Outline

1 Introduction and a first qualitative picture of the Anderson localization

2 quantitative criteria for Anderson localization
   a) return probability and inverse participation number
   b) decay of the wavefunction in the spatial limit to infinity
      i) transfer matrix method
      ii) phase formalism and speckle potential

3 experimental observation and quantitative understanding of AL in Bose–Einstein–condensates

4 theoretical background (additional, not included in the handout version)
   a) second quantization
   b) tight binding approximation and Anderson–model

5 conclusion and references
Introduction

• Quantum mechanics (QM) merges two basic concepts of physics: *particles* and *waves*.

• **Particle** picture: assigns characteristic properties like mass, charge, flavour etc. to a space-time point and describe its dynamics with a certain physical principle (e.g. *least action* $\delta S = \delta \int dtL = 0$)

• **Wave**: object that assigns some values to the *whole* space. The evolution in time follows the same physical principle like for particles. → Crucial feature: *interference*, i.e. superposition of solutions due to linear equations\(^1\)

QM states that the possible values of the dynamic variables (position, momentum, energy, ...) of a particle follow a probability distribution $|\Psi|^2$ that evolves in time according to a (complex) wave equation.

\(^1\)in cases of non-linearities (e.g. **GROSS–PITAЕVSKII equation**) my following argumentation is of course not valid! A more careful treatment is necessary.
Introduction

first qualitative statement

The **Anderson** localization phenomenon is an interference effect of the wavefunction $\Psi$ that yields non-classical (unexpected) behaviour for an elementary particle moving through a disordered potential.

**Anderson’s original definition [Anderson1958]**

”Absence of Diffusion in Certain Random Lattices”
quantitative criteria for **Anderson** localization

a) *return probability* and *inverse participation number* [van Tiggelen 1998]

- time evolution of spatial eigenvector $|x\rangle$: $|x(t)\rangle = e^{iHt} |x\rangle$
- probability to measure eigenvalue $x$ again: $p[x(t)] = |\langle x|x(t)\rangle|^2$
- expectation value of $p$ (time average): $\mathcal{P}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T p dt$

\[
\rightarrow p(x) = \left|\langle x| 1 e^{iHt} 1 |x\rangle\right|^2 = \left|\sum_{n,m} e^{iE_m t} \phi_n^*(x) \phi_m(x) \delta_{n,m}\right|^2
\]

\[
\mathcal{P}(x) = \sum_{m,n} |\phi_m|^2 |\phi_n|^2 \lim_{T \to \infty} \frac{1}{T} \int_0^T dt e^{i(E_m - E_n)t}
\]

\[
= \sum_n |\phi_n(x)|^4
\]

- *inverse participation number*: $P_n^{-1} = \sum_j |\phi_n(x_j)|^4$, $x_{j+1} = x_j + \triangle x$

**notation**: $1 = \sum_n |n\rangle \langle n|$, $H |n\rangle = E_n |n\rangle$, $\phi_n(x) = \langle x|n\rangle$
quantitative criteria for \textbf{Anderson} localization

\textit{a)} return probability and inverse participation number

\textbf{Interpretation of} $P^{-1}$

- $\phi(x)$ equal distributed/maximal extended:
  \[
P^{-1} = \lim_{N \to \infty} \frac{1}{N^2} = \lim_{N \to \infty} \frac{1}{N} = 0
  \]

- $\phi(x)$ maximal localized: $P^{-1} = \sum_{i=1}^{\infty} \delta_{k,i} \Rightarrow 0 \leq P^{-1} \leq 1$

- further illustration: $|\phi|^4 \Delta x \ll |\phi|^2 \Delta x$  
  $\leftrightarrow P^{-1}$ 'measures' amount of space (in units of $\Delta x$ where $|\phi|^2$ is significantly larger than zero)
quantitative criteria for Anderson localization

b) decay of the wavefunction in the spatial limit to infinity [Sanchez-Palencia et al. 2008]

\[ |\phi(x)|^2 \propto e^{-|x|/\lambda}, \quad \lambda \ldots \text{localization length} \]

\[ \frac{1}{\lambda} = -\lim_{|x| \to \infty} \frac{\ln |\phi(x)|^2}{|x|} \]

we now consider two methods to investigate \( \lambda \):

- transfer matrix method (tight binding approximation)
- phase formalism (single particle Schrödinger equation with random potential)

\(^2\)the following refers to the simplest case of one-dimensional, non-interacting and stationary single-particle physics
quantitative criteria for **Anderson** localization

b.1) transfer matrix method [van Tiggelen1998]

- **second quantization** Hamiltonian under consideration:
  \[ H = \sum_m \epsilon_m a_m^\dagger a_m + J \sum_{\langle m,n \rangle} a_m^\dagger a_n + h.c. \]

- time evolution of the system:
  \[ i \partial_t a_m = [a, H] = \ldots = \epsilon_n a_n + J(a_{n+1} + a_{n-1}) \]
  \[ [a, a^\dagger] = 1 \]

- stationary case: \[ a_n(t) = e^{-iEt} \Rightarrow E a_n = \partial_t a_n \]

\[ \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} \rightarrow \begin{pmatrix} E-\epsilon_n \\ J \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \prod_{i=1}^N t_i \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \]

\[ T_N = \prod_{i=1}^N t_i \]

---

\(^3\) (discrete) **tight binding** approximation using **Wannier**–eigenfunctions: \( a_i^\dagger \) creates a particle at site \( i \), convention: \( \hbar = 1 \)
quantitative criteria for Anderson localization

b.1) transfer matrix method

proven facts:

Fürstenberg theorem

\[ |y| = |T_N(E)x| \leq |T_N(E)||x|, \quad \det t_i = 1 \]

⇒ for almost every\(^a\) \(y\) :

\[ |T_N| \propto e^{N/\lambda(E)}, \quad \lambda^{-1} > 0 \]

\(^a\)restricts the result to a number \(\delta n\) of realizations \(x\) with zero measure (\(\delta n/N = 0\) with respect to the total number \(N\))

Ossedelec theorem

Fürstenberg theorem ⇒ (\(\exists! \ x \Rightarrow |T_N| \propto e^{-N/\lambda}\))
quantitative criteria for **Anderson** localization

(my first simulation results for the **Anderson** model)

![Graph showing trapped situation](image)

**trapped situation:**

\[ E_3 \approx 0.00 \]

\[ \epsilon_n \in [0, 1] = \Delta \]

\[ J = 10 \]

\[ (\kappa = |\Delta| / J \ll 1) \]

**note:**

\[ H |n\rangle = E_n |n\rangle \] with \( n = 1 \ldots 2000 \), i.e. 2000 lattice sites correspond to the system’s length

\( \kappa \) is the only ’free’ parameter of the system (scaling invariance of \( H \))
quantitative criteria for Anderson localization

(my first simulation results for the Anderson model)

barely trapped:

\[ E_{189} \approx 0.86 \]

\[ \epsilon_n \in [0, 1] = \Delta \]

\[ J = 10 \]

\[ \kappa \ll 1 \]
quantitative criteria for **Anderson** localization

(my first simulation results for the **Anderson** model)

classically free:

\[ E_{275} \approx 1.83 \]

\[ \epsilon_n \in [0, 1] = \Delta \]

\[ J = 10 \]

\[ \kappa \ll 1 \]
quantitative criteria for Anderson localization

(my first simulation results for the Anderson model)

border effects:

\[ E_{450} \approx 4.47 \]

\[ \epsilon_n \in [0, 1] = \Delta \]

\[ J = 10 \]

\[ \kappa \ll 1 \]
quantitative criteria for **Anderson** localization

b.2) phase formalism [Sanchez-Palencia et al.2008]

- starting point: 
  \[
  \left[ \frac{-\hbar^2}{2m} \partial_x^2 + V(x) \right] \phi(x) = E \phi(x)
  \]
  with a disordered potential \( V(x) \)

- Solution of second-order ODE via standard reduction to first-order system of ODEs and transforming \( \phi \) and \( \phi' \) to physical meaningful quantities\(^4\) (\( E = \hbar^2 k^2 / 2m \)):

  \[
  \phi(x) = r(x) \sin \theta(x), \quad \phi'(x) = kr(x) \cos \theta(x)
  \]

  \( \hookrightarrow \) resulting differential equations:

  \[
  \theta'(x) = k - \frac{k}{E} V(x) \sin^2 \theta(x) \quad (1)
  \]

  \[
  \frac{r'}{r}(x) = \frac{2k}{E} V(x) \sin[2\theta(x)] \quad (2)
  \]

\(^4\)envelope \( r \) and phase \( \theta \) of the wavefunction
quantitative criteria for Anderson localization

b.2) phase formalism

- for a weak disordered potential $V \ll 1$ we can treat the phase $\theta$ according to its differential equation (1) as slowly varying with respect to $x$, i.e. $\theta(x) = \theta_0 + kx + \delta\theta(x)$ in the language of the Born–approximation with $\delta\theta(x) = -\int_0^x dz \frac{E}{k} V(z) \sin^2(\theta_0 + k z)$

- the integration of (2) yields a factorization\(^5\) to

$$\ln \left[ \frac{r(x)}{r(0)} \right] = |x| / \lambda(k) \quad \text{with} \quad \lambda^{-1} = \frac{m}{4\hbar E} \int_{-\infty}^{\infty} dx C(x) \cos(2 k x)$$

in the limit $|x| \to \infty$ that demonstrates the exponential decay of the wavefunction

\(^5\) this result needs some steps of calculation, $C(x) = \langle V(x - x') V(x) \rangle - \langle V \rangle^2$
quantitative criteria for \textsc{Anderson} localization

b.2) phase formalism

\textbf{crucial point}

from equation (3) it becomes clear that the correlation function of the disordered potential $V$ determines the localization length of the wavefunction, i.e. for weak disorder there is \textsc{Anderson}-localization

$\rightarrow$ in the case of a \textit{speckle potential} we arrive at:

$$
\lambda \propto \frac{k_{\text{max}}^2}{V_0^2 \sigma_c (1 - k_{\text{max}} \sigma_c)^6}
$$

\footnote{the correlation length $\sigma_c$ is defined via the decay of the correlation function $C(x)$}
experimental observation of AL in BECs

**speckle potential** $^{7} V$: 

- $V > 0 \ldots$ blue detuned speckle
- $V < 0 \ldots$ red detuned speckle

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experimentalist’s language:

$V > 0 \ldots$ blue detuned speckle

$V < 0 \ldots$ red detuned speckle

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$^{7}$image source [Lye et al.2005]
experimental observation of AL in BECs
(speckle potential)

properties of speckle potentials\(^8\) \( V \):  

- \( \mathcal{P}(\nu) = e^{-\nu^{-1}}, \nu + 1 \geq 0 \)  
- \( c(x) = \left( \frac{\sin(x)}{x} \right)^2 \)

experimental realization:

- \( V \) ... light intensity with sign from the polarizability \( \alpha(\omega) \)  
  \( V_{em}(x) = -\frac{\alpha(\omega)}{2} \langle E^2(x, t) \rangle_t, \alpha(\omega) \propto (\omega_r - \omega)^{-1} \)  
  \( \omega_r \) is the BEC atoms’ resonancy frequency → red/blue tuning  
- \( C \) ... determined by the transmission function of the diffuse plate

\(^8\)\( V(x) = V_0 \nu(x/\sigma_c), \ C(x) = V_0^2 c(x/\sigma_c), \ C \) ... correlation function
experimental observation of AL in BECs [Sanchez-Palencia et al.2008], [Billy et al.2008]

**experimental setup**

![Diagram of experimental setup](image)

9 images from [Billy et al.2008], modified
experimental observation of AL in BECs

experimental conditions vs. theoretical model:

- kinetic energy of BEC atoms $\gg V_0$ (disorder magnitude)
  $\Rightarrow$ no classical trapping

- low atom density of atoms in the condensate’s wings
  $\Rightarrow$ almost no mutual interaction

- cutoff in $k$ – wavevectors at $k_{max}$
  $\Rightarrow$ observing wavefunction decay transition at $\beta = k_{max} \sigma_c = 1$
  (for $\beta > 1$ algebraic and for $\beta < 1$ exponential wings)
Experimental observation of AL in BECs
(measurement results [Billy et al.2008])

- Exponential wings \((\beta < 1)\)
- Algebraic \((\propto |z|^{-r}, \ r \approx 2)\) wings \((\beta > 1)\)

Remember: Localization length \(\lambda \propto \frac{k_{max}^2}{V_0^2 \sigma_c (1-k_{max} \sigma_c)}\) \(\Rightarrow\) transition at \(\beta = 1\)
experimental observation of AL in BECs

qualitative understanding

Due to the low density of the condensate in the wings the atoms can be treated as almost independent. The wavefunction of each single atom localizes in the random potential $V$ and the superposition of all localizations lengths yields the resulting decay of the condensate’s wings.
Conclusion

1. The wavecharacter of quantum mechanics yields non-classical behaviour of matter, e.g. the phenomenon of Anderson localizations.

2. The quantitative formulation of the AL appears along with several measures, but the exponential-decaying-wavefunction-criterion is a common and useful property which can be explored via different analytical techniques.

3. The experiments with BECs give a direct access to observe $|\Psi|^2$ which was impossible with solid probes before.

4. A careful treatment of the background theory is essential for the result’s interpretation.
References


References

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