Introduction to General Relativity

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This is an outline of the course based on

B. Schutz, A First Course in General Relativity and on S. Carroll, Spacetime and Geometry.

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Contents

Part I

From Special Relativity to Curvature

Chapter 1

Special Relativity

1.1 General definitions

- Special Relativity is basd on two postulates: 1) No experiment can measure the absolute velocity of an observer (Galilean invariance); 2) the speed of light relative to any unaccelerated observer is constant $c = 3 \times 10^8$ m s^{-1} (so c is independent of the velocity of the observer)
- For instance, Newton's second law $\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$ remains the same if the velocity changes into $v - V$, i.e. if observed by an observer with constant velocity V; at the same time we must assume that $\mathbf{F}, \mathbf{m}$ do not change. The same applies to the first law (a body moves at constant velocity even if we add another constant velocity) and the third law.
- Is there an absolute acceleration? In Newtonian physics, and in SR, we assume there is. So observers with constant velocity are special, an we call them *inertial observers* (or Lorentz observers).
- The second postulate changes Newtonian physics: ad es., we cannot longer add velocities as $v_1 + v_2$ because otherwise we could observe a different speed of light.
- An inertial observer is such that: 1) the distance between any two point in its system of coordinate is indep. of time (rigid frame); 2) clocks at every point are and remain synchronized; 3) the geometry of space is Euclidian at any time.
- We will see later on that gravity does not allow to realize such an observer!
- To every event we assign the location x, y, z where it occurs and the time t read by the clock at that point (not at the observer's own clock at the origin).
- A spacetime diagram represents events and world lines $x(t)$. If the units are such that $c = 1$ then a light ray propagates always at 45^0. We use Greek indices x^α $\alpha = 0, 1, 2, 3$ to denote time and space coordinates, and Latin indices x^i,i=1,2,3 to denote spatial ones. The x axis can be defined as the axis such that light emitted Δt before $t = 0$ and reflected back by mirrors on the x axis arrives Δt after $t = 0$ (see Fig.). Events on this axis are *simultaneous* for O.
- A second observer O', moving wrt to O at velocity v along x, has a worldline given by a straight line tilted by an angle $\theta = \arctan v$ wrt to the t axis of O. This is the $\bar{t}$ axis of O' . Its $\bar{x}$ axis can be defined as before: one sees then that the $\bar{x}$ axis is tilted wrt to the x axis by the same angle θ. This show that events that are simultaneous for O are not for O'.
- We define the spacetime interval between any two events

  $$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{1.1.1}$$

  We see that $\Delta s^2 = 0$ for a light ray and the same must be true for every inertial observer. If we assume now that the relation between the coordinates of O and O' is linear and that their origin is the same, one can show (see eg S 1.6) that

  $$\Delta s^2 = \Delta \bar{s}^2 \tag{1.1.2}$$

- Events such that $\Delta s^2 > 0, < 0, = 0$ are called spacelike, timelike, lightlike, respectively. Physical objects only travel along timelike trajectories. The null light cone defines an absolute past and future for the observer at the origin.

- The hyperbola $-t^2 + x^2 = a^2$ is an invariant, since the spacetime distance of any point on this curve from the origin is a, and it is the same for every inertial observer. Tangents to the hyperbola at point P define the line of simultaneity (the $t = const$ axes) for that inertial observer whose time axis joins P to the origin.

- Comparing the time read by a clock moving along the world line from the origin to a point P on the $\bar{t}$ axis, an the the time read by a clock standing still at P, one can derive the effect of *time dilation (*S 1.8).

- We define the propert time $\Delta\tau$ between two events as the time measured by a clock that moves between the points. For clocks at rest in one frame $\Delta\tau = \Delta\bar{t}$ and

$$\Delta s^2 = -\Delta\tau^2 = -\Delta\bar{t}^2 \tag{1.1.3}$$

  Since Δs^2 is invariant, we have

$$\Delta\tau = (\Delta t^2 - \Delta x^2)^{1/2} = \Delta t\sqrt{1 - v^2} \tag{1.1.4}$$

  which is again the time dilatation.

- Similarly, one derive the Lorentz contraction.

1.2 Lorentz transformation

- If we assume that the transformation between x, t and $\bar{x}, \bar{t}$ is linear, we have

$$\bar{t} = \alpha t + \beta x \tag{1.2.1}$$
$$\bar{x} = \gamma t + \sigma x \tag{1.2.2}$$

  while y, z remain unaffected. The constants αetc will depend in general on v. Imposing the same origin, the defintiion of the $\bar{x}$ axis, and the invariance of the interval, we find the expression of the *boost in the x direction*

$$\bar{t} = \gamma(t - vx) \tag{1.2.3}$$
$$\bar{x} = \gamma(x - vt) \tag{1.2.4}$$
$$\gamma = (1 - v^2)^{-1/2} \tag{1.2.5}$$

  Together with rotation and translation, these are the general transformations that leave the interval invariant.

- The law of composition of velocities can be derived directly from these trasnformations.

Chapter 2

Vector and tensor analysis

2.1 Vectors

- Vector notation in two different frames:

$$\Delta\vec{x}_O \rightarrow \{\Delta x^{\alpha}\} \tag{2.1.1}$$
$$\Delta\vec{x}_{\bar{O}} \rightarrow \{\Delta x^{\bar{\alpha}}\} \tag{2.1.2}$$

The vector is the same geometrical object; only the components change from frame to frame.

- In an Lorentz transformation we have (using Einstein summation convention)

$$\Delta x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta}\Delta x^{\beta} \tag{2.1.3}$$

Any object that transforms this way is called a *vector*.

- In any frame there are 4 special vectors, called basis vectors

$$\vec{e}_0 \rightarrow (1,0,0,0) \tag{2.1.4}$$
$$\vec{e}_1 \rightarrow (0,1,0,0) \tag{2.1.5}$$
$$... \tag{2.1.6}$$

or

$$(\vec{e}_{\alpha})^{\beta} = \delta^{\beta}_{\alpha} \tag{2.1.7}$$

Then any vector can be written in terms of the basis vectors:

$$\vec{A} = A^{\alpha}\vec{e}_{\alpha} \tag{2.1.8}$$

- In another frame, $\vec{A} = A^{\bar{\alpha}}\vec{e}_{\bar{\alpha}}$, but the $A's$ are the same objects. Moreover, it is also true that $A^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta}A^{\beta}$. From this we deduce (S 2.2) the law of trasnformation of the basis vectors:

$$\vec{e}_{\alpha} = \Lambda^{\bar{\beta}}_{\alpha}\vec{e}_{\bar{\beta}}$$

Notice that this law is different from (2.1.3).

- By considering that if $\bar{O}$ moves with velocity v wrt O then O moves with $-v$ wrt $\bar{O}$, we have

$$\vec{e}_{\bar{\beta}} = \Lambda^{\alpha}_{\bar{\beta}}(-v)\vec{e}_{\alpha} \tag{2.1.9}$$

which implies

$$\Lambda^{\nu}_{\bar{\beta}}(-v)\Lambda^{\bar{\beta}}_{\alpha}(v) = \delta^{\nu}_{\alpha}$$

That is, the Lorentz matrices for v and $-v$ are the inverse of each other.

- Let us define the four velocity $\vec{U}$ as the vector tangent to the worldline of a particle and of a unit length. In the inertial frame in which the partile is at rest, the four velocity points along the time axis, and is therefore identical to $\vec{e}_0$. This can be extended also to accelerated particles by defining the momentarily comoving reference frame (MCRF).
- The four momentum is defined as

$$\vec{p} = m\vec{U} \tag{2.1.10}$$

where m is the rest mass, ie the mass estimated in the rest frame. In this frame, $\vec{p} = (E, 0, 0, 0)$.

- Performing a boost of v along x from the rest frame, we see that

$$p^{\alpha} = m\Lambda^{\alpha}_{\bar{0}} \tag{2.1.11}$$

and therefore

$$p^0 = m(1 - v^2)^{-1/2} \approx m + \frac{1}{2}mv^2 \tag{2.1.12}$$

$$p^1 = mv(1 - v^2)^{-1/2} \approx mv \tag{2.1.13}$$

which justify calling $p^0 = E$ the energy and p^i the spatial momentum.

- In SR one assumes that the conservation of momentum can be interpreted as the conservation of four-momentum.
- One can easily prove that the scalar product of two vectors

$$\vec{A} \cdot \vec{B} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \tag{2.1.14}$$

is invariant under Lorentz transformations. The vectors are said to be orthogonal if $\vec{A} \cdot \vec{B} = 0$. Geometrically, two vectors are orthogonal if they make equal angle wrt the light ray.

- For the basis vectors we have

$$\vec{e}_{\alpha} \cdot \vec{e}_{\beta} = \eta_{\alpha\beta} \tag{2.1.15}$$

- The vector $d\vec{x}$ is tangent to the wordline, and so it $d\vec{x}/d\tau$, which is also of unit magnitude. It follows that in an inertial frame

$$\frac{d\vec{x}}{d\tau} = (1, 0, 0, 0) = \vec{e}_0 \tag{2.1.16}$$

like the four-velocity. Then

$$\vec{U} = \frac{d\vec{x}}{d\tau} \tag{2.1.17}$$

Notice that $\vec{U} \cdot \vec{U} = -1$. The acceleration vector $\vec{a} = d\vec{U}/d\tau$ is orthogonal to $\vec{U}$.

- The magnitude of the momentum vector is

$$\vec{p} \cdot \vec{p} = -m^2 \tag{2.1.18}$$

from which

$$E^2 = m^2 + \sum_i (p^i)^2 \tag{2.1.19}$$

- In the frame of the observer $\vec{U}_{obs} = (1, 0, 0, 0) = \vec{e}_0$ so

$$-\vec{p} \cdot \vec{U}_{obs} = \bar{E} \tag{2.1.20}$$

That is, the energy of a particle with momentum $\vec{p}$ wrt the observer moving with $\vec{U}_{obs}$ is given by $-\vec{p} \cdot \vec{U}_{obs}$ in any frame. This is a frame-independent quantity.

- The four-velocity of photons cannot be defined since $d\tau = 0$. That is, there is no frame in which light is at rest. However photons do have four-momentum. Since this is parallel to the world line, i.e. to $d\vec{x}$, it is a null vector. Then if the photon energy is E and it moves along x, $p^x = E$. Photon's spatial momentum equals their energy. This also implies that photon's mass must vanish.

- Since $E = h\nu$, we have in another frame

$$\bar{E} = \gamma E - \gamma p^x v = \gamma h\nu - \gamma h\nu v = h\bar{\nu} \tag{2.1.21}$$

from which

$$\frac{\bar{\nu}}{\nu} = \left(\frac{1-v}{1+v}\right)^{1/2} \tag{2.1.22}$$

which is the equation for the relativistic Doppler shift.

2.2 Tensors

- Two vectors on a given basis can be written as

$$\vec{A} = A^\alpha \vec{e}_\alpha \, , \vec{B} = B^\beta \vec{e}_\beta \tag{2.2.1}$$

and their scalar product, using (2.1.15)

$$\vec{A} \cdot \vec{B} = A^\alpha B^\beta (\vec{e}_\alpha \cdot \vec{e}_\beta) = A^\alpha B^\beta \eta_{\alpha\beta} \tag{2.2.2}$$

The numbers $\eta_{\alpha\beta}$ are the component of the *metric tensor*. They allow to associate a single number to two vectors.

- General definition of tensor: A tensor of type $\begin{pmatrix} 0 \\ N \end{pmatrix}$ is a function of N vectors into the real numbers which is linear in each of its N arguments. So e.g. η is a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor. Linearity means that

$$(\alpha\vec{A}) \cdot \vec{B} = \alpha(\vec{A} \cdot \vec{B}) \tag{2.2.3}$$
$$(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \tag{2.2.4}$$

and similarly for $\vec{B}$.

- We can denote the metric tensor with $\mathbf{g}$ and write by definition that

$$\mathbf{g}(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} \tag{2.2.5}$$

Linearity means that

$$\mathbf{g}(\alpha\vec{A} + \beta\vec{B}, \vec{C}) = \alpha\mathbf{g}(\vec{A}, \vec{C}) + \beta\mathbf{g}(\vec{A}, \vec{B}) \tag{2.2.6}$$

- A function $f(t, x, y, z)$ associate a number to no vector at all, and therefore is a $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ tensor, or a *scalar*.

- The components in a frame O of a tensor $\begin{pmatrix} 0 \\ N \end{pmatrix}$ are the values of the tensor when its arguments are the basis vectors $\{\vec{e}_\alpha\}$ in that frame. Therefore as already noticed the components of the metric tensor are

$$\mathbf{g}(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \tag{2.2.7}$$

- The tensor $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is called a covector, a covariant vector, or a one-form. We often use the notation $\tilde{p}$. So $\tilde{p}(\vec{A})$ is a number.

- Given another one-form, we can form new one-forms $\tilde{p}+\tilde{q}, \alpha\tilde{p}$: one-forms form therefore a vector space, sometimes called a dual vector space. The components of $\tilde{p}$ are

$$p_\alpha = \tilde{p}(\vec{e}_\alpha) \tag{2.2.8}$$

and are written with subscript index to distinguish them from superscript indexes of vectors. Similarly, the components of $\tilde{p}(\vec{A})$ are

$$\tilde{p}(\vec{A}) = \tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha = A^0 p_0 + A^1 p_1 + A^2 p_2 + A^3 p_3 \tag{2.2.9}$$

(Notice the positive signs). This is called a *contraction*. Contrary to the the scalar product, this operation does not require a tensor like the metric tensor.

- In a tranformed basis $\vec{e}_{\bar{\beta}}$ we have that the components of $\tilde{p}$ are

$$p_{\bar{\beta}} = \Lambda^{\alpha}_{\bar{\beta}} p_\alpha \tag{2.2.10}$$

which is the same law as those of $\vec{e}_{\bar{\beta}}$

$$\vec{e}_{\bar{\beta}} = \Lambda^{\alpha}_{\bar{\beta}} \vec{e}_\alpha \tag{2.2.11}$$

(which justify the term "covector") and opposite to the components of vectors (which are then called "contravariant"). This property makes the product $A^\alpha p_\alpha$ independent of the transformations.

- Now we want to choose four one-forms $\tilde{\omega}^\alpha$ as a dual basis, such that any one-form can be written as

$$\tilde{p} = p_\alpha \tilde{\omega}^\alpha \tag{2.2.12}$$

Sinc we already know that

$$\tilde{p}(\vec{A}) = p_\alpha A^\alpha \tag{2.2.13}$$

we have, by comparing with

$$\tilde{p}(\vec{A}) = p_\alpha \tilde{\omega}^\alpha(\vec{A}) = p_\alpha \tilde{\omega}^\alpha(A^\beta \vec{e}_\beta) = p_\alpha A^\beta \tilde{\omega}^\alpha(\vec{e}_\beta) \tag{2.2.14}$$

that the dual basis must fulfill the condition

$$\tilde{\omega}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta \tag{2.2.15}$$

which indeed defines the basis one-form. We have then

$$\tilde{\omega}^0 \rightarrow (1,0,0,0) \tag{2.2.16}$$
$$\tilde{\omega}^1 \rightarrow (0,1,0,0) \tag{2.2.17}$$

etc.

- The transformation law for the basis one-form is

$$\tilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\beta} \tilde{\omega}^\beta \tag{2.2.18}$$

- The derivative of a function is a one-form. In fact, let $\phi(t,x,y,z)$ be a scalar field defined at every event $\vec{x}$. Given a curve

$$t = t(\tau)\,, x = x(\tau)\,.. \tag{2.2.19}$$

parametrized by τ (proper time, ie a clock moving on he line) at each point, we have the four-velocity $\vec{U} = (\frac{dt}{d\tau}, \frac{dx}{d\tau}, ..)$. The rate of change of the field ϕ along the curve is

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + .. = \frac{\partial\phi}{\partial t}U^0 + \frac{\partial\phi}{\partial x}U^1 + .. \tag{2.2.20}$$

The vector $\vec{U}$ is transformed into a number $\frac{d\phi}{d\tau}$ linear in $\vec{U}$: we have therefore defined a one-form whose components are $(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{dx}, ...)$, denoted as gradient of ϕ, or $\tilde{d}\phi$ (but ofetn written simply as $d\phi$)

$$\tilde{d}\phi \rightarrow (\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{dx}, ...) \tag{2.2.21}$$

- It is easy to show that indeed the components of the gradient transform as the inverse of the components of tensors:

$$(\tilde{d}\phi)_\alpha = \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}}(\tilde{d}\phi)_\beta = \Lambda^\beta_{\bar{\alpha}}(\tilde{d}\phi)_\beta \tag{2.2.22}$$

  Clearly $\tilde{d}x^\alpha \rightarrow (\frac{\partial x^\alpha}{\partial t}, \frac{\partial x^\alpha}{\partial x}, ..)$ has excatly the same components as the basis one-form. Therefore

$$\tilde{d}x^\alpha = \tilde{\omega}^\alpha \tag{2.2.23}$$

  The gradient of the coordinates is the basis one-form.

- A one-form is said to be normal to a surface if its value is zero on every vector tangent to the surface. This defines "orthogonality" without the use of a metric.

- Given two one-forms we can form a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor by forming the *outer product*

$$\tilde{p} \otimes \tilde{q} \tag{2.2.24}$$

  defined as the tensor such that given two vectors $\vec{A}, \vec{B}$ produces $\tilde{p}(\vec{A}) \cdot \tilde{q}(\vec{B})$. Notice that this is different from $\tilde{q} \otimes \tilde{p}$. The outer product is not commutative.

- One can show (S 3.4) that the basis $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor is

$$\tilde{\omega}^{\alpha\beta} = \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \tag{2.2.25}$$

  and we can write for every $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor

$$\mathbf{f} = f_{\alpha\beta}\tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \tag{2.2.26}$$

- Given any $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor one can always form symmetric and antisymmetric tensors. In component notation

$$h_{(\alpha\beta)} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) \tag{2.2.27}$$

$$h_{[\alpha\beta]} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) \tag{2.2.28}$$

  Clearly any tensor can always be written as the sum of its symmetric and antisymmetric parts. The metric tensor is symmetric.

2.2.1 The metric tensor

- The metric tensor is a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor that maps vector into one-forms.

- In fact, consider the one form $\tilde{V} = g(\vec{V}, \cdot)$. Putting any vector $\vec{A}$ in the free slot this produces a number $\vec{V} \cdot \vec{A}$ and therefore is indeed a one-form. The components of $\tilde{V}$ are

$$V_\alpha = \tilde{V}(\vec{e}_\alpha) = \vec{V} \cdot \vec{e}_\alpha = (V^\beta \vec{e}_\beta) \cdot \vec{e}_\alpha = \eta_{\alpha\beta}V^\beta \tag{2.2.29}$$

  This shows that the metric tensor can be used to raise or lower the indexes of any vector, ie to convert a vector into a one-form. In practice, this means that the components of $\tilde{V}$ are identical to those of $\vec{V}$ except for a change in sign of the time component. Similarly, it follows that

$$A^\alpha = \eta^{\alpha\beta}A_\beta \tag{2.2.30}$$

  where $\eta^{\alpha\beta}$ is the inverse of $\eta_{\alpha\beta}$.

- The magnitude of associates one-forms and vectors $\tilde{p}, \vec{p}$ is the same

$$\tilde{p}^2 = \vec{p}^2 = \eta_{\alpha\beta} p^\alpha p^\beta = \eta^{\alpha\beta} p_\alpha p_\beta \tag{2.2.31}$$

The inner product of one-forms can be defined by using simply the sum of one-forms and their magnitude as

$$\tilde{p} \cdot \tilde{q} = \frac{1}{2}[(\tilde{p} \cdot \tilde{q}) - \tilde{p}^2 - \tilde{q}^2] = -p_0 q_0 + p_1 q_1 + ... = \eta^{\alpha\beta} p_\alpha q_\beta \tag{2.2.32}$$

- Vectors are said to be normal to a surface if they are orthogonal to all tangent vectors; equivalently, if their associate one-forms are normal. A surface is said to be timelike, spacelike, null if their normal vectors are timelike, spacelike, null.

2.2.2 General tensors

- A $\begin{pmatrix} M \\ N \end{pmatrix}$ tensor is a linear function of M one-forms and N vectors into the real numbers. Its components are given when the one-forsm and vectors are the basis one-forms and vectors.

- For instance, the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor $\mathbf{R}$ has components

$$R^\alpha_\beta = \mathbf{R}(\tilde{\omega}^\alpha, \vec{e}_\beta) \tag{2.2.33}$$

In a new frame these become

$$R^{\bar{\alpha}}_{\bar{\beta}} = \Lambda^{\bar{\alpha}}_\mu \Lambda^\nu_{\bar{\beta}} R^\mu_\nu \tag{2.2.34}$$

- The derivative of a $\begin{pmatrix} M \\ N \end{pmatrix}$ tensor produces a $\begin{pmatrix} M \\ N+1 \end{pmatrix}$ tensor, just as the gradient of a function $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ produces a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor. In fact we can write for a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor $\mathbf{T}$

$$\frac{d\mathbf{T}}{d\tau} = (T^\alpha_{\beta,\gamma} \tilde{\omega}^\beta \otimes \tilde{e}_\alpha) U^\lambda \tag{2.2.35}$$

from which we see that

$$\nabla T = (T^\alpha_{\beta,\gamma} \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \otimes \tilde{e}_\alpha) \tag{2.2.36}$$

is a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor. Notice that this derivation requires the basis one-forms to be constant everywhere. If this is not so, the derivative has to be defined in a different way.

Chapter 3

The Energy-Momentum tensor

- We have seen that the momentum four-vector of a particle of mass m in a moving frame are

$$p^{\mu} = (\gamma m, v\gamma m, 0, 0) \tag{3.0.1}$$

from which

$$p^{\mu} p_{\mu} = -m^2 \tag{3.0.2}$$

or $E^2 = m^2 + p^2$, where $p^2 = \delta_{ij} p^i p^j$.

- Newton's second law can be extended to SR by defining a four vector f^{μ}

$$f^{\mu} = m \frac{d^2}{d\tau^2} x^{\mu}(\tau) = \frac{d}{d\tau} p^{\mu}(\tau) \tag{3.0.3}$$

For instance, for the electromagnetic forces, one can show that the force

$$f^{\mu} = -qU^{\lambda} F^{\mu}_{\lambda}$$

- For a collection of particles however the momentum is insufficient to describe the full dynamics. We need to introduce the energy momentum tensor $T^{\mu\nu}$, which is a $(2, 0)$ tensor. This is efined as the flux of p^{μ} across a surface of constant x^{ν}.

- so for instance in a fluid at rest, the flux of p^0 in the x^0 (time) direction is just the energy density, $T^{00} = \rho$, while T^{11} is the flux momentum in the direction x, i.e. the pressure

- For a collection of non-interacting pressureless massive particles (dust) one can show (C 1.9) that

$$T^{\mu\nu} = \rho U^{\mu} U^{\nu} \tag{3.0.4}$$

- A perfect fluid can be completey described by the rest-frame energy density and pressure and by the fluid velocity field. One can show that

$$T^{\mu\nu} = (\rho + p) U^{\mu} U^{\nu} + p\eta^{\mu\nu} \tag{3.0.5}$$

- The energy density is constant and the pressure gradient vanish in the rest frame of a fluid; in tensor form, these conditions ensure that

$$T^{\mu\nu} = \rho U^{\mu} U^{\nu} = 0 \tag{3.0.6}$$

- It is convenient to extract from these four equations the component along the velocity U^{μ} and the component orthogonal to it. The first one is obtained by contracting $U_{\nu} T^{\mu\nu} = \rho U^{\mu} U^{\nu}$. We obtain, in the non-relativistic limit

$$\dot{\rho} + \nabla(\rho \mathbf{v}) = 0 \tag{3.0.7}$$

- The second one, orthogonal to U^μ can be obtained by applying the projection operator

$$P^\mu_\nu = \delta^\mu_\nu + U^\mu U_\nu \tag{3.0.8}$$

and gives, again in the NR limit

$$\rho[\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla p \tag{3.0.9}$$

which is the Euler equation.

- Additional material: Introduction to classical field theory (C 1.10)

Chapter 4

Manifolds

4.1 Definitions

- The Equivalence principle forces us to consider the effect of general coordinate transformation. In this case, the notion of Euclidian geometry need to be revised and we need to study the geometry of general hypersurfaces, called manifolds.
- Intuitive definition: A manifold is a space of dimensionality n that locally resembles the n-Euclidian space $\mathbf{R}^n$. I.e. a manifold is a smooth (infintely differentiable) space.
- For the exact definition, see C 2.2
- On a manifold, we can construct curves $\gamma : \mathbf{R} \to M$; we can visualize these curves as paths on M with a parameter γ that labels points along the curve. A curve with the same path but different γ is a different curve.
- Each curve though a point p defines a directional derivative $\frac{df}{d\lambda}$ at p. We denote as *tangent space* T_p the space of all the directional derivatives along curves though p. This is a vector space: in fact every linear combination of directional derivatives is another directional derivative.
- so in a manifold, vectors are no longer defined in terms of how they transform, but as elements of the tangent space.
- An obvious basis for this space is ∂_μ, i.e. the partial derivative along the coordinates of the map. Infact we have
$$\frac{d}{d\lambda} = \frac{d\mu^\mu}{d\lambda}\partial_\mu \tag{4.1.1}$$
Then we denote the basis $\vec{e}_\mu = \partial_\mu$ as coordinate basis for T_p. Clearly the vectors ∂_μ transform as
$$\partial_{\bar{\mu}} = \frac{\partial x^\nu}{\partial x^{\bar{\mu}}}\partial_\nu \tag{4.1.2}$$
and demanding that the vector $\vec{V} = V^\mu \partial_\mu$ remains constant we obtain the usual transformation law of vectors.
- The commutator of two vectors is another vector whose components are
$$[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu \tag{4.1.3}$$
- Similarly, one-forms are now defined as elements of a cotangent space T_p^* such that the action on a vector $d/d\lambda$ is the directional derivative
$$df(\frac{d}{d\lambda}) = \frac{df}{d\lambda} \tag{4.1.4}$$

- The basis $\tilde{\omega}^\alpha$ for the one-forms can be obtained as before by demanding that

$$\tilde{\omega}^\alpha \vec{e}_\beta = \delta^\alpha_\beta \tag{4.1.5}$$

from which we get that the gradient $\tilde{dx}^\mu$ are the coordinate basis one-forms, since

$$\tilde{dx}^\mu(\partial_\nu) = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \tag{4.1.6}$$

- From these, one can build tensors of all ranks just as before
- In a manifold, the metric is assigned a new symbol, $g_{\mu\nu}$, which is again a symmetric real $(0, 2)$ tensor, possibly nondegenerate.
- If we take the usual spacetime interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{4.1.7}$$

and perform a general transformation of coordinates we obtain a new form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{4.1.8}$$

in which the components $g_{\mu\nu}$ are coordinate dependent. Eg changing from a 3D Euclidian space to polar coordinates we obtain

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{4.1.9}$$

- It can be shown that it is alwasy possible *locally* to put the metric into its canonical form, ie a diagonal with only $\pm 1, 0$ along the diagonal. The metric is called Euclidian or Riemannian if all the entries are $+1$, Lorentzian (or pseudo-Riemannian) if one entry is -1 and all the others $+1$, degenerate if there is at least a 0, and indefinite in the other cases.
- locally here means that at any point p there exists a coordinate transformation such that $g_{\mu\nu}$ is canonical and their first derivatives vanish: this is called locally inertial. Ie, any Lorentzian metric is locally Minkowskian. The signature of a nondegenerate metric (sum of the entries on the diagonal of the canonical form) is the same everywhere.
- Other important concepts are tensor densities and differential forms (See C 2.8, 2.9)

Chapter 5

Curvature

5.1 Covariant derivative

- In a general manifold, we cannot any longer use the standard derivative since it is no longer a tensor. In fact, transforming $\partial_\mu W_\nu$ we obtain

$$\partial_{\bar\mu} W_{\bar\nu} = \frac{\partial x^\mu}{\partial x^{\bar\mu}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x^{\bar\nu}} W_\nu\right) \tag{5.1.1}$$

$$= \frac{\partial x^\mu}{\partial x^{\bar\mu}} \frac{\partial x^\nu}{\partial x^{\bar\nu}}\left(\frac{\partial}{\partial x^\mu} W_\nu\right) + W_\nu \frac{\partial x^\mu}{\partial x^{\bar\mu}} \frac{\partial}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\bar\nu}} \tag{5.1.2}$$

which differs from the tensor trasnformation rule

- We need therefore to correct the standard derivative. To maintain linearity, we search therefore for an expression such that

$$\nabla_\mu V^\mu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{5.1.3}$$

where $\Gamma^\nu_{\mu\lambda}$ are the connection coefficients. We need to impose now the transformation law of a $(1,1)$ tensor:

$$\nabla_{\bar\mu} V^{\bar\mu} = \frac{\partial x^\mu}{\partial x^{\bar\mu}} \frac{\partial x^{\bar\nu}}{\partial x^\nu} \nabla_\mu V^\nu \tag{5.1.4}$$

This implies (see C 3.1)

$$\Gamma^{\bar\nu}_{\bar\mu\bar\lambda} = \frac{\partial x^\mu}{\partial x^{\bar\mu}} \frac{\partial x^\lambda}{\partial x^{\bar\lambda}} \frac{\partial x^{\bar\nu}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\bar\mu}} \frac{\partial x^\lambda}{\partial x^{\bar\lambda}} \frac{\partial^2 x^{\bar\nu}}{\partial x^\mu \partial x^\lambda} \tag{5.1.5}$$

(which shows that the Γs are not tensors).

- If we require moreover that the covariant derivative also obey the two rules

$$\nabla_\mu (T^\lambda_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho} \tag{5.1.6}$$

$$\nabla_\mu \phi = \partial_\mu \phi \tag{5.1.7}$$

then one can show that for one-forms

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \tag{5.1.8}$$

- For any (M,N) tensor, the covariant derivative will include $+\Gamma$ for every upper index, and a $-\Gamma$ for every lower index.

- In any manifold one can have an infinite number of connections. However the difference of any two connection is a tensor (C 3.2).

- To fix one connection, we add two more properties: we require the connection to be torsion-free (ie symmetric) and metric-compatible:

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)} \tag{5.1.9}$$
$$\nabla_\rho g_{\mu\nu} = 0 \tag{5.1.10}$$

  This fixes the connection to the standard form:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}) \tag{5.1.11}$$

  also called Christoffel symbols. The metric compatibility also ensures that the metric commutes with the derivative and therefore with raising and lowering of indexes:

$$g_{\mu\lambda}\nabla_\rho V^\lambda = \nabla_\rho V_\mu \tag{5.1.12}$$

- Notice that covariant derivatives do not commute.
- Christoffel symbols vanish in Cartesian coordinates but not in general coordinates, eg in polar coordinates. Using the symbols we can derive forumals for divergence, curl gradient in curvilinear coordinates. For instance we have

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu) \tag{5.1.13}$$

  where g is the determinant of $g_{\mu\nu}$.
- Stokes's theorem becomes

$$\int_\Sigma \nabla_\mu V^\mu \sqrt{|g|} d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|} d^{n-1} x \tag{5.1.14}$$

  where n_μ is normal to $\partial\Sigma$ and γ_{ij} is the induced metric on $\partial\Sigma$.

5.2 Geodesics and parallel transport

- Given a curve $x^\mu(\lambda)$, a vector (or a tensor) remains constant when transported along the curve if

$$\frac{d}{d\lambda} V^\mu = \frac{dx^\nu}{d\lambda} \frac{\partial}{\partial x^\nu} V^\mu = 0 \tag{5.2.1}$$

- To extend this to a general manifold, we use the cov. derivative:

$$\frac{D}{d\lambda} = \frac{dx^\nu}{d\lambda} \nabla_\nu = 0 \tag{5.2.2}$$

  and define the parallel transport along the path $x^\mu(\lambda)$ the requirement that

$$\frac{D}{d\lambda} V^\mu = \frac{dx^\nu}{d\lambda} \nabla_\nu V^\mu = 0 \tag{5.2.3}$$

  If the covariant derivative of a vector does not vanish, it means the vector is not being parallel-transported. The covariant derivative measures therefore the change of a vector along a given direction relative to the case in which it is parallel-transported
- For a vector it becomes

$$\frac{d}{d\lambda} V^\mu + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} V^\rho = 0 \tag{5.2.4}$$

  Therefore, given a vector at some point along the path $x^\mu(\lambda)$, this equation gives a unique parallel transport along the curve. Notice that the parallel transport depends on the connection. The metric is of course parallel transported by a metric-compatible connection. Moreover, for a metric-compatible connection, the norm of vectors is preserved when the vectors are parallel transported. This ensures that vectors that are timelike, spacelike or null remain so when parallel transported.

- geodesics, the paths of shortest distance between two points, can also be defined as the paths along which the tangent vector is parallel-transported:

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \tag{5.2.5}$$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{5.2.6}$$

- To show that this curve is also the one of shortest distance we should show that the proper time

$$\tau = \int (-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda})^{1/2} d\lambda \tag{5.2.7}$$

is extremized (we consider here timelike trajectories). In fact it turns out to be maximized. The variation gives

$$\delta\tau = -\int \frac{1}{2}(-f)^{-1/2}\delta f d\lambda \tag{5.2.8}$$

where $f = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$. If the parameter λ is chosen to be the proper time τ, then $f = -1$ and we can simply maximize

$$I = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau \tag{5.2.9}$$

Expanding $x^\mu + \delta x^\mu$ and $g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\delta x^\sigma$ one finds that (C p107)

$$\frac{d^2x^\sigma}{d\lambda^2} + [\frac{1}{2}g^{\sigma\rho}(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho})]\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{5.2.10}$$

i.e. the geodesic equation with the metric-compatible connection. So the shortest path in a manifold (with metric) defines the metric-compatible connection.

- However we had to use a particular parametrization, the proper time. We could have used any other related to τ by an affine transformation

$$\tau \to \lambda = a\tau + b \tag{5.2.11}$$

which leaves the geodetic equation unchange.

- The geodesic equation can also be written as

$$p^\lambda \nabla_\lambda p^\mu = 0 \tag{5.2.12}$$

which shows that free-falling particles (i.e test-particles moving only under the action of gravity) keep moving in the direction where their momenta are pointing.

- For light rays, the proper time is zero and cannot be used as affine parameter. We can still parametrize the curve in some other way. For instance, we can choose the parameter λ such that the tangent vector $dx^\mu/d\lambda$ equals the momentum

$$p^\mu = \frac{dx^\mu}{d\lambda} \tag{5.2.13}$$

Solving for p^μ using the geodesic equation and then employing the relation (2.1.20) $E = -p_\mu U^\mu$ one can obtain the frequency of light rays for any observer.

5.3 The cosmological metric

- We can now use all this technology to study the simple but fundamental metric of an expanding universe (see C 3.5)

5.4 The Riemann curvature tensor

- The curvature of a manifold can be quantified by the change of a vector $\vec{V}$ on a infinitesimal loop defined by two vectors $\vec{A}, \vec{B}$. Since the change δV^ρ will be a linear combination of V^σ, A^μ, B^ν, we should expect a relation like

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\rho A^\mu B^\nu \tag{5.4.1}$$

  where $R^\rho{}_{\sigma\mu\nu}$ is called the Riemann tensor. We also expect it to be antisymmetric wrt μ, ν since exchanging A^μ, B^ν one traverse the loop in the opposite direction and shoudl obtain the opposite change $-\delta V^\rho$:

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu} \tag{5.4.2}$$

- Similarly, the Riemann tensor can be defined in terms of the commutator of covariant derivatives: the covariant derivative measures the change of a vector along a given direction relative to the case in which it is parallel-transported. The commutator of two derivatives will measure the change when a vector is transported along two sides of the loop wrt to the change when transporting along the other two sides. If there is no change, the manifold is flat.

- We have then

$$[\nabla_\mu, \nabla_\nu]V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \tag{5.4.3}$$

$$= R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} \nabla_\lambda V^\rho \tag{5.4.4}$$

  where the last term is the torsion tensor, and

$$R^\rho{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{5.4.5}$$

  It can be shown that the Riemann tensor here found is the same as in (5.4.1). Although each single term in $R^\rho{}_{\sigma\mu\nu}$ is not a tensor, the resulting combination is. The definition of the Riemann tensor is independent of the metric: it remains true for any connection.

- The Riemann tensor includes second-order derivatives of the metric: it does not vanish therefore in a locally inertial frame. It vanish if and only if a manifold is flat. It is therefore the curvature tensor. In particular, if the Riemann tensor vanish, we can always construct a coordinate system in which the metric components are constant (demonstration in C 3.6).

- Properties of the Riemann tensor. Antisymmetry

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \tag{5.4.6}$$

$$R_{\rho\sigma\mu\nu} = -R_{\rho\rho\nu\mu} \tag{5.4.7}$$

$$\tag{5.4.8}$$

  Symmetry

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \tag{5.4.9}$$

  Sum of cyclic permutation

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\mu\sigma\nu} = 0 \tag{5.4.10}$$

  It can be shown that these constraints reduce the number of independent components of the Riemann tensor in n dimensions from n^4 to $n^2(n^2-1)/12$, i.e. 20 in 4 dimensions, and only 1 in two dimensions.

- Bianchi identities:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \tag{5.4.11}$$

- Invariant parts of the Riemann tensor are the Ricci tensor (a symmetric tensor) and Ricci or curvature scalar:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \tag{5.4.12}$$

$$R = R^\mu_\mu \tag{5.4.13}$$

- The trace-free part of the Riemann tensor is the Weyl tensor

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) \tag{5.4.14}$$

$$+ \frac{2}{(n-1)(n-2)}g_{\rho[\mu}g_{\nu]\sigma}R \tag{5.4.15}$$

All possible contractions of the Weyl tensor vanish.

- The contraction of the Bianchi identities gives

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \tag{5.4.16}$$

This allos to define a "conserved" tensor, the Einstein tensor:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{5.4.17}$$

such that

$$\nabla^\mu G_{\mu\nu} = 0 \tag{5.4.18}$$

5.5 Symmetries

- Since $p^\mu = mU^\mu$, the geodesic equation $p^\lambda\nabla_\lambda p^\mu = 0$ can be written as

$$m\frac{dp_\mu}{d\tau} = \frac{1}{2}(\partial_\mu g_{\nu\lambda})p^\lambda p^\nu \tag{5.5.1}$$

If the metric is independent of some coordinate x^μ one has $\partial_\mu g_{\nu\lambda}$ and therefore the momentum along x^μ will be conserved. This is an example of a *isometry,* a symmetry of the metric tensor

- There are many possible other forms of symmetries, i.e. the Lorenz group.
- If we introduce the vector $K = \partial_{\sigma*}$ whose components are $K^\mu = (\partial_{\sigma*})^\mu = \delta^\mu_{\sigma*}$, and if the metric is independent of $\sigma*$, we have $p_{\sigma*} = K^\nu p_\nu$ and (C 3.8)

$$p^\mu\nabla_\mu(K^\nu p_\nu) = p^\mu p^\nu\nabla_{(\mu}K_{\nu)} = 0 \tag{5.5.2}$$

- Therefore any vector K_μ such that $\nabla_{(\mu}K_{\nu)} = 0$ generates the conservation of $K^\nu p_\nu$ along the geodesic trajectories. These vectors are called *Killing vectors.* Momentum is conserved along the direction of the Killing vectors. One can also define *Killing tensors for which*

$$\nabla_{(\mu}K_{\nu_1\ldots\nu_\ell)} = 0 \;\to\; p^\mu\nabla_\mu(K^\nu p_\nu) = p^\mu\nabla_\mu(K_{\nu_1\nu_2\ldots}p^{\nu_1}p^{\nu_2}\ldots) = 0 \tag{5.5.3}$$

- An first important property of the Killing vectors is that

$$K^\lambda\nabla_\lambda R = 0 \tag{5.5.4}$$

i.e. the directional derivative of R along K^λ vanishes.

- A second property is that if we define the current

$$J^\mu_T = K_\nu T^{\mu\nu} \tag{5.5.5}$$

where $T^{\mu\nu}$ is the energy-momentum tensor, then we have

$$\nabla_\mu J^\mu_T = (\nabla_\mu K_\nu)T^{\mu\nu} + K_\nu(\nabla_\mu T^{\mu\nu}) = 0 \tag{5.5.6}$$

since $T^{\mu\nu}$is conserved and symmetric. If K^μ is timelike, this allows us to define a conserved total energy over a spacelike hypersurface Σ

$$E_T = \int_\Sigma J^\mu_T n_\mu\sqrt{|\gamma|}d^3x \tag{5.5.7}$$

where γ_{ij} is the induced metric on $\Sigma$and n_μ the normal vector to Σ. More information on this in (C App. E).

- A space with maximal number of symmetries (i.e independent Killing vectors) is $\mathbf{R}^n$ with a flat Euclidian metric. There are n translation and $\frac{1}{2}n(n-1)$ rotation at any point p. Therefore
$$\frac{1}{2}n(n+1) \tag{5.5.8}$$
is the maximal amount of independent Killing vectors one can have in a n-dimensional spacetime. These are called maximally symmetric spaces. We are interested here in Euclidian spaces of maximal symmetry, i.e. the spatial section of spacetimes.
- In these spaces, the curvature is the same everywhere (ie does not change with translations) and in every direction (i.e does not change with rotations). Therefore all the components of the Riemann tensor should be derived from just the curvature scalar.
- We can search for the maximally symmetric Riemann tensor in the neighborhood of a point p since all points should have the same property in a maximmally symmetric space. If the Riemann tensor in local inertial coordinates does not change under general Lorentz transformation, it has to be composed by the only tensor with this property: the Minkowsky metric, the Kronecker delta and the Levi-Civita tensor. It turns out that there is only a combination that matches the symmetries of the Riemann tensor:
$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}[g_{\mu\nu}g_{\rho\sigma} - g_{\rho\nu}g_{\sigma\mu}] \tag{5.5.9}$$
where the proportionality constant must match the contraction of $R_{\mu\nu\rho\sigma}$. So indeed the full Riemann tensor depends only on the constant curvature R.
- The classification of maximal spaces is therefore very simple: R can be positive, negative or zero. Correspondingly we denote these spaces as spherical, hyperbolical or flat.
- Maximal spacetimes with Lorentzian signature are called de Sitter space (positive), anti-de Sitter space (negative) and Minkoswky (null).

5.6 Geodesic deviation

- Let us assume that curves on a two-dimensional manifold can be parametrized by a family $\gamma_s(t)$ such that for each $s \in \mathbf{R}$, γ_s is a curve parametrized by the affine parameter t. The coordinates on this manifold can be chosen to be s, t since they univocallly define the points that belong to the manifold. Then we have two vector fields: the tangent vector to the manifold
$$T^\mu = \frac{\partial x^\mu}{\partial t} \tag{5.6.1}$$
and the deviation vector
$$S^\mu = \frac{\partial x^\mu}{\partial s} \tag{5.6.2}$$
which points from one geodesic to the neighboring ones.
- We can define therefore the relative velocity of geodesics
$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu \tag{5.6.3}$$
and the relative acceleration
$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu \tag{5.6.4}$$
- Since S, T are coordinate basis vectors, their commutator vanishes. Then we have
$$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu \tag{5.6.5}$$
From this it follows (C p.146) that
$$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \tag{5.6.6}$$
This gives the **geodesic deviation equation**. The relative acceleration between two neighboring geodesics is proportional to the curvature. This is a manifestation of tidal gravitational forces.

Part II

Gravitation

Chapter 6

Einstein's equations

6.1 The weak-field limit

- General Relativity is based on the assumption of the Einstein Equivalence Principle: "In small enough regions of spacetime the laws of physics reduce to special relativity: it is impossible to detect the existence of a gravitational field by means of local experiments".
- This of course implies that gravity is a universal force that applies equally to all forms of mass and energy. The idea of GR is therefore that gravity can be described in geometrical terms, i.e. as a property of spacetime geometry rather than as a force. The statement that physics in a small region is special-relativistic is then translated as the geometric property that a manifold can be described as locally inertial.
- Therefore, a perfectly valid law in GR is identical to the same law in SR just changing $\eta_{\mu\nu}$ with $g_{\mu\nu}$ and derivatives with covariant derivatives (**minimal coupling principle**). Although the equations of nature could be more complicated than this, this procedure gives valid tensorial equation with the correct SR limit and therefore can be considered as viable candidates as GR physical laws.
- For instance, let's take the equation of freely-falling particles in flat space

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \tag{6.1.1}$$

and rewrite is as

$$\frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = \frac{d}{d\lambda}\left(\frac{\partial x^\mu}{\partial x^\nu}\frac{dx^\nu}{d\lambda}\right) = \frac{dx^\nu}{d\lambda}\partial_\nu\frac{dx^\mu}{d\lambda} + \delta^\mu_\nu\frac{d^2x^\nu}{d\lambda^2} \tag{6.1.2}$$

$$= \frac{dx^\nu}{d\lambda}\partial_\nu\frac{dx^\mu}{d\lambda} \tag{6.1.3}$$

Then by applying the minimal coupling principle, we obtain the geodesic equation

$$\frac{dx^\nu}{d\lambda}\nabla_\nu\frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{6.1.4}$$

- As another example, we can convert the conservation law

$$\partial_\mu T^{\mu\nu} = 0 \tag{6.1.5}$$

into its GR counterpart

$$\nabla_\mu T^{\mu\nu} = 0 \tag{6.1.6}$$

- One can show easily that the geodesic equation really describes Newtonian gravity for slow particles and weak field (this is called **Newtonian limit**), i.e. small deviation from Minkowski (see C 4.1)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},\ |h_{\mu\nu}| \ll 1 \tag{6.1.7}$$

- One finds that to recover Newtonian gravity one has to assume

$$g_{00} = -(1+2\Phi) \tag{6.1.8}$$

This shows indeed that the gravitational force can be described purely geometrically.

6.2 Einstein's equations

- To derive Einstein equation we can proceed by observing that we need to recover Poisson equation

$$\nabla^2\Phi = 4\pi G\rho \tag{6.2.1}$$

where here $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian in space and ρ the mass density, in the Newtonian limit.

- Since we know that $g_{00} = -(1+2\Phi)$ in the Newtonian limit, we see that we need a tensor equation which contains second derivatives (at least) on the rhs and the tensor generalization of the mass density on the lhs, i.e. something like

$$[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu} \tag{6.2.2}$$

- An obvious non-vanishing symmetric quantity for the rhs would be $R_{\mu\nu}$. However $T_{\mu\nu}$ is conserved, while $R_{\mu\nu}$is not. The next possible choice is now Einstein's tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{6.2.3}$$

- The constant κ can be fixed by obtaining the Newtonian limit. It is not difficult to see that one reduces to

$$R_{00} = \frac{1}{2}\kappa\rho \tag{6.2.4}$$

and that

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00} \tag{6.2.5}$$

where $h_{00} = -2\Phi$. Finally, we see from Poisson equation that $\kappa = 8\pi G$ so that we can write down Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{6.2.6}$$

- We immediately see that by contraction $R = -8\pi GT$ so that in empty space $T_{\mu\nu} = 0$ we have $R = 0$ and

$$R_{\mu\nu} = 0 \tag{6.2.7}$$

- One can also obtain the Lagrangian that produces Einstein equation under variation of the field $g_{\mu\nu}$. It turns out that in empty space this is the Hilbert-Einstein Action (C 4.3)

$$S_H = \int \sqrt{-g}Rd^4x \tag{6.2.8}$$

(g being the determinant of the metric). In presence of matter the action is $S = \frac{1}{16\pi G}S_H + S_M$ where the matter Action is such that

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{6.2.9}$$

- Other interesting topics related to the general formulation of Einstein's equations are the cosmological constant (C 4.5), the energy conditions (C 4.6) , and alternative theories of gravity (C 4.7).

Chapter 7

The Schwarzschild solution

7.1 The Schwarszschild metric

- The simplest non-trivial application of Einstein's equation is to a spherical static isolated object in empty space. As we will see, there is a unique metric that describes the spacetime *outside* a spherical body, the **Schwarzschild metric**

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{7.1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2 d\phi^2$ is the metric in the unit two-sphere. The constant M can be identified with the mass of the spherical object. Notice that this metric is asymptotically flat.

- This metric is a solution of the equation in vacuum

$$R_{\mu\nu} = 0 \tag{7.1.2}$$

- A general metric which is spherically symmetric and static can be written as

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega^2 \tag{7.1.3}$$

- We can however redefine r such that $r' = e^{\gamma(r)}r$ and contemporarily redefine $\beta(r)$ so that at the end we can adopt the simpler but equally general metric

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{7.1.4}$$

where after the manipulation we use again the notation r instead of r'.

- The procedure now is to derive the Christoffel symbols and the corresponding elements of the Ricci tensor (C p. 195). Putting $e^{2(\beta-\alpha)}R_{tt} + R_{rr} = 0$ one gets

$$\alpha = -\beta \tag{7.1.5}$$

while from $R_{\theta\theta} = 0$ we obtain

$$e^{2\alpha} = 1 - \frac{R_S}{r} \tag{7.1.6}$$

where R_S is an integration constant, called the Schwarzschild radius. The metric is now fully determined up to R_S.

- We can again make use of the weak-field limit

$$g_{00} = -(1 + 2\Phi) \tag{7.1.7}$$

where, for a spherical body $\Phi = -\frac{GM}{r}$ to identify

$$R_S = 2GM \tag{7.1.8}$$

which indeed allows us to confirm Eq. (7.1.1).

- Schwarzschild metric has apparent singularities at $r = 0$ and $r = 2GM = R_S$. However, it is not enough for some elements of the metric to vanish for defining a real singularity; one should also prove that the singularity shows up in every coordinate frame.
- It is easy to see that the curvature R or some other scalar formed out of the Riemann tensor becomes infinite when $r \to 0$ but not when $r \to R_s$. For all normal bodies, i.e. expect black holes, R_S is inside the body (where the Schwarzschild metric is no longer valid) and therefore cannot be reached.
- The uniqueness of Schwarzschild solution can be proven by first proving Birkhoff's theorem (C 5.2)

7.2 Geodesics in Schwarzschild metric

- Schwarzschild metric possesses 4 Killing vactors, three due to the radial symmetry and one for time translations. For each of these

$$K_\mu \frac{dx^\mu}{d\lambda} = const \tag{7.2.1}$$

Moreover, for any geodesic with an affine parameter we know that

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\mu}{d\lambda} = const \tag{7.2.2}$$

- These imply that we have two conserved quantitities, energy and angular momentum (or angular momentum per unit mass for massive particles)

$$E = (1 - \frac{2GM}{r}) \frac{dt}{d\lambda} \tag{7.2.3}$$

$$L = r^2 \frac{d\phi}{d\lambda} \tag{7.2.4}$$

Inserting these in (7.2.2) with $const = -\epsilon$ we obtain

$$-E^2 + \left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right) = 0 \tag{7.2.5}$$

or

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E} \tag{7.2.6}$$

where the one-dimensional "potential" is

$$V(r) = \frac{1}{2}\epsilon - \epsilon \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{7.2.7}$$

and $\mathcal{E} = E^2/2$. In this way the problems of the orbits around a Schwarzschild body can be transformed into a one-dimensional problem (see C 5.4). One of the most interesting properties is that circular orbits are stable only for $r > r_c$ where

$$r_c = 6GM \tag{7.2.8}$$

7.3 Tests of General relativity

- See C 5.5 or S 11.1
- The proper radial distance from $r_1$to $r_2$is ($dt = d\theta = d\phi = 0$)

$$l_{i2} = \int ds = \int \sqrt{g_{rr}} dr = \int_{r_1}^{r_2} e^{\beta(r)} dr \tag{7.3.1}$$

using the metric (7.1.4)

- Since the metric is static, p_0 is conserved and we can put $p_0 = -E$, the energy of the particle as measured far away from the star. For any inertial observer at rest at r one has $U^i = 0$ and therefore $U^0 = e^{-\alpha}$ and therefore
$$E^* = -\vec{p} \cdot \vec{U} = e^{-\alpha(r)} E \tag{7.3.2}$$
which indeed reduces to E for $r \to \infty$. Since $E > E^*$.

- For a photon emitted at r_1 by a inertial observer we have $E^* = h\nu_{em}$ and the energy received at infinity is
$$E = h\nu_{rec} = E^* e^{\alpha} = h\nu_{em} e^{\alpha} \tag{7.3.3}$$
so that there is a measurable redshift
$$z = \frac{\nu_{em}}{\nu_{rec}} - 1 = e^{-\alpha} - 1 \approx -\alpha \tag{7.3.4}$$
if $\alpha \ll 1$.

7.4 Black Holes

- Null lines in Schwarzschild metric are given by
$$\frac{dt}{dr} = \pm(1 - \frac{2GM}{r})^{-1} \tag{7.4.1}$$
So for $r \to R_S$ the null cone gets narrower and narrower. This equation can be solved to give
$$t = \pm r^* + const \tag{7.4.2}$$
$$r* = r + 2GM \ln(\frac{r}{2GM} - 1) \tag{7.4.3}$$

- We can now introduce the Eddington-Finkelstein coordinates, such that
$$v = t + r^* \tag{7.4.4}$$
$$u = t - r^* \tag{7.4.5}$$
Then we obtain
$$ds^2 = -(1 - \frac{2GM}{r})dv^2 + (dvdr + drdv) + r^2 d\Omega^2 \tag{7.4.6}$$
and now the radial null curves are $dv/dr = 0$ if infalling and
$$\frac{dv}{dr} = 2(1 - \frac{2GM}{r})^{-2} \tag{7.4.7}$$
if outgoing. If $r < 2GM$, $dv/dr < 0$ i.e. future directed paths are directed towards decreasing r. This means that light rays cannot escape the Schwarzschild event horizon $r = 2GM$ once they are inside, from which the name black hole. Notice that this includes all timelike trajectories, even if accelerated.

- Even this metric however does not cover th full manifold. The Kruskal-Szekeres coordinates are valid throughout the manifold. they are defined as
$$T = (\frac{r}{2GM} - 1)^{1/2} e^{r/4GM} \sinh(\frac{t}{4GM}) \tag{7.4.8}$$
$$R = (\frac{r}{2GM} - 1)^{1/2} e^{r/4GM} \cosh(\frac{t}{4GM}) \tag{7.4.9}$$
so that the metric is
$$ds^2 = \frac{32}{r} G^3 M^3 e^{-r/2GM}(-dT^2 + dR^2) + r^2 d\Omega^2 \tag{7.4.10}$$
where r is implicitly defined by
$$T^2 - R^2 = (1 - \frac{r}{2GM}) e^{r/2GM} \tag{7.4.11}$$

- In this coordinate system, radial null curves are, as in flat space, at

$$T = \pm R + const \tag{7.4.12}$$

and the event horizon $r = 2GM$ is at $T = \pm R$, as it should be for a null surface. A full description of the coordinate properties can be obtained with a Kruskal diagram (see C p 226-229).

7.5 Interior solution

- The Schwarzschild metric applies to vacuum regions outside spherical objects. The interior solution for a star is the generalization when the gravitational equations are sources by a fluid. Let us start with the same metric as before

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2 \tag{7.5.1}$$

and let us assume that the star is composed of a perfect fluid with a radial density $\rho(r)$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{7.5.2}$$

for $r < R$. Since we look for a static solution we can take $U_\mu = (e^\alpha, 0, 0, 0)$, which ensures $U^\mu U_\mu = -1$ and the spatial velocity is zero. If moreover we assume $\rho(r) = \rho_0 = const$ we obtain

$$ds^2 = -e^{2\alpha(r)}dt^2 + \left[1 - \frac{2Gm(r)}{r}\right]^{-1} dr^2 + r^2 d\Omega^2 \tag{7.5.3}$$

where

$$e^{\alpha(r)} = \frac{3}{2}\left(1 - \frac{2GM}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GMr^2}{R^3}\right)^{1/2} \tag{7.5.4}$$

and

$$m(r) \equiv 4\pi \int_0^r \rho(r') r'^2 dr' = \frac{4}{3}\pi r^3 \rho_0 \tag{7.5.5}$$

inside R and $M_0 = 4\pi R^3 \rho_0/3$ outside. The pressure of the fluid can be obtained via the Tolman-Oppenheimer-Volkoff solution (hydrostatic equilibrium)

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]} \tag{7.5.6}$$

If we can assign an equation of state $p = p(\rho)$ we can obtain the profile $p(r)$and $\rho(r)$. Equivalently, if we assign $\rho(r)$ (eg, $\rho = const$), we obtain $p(r)$. In this case one finds that the central pressure remains finite only if (Buchdahl limit)

$$M_{max} = \frac{4R}{9G} \tag{7.5.7}$$

7.6 Rotating black-holes

- For a rotating BH the spherical symmetry is broken and a term $g_{t\phi}$arises. The general metric for a rotating BH is the Kerr metric (B 11.3)

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dt d\phi \tag{7.6.1}$$

$$+ \frac{(r^2 + a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{7.6.2}$$

where

$$\Delta = r^2 - 2Mr + a^2 \tag{7.6.3}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{7.6.4}$$

$$a = \frac{J}{M} \tag{7.6.5}$$

where J is an analog of an angular momentum parameter (dimension $mass^2$) which characterizes the solution along with M (for $a = 0$ we reduce to Schwarzschild and for $r \to \infty$to Minkowsky). The angle ϕ is around the axis of rotation. Note that surfaces of constant t, r do not have the metric of 2-spheres.

- Consider a particle with zero angular momentum, $p_\phi = 0$; this is conserved since $g_{\alpha\beta}$does not depend on ϕ. Then we have

$$\frac{d\phi}{dt} = \frac{p^\phi}{p^t} = \frac{g^{\phi t}}{g^{tt}} = \omega(r, \theta) \neq 0 \tag{7.6.6}$$

This shows that a particle with zero angular momentum at infinity (ie that fall initially radially) is dragged into rotation by the star (Lens-Thirring effect).

- Considering photons on equatoarial orbits, $\theta = \pi/2$, one sees that θ remains constant and that $ds = 0$ implies

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \left[\left(\frac{g_{t\phi}}{g_{\phi\phi}} \right)^2 - \frac{g_{tt}}{g_{\phi\phi}} \right]^{1/2} \tag{7.6.7}$$

For the surface at which $g_{tt} = 0$ this has two solutions

$$\frac{d\phi}{dt} = 0 \tag{7.6.8}$$

$$\frac{d\phi}{dt} = -\frac{2g_{t\phi}}{g_{\phi\phi}} \tag{7.6.9}$$

The first solution is for the photon sent backwards, i.e. againts the sense of rotation and it shows that the photon does not move at all wrt the star. Any massive particle will move slower and therefore corotate with the star. The surface at $g_{tt} = 0$ defines a region that is called *ergosphere*. Inside the ergosphere all matter must rotate with the star, no matter how large is their angular momentum. We have that $g_{tt} = 0$ for

$$r_0 = M + \sqrt{M^2 - a^2 \cos^2 \theta} \tag{7.6.10}$$

(notice it is not a sphere).

- The horizon of Kerr metric is at

$$r_h = M + \sqrt{M^2 - a^2} \tag{7.6.11}$$

The ergosphere lies then outside the horizon. The metric of this surface can be obtained by putting $dt = dr = 0$ in Kerr metric. The area of any $r = const$ surface is given by integrating over $d\Omega$ the determinant

$$A(r) = \int d\phi \sin\theta d\theta \sqrt{(r^2 + a^2)^2 - a^2\Delta} = 4\pi\sqrt{(r^2 + a^2)^2 - a^2\Delta} \tag{7.6.12}$$

and therefore the horizon area ($\Delta = 0$) is

$$A(r_h) = 4\pi(r_h^2 + a^2) \tag{7.6.13}$$

- For the motion of photons in the Kerr metric see S 11.3.

Chapter 8

Gravitational waves

8.1 Gravitational waves

- We have seen that Einstein's equation in the weak field limit and in vacuum are

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) \bar{h}^{\alpha\beta} = 0 \tag{8.1.1}$$

when one imposes the gauge condition

$$\bar{h}^{\alpha\beta}_{\ \ ,\beta} = 0 \tag{8.1.2}$$

Let us assume now that the solution can be written as a *plane-wave*

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \exp(ik_\alpha x^\alpha) \tag{8.1.3}$$

where $k_\alpha$are the components of some one-form and $A^{\alpha\beta}$ the constant components of some tensor.

- Writing the equations as

$$\eta^{\mu\nu}\bar{h}^{\alpha\beta}_{\ \ ,\mu\nu} = 0 \tag{8.1.4}$$

and inserting Eq. (8.1.3) we see that

$$k^\nu k_\nu = 0 \tag{8.1.5}$$

i.e. the four-vector k^ν associated to the one-form k_ν is null. Therefore Eq. (8.1.3) describes a wave-like propagation with the speed of light.

- The value of $\bar{h}^{\alpha\beta}$ is constant when $k_\alpha x^\alpha$ is constant. Usually one writes

$$\vec{k} \to (\omega, \mathbf{k}) \tag{8.1.6}$$

Since $\vec{k}$ is null, we have the so-called dispersion relation

$$\omega^2 = |\mathbf{k}|^2 \tag{8.1.7}$$

- The gauge condition imposes the constraint

$$A^{\alpha\beta}k_\beta = 0 \tag{8.1.8}$$

i.e. the tensor A must be orthogonal to $\vec{k}$.

- The reminaing gauge freedom can be employed to constrain the solution further. One can show in fact that we can require also

$$A^\alpha_{\ \alpha} = 0 \tag{8.1.9}$$

$$A_{\alpha\beta}U^\beta = 0 \tag{8.1.10}$$

where $\vec{U}$ is some fixed four-velocity. The conditions are called traceless-transverse (TT) conditions.

- Putting all these conditions together, one can show that there exists a frame in which the tensor A has only two independent components

$$A_{\alpha\beta}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.1.11}$$

- We need now to discuss what happens to particles when a gravitational waves passes by. First of all, one can show writing down the geodesic equation

$$\frac{d}{d\tau}U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0 \tag{8.1.12}$$

that a particle initially at rest remains at rest, i.e. at the same value of the coordinates.

- then the *proper* distance between two points separated by a coordinate distance ϵ along, say, x is

$$\Delta\ell = \int |ds^2|^{1/2} = \int_0^\epsilon |g_{xx}|^{1/2} dx \approx [1 + \frac{1}{2}h_{xx}^{TT}(x=0)]\epsilon \tag{8.1.13}$$

Since $h_{xx}^{TT} \neq 0$, we see that the proper distance changes in time.

- The resulting picture is then that as the waves passes by, a spherical distribution of particles will oscillate along the x, y axes if $h_{xx}^{TT} \neq 0, h_{xy}^{TT} = 0$, and in the 45^0 direction if $h_{xx}^{TT} = 0, h_{xy}^{TT} \neq 0$. The polarization states of a gravitational wave are rotated by 45^0 wrt each other.

8.2 Generation of gravitational waves

- We now want to solve the equation

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{8.2.1}$$

i.e. no longer in vacuum but near a source. Let us assume then that the source oscillates with frequency Ω,

$$T_{\mu\nu} = S_{\mu\nu}(x^i)e^{-i\Omega t} \tag{8.2.2}$$

and that it is a compact object of size $\ll 2\pi/\Omega$. Let us look for solutions of the form

$$\bar{h}_{\mu\nu} = B_{\mu\nu}(x^i)e^{-i\Omega t} \tag{8.2.3}$$

Then we obtain

$$(\nabla^2 + \Omega^2)B_{\mu\nu} = -16\pi S_{\mu\nu} \tag{8.2.4}$$

- Outside the source, i.e. for $S_{\mu\nu} = 0$, we have the solution

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r}e^{i\Omega r} \tag{8.2.5}$$

where we dropped the second solution proportional to $e^{-i\Omega r}$ because it represents a wave traveling towards the source.

- Integrating Eq. (8.2.4) over a spherical volume that contains the source we obtain

$$A_{\mu\nu} = 4\int S_{\mu\nu}d^3x \equiv 4J_{\mu\nu} \tag{8.2.6}$$

from which

$$\bar{h}_{\mu\nu} = 4J_{\mu\nu}\frac{e^{i\Omega(r-t)}}{r} \tag{8.2.7}$$

- By employing the equations of energy-momentum conservation for $T^{\mu\nu}$ we obtain $J^{\mu 0} = 0$ and $\bar{h}^{\mu 0} = 0$. Then finally we obtain

$$\bar{h}_{jk} = -2\Omega^2 D_{jk} \frac{e^{i\Omega(r-t)}}{r} \tag{8.2.8}$$

where we introduced the quadrupole moment of the source distribution

$$D^{\ell m} \equiv e^{i\Omega t} \int T^{00} x^{\ell} x^{m} d^3x \tag{8.2.9}$$

This is the quadrupole approximation for the generation of grav. waves.

8.3 Energy of gravitational waves

See S 9.4

Chapter 9

Cosmology

9.1 The cosmological metric

- The Cosmological Principle requires the universe to be homogeneous and isotropic. The most general metric in a frame in which every galaxy is at rest is

$$ds^2 = -dt^2 + R^2(t)h_{ij}dx^i dx^j \tag{9.1.1}$$

where $R(t)$ is called scale factor. Because of isotropy, the spatial part has to have the form

$$d\ell^2 = e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \tag{9.1.2}$$

(here $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$) If we impose also the condition that the spatial scalar curvature is constant we obtain (the prime means $\partial/\partial r$)

$$G \equiv G_{ij}g^{ij} = -\frac{1}{r}^2[1-(re^{-2\Lambda})]' = \kappa = const$$

- This gives

$$e^{2\Lambda} = (1 + \frac{1}{3}\kappa r^2 - \frac{A}{r})^{-1} \tag{9.1.3}$$

where A is an integration constant. Demanding regularity at $r = 0$ we put $A = 0$. Redefining $k = -\kappa/3$ we obtain

$$g_{rr} = (1 - kr^2)^{-1} \tag{9.1.4}$$

- The most general homogeneous and isotropic metric is therefore the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 - dt^2 + R^2(t)[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2] \tag{9.1.5}$$

- Althouch every value of k is acceptable, we can always rescale the radial distance r and the function R in such a way that there are only three qualitatively different values, $k = 0, \pm 1$.
- For $k = 0$ we have a *flat* Euclidian metric.
- For $k = +1$ we can define a new coordinate

$$d\chi^2 = \frac{dr^2}{1-r^2} \tag{9.1.6}$$

for which

$$d\ell^2 = d\chi^2 + \sin^2\chi d\Omega^2 \tag{9.1.7}$$

which is the metric of a 3-sphere. This is the *closed* or *spherical* FRW metric.

- For $k = -1$ we have analogously

$$d\ell^2 = d\chi^2 + \sinh^2\chi d\Omega^2 \tag{9.1.8}$$

which is the metric of a 3-sphere. This is the *open* or *hyperbolic* FRW metric.

9.2 Cosmological Distances

- The proper distance $d_0$of a galaxy at location χ from us is, at the present instant t_0

$$d_0 = R(t_0)\chi \tag{9.2.1}$$

Since χ is the fixed coordinate, differentiating we obtain

$$v \equiv \dot{d}_0 = \frac{\dot{R}}{R}|_0 d_0 = H_0 d_0 \tag{9.2.2}$$

where we have introduced the present value of the Hubble parameter, or *expansion rate*

$$H(t) = \frac{\dot{R}}{R}$$

- For this it follows

$$R(t) = R_0 \exp \int_{t_0}^{t} H(t')dt' \tag{9.2.3}$$

This can be expanded for $t \sim t_0$

$$R(t) \quad = \quad R_0[1 + H_0(t - t_0) + \frac{1}{2}(H_0^2 + \dot{H}_0)(t - t_0)^2 + ...] \tag{9.2.4}$$

- If light is emitted by a source when its scale factor was $R(t)$ and arrives at time t_0 when the scale factor was $R(t_0)$, it can be shown that the redshift due to the recession velocity of the source is

$$1 + z = \frac{R(t_0)}{R(t)} \tag{9.2.5}$$

This equation allows us to connect the observations of z with the value of the scale factor. We usually take the arbitrary normalization of $R(t)$ such that $R(t_0) = 1$.

- The relation between distance and reshift is a fundamental one in cosmology. We can measure distances by using the cosmological flux-luminosity relation

$$F = \frac{L}{4\pi d_L^2} \tag{9.2.6}$$

where we defined the luminosity distance

$$d_L = r(1 + z) \tag{9.2.7}$$

An expansion for small redshifts gives

$$d_L = \left(\frac{z}{H_0}\right)\left[1 + \left(1 + \frac{1}{2}\frac{\dot{H}_0}{H_0^2}\right)z\right] + ... \tag{9.2.8}$$

- Similarly, one can define the *angular diameter distance*

$$d_A = D/\theta \tag{9.2.9}$$

where θ is the angle that subtends the proper diameter D of an object transverse to the line of sight.

- The relation between the coordinate distance r and z is obtained by putting $d\theta = d\phi = 0$ in the metric (radial propagation) and by solving for $ds = 0$:

$$\int \frac{dr}{(1 - kr^2)^{1/2}} = \int \frac{dt}{a} = \int \frac{dz}{H(z)} \tag{9.2.10}$$

This requires to know $H(z)$.

9.3 Dynamics of the FRW metric

- The only non-trivial conservation equation in a FRW universe is

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{9.3.1}$$

- For pressureless matter (i.e. dust) we have

$$\rho_m = \rho_0 R^{-3} \tag{9.3.2}$$

while for a relativistic fluid for which $p = \rho/3$ one has

$$\rho_\gamma = \rho_0 R^{-4} \tag{9.3.3}$$

For a generic fluid with constant equation of state $w \equiv p/\rho$ we have

$$\rho_\gamma = \rho_0 R^{-3(1+w)} \tag{9.3.4}$$

- These behaviors imply that there was a time in the past in which the universe energy density was dominated by ρ_γ (radiation-dominated epoch, RDE), followed by a matter-dominated epoch (MDE)
- These equations should be combined with the solutions of Einstein's equations:

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{R^2} \tag{9.3.5}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}(\rho + 3p) \tag{9.3.6}$$

called *Friedmann equations.*

- If we consider a flat space $k = 0$, we obtain $R(t) \sim t^{2/3}$ during MDE and $R(t) \sim t^{1/2}$ during RDE and $\sim t^{2/3(1+w)}$ in general. In both cases, we have that $R = 0$ at some time in the past.
- This shows that in order to have an accelerated universe, one should have $\rho + 3p < 0$.
- If we modify Einstein's equation by adding a term, the famous *cosmological constant*

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{9.3.7}$$

then we would obtain a sort of fluid with effective equation of state $w = -1$. We see that this implies $\rho_\Lambda = const$ and

$$\rho_\Lambda + 3p_\Lambda = -2\rho_\Lambda < 0 \tag{9.3.8}$$

and therefore acceleration.

- It is convenient to define the density parameters

$$\Omega_m = \frac{\rho_m}{\rho_c} \tag{9.3.9}$$

$$\Omega_\gamma = \frac{\rho_\gamma}{\rho_c} \tag{9.3.10}$$

and similarly for every component, where the critical density is

$$\rho_c = \frac{3H^2}{8\pi} \tag{9.3.11}$$

- The first Friedmann equation becomes then

$$\sum_i \Omega_i = 1 \tag{9.3.12}$$

We can describe even the curvature as a "fluid component"

$$\Omega_k = -\frac{k}{R^2 H^2} \tag{9.3.13}$$

- The Friedmann equation can be written also in a form that contains only observable quantities at the present:

$$H^2 = H_0^2 \left[\Omega_{m0}(1+z)^3 + \Omega_{\gamma 0}(1+z)^4 + \Omega_{k0}(1+z)^2 + \Omega_{\Lambda 0} + ...\right] \tag{9.3.14}$$

with the constraint $\sum_i \Omega_{i0} = 1$.

- Various cosmological observation can be summarized by

$$\Omega_{m0} \approx 0.26 \tag{9.3.15}$$
$$\Omega_{\Lambda 0} \approx 0.74 \tag{9.3.16}$$
$$|\Omega_{k0}| \leq 10^{-2} \tag{9.3.17}$$
$$\Omega\gamma 0 \approx 10^{-4} \tag{9.3.18}$$