The naive guess would be to require

$$\partial\cdot A\,|\phi\rangle \overset{!}{=} 0 \tag{4.44}$$

for $|\phi\rangle$ to be a physical state. Let us decompose

$$A(x) = A^+(x) + A^-(x), \tag{4.45}$$

with

$$\begin{aligned} A^+(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_\lambda \epsilon(\vec{p},\lambda)a_\lambda(\vec{p})e^{-ip\cdot x},\\ A^-(x) &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_\lambda \epsilon(\vec{p},\lambda)a^\dagger_\lambda(\vec{p})e^{ip\cdot x} \end{aligned} \tag{4.46}$$

so that the physical state condition would be

$$(\partial\cdot A^+ + \partial\cdot A^-)\,|\phi\rangle \overset{!}{=} 0. \tag{4.47}$$

But this is still too strong because then not even the vacuum $|0\rangle$ would be such a physical state - after all

$$\partial \cdot A^+ |0\rangle = 0, \quad \text{but} \quad \partial \cdot A^- |0\rangle \neq 0. \tag{4.48}$$

However, the milder constraint

$$\partial \cdot A^+ |\phi\rangle \stackrel{!}{=} 0 \stackrel{!}{=} \langle\phi| \partial \cdot A^- \tag{4.49}$$

suffices to guarantee

$$\langle\phi| \partial \cdot A |\phi\rangle = \langle\phi| \left(\partial \cdot A^+ |\phi\rangle\right) + \left(\langle\phi| \partial \cdot A^-\right) |\phi\rangle = 0. \tag{4.50}$$

So in the spirit of Ehrenfest's theorem the classical relation $\partial \cdot A = 0$ is realised as a statement about the expectation value $\langle\phi| \partial \cdot A |\phi\rangle = 0$ in the quantum theory.
To summarise: Out of the naive Fock space we define the physical Hilbert space by

$$\boxed{\phi \in \mathcal{H}_{\text{phys}} \;\leftrightarrow\; \partial \cdot A^+ |\phi\rangle = 0.} \tag{4.51}$$

This is the **Gupta-Bleuler condition**.

It suffices to construct the physical 1-particle states of definite momentum since $\mathcal{H}_{\text{phys}}$ is spanned by the tensor product of these. Consider a state $|\vec{p}, \zeta\rangle$, i.e. 1 photon of polarisation ζ^μ, where we define a general polarisation 4-vector

$$\zeta^\mu = \sum_{\lambda,\lambda'} \alpha_\lambda \eta_{\lambda\lambda'} \epsilon^\mu(\vec{p}, \lambda). \tag{4.52}$$

The state $|\vec{p}, \zeta\rangle$ is defined as

$$|\vec{p}, \zeta\rangle := \sqrt{2|\vec{p}|} \sum_\lambda \alpha_\lambda a^\dagger_\lambda(\vec{p}) |0\rangle . \tag{4.53}$$

Therefore

$$\langle\vec{q}, \zeta| \vec{p}, \zeta'\rangle = -(2\pi)^3 \cdot 2|\vec{p}| \delta^{(3)}(\vec{p} - \vec{q}) \zeta \cdot \zeta'. \tag{4.54}$$

The physical state condition $\partial_\mu A^{+\mu} |\vec{p}, \zeta\rangle \stackrel{!}{=} 0$ implies that $p_\mu \zeta^\mu = 0$ because

$$\begin{aligned} \partial_\mu A^{+\mu} |\vec{p}, \zeta\rangle = & \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\sqrt{2|\vec{p}|}}{\sqrt{2|\vec{q}|}} \sum_\lambda (-iq_\mu) \epsilon^\mu(\vec{q}, \lambda) e^{-iq\cdot x} \underbrace{a_\lambda(\vec{q}) \sum_\gamma \alpha_\gamma a^\dagger_\gamma(\vec{p}) |0\rangle}_{= -(2\pi)^3 \sum_\gamma \alpha_\gamma \eta_{\lambda\gamma} \delta^{(3)}(\vec{p} - \vec{q}) |0\rangle} \\ & = i p_\mu \zeta^\mu |0\rangle . \end{aligned} \tag{4.55}$$

Therefore

$$\boxed{|\vec{p}, \zeta\rangle \in \mathcal{H}_{\text{phys}} \;\leftrightarrow\; p^\mu \zeta_\mu = 0.} \tag{4.56}$$

Since $p^2 = 0$ for a massless photon, such ζ^μ can be decomposed as

$$\zeta^\mu = \zeta^\mu_T + \zeta^\mu_S, \tag{4.57}$$

with $\zeta_S = c \cdot p$, $\zeta_S^2 = 0$ and $\vec{\zeta}_T \cdot \vec{p}_T = 0$, $\zeta_T^2 < 0$,. So $|\vec{p}, \zeta\rangle \in \mathcal{H}_{\text{phys}}$ can be written as

$$|\vec{p}, \zeta\rangle = |\vec{p}, \zeta_T\rangle + |\vec{p}, \zeta_s\rangle \tag{4.58}$$

where

- $|\vec{p}, \zeta_T\rangle$ describes 2 transverse degrees of freedom of positive norm, $\| |\vec{p}, \zeta_T\rangle \| > 0$,
- $|\vec{p}, \zeta_S\rangle$ describes 1 combined timelike and longitudinal degree of freedom of zero norm, $\| |\vec{p}, \zeta_S\rangle \| = 0$.

For example consider

$$p^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \zeta_T = \begin{pmatrix} 0 \\ \zeta^1 \\ \zeta^2 \\ 0 \end{pmatrix}, \quad \zeta_s = c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.59}$$

and $|\vec{p}, \zeta_S\rangle \propto (a_0^\dagger(\vec{p}) - a_3^\dagger(\vec{p}))\,|0\rangle$. Up to now, the Gupta-Bleuler procedure has eliminated the negative norm states and left us with 3 polarisation states, but in fact one can prove the following theorem:

The state $|\vec{p}, \zeta_S\rangle$ decouples from all physical processes.

Such a zero-norm state that decouples from all physical processes is called **spurious**, hence the subscript S. The meaning of this decoupling of null states is as follows:

- For the free theory decoupling means that

  $$\langle \vec{p}, \zeta_S | \mathcal{O} | \vec{p}, \zeta_S \rangle = 0 \tag{4.60}$$

  for all observables $\mathcal{O}$. As an example it is easy to check that

  $$\langle \vec{p}, \zeta_S | H | \vec{p}, \zeta_S \rangle = 0. \tag{4.61}$$

  Without loss of generality we take $p^\mu = (1, 0, 0, 1)^T$ and $|\vec{p}, \zeta_S\rangle \propto (a_0^\dagger(\vec{p}) - a_3^\dagger(\vec{p}))\,|0\rangle$ and confirm $\langle \vec{p}, \zeta_S | H | \vec{p}, \zeta_S \rangle = 0$ by noting the structure of the Hamiltonian

  $$H \sim \sum_i a_i^\dagger a_i - a_0^\dagger a_0 \tag{4.62}$$

  and the relative minus sign in the commutation relations for timelike and spacelike modes.

  Furthermore the spurious states decouple in the sense that $|\vec{p}, \zeta_S\rangle$ has zero overlap with $|\vec{p}, \zeta_T\rangle$ because $\zeta_S \cdot \zeta_T = 0$.

- The decoupling statement becomes actually non-trivial in the presence of interactions: As long as the interactions respect gauge invariance, a spurious state $|\vec{p}, \zeta_S\rangle$ decouples from the S-matrix as an external (in or out) state. This follows from the **Ward identities** as will be discussed later.

The conclusion is that only the 2 transverse polarisations are physically relevant as external states. These have positive norm.
This does not mean that spurious states play no role at all:

- Spurious states do appear as internal states in S-matrix processes and are important for consistency of the amplitudes.
- Spurious states are required to establish Lorentz-invariance of the theory because ζ_T may pick up a spurious component ζ_S by going to a different Lorentz frame.

Let $|\psi_S\rangle$ denote any multi-photon state constructed entirely out of spurious photons with polarization ζ_S. Since it decouples in the above sense, we can add and substract it without affecting any physical properties of a state. This establishes an equivalence relation on $\mathcal{H}_{\text{phys}}$:

$$\boxed{|\phi_1\rangle \sim |\phi_2\rangle \;\text{ if } \exists\, |\psi_s\rangle \;:\; |\phi_1\rangle = |\phi_2\rangle + |\psi_S\rangle\,.} \tag{4.63}$$

This is the analogue of the residual gauge symmetry in classical theory: Indeed, one can show that[1]

$$\langle\psi_s|\, A_\mu(x)\, |\psi_s\rangle = \partial_\mu \Lambda(x) \tag{4.64}$$

for a function $\Lambda(x)$ with

$$\Box\Lambda(x) = 0. \tag{4.65}$$

Therefore:

Adding $|\psi_s\rangle$ in $|\phi\rangle \to |\phi\rangle + |\psi_s\rangle$ is the quantum version of the residual transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda \;\; \textit{with} \;\; \Box\Lambda = 0 \tag{4.66}$$

in Lorenz gauge.

A more general perspective on massless Spin 1 fields

Our starting point was not an arbitrary massless vector field $A^\mu(x)$, but the very specific gauge potential that arises in Maxwell-theory. More generally we might ask: Given a general massless vector field $A^\mu(x)$, how can we quantise it?

The definitive treatment of this question can be found in Weinberg, Vol. I, Chapter 8.1, which we urgently recommend. Following this reference, the arguments are:

- By Lorentz invariance alone, any massless vector field $A_\mu(x)$ must describe precisely two helicity or polarization states (the two transverse degrees of freedom we found above).

[1] See Itzykson/Zuber, p.132 for a proof.

- On general grounds one can show that Lorentz vector fields describing two polarization states transform under a Lorentz transformation as

$$A^\mu(x) \to \Lambda^\mu_{\ \nu} A^\nu(\Lambda^{-1}x) + \partial^\mu \epsilon(x,\Lambda) \tag{4.67}$$

  for a spacetime-dependent function $\epsilon(x,\Lambda)$. Therefore Lorentz invariance requires invariance of the action under gauge transformations.

- The Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the unique Lorentz invariant and gauge invariant Lagrangian for a massless free vector field (i.e. up to quadratic order).

This proves the general statement:

Massless vector field theories must be gauge theories.

4.4 Massive vector fields

Theories of massive vector bosons on the other hand are consistent despite the lack of gauge invariance: Consider the Lagrangian for "massive electrodynamics"

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu \tag{4.68}$$

with the classical equation of motion, the so-called **Proca equation**,

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = 0. \tag{4.69}$$

One can show that this Lagrangian is the unique Lorentz invariant Lagrangian for a free massive spin-1 field (without any spin-0 components - see Weinberg I, 7.5 for a proof). The mass term explicitly breaks gauge invariance. So it is not possible to arrange for $\partial \cdot A = 0$ by gauge-fixing. However, the Proca equation implies

$$0 = \underbrace{\partial_\mu \partial_\nu F^{\mu\nu}}_{=0} + \mu^2 \partial_\nu A^\nu \ \Rightarrow\ \partial \cdot A = 0. \tag{4.70}$$

Thus, if $m \neq 0$, the constraint $\partial \cdot A = 0$ arises classically as a consequence of the equations of motion, not of gauge invariance. In the quantum theory it can indeed be justified to impose $\partial \cdot A = 0$ as an operator equation.[2] The physical Hilbert space now exhibits 3 positive-norm degrees of freedom corresponding to the polarisations

$$|\vec{k}, \zeta\rangle \ \text{ with } \ \zeta^\mu k_\mu = 0 \ \text{ and } \ k^2 - \mu^2 = 0. \tag{4.71}$$

Since no residual gauge transformation is available, there is no further decoupling of one degree of freedom. To summarise:

[2] See Weinberg I, 7.5 for details. The main difference to the massless theory is that $\Pi_0 \equiv 0$ now poses no problems because, unlike in the massless case, we can solve A^0 for the spacelike degrees of freedom and simply proceed with the quantisation of (A^i, Π_i). In more sophisticated terms, the system is amenable to quantisation with Dirac constraints, see again Weinberg.

Massive vector fields have 3 *physical degrees of freedom.*
Massless vector fields have 2 *physical degrees of freedom.*

Some comments are in order:

- One might be irritated that the constraint $\partial \cdot A = 0$, which in this case rests on the equations of motion, is imposed as a constraint in the quantum theory - quantisation must hold off-shell. For the free theory this is not really a problem: We can take the classical on-shell relation $\partial \cdot A = 0$ as a motivation to simply declare the physical Hilbert space to consist of the transverse polarisations only, thereby defining the quantum theory. That this remains correct in the presence of interactions rests again on the Ward identities, which, for suitable interactions, still guarantee that the - now negative norm - states with polarisation $\zeta \sim k$ decouple from the S-matrix. We will discuss this in detail in the context of the Ward identities.

- The procedure for quantising the Proca action breaks down if we set $\mu \to 0$. A framework where a smooth limit $m \to 0$ is possible is provided by the Stückelberg Lagrangian[3]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu - \frac{\lambda}{2}(\partial A)^2. \tag{4.72}$$

4.5 Coupling vector fields to matter

We would now like to couple a vector field $A^\mu(x)$ to a matter sector, e.g. a Dirac fermion or a scalar field. Let us collectively denote the matter fields by ϕ and the pure matter action (excluding any term involving the gauge fields) by $S^{\text{rest}}_{\text{matter}}[\phi]$. Then we would like to construct an action

$$S = S^0_A[A] + S_{\text{int}}[A,\phi] + S^{\text{rest}}_{\text{matter}}[\phi] \tag{4.73}$$

that describes the coupling of this matter sector to a vector field $A^\mu(x)$ with free action (prior to coupling) $S^0_A[A]$ and interaction terms $S_{\text{int}}[A,\phi]$. Naively, we would think that we can simply write down all possible Lorentz invariant terms in $S_{\text{int}}[A,\phi]$ involving A^μ and ϕ and then organize these as a series in derivatives and powers of fields, as we have done when writing down the most general interaction for a scalar field. This time, however, $S_{\text{int}}[A,\phi]$ is subject to an important constraint: It must be chosen such that the successful decoupling of negative norm states and zero norm states (in the case of a massless vector) in the free vector theory is not spoiled by the interaction. More precisely, if we denote by

$$\mathcal{M} = \zeta_\mu(k)\mathcal{M}^\mu \tag{4.74}$$

the scattering amplitude involving an external photon of polarisation vector $\zeta_\mu(k)$ and momentum k, then consistency of the interaction requires - at the very least - that

$$k_\mu \mathcal{M}^\mu = 0. \tag{4.75}$$

[3]Cf. Itzykson, Zuber, p. 136 ff. for details. Note that a priori this action does include spin-0 components in agreement with the above claim that the Proca action is the most general action describing spin-1 degrees of freedom only.

This is equivalent to the requirement that external photons of polarisation $\zeta_\mu(k) = k_\mu$ decouple from the interactions. As we will see this is necessary and also sufficient to ensure that if we start with a physical photon state of positive norm, no negative or zero norm states (in the case of a massless vector theory) are produced via the interaction.
Now, one can (and we will somewhat later in this course) prove very generally the following two important theorems:

1.) The decoupling of unphysical photon states with polarisation $\zeta_\mu(k) = k_\mu$, i.e. equ. (4.75), is equivalent to the statement that

$$\frac{\delta S_{\text{int}}[A,\phi]}{\delta A^\mu} = -j^\mu \tag{4.76}$$

for j^μ the conserved current associated with a global continuous $U(1)$ symmetry of the full action S under which

$$\phi(x) \to \phi(x) - e\,\alpha\,\delta\phi(x), \qquad \alpha \in \mathbb{R} \tag{4.77}$$

infinitesimally. Here we have rescaled the symmetry parameter α by a coupling constant e to comply with later conventions. In particular, $\partial_\mu j^\mu = 0$ on-shell. This is the statement that the vector theory must couple to a conserved current. That coupling to a conserved current is equivalent to (4.75) is ensured by the **Ward identities** to be discussed in detail later in this course.

2.) If the vector theory is massless, i.e. if $S^0_A[A]$ is invariant under the gauge symmetry

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x), \tag{4.78}$$

then (4.76) is equivalent to the statement that the full action S is invariant under the **combined gauge transformation**

$$\phi(x) \to \phi(x) - e\,\alpha(x)\,\delta\phi(x) \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x), \tag{4.79}$$

where in particular $\phi(x)$ now transforms under a local symmetry since we have promoted the constant α appearing in (4.77) to a function $\alpha(x)$. This process of promoting the global continuous symmetry (4.77) to a combined gauge symmetry as above is called **gauging**.

As indicated, for pedagogical reasons we postpone a proof of both these assertions and first exemplify the consistent coupling of a massless vector theory to matter via gauging by discussing interactions with a Dirac fermion and a complex scalar theory.

4.5.1 Coupling to Dirac fermions

Let us start with the free Dirac fermion action

$$S^{\text{rest}}_{\text{matter}} = \int d^4x\,\bar\psi(i\gamma^\mu\partial_\mu - m_0)\psi. \tag{4.80}$$

The only available vector Noether current of this action is due to the a priori global $U(1)$ symmetry

$$\psi(x) \to e^{-ie\,\alpha}\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{ie\,\alpha} \quad \alpha \in \mathbb{R}, \tag{4.81}$$

where $\alpha \in \mathbb{R}$ is the symmetry parameter and as above we have introduced a dimensionless coupling constant e.
With the normalisation conventions of eq. (1.37), the corresponding Noether current is

$$j^\mu = e\,\bar{\psi}\gamma^\mu\psi \;\text{ and }\; \partial_\mu j^\mu = 0 \tag{4.82}$$

on-shell for $\psi(x)$. As prescribed by the above theorem we proceed by gauging this global $U(1)$ symmetry. This means that we promote the global $U(1)$ symmetry to a local one, i.e. we promote the constant $\alpha \in \mathbb{R}$ to a function $\alpha(x)$. Now the kinetic term is no longer invariant because

$$\begin{aligned}\bar{\psi}(i\gamma\cdot\partial - m)\psi \mapsto \bar{\psi}e^{ie\alpha(x)}(i\gamma\cdot\partial - m_0)e^{-ie\alpha(x)}\psi \\ &= \bar{\psi}\left(i\gamma^\mu(\partial_\mu - ie\,\partial_\mu\alpha(x)) - m_0\right)\psi \\ &= \bar{\psi}(i\gamma\cdot\partial - m_0)\psi + [\partial_\mu\alpha(x)]j^\mu(x).\end{aligned} \tag{4.83}$$

However, we observe that the interaction term

$$\bar{\psi}(i\gamma^\mu\partial_\mu - m_0)\psi - A_\mu j^\mu, \qquad j^\mu = e\bar{\psi}\gamma^\mu\psi \tag{4.84}$$

is invariant under the combined gauge transformation

$$\psi(x) \to e^{-ie\,\alpha(x)}\psi(x), \quad A_\mu \to A_\mu + \partial_\mu\alpha(x). \tag{4.85}$$

Thus the interaction is gauge invariant off-shell and fully consistent. One can rewrite the interaction in terms of the **covariant derivative**

$$\boxed{D_\mu := \partial_\mu + ieA_\mu.} \tag{4.86}$$

Under a combined gauge transformation

$$\psi(x) \to e^{-ie\alpha(x)}\psi(x), \quad A_\mu \to A_\mu + \partial_\mu\alpha(x) \tag{4.87}$$

the covariant derivative transforms as

$$\boxed{D_\mu\psi(x) \to e^{-ie\alpha(x)}D_\mu\psi(x).} \tag{4.88}$$

Thus the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi = -\frac{1}{4}F^2 + \bar{\psi}(i\gamma\cdot\partial - m_0)\psi - A_\mu j^\mu, \qquad j^\mu = e\bar{\psi}\gamma^\mu\psi \tag{4.89}$$

is manifestly gauge invariant off-shell. In QED we set $e = -|e|$ equal to the elementary charge of one electron such that

$$\begin{aligned}Q\,a_s^\dagger(\vec{p})\,|0\rangle &= -|e|\,a_s^\dagger(\vec{p})\,|0\rangle \quad \text{for an electron,} \\ Q\,b_s^\dagger(\vec{p})\,|0\rangle &= +|e|\,b_s^\dagger(\vec{p})\,|0\rangle \quad \text{for a positron,}\end{aligned} \tag{4.90}$$

.

where Q is the Noether charge associated with j^μ.

Let us stress that global and gauge symmetries are really on very different footings:

- A gauge symmetry is a redundancy of the description of the system.
- A global symmetry is a true symmetry between different field configurations.

Note furthermore that, by construction, the gauge symmetry (4.87) reduces for $\alpha(x) = \alpha = \text{const.}$ to the global symmetry

$$\psi \to e^{-ie\alpha}\psi, \quad A \to A, \tag{4.91}$$

from which, in turn, charge conservation follows. In particular the appearance of a combined $U(1)$ gauge symmetry of the system matter plus gauge fields requires an underlying global $U(1)$ symmetry of the matter system.

4.5.2 Coupling to scalars

Conserved vector currents are available only for **complex** scalars. Put differently, real scalars are uncharged. Let us therefore consider a complex scalar theory, whose action prior to coupling to the Maxwell field reads

$$S^{\text{rest}}_{\text{matter}} = \partial_\mu\phi^\dagger(x)\partial^\mu\phi(x) - m^2\phi^\dagger(x)\phi(x). \tag{4.92}$$

The Noether current of the free theory associated with the global $U(1)$ symmetry $\phi \to e^{-ie\alpha}\phi$ with $\alpha \in \mathbb{R}$ is $j^\mu_{\text{free}} = ie\,(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi)$.
The naive guess for the coupling to the gauge sector would therefore be

$$\mathcal{L}^{\text{naive}}_{\text{int}} = -A_\mu j^\mu_{\text{free}}. \tag{4.93}$$

However - unlike in the fermionic case - this coupling does not exhibit off-shell gauge invariance under

$$\phi \to e^{-ie\alpha(x)}\phi, \quad A_\mu \to A_\mu + \partial_\mu\alpha(x). \tag{4.94}$$

Let us instead follow the general route of replacing the usual derivative $\partial_\mu\phi$ by the covariant derivative

$$D_\mu\phi(x) = (\partial_\mu + ieA_\mu)\phi. \tag{4.95}$$

and consider the action

$$\boxed{S = \int d^4x\Big(-\frac{1}{4}F^2 + (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi\Big).} \tag{4.96}$$

The interaction which follows by expanding $D^\mu = \partial^\mu + ieA^\mu$ is

$$S_{\text{int}}[A,\phi] = -\int d^4x\Big(ie(\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi)A_\mu - e^2A_\mu A^\mu\phi^\dagger\phi\Big). \tag{4.97}$$

The last term quadratic in A^μ is required for gauge invariance and was missed in the naive guess (4.93).

To see what had gone wrong in the naive guess (4.93) note that for constant gauge parameter $\alpha \in \mathbb{R}$ the combined gauge transformation (4.94) reduces to a $U(1)$ global symmetry of the full action (4.96). One can check that the Noether current associated with this global $U(1)$ symmetry of (4.96) is just

$$j^\mu = -2e^2 A^\mu \phi^\dagger \phi + ie\left(\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi\right) = ie(\phi^\dagger D^\mu \phi - (D^\mu \phi)^\dagger \phi) \tag{4.98}$$

and this does agree with the formula

$$\frac{\delta S_{\text{int.}}[A,\phi]}{\delta A_\mu} = -j^\mu. \tag{4.99}$$

The point is that in general the Noether current may itself depend on A_μ once the coupling to the gauge field is taken into account because $S_{\text{int.}}[A,\phi]$ may depend not just linearly on A. Therefore merely writing $-A_\mu j^\mu_{\text{free}}$ with j^μ_{free} the conserved current associated with $S^{\text{rest}}_{\text{matter}}$ only is in general not the right thing to do.

We conclude that generally the correct way to define gauge invariant interactions is by replacing $\partial_\mu \to D_\mu$. This is called **minimal coupling**. In section 5.2 we will show that gauge invariance of the combined matter and gauge sector is necessary and sufficient to define a consistent theory with in particular consistent interactions.

4.6 Feynman rules for QED

We are finally in a position to study the interactions of Quantum Electrodynamics (QED), whose Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial \cdot A)^2 + \bar{\psi}(i\gamma \cdot \partial - m_0)\psi - eA_\mu \bar{\psi}\gamma^\mu \psi. \tag{4.100}$$

This theory describes the coupling of the Maxwell $U(1)$ gauge potential to electro-magnetically charged spin $\frac{1}{2}$ particles of free mass m_0.

The Feynman rules for the $U(1)$ gauge field are simple to state:

- The Feynman propagator for the gauge field in Feynman gauge ($\lambda = 1$) can easily be computed as

$$\langle 0|\, TA^\mu(x)A^\nu(y)\,|0\rangle = -\eta^{\mu\nu} D_F^{(0)}(x-y)\Big|_{m_0^2=0} = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-i\,\eta^{\mu\nu}}{p^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{4.101}$$

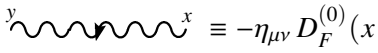
  by plugging in the mode expansion for the quantised spin 1 field in the Heisenberg picture and proceeding as in the scalar field case.

- For completeness we note that for arbitrary λ the propagator is

$$\langle 0|\, TA^\mu(x)A^\nu(y)\,|0\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-i\left(\eta^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}(\frac{1}{\lambda} - 1)\right)}{p^2 + i\epsilon} e^{-ip\cdot(x-y)}. \tag{4.102}$$

This is proven most easily in path-integral quantisation as will be discussed in detail in the course QFT II.

- Graphically we represent the propagator of a gauge field as follows:



$$\equiv -\eta_{\mu\nu} D_F^{(0)}(x-y).$$

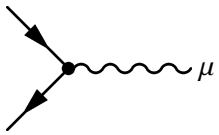
- To determine the Feynman rules we must go through the LSZ analysis for gauge fields. By arguments similar to the ones that lead to the appearance of the spinor polarisation in the spin $1/2$ case one finds that external photon states $|\vec{p},\lambda\rangle$ come with polarisation factors

$$\left.\begin{array}{ll} \epsilon_\mu(\vec{p},\lambda) & \text{for ingoing} \\ \epsilon_\mu^*(\vec{p},\lambda) & \text{for outgoing} \end{array}\right\} |\vec{p},\lambda\rangle \tag{4.103}$$

Here we allowed for complex polarisation vectors.[4]

This leads to the following **Feynman rules of QED**:
To compute the scattering amplitude $i\mathcal{M}_{fi}$ defined in equ. (2.165) of a given process we draw all relevant fully connected, amputated Feynman diagrams to the given order in the coupling constant e and read off $i\mathcal{M}_{fi}$ as follows:

- Each interaction vertex has the form



(with the arrows denoting fermion number flow) and carries a factor $-ie\gamma^\mu$,

- each internal photon line  carries a factor $-\frac{i\eta^{\mu\nu}}{p^2+i\epsilon}$,

- each internal fermion line with the arrow denoting fermion (as opposed to anti-fermion) number flow carries a factor $\frac{i(\gamma\cdot p+m_0)}{p^2-m_0^2+i\epsilon}$,

- momentum conservation is imposed at each vertex,

- we integrate over each (undetermined) internal momentum with the usual measure $\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$,

- each ingoing photon of polarisation λ carries a factor $\epsilon^\mu(\vec{p},\lambda) Z_A^{\frac{1}{2}}$,
  each outgoing photon of polarisation λ carries a factor $\epsilon^{\mu *}(\vec{p},\lambda) Z_A^{\frac{1}{2}}$,

- each ingoing fermion of spin s carries a factor $u_s(\vec{p}) Z_e^{\frac{1}{2}}$,
  each ingoing anti-fermion of spin s carries a factor $\bar{v}_{-s}(\vec{p}) Z_e^{\frac{1}{2}}$,

[4] For instance, sometimes it is useful to consider $\epsilon_\mu^\pm(p) = \pm\frac{1}{\sqrt{2}}(\epsilon_\mu(p,1) \pm i\,\epsilon_\mu(p,2))$ to describe photon states of circular polarisation, which coincide with helicity ± 1 states).

- each outgoing fermion of spin s carries a factor $\bar{u}_s(\vec{p})Z_e^{\frac{1}{2}}$,
  each outgoing anti-fermion of spin s carries a factor $v_{-s}(\vec{p})Z_e^{\frac{1}{2}}$.
- The overall sign of a given diagram is easiest determined directly in the interaction picture. If 2 diagrams are related by the exchange of n fermion lines, then the relative sign is $(-1)^n$. If we are only interested in $|i\mathcal{M}_{fi}|^2$ this is often enough to determine the cross-section.

We will now give some important examples of leading order QED processes:

Electron scattering

Electron-scattering corresponds to the process

$$e^-e^- \to e^-e^-. \tag{4.104}$$

To leading order in the coupling e this process receives contributions from the two following fully connected, amputated diagrams:

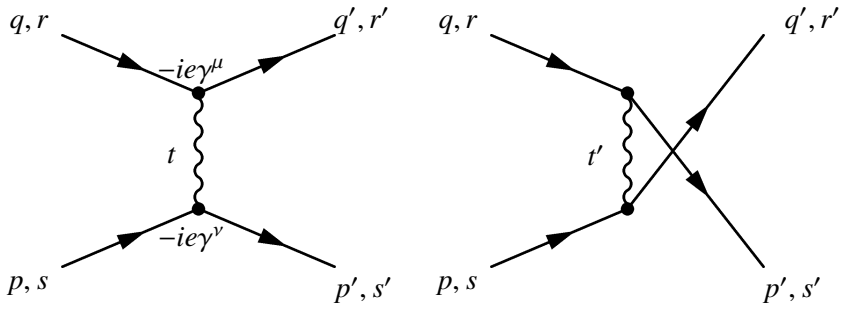


Figure 4.1: Leading order Feynman diagrams for electron scattering.

According to the Feynman rules

$$\begin{aligned} i\mathcal{M}_{fi} = (ie)^2\Big[&\bar{u}_{r'}(q')\gamma^\mu u_r(q)\frac{-i\eta_{\mu\nu}}{t^2+i\epsilon}\bar{u}_{s'}(p')\gamma^\nu u_s(p) \\ &-\bar{u}_{s'}(p')\gamma^\mu u_r(q)\frac{-i\eta_{\mu\nu}}{t'^2+i\epsilon}\bar{u}_{r'}(q')\gamma^\nu u_s(p)\Big], \end{aligned} \tag{4.105}$$

where

$$q-t-q'=0 \;\Rightarrow\; t=q-q' \text{ and } t+p-p'=0 \;\Rightarrow\; p'=p+q-q' \tag{4.106}$$

in the first diagram and

$$t'=q-p' \text{ and } q'=q-p'+p \tag{4.107}$$

in the second diagram. Note the relative minus sign between both diagrams due to the crossing fermion line!

Electron-positron annihilation

Electron-positron annihilation is the process

$$e^+e^- \to 2\gamma. \tag{4.108}$$

The two leading-order diagrams are

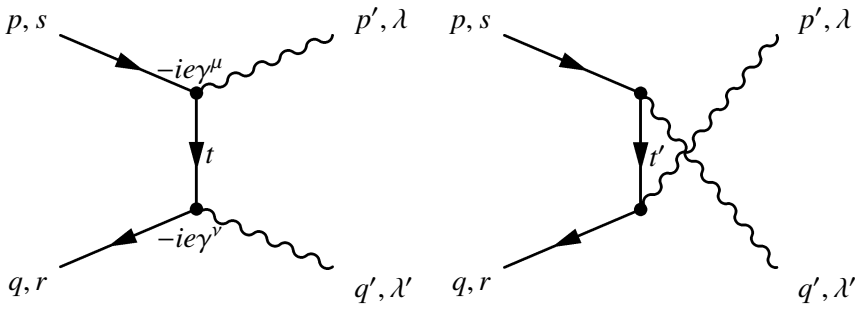


Figure 4.2: Electron-positron annihilation leading-order diagrams.

Then the Feynman rules yield

$$\begin{aligned} i\mathcal{M}_{fi} =& (ie)^2\Big[\bar{v}_{-r}(q)\gamma^\nu \frac{i(\gamma t - m_0)}{t^2 - m_0^2 + i\epsilon}\gamma^\mu u_s(p)\epsilon_\nu^*(q',\lambda')\epsilon_\mu^*(p',\lambda) \\ &+ \bar{v}_{-r}(q)\gamma^\nu \frac{i(\gamma t' + m_0)}{t'^2 - m_0^2 + i\epsilon}\gamma^\mu u_s(p)\epsilon_\nu^*(p',\lambda)\epsilon_\mu^*(q',\lambda')\Big], \end{aligned} \tag{4.109}$$

where momentum conservation implies

$$t = p - p', \quad t' = p - q', \quad q' = p - p' + q, \quad p' = p - q' + q. \tag{4.110}$$

e^+e^-- scattering (Bhabha scattering)

The process

$$e^+e^- \to e^+e^- \tag{4.111}$$

is described by the leading order diagram 4.3.

Compton scattering

Compton scattering corresponds to

$$e^- + \gamma \to e^- + \gamma \tag{4.112}$$

with leading order diagram 4.4.

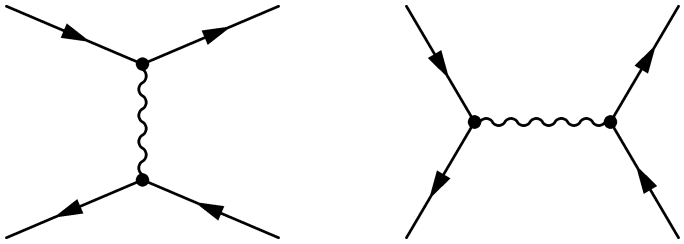


Figure 4.3: Bhabha scattering at leading order.

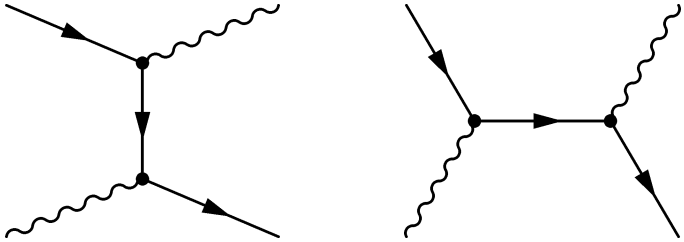


Figure 4.4: Compton scattering at leading order.

Non-linearities

As an interesting new quantum effect, loop diagrams induce an interaction between photons. For example, at one loop two photons scatter due to a process of the form

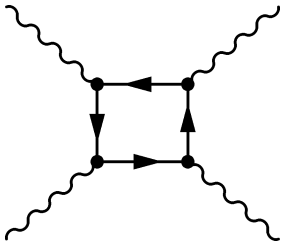


Figure 4.5: Photon scattering is not allowed in classical Electrodynamics, but it is in QED.

Such effects are absent in the classical theory, where light waves do not interact with each other due to the linear structure of the classical theory. It is a fascinating phenomenon that such QED effects break the linearity of classical optics. In high intensity laser beams, where quantum effects are quantitatively relevant, such non-linear optics phenomena can indeed be observed.

4.7 Recovering Coulomb's potential

As we have seen QED interactions are mediated by the exchange of gauge bosons. It is an interesting question to determine the physical potential $V(\vec{x})$ induced by the exchange of such gauge bosons. This will teach us how the concept of classical forces emerges from the QFT framework of scattering.

To derive $V(\vec{x})$ the key idea is to consider the non-relativistic limit of the electron scattering process

$$e^- + e^- \to e^- + e^- \tag{4.113}$$

with distinguishable out-states corresponding to the Feynman diagram 4.6.

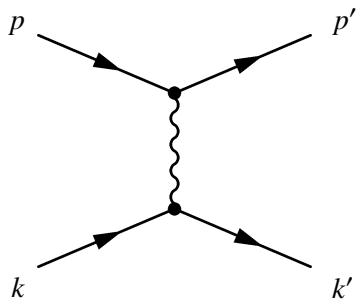


Figure 4.6: Electron scattering with distinguishable out-states.

This is then compared to the non-relativistic scattering process $|\vec{p}\rangle \to |\vec{p}'\rangle$ of two momentum eigenstates off a potential $V(\vec{x})$.

- In non-relativistic quantum mechanics, the scattering amplitude $\mathcal{A}(|\vec{p}\rangle \to |\vec{p}'\rangle)$ for scattering of an incoming momentum eigenstate $|\vec{p}\rangle$ to an outgoing momentum eigenstate $|\vec{p}'\rangle$ in the presence of a scattering potential V is computed in the interaction picture as

$$\mathcal{A}(|\vec{p}\rangle \to |\vec{p}'\rangle) - 1 = \langle \vec{p}' | \, e^{-i\int_{-\infty}^{\infty} \mathrm{d}t' V_I(t')} \, |\vec{p}\rangle - 1 \cong -i \langle \vec{p}' | \int_{-\infty}^{\infty} \mathrm{d}t' V_I(t') \, |\vec{p}\rangle + \ldots,$$

  where we take the potential to be constant in time. As derived in standard textbooks on Quantum Mechanics this equals to first order

$$\mathcal{A}(|\vec{p}\rangle \to |\vec{p}'\rangle) - 1 = (2\pi)\delta(E_p - E_{p'})(-i) \, \langle \vec{p}' | \, V \, |\vec{p}\rangle \, . \tag{4.114}$$

  This result goes by the name **Born approximation** and reads more explicitly in position space

$$\boxed{\mathcal{A}(|\vec{p}\rangle \to |\vec{p}'\rangle) - 1 = (2\pi)\delta(E_p - E_{p'})(-i) \int \mathrm{d}^3 r \, V(\vec{r}) \, e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}}.} \tag{4.115}$$

- This is to be compared with the non-relativistic limit of

$$\int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}_0} \, \langle p', k' | \, S \, |p, k\rangle \Big|_{\text{conn.}} , \tag{4.116}$$

where $\langle p', k' | S | p, k \rangle \Big|_{\text{conn.}}$ is the connected part of the S-matrix element of the scattering process (4.113). Here the idea is to identify the electron with momentum p and p' with the scattering states in the Quantum Mechanics picture and to view the other electron as a fixed target at $\vec{r}_0$ off which $|\vec{p}\rangle$ scatters. The localisation of the fixed target at $\vec{r}_0$ in space is achieved by integrating over all Fourier modes, hence explaining the factor $\int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}_0}$. Without loss of generality we will set $\vec{r}_0 \equiv 0$ in the sequel. Recalling the connection between the S-matrix element and the scattering amplitude $i\mathcal{M}$, for which our Feynman rules are formulated, we find

$$\begin{aligned} \langle p', k' | S | p, k \rangle \Big|_{\text{conn.}} &= (2\pi)^4 \delta^{(4)}(p_f - p_i)(-ie)^2 \\ &\quad \times \Big[\bar{u}_{r'}(p') \gamma^\mu u_r(p) \frac{-i\eta_{\mu\nu}}{(p-p')^2 + i\epsilon} \bar{u}_{s'}(k') \gamma^\nu u_s(k) \Big]. \end{aligned} \tag{4.117}$$

- In the the non-relativistic limit we have $m_0 \gg |\vec{p}|$ and $m_0 \gg |\vec{k}|$ and thus approximate $p \cong (m_0, \vec{p}) + O(\vec{p}^2)$. Then

$$u_r(\vec{p}) = \begin{pmatrix} \sqrt{\sigma^\mu p_\mu}\, \xi_r \\ \sqrt{\bar{\sigma}^\mu p_\mu}\, \xi_r \end{pmatrix} \to \begin{pmatrix} \sqrt{\sigma^0 m_0}\, \xi_r \\ \sqrt{\sigma^0 m_0}\, \xi_r \end{pmatrix} = \sqrt{m_0} \begin{pmatrix} \xi_r \\ \xi_r \end{pmatrix} \tag{4.118}$$

and

$$\bar{u}_{r'}(\vec{p}')\gamma^\mu u_r(\vec{p}) \to \begin{cases} 2m_0\, \delta_{rr'} & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.119}$$

Furthermore $(p-p')^2 = -|\vec{p} - \vec{p}'|^2 + O(|\vec{p}|^4)$ and therefore in the non-relativistic limit

$$\langle p', k' | S | p, k \rangle \Big|_{\text{conn.}} \simeq -i \frac{e^2}{|\vec{p} - \vec{p}'|^2 - i\epsilon} (2m)^2 \delta(E_f - E_i)(2\pi)^4 \delta^{(3)}(\vec{p}_f - \vec{p}_i). \tag{4.120}$$

The factor $(2m)^2$ is due to the different normalisation of momentum eigenstates in QFT and in QM and must therefore be neglected in comparing the respective expressions for the amplitude.

Putting all pieces together we can now identify

$$\int \mathrm{d}^3 r\, V(\vec{r})\, e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}} = \frac{e^2}{|\vec{p} - \vec{p}'|^2 - i\epsilon}. \tag{4.121}$$

Thus the Coulomb potential is

$$\begin{aligned} V(\vec{r}) &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{e^2}{\vec{q}^2 - i\epsilon} e^{i \overbrace{\vec{q}\cdot\vec{r}}^{\equiv qr\cos(\theta)}} = \frac{e^2}{4\pi^2} \int_0^\infty \mathrm{d}q \frac{q^2}{q^2 - i\epsilon} \frac{e^{iqr} - e^{-iqr}}{iqr} \\ &= \frac{e^2}{4\pi^2 ir} \int_{-\infty}^{\infty} \mathrm{d}q \frac{q}{(q + \sqrt{i\epsilon})(q - \sqrt{i\epsilon})} e^{iqr}. \end{aligned} \tag{4.122}$$

Note that $\sqrt{i\epsilon} = e^{i\pi/4}\sqrt{\epsilon}$ lies in the upper complex half-plane. We interpret the integral as a complex contour integral in the complex upper half-plane because we can close the contour in the upper half-plane as the contribution above the real axis vanishes in the limit $|q| \to \infty$. We then pick up the residue at $q = \sqrt{+i\epsilon}$ and afterwards take the limit $\epsilon \to 0$. This leads to

$$\boxed{V(\vec{r}) = \frac{e^2}{4\pi^2 ir}\, 2\pi i\, \frac{\sqrt{i\epsilon}}{2\sqrt{i\epsilon}}\, e^{i\sqrt{i\epsilon}r}\Big|_{\epsilon=0} = \frac{e^2}{4\pi r}.} \tag{4.123}$$

Since V is positive the potential for scattering of two electrons is repulsive as expected. If we replace one e^- by e^+ and consider instead the process

$$e^- + e^+ \to e^- + e^+, \tag{4.124}$$

this amounts to replacing

$$\bar{u}_{r'}(p')\gamma^\mu u_r(p) \;\text{ by }\; \bar{v}_{r'}(p')\gamma^\mu v_r(p) \to \begin{cases} 2m_0\delta_{rr'} & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.125}$$

However, a careful analysis of the scattering amplitude in the interaction picture shows that one picks up one relative minus sign in the amplitude. Indeed, establishing the absolute sign of the amplitude will be the task of Assignment 11. Thus we confirm the expected result that the potential mediated by exchange of a spin-1 particle yields an attractive potential for oppositely charged particles and a repulsive potential for identically charged particles.

4.7.1 Massless and massive vector fields

It is instructive to compare the interaction of massless and massive vector fields:

- The Coulomb potential due to **exchange of a massless spin-1 particle leads to a long-range force as the potential dies off only like** $1/r$. If on the other hand the exchanged spin-1 particle is massive, the interaction is short-ranged. To see this suppose the photon has mass μ. All that changes is a mass term in the phton propagator. Thus

$$V(\vec{r}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{e^2}{\vec{q}^2 + \mu^2 - i\epsilon} e^{i\vec{q}\cdot\vec{r}} = \frac{e^2}{4\pi r} e^{-\mu r}. \tag{4.126}$$

  Therefore **massive vector bosons lead to short-range forces of range** $\sim \frac{1}{\mu}$.

  While the vector boson of QED is massless[5], the weak nuclear forces in the Standard Model are mediated by massive vector bosons (the three massive vector bosons W^+, W^- and Z of spontaneously broken $SU(2)$ gauge symmetry of mass of order 100 GeV). So this effect is

[5] A Yukawa-type deviation of the electromagnetic potential would be directly measurable and constrains the mass of the photon to be below 10^{-14} eV, see E. R. Williams, J. E. Faller, and H. A. Hill, Phys. Rev. Lett. 26, 721-724 (1971). In addition, a variety of cosmological and astrophysical constraints imply that the photon mass is at best 10^{-18}eV, as reviewed e.g. in http://arxiv.org/pdf/0809.1003v5.pdf

indeed highly relevant in particle physics and explains why in everyday physics at distances bigger than $(100\,\mathrm{GeV})^{-1}$ only the electromagnetic force can be experienced.[6]

- The concept of forces mediated by exchange bosons is not restricted to spin-1 theories.

  If we replace the vector boson of QED by a scalar boson ϕ of mass μ we arrive at **Yukawa theory** defined by

$$\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m_0)\psi - \underbrace{e\phi\bar{\psi}\psi}_{\text{Yukawa int.}} . \tag{4.127}$$

  The Feynman rules are very similar to those in QED except for two important changes: First, the interaction vertex carries no γ^μ factor and second the scalar boson propagator must be modified from

$$\frac{-i\eta_{\mu\nu}}{p^2-\mu^2+i\epsilon} \;\rightarrow\; \frac{i}{p^2-\mu^2+i\epsilon}. \tag{4.128}$$

  This amounts to a crucial sign change in the non-relativistic limit because

$$\frac{-i\eta_{00}}{-|\vec{p}|^2-\mu^2+i\epsilon} \;\rightarrow\; \frac{i}{-|\vec{p}|^2-\mu^2+i\epsilon}. \tag{4.129}$$

  Combining these two changes yields the **universally attractive Yukawa potential**

$$V(r) = -\frac{e^2}{4\pi r}e^{-\mu r}. \tag{4.130}$$

- Perturbative gravity can be understood at the level of field theory as a theory of massless spin-2 particles, the gravitons, whose exchange likewise yields a universally attractive force. Since the gravitational potential is of the form $1/r$ (at least for conventional Einstein gravity), gravitons must be massless.[7]

To summarise the potential induced by the exchange of bosons of different spin acts on fermions (f) and anti-fermions ($\bar{f}$) as follows:

	ff	$f\bar{f}$	$\bar{f}\bar{f}$
Spin 0 (Yukawa theory)	attractive	attractive	attractive
Spin 1 (Gauge theory)	repulsive	attractive	repulsive
Spin 2 (Gravity)	attractive	attractive	attractive

[6] The remaining Standard Model forces mediated by the eight massless gluons of $SU(3)$ gauge theory is also short-ranged, but this is because of confinement.

[7] Current constraints imply that the graviton mass must be smaller than $10^{-20}eV$, see http://arxiv.org/pdf/0809.1003v5.pdf.

Chapter 5

Quantum Electrodynamics

5.1 QED process at tree-level

As an example for a typical tree-level QED process we compute the differential cross-section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ for the scattering $e^+e^- \to \mu^+\mu^-$. We describe both the electron/positron and the muon/anti-muon by a Dirac spinor field of respective free mass m_e and m_μ which we couple to the $U(1)$ gauge field.

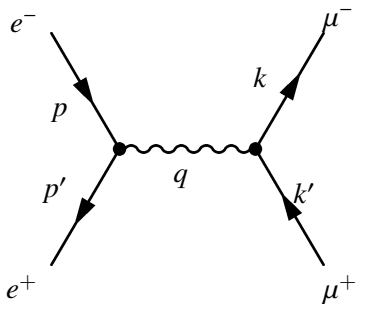


Figure 5.1: $e^+e^- \to \mu^+\mu^-$ reaction.

In this protypical example and in many similar processes one proceeds as follows:

5.1.1 Feynman rules for in/out-states of definite polarisation

Apply the rules from the previous chapter and obtain:

$$i\mathcal{M}(p,s;p',-s' \to k,r;k',-r') = (ie)^2 \bar{v}_{s'}(p')\gamma^\mu u_s(p)\frac{-i\eta_{\mu\nu}}{q^2+i\epsilon}\bar{u}_r(k)\gamma^\nu v_{r'}(k'), \tag{5.1}$$

where $q = p + p' = k + k'$. Thus

$$i\mathcal{M} = \frac{ie^2}{q^2}\bar{v}_{s'}(p')\gamma^\mu u_s(p)\bar{u}_r(k)\gamma_\mu v_{r'}(k'). \tag{5.2}$$

5.1.2 Sum over all spin and polarisation states

Often we do not keep track of the polarisation states of in and out states but are interested in the unpolarised amplitude-square

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'}\left|i\mathcal{M}(p,s;p',-s'\to k,r;k',-r')\right|^2, \tag{5.3}$$

where $\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}$ averages over the initial state polarisation as is appropriate if the incoming beam is not prepared in a polarisation eigenstate. The sum $\sum_r\sum_{r'}$ over the final state polarisations is required in addition if the detector is blind to polarisation. With

$$(\bar v_{s'}\gamma^\mu u_s)^* = u_s^\dagger \underbrace{\gamma^{\mu\dagger}\gamma^{0\dagger}}_{=\gamma^0\gamma^\mu} v_{s'} = \bar u_s\gamma^\mu v_{s'} \tag{5.4}$$

one finds

$$\frac{1}{4}\sum_{\text{Spins}}|\mathcal{M}|^2 = \frac{1}{4}\frac{e^2}{q^4}\left[\sum_{s,s',r,r'}(\bar u_s\gamma^\mu v_{s'})(\bar v_{s'}\gamma^\nu u_s)(\bar v_{r'}\gamma_\mu u_r)(\bar u_r\gamma_\nu v_{r'})\right]. \tag{5.5}$$

Now we want to make use of the completeness relations

$$\begin{aligned}\sum_s u_s(\vec p)\bar u_s(\vec p) &= \gamma\cdot p + m_e\\ \sum_s v_s(\vec p)\bar v_s(\vec p) &= \gamma\cdot p - m_\mu.\end{aligned} \tag{5.6}$$

To do so it is first useful to make the spinor indices explicit[1], e.g.

$$\bar u_s\gamma^\mu v_{s'} = \sum_{A,B}(\bar u_s)_A\gamma^{\mu A}{}_B v_{s'}{}^B, \tag{5.7}$$

which shows that the expression can be reordered as

$$\begin{aligned}\frac{1}{4}\sum_{\text{Spins}}|\mathcal{M}|^2 =& \frac{1}{4}\frac{e^2}{q^4}\Big[\sum_{s,s',r,r'} u_s(p)^D\bar u_s(p)_A\gamma^{\mu A}{}_B v_{s'}(p')^B\bar v_{s'}(p')_C\gamma^{\nu C}{}_D\\ &\times v_{r'}(k')^H\bar v_{r'}(k')_E\gamma_\mu{}^E{}_F u_r(k)^F\bar u_r(k)_G\gamma_\nu{}^G{}_H\Big]\\ =& \frac{1}{4}\frac{e^2}{q^4}\Big[\operatorname{tr}\Big((\gamma\cdot p - m_e)\gamma^\mu(\gamma\cdot p + m_e)\gamma^\nu\Big)\\ &\times\operatorname{tr}\Big((\gamma\cdot k' + m_\mu)\gamma_\mu(\gamma\cdot k - m_\mu)\gamma_\nu\Big)\Big].\end{aligned} \tag{5.8}$$

In order to proceed further we need to evaluate the traces over the γ matrices.

[1] Recall that the indices μ, ν are completely unrelated to the spinor indices, c.f. section 3.2.

5.1.3 Trace identities

The following identities are very important to proceed further. We will prove them in general fashion on exercise sheet 10. The only thing to do is: insert a $\mathbb{1}$ cleverly, e.g. $\gamma^5\gamma^5$ and use the properties of the Clifford algebra as well as the cyclicity of the trace.

- $\mathrm{tr}\gamma^\mu = 0$, because:

$$\mathrm{tr}\gamma^\mu = \mathrm{tr}\underbrace{\gamma^5\gamma^5}_{\equiv\mathbb{1}}\gamma^\mu = -\mathrm{tr}\gamma^5\gamma^\mu\gamma^5 = -\mathrm{tr}\gamma^\mu\gamma^5\gamma^5 = -\mathrm{tr}\gamma^\mu. \tag{5.9}$$

- The trace of an odd number of γ-matrices vanishes, since

$$\mathrm{tr}\gamma^{\mu_1}...\gamma^{\mu_n} = \mathrm{tr}\gamma^5\gamma^5\gamma^{\mu_1}...\gamma^{\mu_n} = (-1)^n\mathrm{tr}\gamma^\mu_1...\gamma^{\mu_n}. \tag{5.10}$$

- $\mathrm{tr}\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$, because

$$\mathrm{tr}\gamma^\mu\gamma^\nu = \mathrm{tr}(2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2\eta^{\mu\nu}\mathrm{tr}\mathbb{1} - \underbrace{\mathrm{tr}(\gamma^\nu\gamma^\mu)}_{=\mathrm{tr}\gamma^\mu\gamma^\nu}. \tag{5.11}$$

- Similarly one finds

$$\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\Big[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\Big]. \tag{5.12}$$

- $\mathrm{tr}\gamma^5 = 0$ since

$$\mathrm{tr}\gamma^5 = \mathrm{tr}\gamma^0\gamma^0\gamma^5 = -\mathrm{tr}\gamma^0\gamma^5\gamma^0 = -\mathrm{tr}\gamma^5. \tag{5.13}$$

- Likewise one can prove

$$\mathrm{tr}\gamma^5\gamma^\mu = \mathrm{tr}\gamma^5\gamma^\mu\gamma^\nu = \mathrm{tr}\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma = 0. \tag{5.14}$$

- $\mathrm{tr}\gamma^5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = -4i\epsilon^{\alpha\beta\gamma\delta}$, because the result must be antisymmetric in all indices and in particular

$$\mathrm{tr}\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3 = -i\,\mathrm{tr}\,\gamma^5\gamma^5 = -4i. \tag{5.15}$$

- Further useful identities are

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu.\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= \ 4\eta^{\nu\sigma},\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu.\end{aligned} \tag{5.16}$$

Applied to the present case these identities yield

$$\begin{aligned}&\mathrm{tr}\Big[(\gamma^\rho p_\rho - m_e)\gamma^\mu(\gamma^\sigma p'_\sigma + m_e)\gamma^\nu\Big]\\ &\qquad= 4p_\rho p'_\sigma\Big(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\rho\sigma}\eta^{\mu\nu} + \eta^{\rho\nu}\eta^{\mu\sigma}\Big)\\ &\qquad\quad - m_e^2 4\eta^{\mu\nu} + \mathrm{tr}(\text{odd number of }\gamma\text{' s})\times m_e\times ...\\ &\qquad= 4\Big[p^\mu p'^\nu + p^\nu p'^\mu - (p\cdot p' + m_e^2)\eta^{\mu\nu}\Big]\end{aligned} \tag{5.17}$$

and similarly

$$\mathrm{tr}\Big[(\gamma\cdot k'+m_\mu)\gamma_\mu(\gamma\cdot k-m\mu)\gamma_\nu\Big]=4\Big[k'_\mu k_\nu+k'_\nu k_\mu-(k\cdot k'+m_\mu^2)\eta_{\mu\nu}\Big]. \tag{5.18}$$

Since the ratio of m_μ and m_e is about 200 we can drop m_e. Then altogether after a few cancellations

$$\boxed{\frac{1}{4}\sum_{\text{Spins}}|\mathcal{M}|^2=8\frac{e^4}{q^4}\Big[(p\cdot k)(p'\cdot k')+(p'\cdot k)(p\cdot k')+(p\cdot p')m_\mu^2\Big].} \tag{5.19}$$

5.1.4 Centre-of-mass frame

Switch to the c.o.m. frame and rotate the coordinate frame such that p points in $\hat{z}$-direction and p' in $-\hat{z}$-direction (see Figure 5.2).

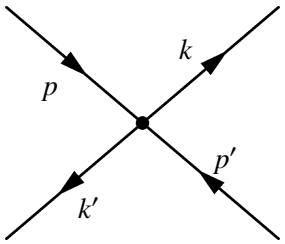


Figure 5.2: Centre-of-mass frame, p in $\hat{z}$-direction.

Introducing the angle θ between p' and k and taking the relativistic limit, i.e. $m_e^2\ll|\vec{p}|^2$, yields

$$p=\begin{pmatrix}E\\E\hat{z}\end{pmatrix},\quad p'=\begin{pmatrix}E\\-E\hat{z}\end{pmatrix},\ k=\begin{pmatrix}E\\\vec{k}\end{pmatrix},\qquad \vec{k}\cdot\hat{z}=|\vec{k}|\cos\theta. \tag{5.20}$$

Therefore

$$\begin{aligned}q^2&=(p+p')^2=2p\cdot p'=4E^2,\\ p\cdot k&=E^2-E|\vec{k}|\cos\theta=p'\cdot k',\\ p\cdot k'&=E^2+E|\vec{k}|\cos\theta=p'\cdot k.\end{aligned} \tag{5.21}$$

Plugging this in and eliminating $|\vec{k}|^2$ using $|\vec{k}|^2=E^2-m_\mu^2$ yields

$$\boxed{\frac{1}{4}\sum_{\text{Spins}}|\mathcal{M}|^2=e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].} \tag{5.22}$$

5.1.5 Cross-section

To this end recall the general formula for a 2-2 scattering event. For this we had derived the expression

$$\mathrm{d}\sigma=\frac{(2\pi)^4}{4E_pE_{p'}|v_p-v_{p'}|}\mathrm{d}\Pi_2\delta^{(4)}(p+p'-k-k')|\mathcal{M}_{fi}|^2 \tag{5.23}$$

with

$$\mathrm{d}\Pi_2 = \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{2E_k 2E_{k'}}. \tag{5.24}$$

Note that no factor of $\frac{1}{2}$ is required here since μ^+ and μ^- are distinguishable. In the c.o.m. frame we generally have

$$\mathrm{d}\Pi_2(2\pi)^4\delta^{(4)}(p+p'-k-k') = \frac{\mathrm{d}|\vec{k}||\vec{k}|^2\mathrm{d}\Omega}{(2\pi)^3 4E_k E_{k'}}\delta(E_{\mathrm{com}}-E_k-E_{k'}) \tag{5.25}$$

with $E_k = \sqrt{|\vec{k}|^2+m_k^2}$ and $E_{k'} = \sqrt{|\vec{k}|^2+m_{k'}^2}$ and

$$\mathrm{d}|\vec{k}|\delta(E_{\mathrm{com}}-E_k-E_{k'}) = \left(\frac{|\vec{k}|}{E_k}+\frac{|\vec{k}|}{E_{k'}}\right)^{-1}. \tag{5.26}$$

Then, altogether

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{4E_pE_{p'}|v_p-v_{p'}|}\frac{|\vec{k}|}{16\pi^2E_{\mathrm{com}}}|\mathcal{M}_{fi}|^2.} \tag{5.27}$$

Note that, if all 4 particles had equal masses and the outstates were indistinguishable, this would correctly reduce to the famous expression encountered earlier

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2!}\frac{1}{64\pi^2}\frac{1}{s}|\mathcal{M}_{fi}|^2 \tag{5.28}$$

with $s = E_{\mathrm{com}}^2$. Applied to the present case (5.27) yields, with

$$E_p = E_{p'} = E = \frac{1}{2}E_{\mathrm{com}}\,, \quad |v_p - v_{p'}| = 2, \tag{5.29}$$

the differential cross-section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{32\pi^2}\frac{1}{E_{\mathrm{com}}^2}\frac{|\vec{k}|}{E_{\mathrm{com}}}\frac{1}{4}\sum_{\mathrm{Spins}}|\mathcal{M}_{fi}|^2. \tag{5.30}$$

Introducing the fine-structure constant

$$\boxed{\alpha := \frac{e^2}{4\pi} \sim \frac{1}{137}} \tag{5.31}$$

leaves us with

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_{\mathrm{com}}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]} \tag{5.32}$$

for the differential cross-section. Thus for the total cross-section

$$\sigma = \frac{4\pi\alpha^2}{3E_{\mathrm{com}}}\sqrt{1+\frac{m_\mu}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right). \tag{5.33}$$

The first term reflects the purely kinematic and thus universal energy dependence, while the second term represents the correction due to specifics of the QED interaction and is thus characteristic of the concrete dynamics involved in this process.

5.2 The Ward-Takahashi identity

Consider a QED amplitude $\mathcal{M}(k)$ involving an external photon of momentum k^μ with $k^2 = 0$ and polarisation $\xi^\mu(k)$. We thus can write the scattering amplitude as

$$\mathcal{M}(k) = \xi^\mu(k)\mathcal{M}_\mu(k). \tag{5.34}$$

Then the **Ward-Takahashi identity for QED** is the statement that

$$\boxed{k^\mu \mathcal{M}_\mu(k) = 0.} \tag{5.35}$$

To appreciate its significance recall from Gupta-Bleuler quantisation that the constraint

$$\partial \cdot A^+ |\psi_{\text{phys}}\rangle = 0 \tag{5.36}$$

implies $\zeta^\mu k_\mu = 0$ for states $|k,\zeta\rangle \in \mathcal{H}_{\text{phys}}$. This left 2 positive norm transverse polarisations ζ_T and 1 zero norm polarisation $\zeta_s = k$. (5.35) proves that the unphysical zero-norm polarization state $|k,\zeta_s\rangle$ decouples from the S-matrix as an external state, as claimed.
(5.35) follows from a more general from of the Ward identity. Consider a theory with arbitrary spin fields $\phi_a(x)$ with a global continuous symmetry

$$\phi_a(x) \to \phi_a(x) + \epsilon\delta\phi_a(x), \qquad \epsilon \in \mathbb{R}, \tag{5.37}$$

and a conserved Noether current $j^\mu(x)$ such that classically

$$\partial_\mu j^\mu = 0 \tag{5.38}$$

holds on-shell. Then in the quantum theory the general **Ward-Takahashi-identity** is a statement about current conservation inside a general n-point function:

$$\boxed{\begin{aligned} 0 = \partial_\mu \langle\Omega| T\, j^\mu(x)\phi_{a_1}(x_1)...\phi_{a_n}(x_n) |\Omega\rangle \\ + i\sum_{j=1}^{n} \langle\Omega| \phi_{a_1}...\hat{\phi}_{a_j}(x_j)\delta\phi_{a_j}(x_j)\delta^{(4)}(x-x_j)...\phi_{a_n}(x_n) |\Omega\rangle, \end{aligned}} \tag{5.39}$$

where $\hat{\phi}_{a_j}(x_j)$ is omitted. (5.39) can be proven as follows:

- For definiteness and simplicity consider a scalar theory. For the free theory, the classical equation of motion is

$$\frac{\delta S}{\delta\phi(x)} := \frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\phi(x))} = -(\partial^2 + m^2)\phi(x) = 0. \tag{5.40}$$

  This equation is satisfied by the Heisenberg quantum fields as an operator equation.

- With the help of this operator equation, one finds that

$$(\partial_x^2 + m^2) i \langle 0| T\phi(x)\phi(x_1) |0\rangle = \delta^{(4)}(x - x_1) \tag{5.41}$$

for a 2-point-function. The δ-distribution occurs due to the action of ∂_{x^0} on $\Theta(x - x_0)$ appearing in the time-ordering prescription symbolized by T. More generally one finds

$$\begin{aligned}(\partial_x^2 + m^2) i \langle 0| T\phi(x)\phi(x_1)...\phi(x_n) |0\rangle \\ = \sum_{j=1}^{n} \langle 0| T\phi(x_1)...\hat{\phi}(x_j)\delta^{(4)}(x - x_j)...\phi(x_n) |0\rangle .\end{aligned} \tag{5.42}$$

This can be written as the **Schwinger-Dyson equation**

$$\boxed{\langle 0| T\frac{\delta S}{\delta\phi(x)}\phi(x_1)...\phi(x_n) |0\rangle = i\sum_{j=1}^{n} \langle 0| T\phi(x_1)...\hat{\phi}(x_j)\delta^{(4)}(x - x_j)...\phi(x_n) |0\rangle .} \tag{5.43}$$

- By Noether's theorem, if $\phi(x) \to \phi(x) + \epsilon\delta\phi(x)$ is a global continuous symmetry, then the Noether current enjoys

$$\partial_\mu j^\mu = -\left(\frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\right)\delta\phi(x) \equiv -\frac{\delta S}{\delta\phi(x)}\delta\phi(x) \tag{5.44}$$

off-shell. Plugging this into the Schwinger-Dyson equation yields:

$$\begin{aligned}\partial_\mu \langle 0|T j^\mu(x)\phi(x_1)...\phi(x_n) |0\rangle \\ + i\sum_{j=1}^{n} \langle 0| T\phi(x_1)...\hat{\phi}(x_j)\delta\phi(x)\delta^{(4)}(x - x_j)...\phi(x_n) |0\rangle = 0.\end{aligned} \tag{5.45}$$

- In interacting theories the reasoning goes through - all that changes is that the equations of motion involve extra polynomial terms due to the interactions, which however do not alter the conclusions. This yields the corresponding statements about the full correlation functions $\langle\Omega| ... |\Omega\rangle$.

The Schwinger-Dyson equation and the Ward-identity show that the classical equation of motion and current conservation hold inside the correlation functions only up to so-called **contact terms**, which are precisely the extra terms we pick up if the insertion point x of the operator $\frac{\delta S}{\delta\phi(x)}$ or $\partial_\mu j^\mu(x)$ coincides with the insertion point x_j of one of the other fields inside the correlator.

Caveat:
It is important to be aware that the n-point correlator $\langle\Omega| ... |\Omega\rangle$ involves in general divergent loop-diagrams to be discussed soon. The Ward identity only holds if the regularisation required to define these divergent intergrals respects the classical symmetry. We will see that for QED such regulators can be found. On the other hand, if no regulator exists that respects the Ward-identity for a classical

symmetry, this symmetry is **anomalous** - it does not hold at the quantum level. Such anomalies will be studied in great detail in QFT II.

It remains to deduce (5.35) from (5.39):

- Following LSZ, the scattering amplitude $\mathcal{M}(k)$ involving an external photon $|k^\mu, \xi^\mu(k)\rangle$ and n further particles is given by

$$\langle f|\, i\rangle = iZ_A^{-1/2}\xi^\mu(k)\int \mathrm{d}^4x\, e^{-ik\cdot x}\partial_x^2 \ldots \langle\Omega|\, TA_\mu(x) \underbrace{\ldots}_{(n \text{ other fields})} |\Omega\rangle \tag{5.46}$$

  In Feynman gauge the full interacting QED Lagrangian reads

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - A_\mu j^\mu + \bar\psi(i\gamma\cdot\partial - m_0)\psi, \qquad j^\mu = e\bar\psi\gamma^\mu\psi. \tag{5.47}$$

  The classical equation of motion for A^μ is

$$\partial^2 A^\mu = j^\mu \tag{5.48}$$

  and therefore

$$\partial_x^2 \langle\Omega|\, TA_\mu(x)\ldots|\Omega\rangle = \langle\Omega|\, T\, j_\mu(x)\ldots|\Omega\rangle + \text{contact terms}, \tag{5.49}$$

  where the contact terms include $(n-1)$ fields inside $\langle\Omega|\ldots|\Omega\rangle$. Being correlators only of $(n-1)$ fields, these contact terms cannot have precisely n poles in the momenta of the n fields and thus do not contribute to $\langle f|\, i\rangle$ according to the LSZ formalism.

- Now we are left with

$$\langle f|\, i\rangle = i\, Z_A^{-1/2}\xi^\mu \int \mathrm{d}^4x e^{-ik\cdot x}\, \langle\Omega|\, T\, j_\mu(x)\ldots|\Omega\rangle\,. \tag{5.50}$$

  For $\xi^\mu = k^\mu$ we find

$$\begin{aligned} k^\mu \int \mathrm{d}^4x\, e^{-ik\cdot x}\, \langle\Omega|\, T\, j_\mu(x)\ldots|\Omega\rangle &= i\int \mathrm{d}^4x\, (\partial^\mu e^{-ik\cdot x}\, \langle\Omega|\, T\, j_\mu(x)\ldots|\Omega\rangle \\ &= -\,i\int \mathrm{d}^4x\, e^{-ik\cdot x}\partial^\mu\, \langle\Omega|\, T\, j_\mu(x)\ldots|\Omega\rangle\,, \end{aligned} \tag{5.51}$$

  where we used that surface terms vanish (because, as in LSZ, we are really having suitable wave-packets in mind). According to (5.39) this is

$$0 + \text{contact terms}. \tag{5.52}$$

  Also in this case the contact terms do not contribute to $\langle f|\, i\rangle$ because we are essentially trading one $A_\mu(x)$ field against one matter field $\delta\varphi(x)$ inside the correlator so that the resulting pole structure is not of the right form, in and (5.35) is proven.

5.2.1 Relation between current conservation and gauge invariance

The proof of $k^\mu \mathcal{M}_\mu(k) = 0$ only requires that $A_\mu(x)$ couples to a conserved current in the sense that

$$S = S_A^0[A] + S_{\text{int}}[A,\phi] + S_{\text{matter}}^{\text{rest}}[\phi] \tag{5.53}$$

with

$$\frac{\delta S_{\text{int}}[A,\phi]}{\delta A_\mu} = -j^\mu \tag{5.54}$$

the Noether current associated with a global continuous symmetry of the full interacting action S.

- For a **massless** vector theory, which, as a free theory, must always be gauge invariant, (5.54) is in fact equivalent to invariance of the theory under combined gauge transformations of the vector and the matter sector.

  To prove that (5.54) implies gauge invariance, our point of departure is the existence of a global continuous symmetry $\phi(x) \to \phi(x) + \epsilon\delta\phi(x)$ with $\epsilon \in \mathbb{R}$ constant, which leaves the full action S invariant. If we perform instead a local transformation $\phi(x) \to \phi(x) + \epsilon(x)\delta\phi(x)$ with varying $\epsilon(x)$, then δS is in general no longer zero. Since it vanishes for constant ϵ, δS must be proportional (to first order) to $\partial_\mu\epsilon(x)$. In fact, Lorentz invariance implies that there exists a 4-vector $\tilde{j}^\mu(x)$ such that $\delta S = \int \tilde{j}^\mu \partial_\mu\epsilon(x) = -\int(\partial_\mu \tilde{j}^\mu)\epsilon(x)$. This identifies $\tilde{j}^\mu$ with j^μ, the conserved current (because for ϵ constant, the Noether current has the property that $-\int(\partial_\mu j^\mu)\epsilon = \int(\frac{\delta S}{\delta\phi}\delta\phi\epsilon) \equiv \delta S$ - see equ. (5.44)). Now, since by assumption $\frac{\delta S_{int}}{\delta A_\mu} = -j^\mu$ we have $\int(\frac{\delta S_{int}}{\delta A_\mu} + j^\mu)\partial_\mu\epsilon(x) = 0$. Writing $S = S_A^0[A] + S_{\text{int}}[A,\phi] + S_{\text{matter}}^{\text{rest}}[\phi]$, the fact that $S_A^0[A]$ is gauge invariant for a massless vector theory implies that this is just the variation of S with respect to the combined gauge transformation

  $$A_\mu \to A_\mu + \partial_\mu\epsilon(x), \qquad \phi(x) \to \phi(x) + \epsilon(x)\delta\phi(x). \tag{5.55}$$

  The theory is thus gauge invariant.

  To prove the other direction, suppose (5.55) leaves the action $S_A^0[A] + S_{\text{int}}[A,\phi] + S_{\text{matter}}^{\text{rest}}[\phi]$ invariant (where again $S_A^0[A]$ is separately gauge invariant). Then there exists some conserved $j^\mu(x)$ such that $\int(\frac{\delta S_{int}}{\delta A_\mu} + j^\mu)\partial_\mu\epsilon(x) = 0$, and we conclude (5.54).

  To conclude, **massless vector theories are consistent if and only if the vector couples to a conserved current**. This is the necessary and consistent condition for gauge invariance and for the Ward identities.

- If we couple a **massive** vector, whose action is never gauge invariant, to a conserved current as in (5.54), then $k^\mu \mathcal{M}_\mu = 0$ still holds. The fact that in a massive vector theory the Ward identities are still satisfied provided that theory couples as in (5.54) is crucial for its consistency: Recall that in a massive vector theory ($k^2 > 0$), the negative norm states are the states

  $$|k^\mu, \xi^\mu\rangle \quad \text{with} \quad \xi^\mu = k^\mu. \tag{5.56}$$

Restriction to ζ^μ with $\zeta^\mu k_\mu = 0$ removes these. This is only justified in the interacting theory as long as no dangerous states $|k^\mu, \zeta^\mu = k^\mu\rangle$ are produced, as is guaranteed thanks to $k^\mu \mathcal{M}_\mu = 0$ for couplings to conserved currents.

5.2.2 Photon polarisation sums in QED

To evaluate the consequences of the Ward identities we can take, without loss of generality, the photon 4-momentum to be

$$k^\mu = k \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.57}$$

and consider the corresponding basis of polarisation vectors

$$\epsilon^\mu(k,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^\mu(k,1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^\mu(k,2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon^\mu(k,3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{5.58}$$

The Ward identity $k^\mu \mathcal{M}_\mu(k) = 0$ then implies $\mathcal{M}^0(k) = \mathcal{M}^3(k)$. In physical applications we often need to sum over the two transverse polarisation vectors $\lambda = 1, 2$ of an external photon involved in a scattering process. By the Ward identity this becomes

$$\begin{aligned} \sum_{\lambda=1}^{2} |\epsilon_\mu(k,\lambda)\mathcal{M}^\mu(k)|^2 &= \sum_{\lambda=1}^{2} \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k,\lambda)\mathcal{M}^\mu(k)(\mathcal{M}^\nu(k))^* = |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2 \\ &= |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2 + |\mathcal{M}^3(k)|^2 - |\mathcal{M}^0(k)|^2 \\ &= -\eta_{\mu\nu}\mathcal{M}^\mu(k)(\mathcal{M}^\nu(k))^*. \end{aligned} \tag{5.59}$$

Thus in sums over polarisations of the above type we can replace $\sum_{\lambda=1}^{2} \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k,\lambda)$ by $-\eta_{\mu\nu}$. This will be used heavily in deriving the Klein-Nishina formula for Compton scattering in the tutorials.

5.2.3 Decoupling of potential ghosts

So far we have merely shown that the zero-norm states with polarization k^μ decouple from the S-matrix and are thus not produced in scattering experiments. But what about the status of timelike polarisation states, which correspond to the even more dangerous negative norm ghosts? Even if we declare $\mathcal{H}_{\text{phys}}$ not to contain them, we must prove that no such states are created as outgoing states from physical in-states in scattering processes. Otherwise the interactions would render the theory inconsistent.[2]

[2]Note that this is only a question for massless vector fields. For massive vector fields the negative norm states are just the photons with polarization k^μ (as $k^2 > 0$) and these obviously decouple by the Ward identity.

In fact, it is again the Ward identity that guarantees that no ghosts are created as external states in interactions. The argument relies on the relation

$$\sum_{\lambda=1}^{2} \epsilon_\mu(k,\lambda)\epsilon_\nu^*(k,\lambda)\mathcal{M}^\mu(k)(\mathcal{M}^\nu(k))^* = -\eta_{\mu\nu}\mathcal{M}^\mu(k)(\mathcal{M}^\nu(k))^* \tag{5.60}$$

derived above and is sketched as follows: It suffices to show that the restriction of the S-matrix S to the space of transverse polarisations is unitary. This means that we lose no information by considering only transverse in and out-states and thus guarantees that no unphysical states can be produced out of transverse incoming states. Let P denote the projector onto these states of transverse polarisations. The relation (5.60) amounts to the statement

$$S^\dagger PS = S^\dagger S. \tag{5.61}$$

Since $S^\dagger S = \mathbb{1}$, this implies that $(PSP)^\dagger(PSP) = P$. This is precisely the statement that the restriction of S to the state of transverse polarisation is unitary.

5.3 Radiative corrections in QED - Overview

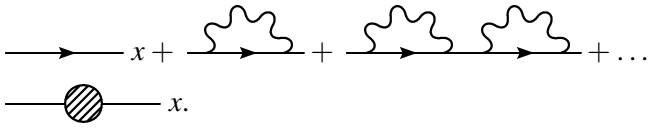
We now enter a quantitative discussion of radiative corrections in Quantum Field Theory as exemplified by loop corrections to QED. As we will see it is sufficient to study the following types of loop corrections modifying the QED building blocks.

Corrections to the fermion propagator

We recall that the Feynman propagator of the free fermion field is

$$S_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)} \underbrace{\frac{\gamma\cdot p + m_0}{p^2 - m_0^2 + i\epsilon}}_{\equiv \frac{i}{\gamma\cdot p - m_0 + i\epsilon}}. \tag{5.62}$$

Taking into account corrections due to QED interactions we have instead

$\langle\Omega|\, T\psi(x)\bar\psi(y)\,|\Omega\rangle =$  (5.63)

Let

$$A \text{ ——(1PI)—— } B \equiv -i\Sigma(\not{p})_{AB} \tag{5.64}$$

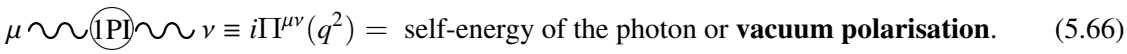
denote the amputated 1PI diagram. By Dyson resummation the full propagator then takes the form

$$y \text{ ——●—— } x = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{\gamma\cdot p - m_0 - \sum(\not{p}) + i\epsilon}, \tag{5.65}$$

where $\sum(\not{p})$ is called **self-energy of the electron**.

Corrections to the photon propagator

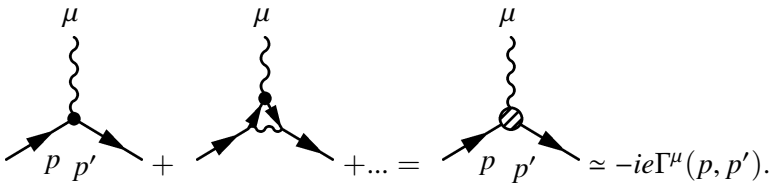
The Fourier transform of the photon propagator, denoted by  , derives by Dyson resummation from the 1-PI diagram



$$\equiv i\Pi^{\mu\nu}(q^2) = \text{ self-energy of the photon or } \textbf{vacuum polarisation}. \tag{5.66}$$

Corrections to the interaction vertex

Loop corrections modifiy the cubic interaction vertex. We will find it useful to define an effective vertex by summing up all loop-corrections. Schematically,



We will compute these corrections to 1-loop order. The diagrams will exhibit

- ultraviolet (UV) divergences from integrating the momenta of particles in the loop up to infinity and
- infra-red (IR) divergences if the diagram contains massless particles - i.e. photons - running in the loop.

The general status of these singularities is as follows:

- IR divergences in loop-diagrams cancel against IR divergences from radiation of soft, i.e. low-energy photons and thus pose no conceptual problem.
- UV divergences require regularisation of the integral and can be absorbed in a clever definition of the parameters via renormalisation.

5.4 Self-energy of the electron at 1-loop

At 1-loop order the electron propagator takes the form

$$\langle\Omega|\, T\psi(x)\bar{\psi}(y)\,|\Omega\rangle = y \longrightarrow x + \underbrace{y \xrightarrow{\quad\quad} x}_{=\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\times(\mathrm{I})} \tag{5.67}$$

where the integrand (I) is given by the diagram

$$(\mathrm{I}) \equiv p \xrightarrow{\quad k \quad} p. \tag{5.68}$$

The photon in the loop carries momentum $p-k$. By means of the Feynman rules,

$$\text{(I)} = \frac{i(\gamma\cdot p + m_0)}{p^2 - m_0^2 + i\epsilon}(-i\Sigma_2(\not{p}))\frac{i(\gamma\cdot p + m_0)}{p^2 - m_0^2 + i\epsilon}. \tag{5.69}$$

The **amputated 1-loop contribution** corresponds to omitting the two outer fermion propagators and is thus given by

$$i\Sigma_2(\not{p}) = (-ie)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\gamma^\mu\frac{i(\not{k}+m_o)}{k^2-m_0^2+i\epsilon}\gamma^\nu\frac{-i\eta_{\mu\nu}}{(p-k)^2+i\epsilon}. \tag{5.70}$$

It will turn out that the integral is divergent near $k=0$ if $p\to 0$. This is an IR divergence. A careful analysis reveals that it will cancel in all amplitudes against similar such IR divergences from other diagrams involving soft photons and thus poses no harm. For the present discussion we could just ignore it, but for completeness we introduce a ficticious small photon mass μ to regulate the IR divergence:

$$i\Sigma_2(\not{p}) = (-ie)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\gamma^\mu\frac{i(\not{k}+m_o)}{k^2-m_0^2+i\epsilon}\gamma_\mu\frac{-i}{(p-k)^2-\mu^2+i\epsilon}. \tag{5.71}$$

The evaluation of such typical momentum integrals proceeds in 3 steps:

5.4.1 Feynman parameters

The integrand contains a fraction of the form $\frac{1}{AB}$ with $A=(p-k)^2-\mu^2+i\epsilon$ and $B=k^2-m_0^2+i\epsilon$. It turns out useful to write this as $\frac{1}{(\ldots)^2}$ and to complete the square in k. To this end we exploit the elementary identity

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\frac{1}{(xA+(1-x)B)^2}. \tag{5.72}$$

x is called a **Feynman parameter**. Applying this identity yields in the present case

$$\begin{aligned}\frac{1}{AB} &= \int_0^1\mathrm{d}x\frac{1}{(x((p-k)^2-\mu^2+i\epsilon)+(1-x)(k^2-m_0^2+i\epsilon))^2}\\ &= \int_0^1\mathrm{d}x\frac{1}{(k^2-2xk\cdot p+xp^2-x\mu^2-(1-x)m_0^2+i\epsilon+x^2p^2-x^2p^2)^2}\\ &= \int_0^1\mathrm{d}x\frac{1}{(l^2-\Delta+i\epsilon)^2},\end{aligned} \tag{5.73}$$

where $l=k-xp$ and $\Delta=-x(1-x)p^2+x\mu^2+(1-x)m_0^2$. Then

$$-i\Sigma_2(\not{p}) = -e^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{\gamma^\mu(\not{k}+m_0)\gamma_\mu}{(l^2-\Delta+i\epsilon)^2}. \tag{5.74}$$

The term in the numerator can be simplified with the help of the gamma-matrix identity

$$\gamma^\mu\gamma^\nu k_\nu\gamma_\mu = (2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu)k_\nu\gamma_\mu = k^\mu\gamma_\mu(2 - \gamma^\nu\gamma_\nu) \tag{5.75}$$

and, in d-dimensions, $\gamma^\mu\gamma_\mu = d$. Therefore altogether (keeping d arbitrary for later purposes)

$$\gamma^\mu(\not{k} + m_0)\gamma_\mu = (2-d)(\not{l} + x\not{p}) + dm_0. \tag{5.76}$$

Now for symmetry reasons

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{l^\mu}{(l^2-\Delta)^2} = 0 \tag{5.77}$$

and thus, if $d \equiv 4$,

$$\boxed{-i\Sigma_2(\not{p}) = -e^2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{-2x\not{p} + 4m_0}{(l^2-\Delta+i\epsilon)^2}.} \tag{5.78}$$

Remark: For more general loop integrals one makes use of the identity

$$\frac{1}{A_1...A_n} = \int_0^1 \mathrm{d}x_1...\mathrm{d}x_n\delta(\sum_i x_i - l)\frac{(n-1)!}{(x_1A_1 + ... + x_nA_n)^n}, \tag{5.79}$$

which can be proven by induction.

5.4.2 Wick rotation

As we have seen, we encounter loop-integrals of the typical form $\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{(l^2-\Delta+i\epsilon)^n}$. This integral would be relatively easy to perform if it were defined in Euclidean space. The Wick rotation relates it to such a Euclidean integral.

Indeed, as the $i\epsilon$ factors remind us, the l^0 integral is in fact a complex contour integral along the real axis. The value of this integral is unchanged if we deform the contour without hitting any pole. Therefore we can rotate the contour by 90° counter-clockwise to lie along the imaginary axis. Introducing the Euclidean 4-momentum $l_E = (l^0_E, \vec{l}_E)$ as

$$l^0 = il^0_E, \quad \vec{l} = \vec{l}_E \tag{5.80}$$

such that $l^2 \equiv -l^2_E \equiv -\sum_i (l^i_E)^2$, we can write the integral as

$$\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{1}{(l^2-\Delta+i\epsilon)^m} = i(-1)^m\int \frac{\mathrm{d}^4 l_E}{(2\pi)^4}\frac{1}{(l^2_E+\Delta-i\epsilon)^m}. \tag{5.81}$$

Since we won't need the $i\epsilon$ any longer we can omit it at this stage. We can now peform the integral as a spherical integral in $\mathbb{R}^4$.

5.4.3 Regularisation of the integral

The integral

$$\mathrm{I}_4 = \int \frac{\mathrm{d}^4 l_E}{(2\pi)^4} \frac{1}{(l_E^2+\Delta)^2} = \int_0^\infty \frac{|l_E|^3 \mathrm{d}|l_E|}{(2\pi)^4} \int \mathrm{d}\Omega_4 \frac{1}{(|l_E|^2+\Delta)^2} \tag{5.82}$$

is divergent due to integration over the UV region $|l_E| \to \infty$. To isolate the divergence we regularise the integral. The 3 most common methods for regularisation are

- **Momentum cutoff**: Isolate the UV divergence as a divergence in the upper momentum limit Λ by writing

$$\int_0^\infty \ldots = \lim_{\Lambda\to\infty} \int_o^\Lambda \ldots. \tag{5.83}$$

  Following this procedure we can evaluate the integral in an elementary fashion as

$$\begin{aligned}
\mathrm{I}_4 &= \int \mathrm{d}\Omega_4 \lim_{\Lambda\to\infty} \int_0^\infty \frac{\mathrm{d}|l_E|}{(2\pi)^4} \frac{|l_E|^3}{(|l_E|^2+\Delta)^2} \\
&= \int \mathrm{d}\Omega_4 \lim_{\Lambda\to\infty} \frac{1}{2} \int_0^\infty \frac{\mathrm{d}|l_E|^2}{(2\pi)^4} \frac{|l_E|^2}{(|l_E|^2+\Delta)^2} \\
&= \int \mathrm{d}\Omega_4 \lim_{\Lambda\to\infty} \frac{1}{2} \frac{1}{(2\pi)^4} \left[\frac{\Delta}{\Delta+|l_E|^2} + \log(|l_E|^2+\Delta) \right]_{l^2=0}^{\Lambda^2} \\
&= \int \mathrm{d}\Omega_4 \lim_{\Lambda\to\infty} \frac{1}{2} \frac{1}{(2\pi)^4} \left[\log\left(\frac{\Lambda^2}{\Delta}\right) - 1 \right].
\end{aligned} \tag{5.84}$$

  The loop leads to a **log-divergence as** $\Lambda \to \infty$. The problem is that this regularisation procedure is not consistent with the Ward identities, as we will see when computing the photon propagator, and is therefore not a useful regularisation method in QED.

- **Dimensional regularisation** (dimReg) is probably the most common method. We first evaluate the integral in d dimensions; writing $d = 4-\epsilon$ then isolates the divergence as a pole in ϵ as $\epsilon \to 0$.

  Let us therefore consider

$$\mathrm{I}_d = \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^2} = \int \frac{\mathrm{d}\Omega_d}{(2\pi)^3} \int_0^\infty \mathrm{d}|l_E| \frac{|l_E|^{d-1}}{(|l_E|^2+\Delta)^2}. \tag{5.85}$$

  The volume of the unit sphere in d-dimensions is

$$\int \mathrm{d}\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \tag{5.86}$$

  where

$$\boxed{\Gamma(z) = \int_0^\infty \mathrm{d}y\, y^{z-1}\, e^{-y}} \tag{5.87}$$

is the **Euler Γ-function**. This is because

$$\begin{aligned}(\sqrt{\pi})^d &= \left(\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2}\right)^d = \int \mathrm{d}^d x e^{-\sum_{i=1}^d x_i^2} \\ &= \int \mathrm{d}\Omega_d \int_0^{\infty} \mathrm{d}|x||x|^{d-1} e^{-|x|^2} \\ &= \int \mathrm{d}\Omega_d \frac{1}{2} \int_0^{\infty} \mathrm{d}(x^2)(x^2)^{d/2-1} e^{-x^2}. \quad \square\end{aligned} \tag{5.88}$$

The Γ-function has the properties

- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}^+$ as can be shown by integration by parts.
- $\Gamma(z)$ has analytic poles at $z = 0, -1, -2, -3, ...$; for a proof we refer to standard textbooks on complex analysis.

Continuing with our integral we write

$$\int_0^{\infty} \mathrm{d}|l_E| \frac{|l_E|^{d-1}}{(|l_E|^2+\Delta)^2} = \frac{1}{2} \int_0^{\infty} \mathrm{d}|l_E|^2 \frac{(|l_E|^2)^{d/2-1}}{(|l_E|^2+\Delta)^2} \tag{5.89}$$

and substitute

$$x = \frac{\Delta}{|l_E|^2+\Delta} \;\Rightarrow\; |l_E|^2 = \frac{\Delta}{x} - \Delta \tag{5.90}$$

with

$$\mathrm{d}x = -\mathrm{d}|l_E|^2 \frac{\Delta}{(|l_E|^2+\Delta)^2}. \tag{5.91}$$

Then the integral (5.89) becomes

$$= \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-d/2} \int_0^1 \mathrm{d}x\, x^{1-d/2}(1-x)^{d/2-1}. \tag{5.92}$$

As a last ingredient we need the **Euler β-function**

$$\mathcal{B}(\alpha,\beta) := \int_0^1 \mathrm{d}x\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{5.93}$$

where the last identity is non-trivial and proven again in textbooks on complex analysis.

We can rewrite (5.92) in terms of $\mathcal{B}(\alpha,\beta)$ with $\alpha = 2-d/2$ and $\beta = d/2$. Together with $\Gamma(2) = 1$ we arrive at

$$\frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-d/2} \int_0^1 \mathrm{d}x\, x^{1-d/2}(1-x)^{d/2-1} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-d/2} \Gamma(2-d/2)\Gamma(d/2). \tag{5.94}$$

Therefore

$$\boxed{\mathrm{I}_d = \frac{\mathrm{d}^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}}\Gamma(2-d/2)\left(\frac{1}{\Delta}\right)^{2-d/2}.} \tag{5.95}$$

We now apply this to I_d with $d = 4$ and note that we encounter a singularity from $\Gamma(0)$. To isolate the singularity we define

$$d = 4-\epsilon \tag{5.96}$$

and would like to write $\mathrm{I}_4 \propto \lim_{\epsilon\to 0}\mathrm{I}_{4-\epsilon}$. However I_4 is dimensionless because

$$[l_E] = [\text{mass}], \tag{5.97}$$

whereas $\mathrm{I}_{4-\epsilon}$ has dimension $[\text{mass}]^{-\epsilon}$. If we introduce a compensating mass scale $\tilde{M}$, we can write

$$\begin{aligned}\mathrm{I}_4 &= \lim_{\epsilon\to 0}\tilde{M}^\epsilon \mathrm{I}_{4-\epsilon} = \lim_{\epsilon\to\infty}\tilde{M}^\epsilon \frac{1}{4\pi^{2-\epsilon/2}}\Gamma(\epsilon/2)\left(\frac{1}{\Delta}\right)^{\epsilon/2}\\ &= \lim_{\epsilon\to 0}\frac{1}{4\pi^2}\Gamma(\epsilon/2)\left(\frac{4\pi\tilde{M}^2}{\Delta}\right)^{\epsilon/2}.\end{aligned} \tag{5.98}$$

Finally we use

- $\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$, where $\gamma \approx 0.5772$ is the Euler-Mascheroni number and we refer again to the complex analysis literature for a proof;
- $x^{\epsilon/2} = 1 + \frac{\epsilon}{2}\log(x) + O(\epsilon^2)$, which is just a Taylor expansion in ϵ since $x^{\epsilon/2} = e^{\frac{\epsilon}{2}\log(x)}$.

With this input

$$\mathrm{I}_4 = \lim_{\epsilon\to 0}\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\tilde{M}^2}{\Delta}\right) + O(\epsilon)\right) \tag{5.99}$$

and therefore

$$\boxed{\mathrm{I}_4 = \lim_{\epsilon\to 0}\frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} + \log\left(\frac{M^2}{\Delta}\right) + O(\epsilon)\right),} \tag{5.100}$$

where $M^2 = 4\pi e^{-\gamma}\tilde{M}^2$.

The final result for the electron self-energy at 1-loop is

$$\boxed{-i\Sigma_2(\not{p}) = (-i)\frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\,((2-\epsilon/2)m_0 - (1-\epsilon/2)x\not{p})\left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta}\right)\Bigg|_{\epsilon\to 0},} \tag{5.101}$$

where M is some as yet arbitrary mass-scale. We will understand its significance soon.

- **Pauli-Villars** (PV) regularisation: We subtract a diagram with a ficiticious massive particle in the loop - here a photon - of mass Λ, i.e. we compute

$$\underbrace{p \xrightarrow{\quad k \quad}}_{\text{involves } \frac{i}{(p-k)^2-\mu^2+i\epsilon}} - \underbrace{p \xrightarrow{\quad k \quad}}_{\text{involves } \frac{i}{(p-k)^2-\Lambda^2+i\epsilon}}. \tag{5.102}$$

Subtracting the second diagram subtracts the divergence because for infinite momenta the mass of the particle in the loop becomes irrelevant and both diagrams asymptote to the same divergent value. If $\Lambda \to \infty$ the contribution of the subtracted vanishes and we recover the actual diagram. The divergence reappears therefore as a divergence in Λ as $\Lambda \to \infty$.

Subtracting both diagrams yields

$$\begin{aligned} -i\Sigma_2(\not{p}) &= \lim_{\Lambda\to\infty} (\text{computation with } \mu - \text{computation with } \Lambda) \\ &= ... \\ -i\Sigma_2(\not{p}) &= \lim_{\Lambda\to\infty} (-i)\frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x(2m_0 - x\not{p})\log\left(\frac{x\Lambda^2}{\Delta}\right), \end{aligned} \tag{5.103}$$

where the omitted steps are elementary and similar to the integration performed in (5.84). We can rewrite this by introducing again an arbitrary mass scale M as

$$\boxed{-i\Sigma_2(\not{p}) = \lim_{\Lambda\to\infty} (-i)\frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x(2m_0 - x\not{p})\left(\log\left(\frac{x\Lambda^2}{M^2}\right) + \log\left(\frac{M^2}{\Delta}\right)\right)\Bigg|_{\Lambda\to\infty}} \tag{5.104}$$

Comparing (5.101) and (5.104) we note that the momentum-dependent terms are in both regularisation approaches the same, while the $\frac{1}{\epsilon}$-divergence in dimReg corresponds to a logarithmic divergence in PV or in momentum cutoff regularisation.

5.5 Bare mass m_0 versus physical mass m

Before dealing with the UV divergence of the electron-propagator, let us recall the non-perturbative information encoded in the 2-point function. By the Källén-Lehmann spectral representation we have

$$\int \mathrm{d}^4x\, e^{ip\cdot x}\langle\Omega|\, T\psi(x)\bar{\psi}(0)\,|\Omega\rangle = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})} \stackrel{!}{=} \frac{iZ_2}{\not{p} - m} + \text{terms analytic at } m. \tag{5.105}$$

The **physical mass** m is the location of the lowest-lying analytic pole and can thus be computed by solving

$$\not{p} - m_0 - \Sigma(\not{p})\big|_{\not{p}=m} = 0, \tag{5.106}$$

while the wavefunction renormalisation of the electron - called Z_2 - is the residue of the propagator at $\not{p} = m$. To find this residue we perform a Taylor expansion of

$$f(\not{p}) := \not{p} - m_0\mathbb{1} - \Sigma(\not{p}) \tag{5.107}$$

around $\not{p}_0 = m\cdot\mathbb{1}$. Then

$$\begin{aligned} f(\not{p}) &= f(\not{p}_0) + (\not{p} - \not{p}_0)\frac{\mathrm{d}f}{\mathrm{d}\not{p}}\bigg|_{\not{p}_0} + O\big((\not{p} - \not{p}_0)^2\big) \\ &= 0 + (\not{p} - m\cdot\mathbb{1})\left(1 - \frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\right)\bigg|_{\not{p}=m\cdot\mathbb{1}} + O\big((\not{p} - m)^2\big). \end{aligned} \tag{5.108}$$

Therefore

$$\text{res}\left[\frac{1}{\not p - m_0 - \Sigma(\not p)}\right]\Bigg|_{\not p = m} = \left(1 - \frac{\mathrm{d}\Sigma(\not p)}{\mathrm{d}\not p}\right)^{-1}\Bigg|_{\not p = m} =: Z_2 \tag{5.109}$$

and thus

$$\boxed{Z_2^{-1} = 1 - \left.\frac{\mathrm{d}\Sigma(\not p)}{\mathrm{d}\not p}\right|_{\not p = m}}. \tag{5.110}$$

We can now compute m to order α, where

$$\Sigma(\not p) = \Sigma_2(\not p) + O(\alpha^2). \tag{5.111}$$

Then to order α

$$m - m_0 - \Sigma_2(\not p = m) = 0 \tag{5.112}$$

and thus

$$\delta m := m - m_o = \Sigma_2(\not p = m) + O(\alpha^2). \tag{5.113}$$

Note that if we trade m by m_0 in $\Sigma_2(\not p = m)$, the error will be $O(\alpha^2)$ and thus we can also write

$$\delta m := m - m_o = \Sigma_2(\not p = m_0) + O(\alpha^2). \tag{5.114}$$

In dimReg we find

$$m - m_0 = \frac{\alpha}{2\pi} m_0 \int_0^1 \mathrm{d}x \left((2-x) + \frac{\epsilon}{2}(x-1)\right)\left[\frac{2}{\epsilon} + \log\left(\frac{M^2}{(1-x)^2 m_0^2 + x\mu^2}\right)\right] \tag{5.115}$$

and in PV

$$m - m_0 = \frac{\alpha}{2\pi} m_0 \int_0^1 \mathrm{d}x (2-x) \underbrace{\left[\log\left(\frac{x\Lambda^2}{M^2}\right) + \log\left(\frac{M^2}{(1-x)^2 m_0^2 + x\mu^2}\right)\right]}_{\equiv \log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right)}. \tag{5.116}$$

Note that $m - m_0 > 0$: This is because what we are computing is literally the self-energy of the electron, i.e. the mass shift due to the (positive!) energy stored in its own electric field. Recall from classical electrodynamics that also classically this quantity is divergent due to the pointlike structure of the electron. Not surprisingly, the divergence remains in QED.[3]

Likewise we can compute Z_2 to order α by evaluating (5.110) at order α, the result being

$$Z_2 = 1 + \frac{\alpha}{2\pi} \int_0^1 \mathrm{d}x \left[-x \log\left(\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right) + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2}\right] + O(\alpha^2). \tag{5.117}$$

[3] Indeed, the fundamental reason for this divergence is because in QFT the fundamental objects are still pointlike, i.e. they have no substructure. In string theory, on the other hand, the fundamental objects do have an intrinsic substructure (as 1-dimensional strings instead of points) and correspondingly this theory is free of UV divergences.

5.5.1 Mass renormalisation

We thus encounter an obvious problem: $m - m_0$ is divergent due to the UV divergence of the propagator. What saves the day are the following crucial observations:

- Only m, the **physical mass**, is a physical observable. Namely m is the rest mass of an electron as measured in experiments.
- By contrast, m_0, the so-called **bare mass**, is merely a parameter that appears in the Lagrangian and per se cannot be measured directly. Rather the Lagrangian produces for us, via the Feynman rules, measurable quantities, the scattering amplitudes, which depend on m_0.

This suggests the following solution to the divergence of δm: The divergence can be absorbed in the definition of m_0 by interpreting the equation (5.113) for δm as an equation for m_0 in terms of the measured physical mass m and the cutoff Λ or ϵ. Concretely in PV regularisation (to be specific)

$$m_0 = m\left(1 - \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x(2-x)\log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right)\right) + O(\alpha^2) = m_0(\Lambda). \tag{5.118}$$

This means that we take the parameter m_0 in the Lagrangian to be divergent. We compute scattering amplitudes etc. in terms of this divergent object $m_0(\Lambda)$ and at the end plug in the above equation for $m_0(\Lambda)$ to express everything via m. If through this procedure all physical quantities in the end are independent of Λ, the theory is said to be renormalisable. Indeed QED is renormalisable.We will discuss how one can see this more systematically later.
Rather than being systematic here, let us exemplify this in the following trivial example: Compute the mass m at 1-loop. Well, this means we take the equation for m in terms of m_0 at 1-loop and plug in (5.118). Obviously we find $m = m$. This demonstrates that absorbing the divergence in the bare mass comes at a price: We lose predicitivity for the physical mass. In other words, we must now take m directly from experiment. In a renormalisable theory we retain, however, predicitivity for all but a finite number of quantities.

5.6 The photon propagator

The building block to compute radiative corrections to the photon propagator is the 1PI amputated diagram

$$:= i\Pi^{\mu\nu}(q). \tag{5.119}$$

On general grounds we can make the following statements:

- By Lorentz invariance its tensorial structure can only depend on $\eta^{\mu\nu}$ and $q^\mu q^\nu$.
- The Ward identity further implies transversality,

$$q^\mu \Pi_{\mu\nu}(q) = 0, \tag{5.120}$$

because we can view $\Pi^{\mu\nu}$ as a $1-1$ scattering amplitude.[4] Thus

$$i\Pi^{\mu\nu}(q) = \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) f(q^2). \tag{5.121}$$

Note in particular that the Ward identity implies that no terms of the form $\frac{q^\mu q^\nu}{m_0^2}$ or the alike arise as these would destroy transversality.

- Finally, $i\Pi^{\mu\nu}(q)$ cannot have an analyic pole at $q^2 = 0$,[5] because this would require a single-particle massless intermediate state (whose propagator vanishes at zero momentum), but no such intermediate states occur for the 1 PI diagram relevant for the photon propagator.

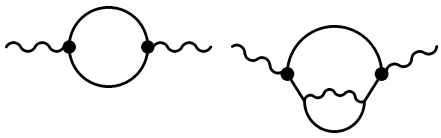


Figure 5.3: Possible diagrams: The first loop carries only massive particles. In the second diagram the photon in the loop is massless, but it does not arise as a single-particle due to the accompanying electron line.

Therefore,

$$i\Pi^{\mu\nu}(q) = \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2), \tag{5.122}$$

such that $\Pi(q^2)$ is regular at $q^2 = 0$. It is useful to define the projection operator onto momenta orthogonal to q^μ,

$$P^{\mu\nu}(q) = \eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \tag{5.123}$$

with

$$q_\mu P^{\mu\nu}(q) = 0 \text{ and } P^{\mu\nu}(q)P_{\nu\rho}(q) = P^\mu_\rho(q). \tag{5.124}$$

By Dyson resummation the Fourier transform of the full propagator takes the form

$$= \frac{-i}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}, \tag{5.125}$$

where the final result follows with the help of (5.124) as will be discussed in the tutorial. Note that this holds in Feynman gauge $\xi = 1$. In general gauge we would get[6]

$$= \frac{-i}{q^2(1-\Pi(q^2))}\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi\frac{-i}{q^2}\frac{q^\mu q^\nu}{q^2}. \tag{5.126}$$

[4] In fact, this argument applies to the fully resummed propagator rather than to $\Pi_{\mu\nu}(q)$. However, given the relation between both via Dyson resummation it is not hard to see that the Ward identity carries over to $q^\mu\Pi_{\mu\nu}(q) = 0$ because it must hold order by order in the coupling constant.

[5] To avoid confusion, think of computing the diagram for finite cut-off Λ or $\epsilon > 0$. The statement is that apart from the UV divergence as the cutoff is removed, $i\Pi^{\mu\nu}(q)$ exhibits no analytic pole at $q^2 = 0$.

[6] We will see this via an easy path-integral proof in the course QFT 2.

This identifies the ξ-dependent term as pure gauge. We can omit it by going to Landau gauge $\xi = 0$, in which

$$\boxed{\left.\sim\!\!\blacksquare\!\!\sim\right|_{\text{Landau}} = \frac{-i}{q^2(1-\Pi(q^2))}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right).} \tag{5.127}$$

This result has two important consequences:

- Since $\Pi(q^2)$ is regular at $q^2 = 0$ (again in the sense that for finite cutoff, there is no analytic pole) the pole at $q^2 = 0$ is unaffected by the radiative corrections. Thus **the photon remains massless as required by gauge invariance**. This is a consequence of the Ward identity, which had lead to this form of $i\Pi^{\mu\nu}(q)$.

- The photon field strength renormalisation is simply

$$\boxed{Z_3 = \frac{1}{1-\Pi(0)}.} \tag{5.128}$$

To obtain the result at order α we procced in a similar manner as for $\Sigma_2(\not{p})$. We denote by $\Pi_2(q^2)$ the 1-loop contribution to $\Pi(q^2)$ (just like $\Sigma_2(\not{p})$ denotes the 1-loop contribution to $\Sigma(\not{p})$). To compute $\Pi_2(q^2)$ we must consider the diagram

(5.129)

and bring the result into the form (5.121) with $\Pi_2(q^2)$ instead of $\Pi(q^2)$.
This computation is most conveniently carried out in dimensional regularization. We merely quote the final result

$$\boxed{\Pi_2(q^2) = -\frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta}\right)\Bigg|_{\epsilon\to 0},} \tag{5.130}$$

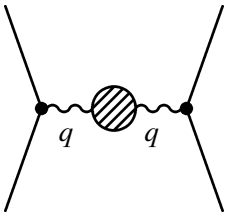
where $\Delta = m_0^2 - x(1-x)q^2$.
To conclude this section we state without proof that if we were to compute the 1-loop corrected propagator with a naive momentum cutoff $\int_0^\infty \mathrm{d}p \to \int_0^\Lambda \mathrm{d}p$, then we would find a divergent term of the form

$$i\Pi_2^{\mu\nu}(q) \sim \eta^{\mu\nu}\Lambda^2 + P^{\mu\nu}(...). \tag{5.131}$$

This would violate $q^\mu\Pi_{\mu\nu}(q) = 0$ and thus the Ward identity. This shows, as claimed, that momentum cutoff regularisation breaks gauge invariance and is thus not useful in QED. Note that from our derivation of the Ward identities, it may not be completely obvious why cutoff regularization leads to a breakdown of the Ward identities. Next term we will get to know a very simple derivation of the Ward identities in the path integral quantization approach and interpret the breakdown of the Ward identities as due to the fact that the cutoff-regularised measure of the path integral is not invariant under the U(1) symmetry of the classical action.

5.7 The running coupling

Radiative corrections to the photon propagator are responsible for an important phenomenon in QED, the running of the electric coupling. To see this we consider a scattering process with an intermediate photon of the form



Taking into account the appearance of e_0 at each vertex, it is clear that the amplitude involves a factor of

$$\frac{(-ie_0)^2}{1-\Pi(q^2)}\frac{-i(\eta^{\mu\nu}-q^\mu q^\nu/q^2)}{q^2}. \tag{5.132}$$

We can absorb the correction term $(1-\Pi(q^2))^{-1}$ into the coupling and define an effective coupling

$$e(q^2) := \frac{e_0}{\sqrt{1-\Pi(q^2)}}. \tag{5.133}$$

This obviously depends on q^2, the energy transferred by the photon in the scattering process. When measuring the physical charge of an electron, we therefore need to specify the energy scale at which the measurement is performed. Let us define the **physical coupling** or **renormalised charge** as the effective charge as measured at $q^2 = 0$, i.e.

$$e := \lim_{q^2\to 0} e(q^2) = \frac{e_0}{\sqrt{1-\Pi(0)}} = e_0\sqrt{Z_3}. \tag{5.134}$$

Note that $\Pi(0)$ is a divergent constant. E.g. at 1-loop in perturbation theory it takes the form in dimensional regularisation

$$\Pi(0) = \Pi_2(0) + O(\alpha^2) = -\frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{2}{\epsilon} + \log\frac{M^2}{m_0^2}\right) + O(\alpha^2). \tag{5.135}$$

Note that at this order in α, m_0 and m can be exchanged (because the difference is itself of $O(\alpha)$). We proceed as we did when defining the physical electron mass and absorb the divergence into the definition of the bare coupling e_0. This can be done because only e is a physical observable, whose finite value we take from experiment. We define the **bare coupling** as

$$e_0 = e\sqrt{1-\Pi(0)} \equiv e_0(\Lambda) \text{ or } e_0(\epsilon), \tag{5.136}$$

depending on the regularisation method we used (Pauli-Villars or dimensional regularisation). The effective coupling at energy q^2 is therefore

$$e^2(q^2) = \frac{e_0^2}{1-\Pi(q^2)} = \frac{e_0^2}{1-\Pi_2(q^2)} + O(\alpha^2) = \frac{e^2(1-\Pi_2(0))}{1-\Pi_2(q^2)} + O(\alpha^2), \tag{5.137}$$

where $1-\Pi_2(0) = \frac{1}{1+\Pi_2(0)} + O(\alpha^2)$. Thus

$$e^2(q^2) = \frac{e^2}{1-(\Pi_2(q^2)-\Pi_2(0))} + O(\alpha^2), \tag{5.138}$$

where $e^2 \equiv e^2(q^2=0)$ is the physical charge at $q^2=0$. We conclude that the effective coupling at $q^2 \neq 0$ is

$$e^2(q^2) = \frac{e^2}{1-\hat{\Pi}_2(q^2)} + O(\alpha^2), \tag{5.139}$$

with $\hat{\Pi}_2(q^2) = \Pi_2(q^2) - \Pi_2(0)$ finite and independent of Λ. Concretely,

$$\hat{\Pi}_2(q^2) = -\frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\log\left(\frac{m^2}{m^2-x(1-x)q^2}\right). \tag{5.140}$$

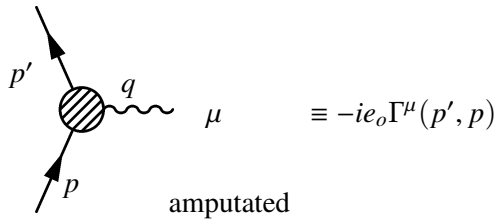
The effective coupling of QED increases logarithmically with the energy scale at which the experiment is performed. The physical interpretation of this is that the charge of an electron is screened by virtual e^+e^- pairs so the effective charge decreases at long distance corresponding to small energy. Thus the QED vacuum appears like a polarisable medium. This explains the name **vacuum polarisation** for the radiatively corrected photon propagator.

A comment on the Landau pole

Note that from this analysis, the effective coupling increases indefinitely as we increase the energy. This phenomenon is called the **Landau pole of QED** and casts doubt on the validity of the theory at arbitrarily high energies. We will find in QFT II that suitable non-abelian gauge theories such as QCD exhibit precisely the opposite phenomenon: they become asymptotically free and are thus perfectly well-defined in the UV. One proposed solution to the Landau pole problem of QED is therefore that the electromagnetic gauge group, which is part of the Standard Model, might in fact emerge as part of a unified non-abelian gauge group, whose dynamics takes over at high energies.

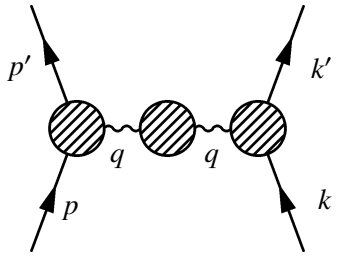
5.8 The resummed QED vertex

We define the amputated resummed cubic QED vertex as



$$\equiv -ie_o\Gamma^\mu(p',p)$$

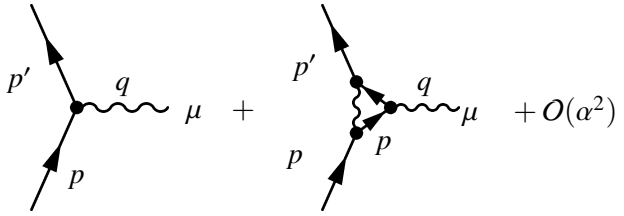
For later purposes we point out that in diagrams with external fermions this resummed vertex is dressed with suitable factors of Z_2. For instance the amplitude associated with the diagram



is of the form

$$i\mathcal{M} \sim \bar{u}(p')\left(-ie_0 Z_2 \Gamma^\mu(p',p)\right) u(p) \frac{-i(\eta_{\mu\nu} - q^\mu q^\nu/q^2)}{q^2(1-\Pi(q^2))} \bar{u}(k')\left(-ie_0 Z_2 \Gamma^\nu(k',k)\right) u(k). \tag{5.141}$$

Perturbatively the cubic vertex can be expanded as



By Lorentz invariance and the Ward identity, which implies that

$$q^\mu \bar{u}(p')\Gamma_\mu(p',p)u(p) = 0, \tag{5.142}$$

one can show that it must take the form

$$\boxed{\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + i\frac{S^{\mu\nu}q_\nu}{2m_0}F_2(q^2),} \tag{5.143}$$

where $F_1(q^2)$ and $F_2(q^2)$ are called **form factors** and $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. The form factors can be computed perturbatively via loop-integrals. We merely quote the following results:

- $F_1(q^2)$ is given by:

$$F_1(q^2) = 1 + \delta F_1(q^2) + O(\alpha^2), \tag{5.144}$$

  where $\delta F_1(q^2)$ is a UV divergent function, which, e.g. in PV regularization, is given by

$$\begin{aligned}\delta F_1(q^2) =& \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(x+y+z-1)\times \\ &\times\left[\log\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}(1-x)(1-y)q^2 + (1-4z+z^2)m_0^2\right],\end{aligned} \tag{5.145}$$

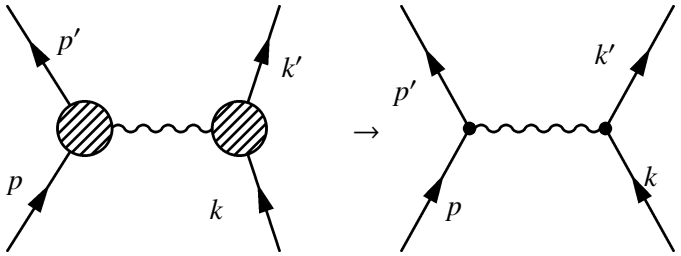
  with $\Delta = -xyq^2 + (1-z)^2 m_0^2 + \mu^2 z$. Here μ shows up to regulate the IR divergences that will cancel eventually.

- For $F_2(q^2) = 0 + \delta F_2(q^2) + O(\alpha^2)$ one finds in PV regularisation

$$\delta F_2(q^2) = \frac{\alpha}{2\pi} \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\delta(x+y+z-1)\frac{2m_0^2 z(1-z)}{\Delta}, \tag{5.146}$$

  which is finite. This finiteness of $F_2(q^2)$ persists to all orders in perturbation theory.

In the limit $p' \to p$, i.e. $q \to 0$, we expect the loop corrections to vanish, i.e. as far as the vertex corrections are concerned e.g.



Indeed one can show that

$$\boxed{\lim_{q\to 0} Z_2 \Gamma^\mu(p+q,p) = \gamma^\mu}. \tag{5.147}$$

Note the appearance of the factor Z_2 due to the external fermion legs, as pointed out before. We can prove (5.147) in two ways:

- By direct inspection in perturbation theory one finds

$$\begin{aligned} Z_2\Gamma^\mu(p+q,p) &= (1+\delta Z_2 + O(\alpha^2))\left(\gamma^\mu\left(1+\delta F_1(q^2+O(\alpha^2)\right) + i\frac{S^{\mu\nu}q_\nu}{2m_0}F_2(q^2)\right) \\ &= (1+\delta Z_2 + \delta F_1(q^2) + O(\alpha^2))\gamma^\mu + \frac{i}{2m_0}S^{\mu\nu}q_\nu F_2(q^2), \end{aligned} \tag{5.148}$$

  where $\delta F_1(q^2)$ and δZ_2 are related as

$$\boxed{\delta F_1(0) = -\delta Z_2.} \tag{5.149}$$

  This implies (5.147) because $F_2(q^2)$ is finite.

- In fact one can prove (5.147) non-perturbatively via the Ward identities. One defines a quantity Z_1 by

$$\lim_{q\to 0} \Gamma^\mu(p+q,p) = Z_1^{-1}\gamma^\mu. \tag{5.150}$$

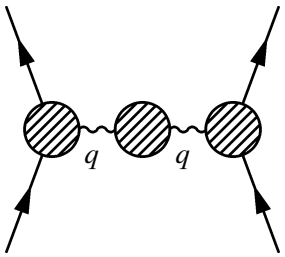
  As we will prove on Assignment 12, the Ward identities imply that $Z_1 = Z_2$ and thus

$$\lim_{q\to 0} Z_2\Gamma^\mu(p+q,p) = \gamma^\mu. \tag{5.151}$$

  In turn this proves that the relation (5.149) must hold order by order in perturbation theory.

5.8.1 Physical charge revisited

Since the cubic vertex contains information about the coupling strength, one might wonder whether the structure of radiative corrections included in $\Gamma^\mu(p',p)$ is consistent with our previous definition of the effective coupling given in section (5.7) based only on the photon propagator.
To investigate this consider a typical fully resummed diagram of the form



$$\sim \frac{-ie_0\Gamma^\mu(p',p)Z_2}{\sqrt{1-\Pi(q^2)}}\frac{1}{q^2}\frac{-ie_0\Gamma_\mu(k',k)Z_2}{\sqrt{1-\Pi(q^2)}}$$

As $q^2 \to 0$ we find $\Gamma^\mu(p+q,p) \to \gamma^\mu Z_1^{-1}$ (see eq. (5.150)) and thus as $q^2 \to 0$ the amplitude reduces to (ignoring polarizations of external fields)

$$\frac{-ie_0}{\sqrt{1-\Pi(0)}}\frac{Z_2}{Z_1}\gamma^\mu\frac{1}{q^2}\gamma_\mu\frac{-ie_0}{\sqrt{1-\Pi(0)}}\frac{Z_2}{Z_1}. \tag{5.152}$$

Since $Z_2/Z_1 = 1$ by the Ward identites, the physical charge at $q^2 = 0$ is $e_0\sqrt{Z_3}$ as we found before.

5.8.2 Anomalous magnetic moment

It is very instructive to study the the non-relativistic limit of the radiatively corrected vertex and compare the interactions it induces with quantum mechanical scattering amplitudes in Born's approximation. This analysis is performed in detail e.g. in Peskin-Schröder, p.187/188. The computation shows, amongst other things, that the coupling of an electron to an external $\vec{B}$-field is described, in the non-relatvistic limit, by a potential

$$V(\vec{x}) = -\vec{\mu}\cdot\vec{B}(\vec{x}), \qquad \vec{\mu} = g\frac{e}{2m}\vec{S}, \tag{5.153}$$

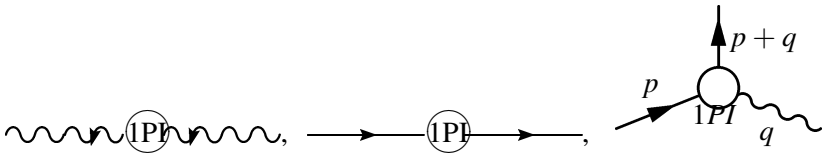
where $\vec{\mu}$ represents the magnetic moment of the electron and $\vec{S}$ denotes the quantum mechanical spin operator. The Landé factor comes out as

$$g = 2((F_1(0)+Z_2-1)+F_2(0)) = 2+\underbrace{2F_2(0)}_{=O(\alpha)}. \tag{5.154}$$

The value $g = 2$ follows already from relativistic Quantum Mechanics, in which the Dirac equation is interpreted as an equation for the wavefunction of the electron. Crucially, $F_2(0)$ yields QED loop corrections to $g = 2$. These can be computed order by order in perturbation theory and are in impressive agreement with experiment.

5.9 Renormalised perturbation theory of QED

For a systematic treatment of UV divergent diagrams, it suffices to consider all amputated, 1PI UV-divergent diagrams. All divergent diagrams in a QFT are given either by these diagrams or possibly by diagrams containing these as subdiagrams.
In QFT II we will find a simple way to classify the divergent 1PI amputated diagrams in a given QFT. Applied to QED, this classification will prove that in QED the UV divergent 1PI amputated diagrams are precisely the three types of diagrams which we have studied in the previous sections:



(5.155)

Let us recap their properties.

- At 1-loop order we have found the following structure of UV divergences:

$$i\Pi^{\mu\nu}(q^2) = i\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi_2(q^2). \tag{5.156}$$

  Expanding $\Pi_2(q^2)$ as a Taylor series in q^2 yields

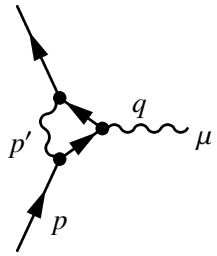
$$\Pi_2(q^2) = c_0^{(1)} \log \frac{\Lambda}{M} + \text{finite} \times O(q^2). \tag{5.157}$$

  That is, the UV divergence appears at order $(q^2)^0$ and is characterized by a constant coefficient $c_0^{(1)}$ independent of q^2.

  The remaining two diagrams at 1-loop order have the following structure:

$$-i\Sigma_2(\not{p}) = a_o^{(1)} m_o \log \tfrac{\Lambda}{M} + a_1^{(1)}\, \not{p} \log \tfrac{\Lambda}{M} + \text{finite terms}$$

  and



$$-i\Gamma_2^\mu(p', p) = b_0^{(1)} \gamma^\mu \log \tfrac{\Lambda}{M} + \text{finite}$$

  Therefore the UV divergences are specified, at 1-loop order, by altogether 4 divergent constants.

- In QFT II we will argue that this structure persists to all orders in perturbation theory, i.e.

$$\begin{aligned}
-i\Sigma(\not{p}) &= a_0 m_0 \log\frac{\Lambda}{M} + a_1\, \not{p} \log\frac{\Lambda}{M} + \text{finite}\times O(\not{p}^{\,2}) \\
a_0 &= \underbrace{a_0^{(1)}}_{O(\alpha)} + \underbrace{a_0^{(2)}}_{O(\alpha^2)} + \ldots \qquad a_1 = a_1^{(1)} + a_1^{(2)} + \ldots \\
\Pi(q^2) &= c_0 \log\frac{\Lambda}{M} + \text{finite}\times O(q^2) + \ldots \\
c_0 &= c_0^{(1)} + c_0^{(2)} + \ldots \\
i\Gamma_2^\mu(p',p) &= b_0\gamma^\mu \log\frac{\Lambda}{M} + \text{finite}\times O((p'-p)^2) \\
b_0 &= b_0^{(1)} + b_0^{(2)} + \ldots
\end{aligned} \tag{5.158}$$

Thus at each order in perturbation theory, we encounter 4 constants that multiply UV divergent terms in the above 1PI diagrams. One can absorb these 4 divergent constants order by order in perturbation theory by a procedure called **renormalisation**.
There are two different, but equivalent ways to perform this procedure, which we now discuss in the context of QED.

5.9.1 Bare perturbation theory

So far we have worked in bare perturbation theory, which works as follows:

- We start with the bare Lagrangian

$$\mathcal{L} = \mathcal{L}_0(e_0, m_0), \tag{5.159}$$

  where e_0 and m_0 are the so-called bare charge and mass.

- We compute the above three 1PI amputated UV divergent amplitudes to a given order in perturbation theory as functions of e_0 and m_0 and a cutoff Λ or ϵ. We keep the cutoff Λ finite (or ϵ non-zero) for the time being so that all computations are perfectly well-defined.

- From these amplitudes we deduce

$$\begin{aligned}
m &= m(m_0, e_0, \Lambda) \quad \text{as the physical electron mass,} \\
e &= e(m_0, e_0, \Lambda) \quad \text{as the physical coupling at } q^2 = 0
\end{aligned} \tag{5.160}$$

  and also

$$\begin{aligned}
Z_2 &= Z_2(m_0, e_0, \Lambda), \quad \text{the wavefunction renormalisation of the electron} \\
Z_3 &= Z_3(m_0, e_0, \Lambda), \quad \text{the wavefunction renormalisation of } A^\mu.
\end{aligned} \tag{5.161}$$

- The **renormalisation step** amounts to interpreting (5.160) as an equation for the bare mass m_0 and the bare coupling e_0 in terms of the finite physical quantities m and e, i.e. we write

$$m_0 = m_0(e, m, \Lambda), \qquad e_0 = e_0(e, m, \Lambda). \tag{5.162}$$

Thus 2 linear combinations of the 4 UV divergent constants are now contained in m_0 and e_0. We plug the expression for e_0 and m_0 back into $\mathcal{L}$. The resulting renormalized Lagrangian is now cuttoff dependent,

$$\mathcal{L} = \mathcal{L}(e_0(m, e, \Lambda), m_0(m, e, \Lambda)) \tag{5.163}$$

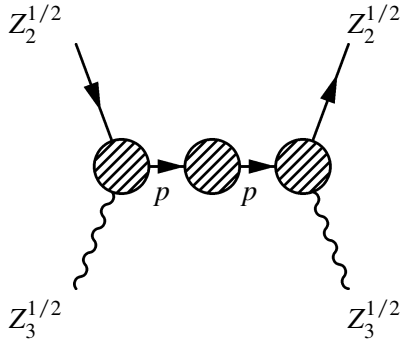
and in particular divergent if we take $\Lambda \to \infty$. This is not a problem because the Lagrangian per se has no physical meaning - only physical observables computed from $\mathcal{L}$ must be finite to make sense.

The remaining 2 linear combinations of divergent constants which are not contained in m_0 and e_0 are contained in the expressions for Z_2, Z_3. Even though these do not appear in $\mathcal{L}$ in the present formulation, they enter the computation of scattering amplitudes via the Feynman rules.

- We now compute a given observable from $\mathcal{L}$ in perturbation theory to given order as functions of e_0 and m_0, Z_2, Z_3 and Λ. The UV divergent terms in Λ cancel in all final expressions and all observables are finite expressions of m and e plus terms in Λ which vanish as we take $\Lambda \to 0$. For this to work it is crucial that we compute the observable to the same order in perturbation theory to which we have computed (5.160) and (5.161). E.g. if we evaluate (5.160) and (5.161) at 1-loop order, then we must compute all remaining scattering amplitudes to 1-loop order as well.

- At the very end we take $\Lambda \to \infty$ or $\epsilon \to 0$. All observables remain finite.

The cancellation of Λ works because in all amplitudes we have just the correct numbers of Z_2, Z_3 etc. so that all divergences drop out. Rather than give a general proof we demonstrate this for one of the diagrams contributing to Compton scattering, including radiative corrections:



Ignoring boring polarisation factors, we can organize the amplitude as follows:

$$(Z_3^{1/2} i e_0 \Gamma^\mu Z_2) \frac{1}{Z_2} \frac{i}{\not{p} - m_0 - \Sigma(\not{p})} (Z_3^{1/2} i e_0 \Gamma_\mu Z_2). \tag{5.164}$$

We note the following crucial points:

- $e_0 Z_3^{1/2} = e$ is finite.

- $\Gamma^\mu(p+q,p)Z_2$ is also finite. To see this recall that divergence in $\Gamma^\mu(p+q,p)$ arises as the q^2-independent term as parametrized in (5.158) and is thus already contained in $\lim_{q^2\to 0}\Gamma^\mu(p+q,p)$. Since Z_2 is independent of q^2 it thus surfices to consider $\lim_{q^2\to 0}\Gamma^\mu(p+q,p)Z_2$. But as discussed around equ. (5.151) this is finite to all orders in perturbation theory by means of the Ward identities.

  Explicitly, this can be confirmed perturbatively from

$$\Gamma^\mu(p+q,p) = \gamma^\mu F_1(q^2) + i\frac{S^{\mu\nu}q_\nu}{2m_0}F_2(q^2). \tag{5.165}$$

  Concerning the second term, $F_2(q^2)$ is finite as function of m_0 and e_0. To a given order in perturbation theory we can replace e_0 by e and m_0 by m as the difference is relevant only at the next order and thus the second term is finite, order by order in perturbation theory. Concerning the first term,

$$F_1(q^2) = 1 + \delta F_1(q^2) + O(\alpha^2) \tag{5.166}$$

  with

$$\delta F_1(q^2) = \delta F_1(0) + f(q^2), \tag{5.167}$$

  where $\delta F_1(0)$ carries all UV divergences, while $f(q^2)$ is finite as a function of m_0. Furthermore we have

$$Z_2 = 1 + \delta Z_2 + O(\alpha^2) \tag{5.168}$$

  and

$$Z_2\Gamma^\mu(p+q,p) = \gamma^\mu(1 + \underbrace{\delta Z_2 + \delta F_1(0)}_{=0} + f(q^2) + O(\alpha^2) + \text{...finite}). \tag{5.169}$$

  Therefore the divergence has cancelled out, because $\delta F_1(0) = -\delta Z_2$. This persists to all orders.

- What remains is the term

$$\frac{1}{Z_2}\frac{i}{\not p - m_0 - \Sigma(\not p)} = \frac{1}{Z_2}\left(\frac{iZ_2}{\not p - m} + \text{terms analytic at } \not p = m\right). \tag{5.170}$$

  Crucially, the terms analytic at $\not p = m$ do not contain any divergence in the cutoff. This follows from the Taylor expansion (5.108), according to which these terms are given by the second derivative of $\Sigma(\not p)$, together with the fact that in $-i\Sigma(\not p)$ all terms quadratic in $\not p$ and higher are UV finite. The Z_2 factors in the first term cancel and the entire expression is finite.

Finally let us outline the generalization and consequences of this renormalisation procedure:

- Consider a general QFT. The theory is called **renormalisable** if only a finite number of resummed amputated 1PI diagrams is UV divergent.

- Suppose the renormalisable QFT contains m different fields and suppose that the UV divergent 1PI diagrams give rise to n divergent constants order by order in perturbation theory. Then $(n-m)$ of these constants can be absorbed in the definition of $(n-m)$ unphysical parameters, the so-called bare couplings. This procedure requires specifying the outcome of $(n-m)$ physical observables as external input. The remaining m constants can be absorbed in the definition of the kinetic terms of the m fields without reducing the predictability of the theory further.

- Thus in a renormalisable theory only a finite number $(n-m)$ of physical observables must be specified order by order in perturbation theory, and predictive power is retained for all remaining observables, which can be computed and are finite as we remove the cutoff.

- However, the price to pay for the appearance of the UV divergences in the first place is that the $(n-m)$ observables cannot be computed by the theory even in principle!For instance, QED cannot make any prediction whatsoever for the absolute value of the electron mass or the charge at $q^2 = 0$.[7] As a result the renormalized QFT necessarily contains free parameters that must be fitted to experiment. Another example for such an observable for which no prediction can be made in a QFT with divergent partition function is the vacuum energy (cosmological constant). From a modern and widely accepted point of view (introduced by K. Wilson), a non-UV finite, but renormalisable QFT is an effective theory: The UV divergences hint at a breakdown of the theory at high energies, where it does not describe the microscopic degrees of freedom correctly. **Renormalisation hides our ignorance about the true physics at high energies in the $(n-m)$ observables and we can fit the theory to experiment as one typically does with a phenomenological model.**

  If we want to go beyond this and describe a truly fundamental (as opposed to effective) theory, we need a theory that is UV finite even before renormalisation. As of this writing the only known theory with this property which also includes gravity is string theory. It has indeed no free parameters.

5.9.2 Renormalised Perturbation theory

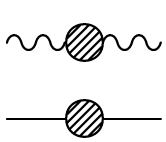
An equivalent treatment is given by so-called **renormalised perturbation theory**. The aim is to organise perturbation theory directly in the physical parameters m and e and without the need of including Z_2 and Z_3 in the Feynman rules. This comes at the expense of certain divergent counterterms in the renormalised Lagrangian. The systematics is as follows:

- Start again with the bare Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{\partial} - m_0)\psi - e_0\bar{\psi}\gamma^\mu\psi A_\mu. \tag{5.171}$$

[7] However, once we take $e(q^2 = 0)$ from experiment, QED does predict the logarithmic running of the effective charge as a function of q^2.

- Recall that the 2-point functions are normalised such that



$$= \frac{-iZ_3(\eta_{\mu\nu} + ...)}{q^2} + \text{analytic terms},$$

$$= \frac{iZ_2}{\not{p} - m} + \text{analytic terms}. \tag{5.172}$$

We can absorb Z_2 and Z_3 into the fields by **renormalising the field strengths** as

$$\begin{aligned} A^\mu &=: Z_3^{1/2} A_r^\mu, \\ \psi &=: Z_2^{1/2} \psi_r \end{aligned} \tag{5.173}$$

such that no factor Z_2 and Z_3 appears in the propagators of ψ_r and A_r^μ,

$$\langle \Omega | \, T A_r^\mu A_r^\nu \, | \Omega \rangle \sim \frac{-i\eta^{\mu\nu} + \dots}{q^2} + ..., \qquad \langle \Omega | \, T \psi_r \bar{\psi}_r \, | \Omega \rangle \sim \frac{i}{\not{p} - m} + \dots \tag{5.174}$$

We can write the same Lagrangian in terms of the renormalised fields as

$$\mathcal{L} = -\frac{1}{4} Z_3 F_r^2 + Z_2 \bar{\psi}_r (i \not{\partial} - m_0) \psi_r - Z_2 Z_3^{1/2} e_0 \bar{\psi}_r \gamma^\mu \psi_r A_\mu. \tag{5.175}$$

To compute an amplitude, we apply the Feynman rules but with m_0 replaced by $Z_2 m_0$ and e_0 replaced by $Z_2 Z_3^{1/2} e_0$, and **no** factors of $Z_2^{1/2}$ or $Z_3^{1/2}$ for external particles in the S-matrix. It is important to appreciate that

> *Renormalising the field strengths does not change any physics if we modify the Feynman rules accordingly.*

- We can further rewrite this same Lagrangian as follows:

$$\begin{aligned} \mathcal{L} &= \left(-\frac{1}{4} F_r^2 + \bar{\psi}_r (i \not{\partial} - m) \psi_r - e \bar{\psi}_r \gamma^\mu \psi_r A_{r\mu} \right) \\ &\quad + \left(-\frac{1}{4} \delta_3 F_r^2 + \bar{\psi}_r (i\delta_2 \not{\partial} - \delta_m) \psi - e\delta_1 \bar{\psi}_r \gamma^\mu \psi_r A_{r\mu} \right) \\ &\equiv \mathcal{L}_r^{(1)} + \mathcal{L}_r^{(2)}, \end{aligned} \tag{5.176}$$

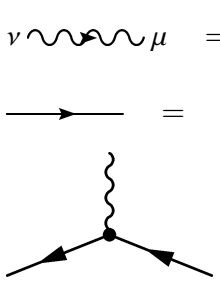
where

$$\delta_3 = Z_3 - 1, \quad \delta_2 = Z_2 - 1, \quad \delta_m = Z_2 m_0 - m, \quad \delta_1 = \frac{e_0}{e} Z_2 Z_3^{1/2} - 1 =: Z_1 - 1. \tag{5.177}$$

This is only a rewriting of $\mathcal{L}$ in the form (5.175) by adding and subtracting $\mathcal{L}_r^{(1)}$. $\mathcal{L}_r^{(2)}$ contains the so-called **counterterms**. Note that we have defined Z_1 by $Z_1 e = e_0 Z_2 Z_3^{1/2}$. At this stage m and e are just arbitrary parameters to be fixed soon.

- The Feynman rules associated with the form (5.176) of the Lagrangian are now as follows: Associated with $\mathcal{L}_r^{(1)}$ are the usual Feynman rules, but with the correct couplings as appearing

in $\mathcal{L}_r^{(1)}$:



$$= \frac{-i\eta_{\mu\nu}}{q^2+i\epsilon} \quad (\text{in Feynman gauge}),$$

$$= \frac{i}{\not{p}-m+i\epsilon} \quad \text{in terms of } m, \text{ not } m_0, \tag{5.178}$$

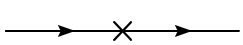
$$= -ie\gamma^\mu, \quad \text{in terms of } e, \text{ not } e_0.$$

The counterterms in $\mathcal{L}_r^{(2)}$ give rise to additional diagrams (**counterterm diagrams**). Their structure becomes evident if we view the terms in $\mathcal{L}_r^{(2)}$ as extra couplings leading to amputated diagrams. In deriving the Feynman rules note that a derivative ∂^μ in position space will give rise to a factor of $-ip^\mu$ in the momentum space Feynman rules.

– Thus the counterterm

$$\bar{\psi}_r(i\delta_2 \not{\partial} - \delta_m)\psi_r \tag{5.179}$$

gives rise to



$$\equiv i(\not{p}\,\delta_2 - \delta_m). \tag{5.180}$$

– The coupling of a photon to two fermions, i.e. $-e\delta_1\psi_r\gamma^\mu\psi_r A_{r\mu}$, gives rise to

$$-ie\gamma^\mu\delta_1. \tag{5.181}$$

– The coupling of two photons

$$-\frac{1}{4}\delta_3 F_{r\mu\nu}F_r^{\mu\nu} = -\frac{1}{2}\delta_3 A_{r\mu}(-\eta^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu)A_{r\nu} \tag{5.182}$$

gives rise to

$$-i(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3. \tag{5.183}$$

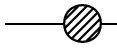
We compute diagrams with the above rules and stress again that no factors of Z_2 and Z_3 appear for external particles because these are already contained in the counterterms.

The above procedure is merely a reorganisation of perturbation theory. The result for an amplitude is the same irrespective of whether the computation is performed

- either starting from $\mathcal{L}$ as given in (5.171) with the original Feynman rules including all Z-factors as before,
- or from $\mathcal{L}_r^{(1)} + \mathcal{L}_r^{(2)}$ in terms of A_r and ψ_r, including counterterm diagrams, and hence without Z-factors for external particles.

This reflects the ambiguity in setting up perturbation theory.
As an example, let us compute the relevant **1-loop 1PI diagrams using the renormalized Feynman rules**.

- The fully resummed propagator of the renormalised electron field takes the form



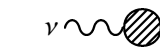
$$= \frac{i}{\not p - m - \Sigma_r(\not p)}. \tag{5.184}$$

At 1-loop level, i.e. at order α in perturbation theory,

$$|_{1-\text{loop}} = -i\Sigma_r(\not p)|_{1-\text{loop}} = -i\Sigma_2^{(1)}(\not p) + i(\not p\,\delta_2 - \delta_m)|_\alpha. \tag{5.185}$$

Here $-i\Sigma_2^{(1)}(\not p)$ is computed from $\mathcal{L}_r^{(1)}$ in terms of m and e, i.e. it has the form (5.101) with e_0 and m_0 replaced by e and m. The term $i(\not p\,\delta_2 - \delta_m)$ is due to the counterterm present in $\mathcal{L}_r^{(2)}$. As we will see momentarily, δ_m and δ_2 depend on α (as do the remaining counterterms δ_1 and δ_3). At 1-loop level only terms up to and including order α are to be included.

- The photon propgator reads (in Landau gauge for simplicity)



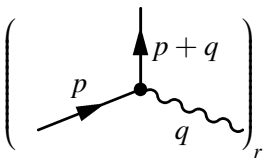
$$= \frac{-i\left(\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)}{q^2(1 - \Pi_r(q^2))}. \tag{5.186}$$

At 1-loop level, $\Pi_r(q^2)$ is computed via

$$|_{1-\text{loop}} = i\Pi_r^{\mu\nu}(q)|_{1-\text{loop}} = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)(\Pi_2^{(1)}(q^2) - \delta_3|_\alpha) \tag{5.187}$$

with $\Pi_2^{(1)}(q^2)$ computed from $\mathcal{L}_r^{(1)}$ and $\delta_3|_\alpha$) the counterterm expanded up to order α.

- The full vertex is

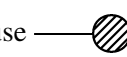


$$= -ie\Gamma_r^\mu(p+q,p). \tag{5.188}$$

Again at 1-loop level it takes the form

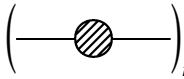
$$-ie\Gamma_r^\mu(p+q,p)|_{1-\text{loop}} = -ie\Gamma^{(1)\mu}(p+q,p) - ie\gamma^\mu\delta_1|_\alpha. \tag{5.189}$$

Finally the 4 counterterm couplings are fixed by **the renormalisation conditions** as follows:

- Two conditions arise because  and  must not involve Z-factors at the physical mass poles. This is because we had defined ψ_r and A_r by this condition, i.e. we had declared that in the renormalised Feynman rules no factors of Z_2 and Z_3 appear.

- Two more conditions arise by specifying the meaning of e and m. This is arbitrary in principle. One possible choice (out of infinitely many) is that m represents the physical mass and e the physical charge measured at $q = 0$.

Combining all of these four conditions translates into the following equations:


$$\overset{!}{=} \frac{i}{\not p - m} + \text{terms analytic at } m. \tag{5.190}$$

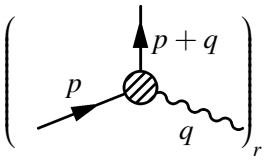
Therefore we find

$$\boxed{\Sigma_r(\not p)\Big|_{\not p m} \overset{!}{=} 0 \;\leftrightarrow\; m \text{ is the physical mass}} \tag{5.191}$$

and

$$\boxed{1 - \frac{\mathrm{d}}{\mathrm{d}\not p}\Sigma_r(\not p)\Big|_{\not p = m} = 1 \;\leftrightarrow\; \text{residue is 1.}} \tag{5.192}$$

Identifying e as the physical charge at $q^2 = 0$ amounts to requiring that



$$= -ie\Gamma_r^\mu(p+q,p) \xrightarrow[q\to 0]{!} -ie\gamma^\mu. \tag{5.193}$$

Therefore

$$\boxed{-ie\Gamma_r^\mu(p,p) \overset{!}{=} -ie\gamma^\mu}. \tag{5.194}$$

Note that (5.192) has already been used. The last condition is


$$\overset{!}{=} \frac{-i\eta_{\mu\nu}}{q^2} + \text{terms analytic at } q = 0 \tag{5.195}$$

and thus

$$\boxed{1 - \Pi_r(0) \overset{!}{=} 1.} \tag{5.196}$$

The conditions (5.191), (5.192), (5.194) and (5.196) are the 4 **renormalisation conditions**. We can solve these perturbatively, order by order in perturbation theory. At 1-loop order we find for

- condition (5.191)

$$\Sigma_2^{(1)}(m) - (m\,\delta_2 - \delta_m) \overset{!}{=} 0 \tag{5.197}$$

  and therefore

$$\begin{aligned}\delta_m =& m\,\delta_2 - \Sigma_2^{(1)} \\ =& m\,\delta_2 - \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x \left(2 - \frac{\epsilon}{2}\right) m - \left(1 - \frac{\epsilon}{2}\right) x\, m \left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta}\right),\end{aligned} \tag{5.198}$$

- conditon (5.192)

$$\delta_2 = \frac{\mathrm{d}}{\mathrm{d}\not p}\Sigma_2^{(1)}(\not p)\Big|_m, \tag{5.199}$$

- condition (5.194)

$$\delta_1 = -\delta F_1^{(1)}(0), \tag{5.200}$$

- condition (5.196)

$$\delta_3 = \Pi_2^{(1)}(0) = -\frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{2}{\epsilon} + \log\frac{M^2}{m^2}\right). \tag{5.201}$$

Thus indeed the counterms depend on α, as anticipated above.
To conclude the **1-loop renormalised Lagrangian** takes the form

$$\begin{aligned}\mathcal{L} = &\left(-\frac{1}{4}F^2 + \bar\psi(i\not\partial - m)\psi - e\bar\psi\gamma^\mu\psi A_\mu\right)\\ &+ \left(-\frac{1}{4}\delta_3 F^2 + \bar\psi(i\delta_2\not\partial - \delta_m)\psi - e\delta_1\bar\psi\gamma^\mu\psi A_\mu\right),\end{aligned} \tag{5.202}$$

where we have dropped the subscript r for the renormalised fields. We summarize:

- e and m are finite quantities to be taken from experiment - the physical coupling and mass.
- δ_1, δ_2, δ_3 and δ_m are cutoff-dependent functions, $\delta_i = \delta_i(e, m, \Lambda)$ in PV (or $\delta_i = \delta_i(\epsilon, e, m)$ in dimReg), which are divergent for $\Lambda \to \infty$ (or $\epsilon \to 0$). Since we have renormalised to 1-loop order, each δ_i so far is of $O(\alpha)$.
- When computing an amplitude, we take into account both the Feynman rules (5.178) and the counterterms. Since we have computed the counterterms only to order $O(\alpha)$ it is only consistent to compute all other diagrams to that same order, where we must take into account that the δ_i are of order $O(\alpha)$. Thus, as in the computation of the 1PI diagrams, a loop diagram is always accompanied by a counterterm diagram, but to order α no counterterm diagrams appear inside a loop (as this would be order α^2).
- The Feynman rules (5.178) give rise to a divergent expression in terms of m, e, Λ. The counter term coefficients $\delta_i(m, e, \Lambda)$ will precisely cancel these divergences. All physical amplitudes are therefore independent of Λ in the end, so we can take $\Lambda \to \infty$.
- If we wish to compute a quantity to order α^2 we first need to compute the divergent 1PI terms to order α^2, and impose again the renormalisation conditions. In this computation we must take into account, in addition to usual 2-loop diagrams from (5.178), also 1-loop diagrams on top of O(α) counterterms, as well as diagrams involving only counterterms with coefficients expanded to order α^2. We will discuss this more systematically in a 2-loop example in QFT II.
- The perturbative cancellation of divergences works only because the counterterms are of the same form as the terms in the bare Lagrangian, i.e. they do not introduce any qualitatively new interactions. This is guaranteed because at each order in perturbation theory, no qualitatively new UV divergent 1PI diagrams arise, but the same 1PI diagrams merely receive higher-loop contributions. This leads to a readjustment of the counterterm coefficients order by order such as to absorb the new divergences.

The renormalisation scheme

- The renormalisation conditions (5.191) and (5.194) are arbitrary: We could define m and e to be any function of the physical mass m or e, thereby changing the δ_i accordingly. This ambiguity will cancel in all final amplitudes.

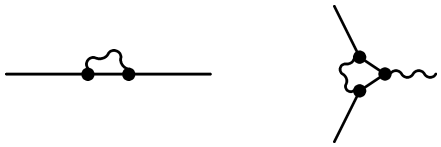
- Conditions (5.192) and (5.196) are fixed by the Feynman rules, which do not contain any Z-factors. However, it is possible to change (5.192) and (5.196) if at the same time we modify the Feynman rules accordingly. I.e. we can include arbitrary $\tilde{Z}$-factors for external states in the Feynman rules, provided we change (5.192) and (5.196). This reflects our freedom in normalising the fields in a manner consistent with the Feynman rules.

The concrete choice of renormalisation condition is called **renormalisation scheme**.

5.10 Infrared divergences

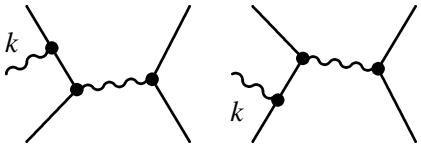
Loop diagrams with massless particles may exhibit infrared divergences from integration over loop momenta $k \to 0$.

- In QED at 1-loop order the IR divergent 1PI diagrams are



  Indeed, we had introduced a small fictitious photon mass μ to regulate the IR divergences.

- These IR divergences cancel in all cross-sections against another source of IR divergences from radiation of soft photons (Bremsstrahlung). Consider e.g.



  As $k \to 0$ these processes are divergent. This is the case already in classical electrodynamics and called **infrared catastrophe of electrodynamics**. The reason why this divergence is not that catastrophic after all is that no detector can measure a photon below a certain threshold. Thus we can include all possible Bremsstrahlung photons for $k \to 0$ to a given process (because as $k \to 0$ the soft photons cannot be measured. It now so happens that the resulting infrared divergences precisely cancel the IR divergences from the above loop diagrams oder by order in α, in the cross-section σ. For details we refer to Peskin-Schröder 6.1, 6.4, 6.5.

Chapter 6

Classical non-abelian gauge theory

6.1 Geometric perspective on abelian gauge theory

We had approached $U(1)$ gauge symmetry from the perspective of massless vector fields:

- A consistent Lorentz invariant quantum theory of free massless spin-1 field $A_\mu(x)$ **must** be a gauge theory - see our discussion around equ. (4.67) based on Weinberg I, 8.1.
- At the level of interactions consistency requires that $A_\mu(x)$ couples to a conserved current $j^\mu(x)$. This is equivalent to the gauging of a global symmetry in the matter sector - see our discussion around (5.54) .

An alternative perspective on $U(1)$ gauge symmetry is as follows:

- Start from a matter theory with a global $U(1)$ symmetry, e.g.

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi, \tag{6.1}$$

  which is invariant under $\psi(x) \mapsto e^{-ie\alpha}\psi(x)$ for constant $\alpha \in \mathbb{R}$.

- In a local QFT it is natural to consider local symmetries, i.e. to promote this to a local transformation

$$\psi(x) \mapsto e^{-ie\alpha(x)}\psi(x) =: U(x)\psi(x). \tag{6.2}$$

  The logic behind such a modification is that a symmetry transformation at spacetime point y far away from x should not affect the field at x. Note that in QFT global symmetries, though, from this perspective, unnatural, are of course fully consistent. By contrast, it is conjectured that in presence of gravity all symmetries must be local and that no global symmetries exist. In any case we are motivated to consider the consequences of the transformation (6.2).

- There is an immediate problem: The ordinary derivative $\partial_\mu\psi(x)$, defined via

$$n^\mu\partial_\mu\psi(x) := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\psi(x^\mu + n^\mu\epsilon) - \psi(x)\right] \tag{6.3}$$

is not a good object with respect to (6.2) because in (6.3) two objects with very different transformation behaviour under (6.2) appear, i.e.

$$\psi(x) \mapsto U(x)\psi(x), \quad \text{but} \quad \psi(x+n\epsilon) \mapsto U(x+n\epsilon)\psi(x+n\epsilon). \tag{6.4}$$

- To define a better notion of derivative we introduce the object $C(x,y)$ - the so-called comparator or **Wilson line** - such that under (6.2)

$$C(y,x)\psi(x) \mapsto U(y)C(y,x)\psi(x). \tag{6.5}$$

Then we can define the **covariant derivative** $D_\mu\psi$ via

$$\boxed{n^\mu D_\mu\psi(x) := \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\psi(x+n\epsilon) - C(x+n\epsilon,x)\psi(x)\right].} \tag{6.6}$$

Its transformation under (6.2) is

$$D_\mu\psi(x) \mapsto U(x)D_\mu\psi(x). \tag{6.7}$$

- Let us now construct the Wilson line starting from the requirement (6.5), which implies that under (6.2)

$$C(y,x) \mapsto U(y)C(y,x)U^{-1}(x). \tag{6.8}$$

Furthermore we impose $C(y,y) = \mathbb{1}$ for obvious reasons. Note that for the $U(1)$ symmetry under consideration $U(x)$ is a pure phase, and it thus suffices to take $C(y,x)$ as a pure phase. We will soon generalize this.

- Taylor expansion of $C(y,x)$ yields

$$C(x+\epsilon n,x) = 1 - ieA_\mu(x)\epsilon n^\mu + O(\epsilon^2) \tag{6.9}$$

for some vector field $A_\mu(x)$. Therefore

$$\boxed{D_\mu\psi = \partial_\mu\psi + ieA_\mu(x)\psi(x).} \tag{6.10}$$

The transformation behaviour of the Wilson line is

$$C(x+\epsilon n,x) \mapsto U(x+\epsilon n)C(x+\epsilon n,x)U^{-1}(x) \tag{6.11}$$

and to order ϵ therefore

$$1 - ieA_\mu(x)n^\mu\epsilon \mapsto \left(U(x) + \epsilon n^\mu\partial_\mu U(x)\right)\left(1 - ieA_\mu(x)n^\mu\epsilon\right)U^{-1}(x). \tag{6.12}$$

Thus the vector field A_μ transforms as

$$\boxed{A_\mu(x) \mapsto U(x)A_\mu(x)U^{-1}(x) + \frac{i}{e}\partial_\mu U(x)U^{-1}(x).} \tag{6.13}$$

For $U(x) = e^{-ie\alpha(x)}$ we recover $A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\alpha(x)$ as expected.

- The vector field $A_\mu(x)$ is therefore a direct consequence of the existence of a local symmetry. It is called a **connection**. $A_\mu(x)$ is a local field and thus has dynamics in its own right.
- Consider $[D_\mu, D_\nu]$ interpreted as acting on $\psi(x)$. Since

$$D_\mu\psi(x) \mapsto U(x)D_\mu\psi(x) \tag{6.14}$$

we have

$$[D_\mu, D_\nu]\psi(x) \mapsto U(x)[D_\mu, D_\nu]\psi(x) = U(x)[D_\mu, D_\nu]U^{-1}(x)U(x)\psi(x) \tag{6.15}$$

and thus

$$\boxed{[D_\mu, D_\nu] \mapsto U(x)[D_\mu, D_\nu]U^{-1}(x).} \tag{6.16}$$

- The term

$$\begin{aligned}[D_\mu, D_\nu]\psi(x) &= [\partial_\mu + ieA_\mu(x), \partial_\nu + ieA_\nu(x)]\,\psi(x) \\ &= ie\Big(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ie[A_\mu(x), A_\nu(x)]\Big)\psi(x)\end{aligned} \tag{6.17}$$

leads to the definition of the **field strength** or **curvature**

$$\boxed{F_{\mu\nu} := \frac{1}{ie}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu].} \tag{6.18}$$

For our $U(1)$ theory with $U(x) = e^{-ie\alpha(x)}$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.19}$$

and $F_{\mu\nu}$ is invariant under $U(1)$. Therefore

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{6.20}$$

is invariant.

6.2 Non-abelian gauge symmetry

We now generalise all of this to a non-abelian symmetry group. Consider a Lie group H of dimension $\dim(H)$. An element $h \in H$ can be written as

$$H \ni h = \exp\left(-ig\sum_{a=1}^{\dim(H)} \alpha^a T^a\right), \tag{6.21}$$

where $g \in \mathbb{R}$ takes the role of e, $\alpha^a \in \mathbb{R}$ and T^a form a basis of the Lie algebra $\mathrm{Lie}(H)$, i.e. the algebra of infinitesimal group transformations. Viewed as an abstract Lie algebra, $\mathrm{Lie}(H)$ is determined by the commutation relations

$$[T^a, T^b] = if^{abc}T^c, \tag{6.22}$$

where a sum over c is understood. Without proof we state here that the **structure constants** f^{abc} are totally antisymmetric in the indices a, b, c and thus in particular invariant under cyclic permutations of a, b, c.
For example:

- The Lie group $H = U(1)$ has $\dim(H) = 1$ and its generator is simply $T^a \equiv T \in \mathbb{R}$. Therefore $[T, T] = 0$ and H is an abelian Lie algebra.

- The Lie group $H = SU(N)$, viewed as an abstract group, is defined as the group of volume-element preserving linear transformations on $\mathbb{C}^N$ which leave the sesqui-linear form

$$\mathbb{C}^N \ni u, v \to \mathbb{C} : (u, v) \mapsto \sum_{i=1}^{N} \bar{u}_i v_i \tag{6.23}$$

  invariant. We can identify H with the group of $V \in \mathbb{C}^{N,N}$ such that

$$V^\dagger = V^{-1} \quad \text{and} \quad \det V = 1 \tag{6.24}$$

  by assigning $\forall h \in H$ a matrix $V(h)$ as above. $\text{Lie}(H)$ is then the algebra of $T^a \in \mathbb{C}^{N,N}$ such that

$$T^{a\dagger} = T^a \quad \text{and} \quad \text{tr} T^a = 0. \tag{6.25}$$

  The dimension of $Lie(H)$ is $N^2 - 1$. For instance, the generators of $H = SU(2)$ are typically normalised to be

$$T^a = \frac{1}{2}\sigma^a \tag{6.26}$$

  with $a = 1, 2, 3$ and σ^a the Pauli matrices. The structure constants of $SU(2)$ are then

$$f^{abc} = \epsilon^{abc}. \tag{6.27}$$

- Other Lie groups of relevance in physics include $O(N), SO(N), Sp(2N), E_6, E_7, E_8$. For example, $O(N)$ ($SO(N)$) is the group of (volume element preserving) linear transformations on $\mathbb{R}^N$ which leave the bilinear form $u^i \delta_{ij} v^j$, $i, j = 1, \ldots N$, invariant and can be identified with real orthogonal $N \times N$-matrices (of determinant one). $Sp(2N)$ is the group of linear transformations on $\mathbb{R}^{2N}$ that leave the anti-symmetric symplectic form $u^i \omega_{ij} v^j$, $i, j = 1, \ldots 2N$ invariant.

Consider now a matter Lagrangian with matter fields $\psi(x)$ transforming in a unitary representation of H such that $\mathcal{L}$ is invariant under global transformations

$$\psi(x) \mapsto R(h) \cdot \psi(x), \qquad R(h)^\dagger = R(h)^{-1} \tag{6.28}$$

with $h \in H$. Two examples are the following:

- Take $H = SU(N)$ and $\psi(x)$ a Dirac spinor field in the **fundamental representation**, i.e. we consider a $\mathbb{C}^N$-valued spinor field such that, suppressing spinor indices,

$$\forall x : \psi(x) \equiv \psi_i(x) = \begin{pmatrix} \psi_1(x) \\ ... \\ \psi_N(x) \end{pmatrix}, \qquad R(h) \cdot \psi(x) \equiv V_{ij}(h)\psi_j(x). \tag{6.29}$$

The Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \sum_{i=1}^{N} \bar{\psi}_i(i\not{\partial} - m)\psi_i \tag{6.30}$$

is invariant under a global $SU(N)$ transformation (6.28) because

$$\bar{\psi}(i\not{\partial} - m)\psi \mapsto \bar{\psi}R^\dagger(h)(i\not{\partial} - m)R(h)\psi = \bar{\psi}\underbrace{R^\dagger(h)R(h)}_{=\mathbb{1}}(i\not{\partial} - m)\psi. \tag{6.31}$$

By the dimension of a representation we mean the dimension of the vector space in which the matter field takes its value. The (complex) dimension of the fundamental representation of $SU(N)$ is thus N.

- Take $H = SU(N)$, but $\psi(x)$ now in the **adjoint representation**, i.e. we consider now a spinor field valued in the Lie algebra Lie(H) viewed as a vector space. This means that, with spinor indices suppressed,

$$\forall x : \psi(x) \equiv \psi_{ij}(x) \in \left\{ \mathbb{C}^{N,N} \middle| \psi^\dagger(x) = \psi(x),\ \mathrm{tr}\,\psi(x) = 0 \right\} \tag{6.32}$$

and the transformation behavior is given by the adjoint action of the Lie group H on its Lie algebra,

$$\begin{aligned} R(h) \cdot \psi &:= V(h)\psi V(h)^{-1} \equiv V(h)_{ij}\psi_{jk}\left(V^{-1}(k)\right)_{kl}, \\ R^\dagger(h) \cdot \psi &:= V^\dagger(h)\psi\left(V^{-1}(h)\right)^\dagger = V^{-1}(h)\psi V(h). \end{aligned} \tag{6.33}$$

The real dimension of the adjoint representation coincides with $\dim(H)$.

Now consider

$$\mathcal{L} = \mathrm{tr}\,\bar{\psi}(i\not{\partial} - m)\psi \equiv \mathrm{tr}\,[\psi_{ij}(i\not{\partial} - m)\psi_{jl}] \equiv \psi_{ij}(i\not{\partial} - m)\psi_{ji}. \tag{6.34}$$

This is invariant because

$$\begin{aligned} \mathrm{tr}\,\bar{\psi}(i\not{\partial} - m)\psi \mapsto \mathrm{tr}\,&\left((V\psi V^{-1})^\dagger\gamma^0(i\not{\partial} - m)V\psi V^{-1}\right) \\ &= \mathrm{tr}\left(V\bar{\psi}V^\dagger(i\not{\partial} - m)V\psi V^{-1}\right) \\ &= \mathrm{tr}\left(\bar{\psi}\underbrace{V^\dagger V}_{=\mathbb{1}}(i\not{\partial} - m)\psi\underbrace{V^{-1}V}_{=\mathbb{1}}\right), \end{aligned} \tag{6.35}$$

where cyclicity of the trace was used to go from the second to the third line.

We can now repeat all the steps involved in the gauging of $U(1)$ in this more general setting. For definiteness we work with a Dirac spinor field in the fundamental representation of $SU(N)$. Consider the gauge transformation

$$\psi(x) \mapsto U(x)\psi(x) \tag{6.36}$$

with

$$U(x) = \exp\left(ig \sum_{a=1}^{\dim(H)} \alpha^a(x)\, T^a \right) \equiv V(h(x)). \tag{6.37}$$

Now it turns out that the compensator must take values in the representation of H. It can therefore be expanded as

$$C(x+\epsilon n, x) = \mathbb{1} - ig \sum_{a=1}^{\dim(H)} A^a_\mu(x) T^a\, \epsilon\, n^\mu + O(\epsilon^2). \tag{6.38}$$

This defines $\dim H$ vector fields $A^a_\mu(x)$. The object

$$\boxed{A_\mu(x) \equiv \sum_a A^a_\mu(x)\, T^a} \tag{6.39}$$

is then an $N \times N$ matrix-valued vector field. The covariant derivative takes the form

$$\boxed{D_\mu\psi(x) = \partial_\mu(x) + ig \sum_a A^a_\mu(x) T^a\, \psi(x),} \tag{6.40}$$

where $T^a\psi(x) \equiv T^a_{ij}\psi_j(x)$. The gauge transformation on $A_\mu(x)$ is still

$$A_\mu(x) \mapsto U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g}\partial_\mu U(x)U^{-1}(x), \tag{6.41}$$

but now $U(x)\, A_\mu(x)\, U^{-1}(x) \neq A_\mu(x)$. Expanding

$$U(x) = \mathbb{1} - ig \sum_a \alpha^a(x) T^a + O(\alpha^a(x)^2), \tag{6.42}$$

one can read off that

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\alpha^a(x)T^a - ig \sum_a \alpha^a(x)\, [T^a, A_\mu(x)], \tag{6.43}$$

where $[T^a, A_\mu(x)] = [T^a, A^b_\mu(x)T^b] = A^b_\mu(x)\, [T^a, T^b] = if^{abc}\, A^b_\mu(x)T^c$. Therefore

$$\boxed{A^c_\mu(x) \mapsto A^c_\mu(x) + \partial_\mu\alpha^c(x) + g\, f^{abc}\, \alpha^a(x)\, A^b_\mu(x).} \tag{6.44}$$

The field strength $F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu] \equiv F_{\mu\nu}(x)^a T^a$ is

$$\boxed{\begin{aligned} F_{\mu\nu}(x) &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)], \\ F^a_{\mu\nu}(x) &= \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) - g f^{abc}[A^b_\mu(x), A^c_\nu(x)]. \end{aligned}} \tag{6.45}$$

It transforms under a gauge transformation as

$$F_{\mu\nu}(x) \mapsto U(x)F_{\mu\nu}(x)U^{-1}(x) \tag{6.46}$$

so $F_{\mu\nu}(x)$ is not invariant. Rather $F_{\mu\nu}(x)$ transforms in the adjoint representation. Thus

$$\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) \mapsto \mathrm{tr}\Big(UF_{\mu\nu}\underbrace{U^{-1}U}_{=\mathbb{1}}F^{\mu\nu}U^{-1}\Big) = \mathrm{tr}\Big(F_{\mu\nu}F^{\mu\nu}U^{-1}U\Big) = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{6.47}$$

Typically one normalises the generators T^a such that

$$\mathrm{tr}T^aT^b = \frac{1}{2}\delta^{ab}. \tag{6.48}$$

Then

$$\boxed{\begin{aligned}\mathcal{L} = & -\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\ \equiv & -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{\mu\nu a} + \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - g\bar{\psi}_i\gamma^\mu A^a_\mu T^a_{ij}\psi_j\end{aligned}} \tag{6.49}$$

defines the so-called **Yang-Mills Lagrangian**. The crucial difference to $U(1)$ gauge theory is that the Yang-Mills gauge field exhibits cubic and quartic self-interactions which are contained in the term

$$-\frac{1}{4}\sum_a F^a_{\mu\nu}F^{\mu\nu a}. \tag{6.50}$$

Diagrammatically, these interactions between the gauge bosons are of the form

In QFT2 we will learn how to quantise such an intrinsically self-interacting theory.

6.3 The Standard Model

As a quick application let us briefly sketch that structure of the the Standard Model (SM) of Particle Physics, which is formulated as a Yang-Mills theory specified by the following data:

- **The gauge group** is

$$G = SU(3)\times SU(2)\times U(1)_Y, \tag{6.51}$$

  where $SU(3)$ represents QCD and its 8 gluons, $SU(2)$ describes the weak interactions via W^+, W^- and Z and $U(1)_Y$ denotes hypercharge. $SU(2)$ and $U(1)_Y$ are broken spontaneously to $U(1)_{\mathrm{e.m}}$, whose gauge field is the photon γ.

- The **representations of fermionic matter** are given as follows: The SM is a chiral theory, i.e. left- and righthanded fermion fields $\psi_L \equiv P_L\psi$ and $\psi_R \equiv P_R\psi$ transform in different representations. The Standard Model comprises 3 families of

	$SU(3)$	$SU(2)$	$U(1)_Y$
lefthanded Q_L	$\square$	$\square \equiv \begin{pmatrix} u \\ d \end{pmatrix}$	$\frac{1}{6}$
L	$\cdot$	$\square \equiv \begin{pmatrix} \nu \\ l \end{pmatrix}$	$-\frac{1}{2}$
righthanded u_R / d_R	$\square$	$\cdot$	$\frac{2}{3} \Big/ -\frac{1}{3}$
ν_R / l_R	$\cdot$	$\cdot$	$0 \Big/ -1$

- In addition there is 1 **complex scalar field** $\phi(x)$ in representation

$$\begin{array}{ccc} SU(3) & SU(2) & U(1)_Y \\ \cdot\ , & \square\ , & \frac{1}{2} \end{array}$$

The interactions are - apart from the Yang-Mills interactions that follow from the above representations - given by

- the **Higgs potential**

$$V(\phi) = \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2, \tag{6.52}$$

  with $\langle \phi \rangle \neq 0$, which leads to spontaneous breaking of $SU(2) \times U(1)_Y \to U(1)_{\text{e.m.}}$,

- and the gauge invariant **Yukawa interactions**, very roughly of the form $\phi \bar{\psi}_i \psi_i$, which yield fermion masses if $\langle \phi \rangle \neq 0$.