General Relativity
v 2.9

Luca Amendola
University of Heidelberg
l.amendola@thphys.uni-heidelberg.de
2024
http://www.thphys.uni-heidelberg.de/~amendola/teaching.html

15th July 2024

UNIVERSITÄT
HEIDELBERG
ZUKUNFT
SEIT 1386

Contents

These notes summarize my course on General Relativity at the University of Heidelberg. **The content and the notation follow quite closely B. Schutz, *A First Course on General Relativity*, (Cambridge UP) which is the textbook recommended for this course.** Another useful text is S. Carroll, *Introduction to Space-Time.* For gravitational waves, see also M. Maggiore, *Gravitational Waves,* Vol I. For Cosmology, see my lecture notes or the book L. Amendola and S. Tsujikawa, *Dark Energy, Theory and Observations.* For other topics like Penrose diagrams and more details on Kerr and other black hole solutions, see R. D'Inverno, *Introducing Einsten's Relativity.*

Not much backgound is needed beyond the standard bachelor courses on Classical Mechanics, Linear Algebra, Calculus, Electromagnetism.

Thanks to Clemens Vittmann and Jakob Ullmann and to several other students for sending me corrections and suggestions. Contact me for comments or typos.

Chapter 1

Special Relativity

1.1 General concepts

1. Special Relativity (SR) is based on two postulates: a) no experiment can measure the absolute velocity of an observer, but only its relative velocity with respect to another observer (Galilean invariance); b) the speed of light relative to any unaccelerated observer is constant, $c = 3 \cdot 10^5$ km/sec.

2. The meaning of a) is that if we have an observer moving with velocity $\mathbf{v}(t)$ (bold face symbols denote vectors) with respect to another one, and we add the constant velocity $\mathbf{V}$, then the Newtonian dynamics remains unchanged. That is, if we replace $\mathbf{v}(t)$ with $\mathbf{v}'(t) = \mathbf{v}(t) + \mathbf{V}$, then all three Newtonian laws remain unchanged. In particular the second law

$$\mathbf{F} = m\mathbf{a} \tag{1.1}$$

   does not change since $\mathbf{a} = d\mathbf{v}(t)/dt$ remains the same. Notice that this assumes also that $\mathbf{F}, m$ do not change as well. The first law also remains the same since if $\mathbf{v}(t)$ is constant then also $\mathbf{v}(t) + \mathbf{V}$ is constant; the third law remains the same because it refers to forces and they by assumption are unchanged.

3. That is, every experiment gives the same result if the entire laboratory moves with constant velocity $\mathbf{V}$

4. On the contrary, we can define an "absolute acceleration" in SR, since we can always measure acceleration by detecting a force. For instance, we feel the acceleration on a train as an extra force acting on us.

5. We define an "inertial observer" as an observer that does not detect any external force; for instance if one has a glass of water, an inertial observer will see the level parallel to ground, while a "non-inertial observer" will see a tilt.

6. The proper definition of inertial observer that will be used in the following is a system (or *frame*) that extends to infinity in a Euclidean space, with a rigid frame of coordinates (the distance between any two points, measured by sending and receiving light signals, is fixed), and with a synchronized clock at any position.

7. Gravity has to be excluded from this world. Whenever there is gravity, as we will see, this rigid system cannot exist over a finite amount of space. When gravity is introduced, we need to move to General Relativity. Other forces can instead be present and we can take them into account.

8. In the inertial observer frame, an event is a location (x, y, z) and an instant of time at which that event takes place (time measured by the clock located at (x, y, z)). An event is then characterized by four numbers, (t, x, y, z).

9. We use this *index notation*

$$x^\alpha = \{t, x, y, z\} \tag{1.2}$$

   where $\alpha = 0, 1, 2, 3$. We also use Latin indexes to define purely spatial coordinates, $x^i = \{x, y, z\}$, $i = 1, 2, 3$.

10. Let us now consider an observer $\mathcal{O}$ and draw its time axis as a vertical line. What this observer will see as the locus of events at $t = 0$, i.e. events all simultaneous with its $t = 0$ instant?

11. If it sends a light signal at $t = -a$ to a point located at $x_0 = c|t| = ac$ (we now fix $y = z = const$) where a mirror is located, and receives the signal back at time $t = a$, it will evidently calculate that the mirror reflected the signal at time $t = 0$ (i.e., when the clock located there had the same time as its own clock at $x = 0$). The event of mirroring is therefore simultaneous with $t = 0$. (Figure 1.1)

12. Sending signals at various times, the observer $\mathcal{O}$ can identify a full line of events simultaneous with $t = 0$. This will define the axis of simultaneity x relative to $\mathcal{O}$.

13. From now on, we will measure velocities in units of c. This means that e.g. $v = 0.01$ means $v = 3,000$ km/sec, $v = 1$ means speed of light, etc. Then, time will be measured in meters: a meter of time means the time it takes for light to run over one meter.

14. On the frame of $\mathcal{O}$, the trajectory of a light ray can then be represented by a line on which $x = x_0 + t$, i.e. a straight line with an angle of 45^0. So for instance a light ray emitted by $\mathcal{O}$ at the origin is a bisector line. The two bisector lines originating from $\mathcal{O}$ define two *light cones*, the future one and the past one (see Fig. 1.3).

15. Postulate b) of SR says that light lines are always tilted at 45^0 with respect to *any* inertial observer, not just $\mathcal{O}$.

16. A particle moving with velocity v will be represented instead by a line with angle $\theta = \arctan v$, where θ is counted clockwise from the t axis.

17. The lines that join subsequent events are called *world lines.* A world line can be described mathematically by a time dependent vector $x^\alpha(t)$.

18. All particles moving with velocity less than c will stay inside the light cone of $\mathcal{O}$ with respect of any point on its world line. That is, their tilt will always be less than 45^0 (see Fig. 1.3).

19. Accelerated particles will have curved world lines. If they move always with speed less than c, as it happens for all the particle we know, the tangent $dx(t)/dt$ of their world lines must always be less than 1.

1.2 Relativity of simultaneity

1. Let us now represent another inertial observer frame, $\mathcal{O}'$, moving with velocity v, on the frame $\mathcal{O}$. Let us assume for simplicity that $\mathcal{O}$ and $\mathcal{O}'$ were both at the origin of $\mathcal{O}$ at time $t = 0$. That is, the origins of their frames coincides. The coordinates as seen by $\mathcal{O}'$ will be denoted as $x'^\alpha = \{t', x', y', z'\}$.

2. The new observer is then following a world line that is a straight line passing through $t = x = 0$ and tilted at angle $\tan\theta = v$.

3. Of course $\mathcal{O}'$ might as well assume it is standing still and $\mathcal{O}$ moves with velocity $-v$. The frame of $\mathcal{O}'$ has indeed a representation identical to what we have already seen for $\mathcal{O}$. But now we want to represent the frame of $\mathcal{O}'$ as seen by $\mathcal{O}$.

4. The time axis t' at $x' = 0$ for $\mathcal{O}'$ seen by $\mathcal{O}$ is then the world line itself. What is the axis of simultaneity at $t' = 0$ of $\mathcal{O}'$?

5. If $\mathcal{O}'$ performs the same experiment that $\mathcal{O}$ employed to define its axis of simultaneity, namely sending light signals to mirrors, it will produce an axis as in Figure 1.2

6. The point is again that light signals travel at 45^0 degrees for any observer, so the light signals sent by $\mathcal{O}'$ will be seen by $\mathcal{O}$ as 45^0 lines.

7. The x' axis of $\mathcal{O}'$, i.e. the axis of simultaneity $t' = 0$, will be tilted wrt to the x-axis of $\mathcal{O}$ by the same angle $\tan\theta = v$. Simultaneity is a concept that depends on the observer!

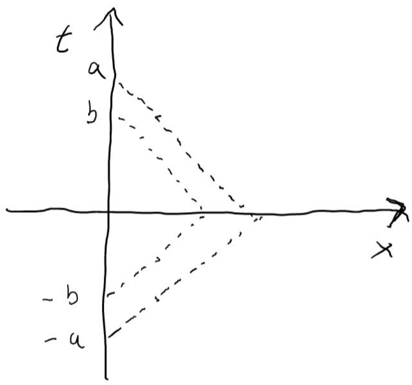

Figure 1.1: Axis of simultaneity.

8. The equation of the t'-axis on the $\mathcal{O}$ plane is then $t = x/v$, while the equation of the x'-axis is $t = vx$. For $v = 1$ they clearly coincide with the light-ray line.

9. The axes y, z are not changed by all this, so we have $y = y'$ and $z = z'$. The simultaneity of events at the same x but at different y, z is not affected by a velocity v directed along x, i.e. perpendicularly to y, z.

1.3 Space-time interval

1. We see that SR can have a direct geometric meaning on the t, x plane. To study geometry we need distances. We define now the important concept of a space-time interval between two events x^α and $x^\alpha + \Delta x^\alpha$:
$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{1.3}$$
(here and in the following, $\Delta x \equiv x_2 - x_1$ is an interval in the coordinate x: this is normally the quantity of observational interest). Notice we could have defined this quantity with opposite signs; all what follows would have been formally identical. The space-time interval defines a Minkowskian space, also called pseudo-Euclidean.

2. The first thing to notice is that, contrary to the usual definition of "distance", Δs^2 is not positive-definite!

3. For a light signal, $(\Delta t)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$, and therefore $\Delta s = 0$. Any two points on a 45^0 line have then zero space-time interval. Light lines are then also called *null* lines.

4. Since postulate b) says that the velocity of light is the same for every inertial observer, if $\Delta s = 0$ for one observer, it remains zero for every observer, i.e. for both $\mathcal{O}$ and $\mathcal{O}'$. We now will show that a natural consequence of postulates a) and b) is that Δs turns out to be equal for every observer even when is not zero. That is, the space-time interval is an invariant quantity in SR.

5. To do so, we must show that Δs does not change under a coordinate transformation that brings us from $\mathcal{O}$ to $\bar{\mathcal{O}}$. (From now on, we use a overbar to define a new observer.) Since observers can only have a

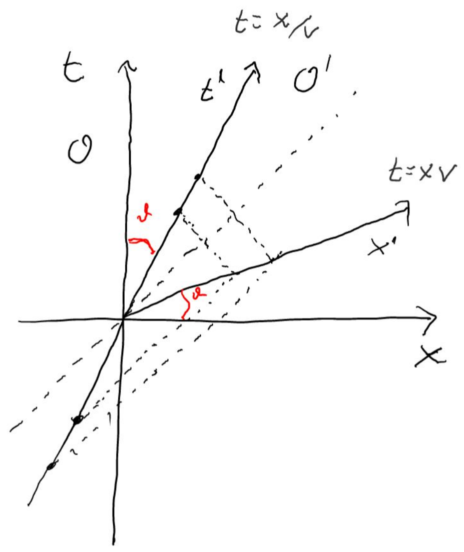

Figure 1.2: Change of frame.

constant relative velocity, we assume a linear general transformation. So, in full generality, for $\bar{\mathcal{O}}$ we write

$$-\Delta\bar{s}^2 = \sum_{\alpha,\beta} M_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) \tag{1.4}$$

(the minus sign is only for later convenience) where $M_{\alpha\beta}$ are the entries of some symmetric matrix (symmetric because if I interchange α, β the result must be the same). This relation must be valid for any space-time interval, so also for light rays.

6. We know that for a light ray, $\Delta s = \Delta\bar{s} = 0$. Also, $\Delta t = \Delta r = [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}$. Then we can write

$$-\Delta\bar{s}^2 = M_{00}(\Delta r)^2 + 2(\Sigma_i M_{0i}\Delta x^i)\Delta r + \Sigma_i\Sigma_j M_{ij}(\Delta x^i)(\Delta x^j) \tag{1.5}$$

As we said above, for a light ray $\Delta\bar{s} = 0$ for every two points. So consider now two points with some given Δx^i and two with $-\Delta x^i$. Then

$$\Delta\bar{s}^2(\Delta x^i) - \Delta\bar{s}^2(-\Delta x^i) = 2\Delta r[\Sigma_i M_{0i}(\Delta x^i + \Delta x^i)] = 0 \tag{1.6}$$

from which we conclude that $M_{0i} = 0$.

7. Now we consider intervals such that $\Delta x = 1$ and $\Delta y = \Delta z = 0$, i.e. $\Delta r = 1$. Then we have

$$-\Delta\bar{s}^2 = M_{00} + M_{11} = 0 \tag{1.7}$$

i.e. $M_{00} = -M_{11}$. Using in turn $\Delta y = 1$ and $\Delta z = 1$ instead of $\Delta x = 1$, we find also $M_{22} = M_{33} = -M_{00}$.

8. Finally, putting $\Delta x = \Delta y = 1$ and $\Delta z = 0$, (so $\Delta r^2 = 2$) we see that

$$-\Delta\bar{s}^2 = 2M_{00} - 2M_{00} + 2M_{12} = 0 \tag{1.8}$$

and therefore $M_{12} = 0$; similarly, one finds $M_{23} = M_{13} = 0$. That is, M_{ij} is a diagonal matrix.

9. We find finally

$$M_{ij} = -M_{00}\delta_{ij} \tag{1.9}$$

where δ_{ij} represents the identity matrix.

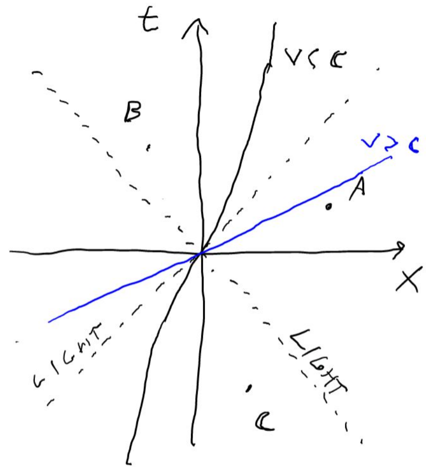

Figure 1.3: World lines.

10. So we obtain
$$\Delta\bar{s}^2 = M_{00}[-(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2] \tag{1.10}$$

11. We still need to find M_{00}. Let us now assume that M_{00} depends on the only variable at hand, namely the relative speed of the observers, $M_{00} = \phi(|v|)$, and not on direction.

12. Consider now a transformation from $\mathcal{O}$ to $\bar{\mathcal{O}}$:
$$\Delta\bar{s}^2 = \phi(|v|)\Delta s^2 \tag{1.11}$$
and now from $\bar{\mathcal{O}}$ back to $\mathcal{O}$:
$$\Delta s^2 = \phi(|v|)\Delta\bar{s}^2 = \phi^2(|v|)\Delta s^2 \tag{1.12}$$
then we see that $\phi(|v|) = \pm 1$. We choose from now on +1 to maintain continuity (i.e., if we had chosen -1, one would have that even for arbitrarily small velocities v the space-time interval jumps from positive to negative values).

13. This demonstrates that $\Delta s = \Delta\bar{s}$ for any inertial observer. That is, the space-time interval is an invariant of SR transformations.

14. The following definitions are therefore also invariant properties:
$$\Delta s^2 = 0, \quad \text{null trajectory} \tag{1.13}$$
$$\Delta s^2 > 0, \quad \text{space-like trajectory} \tag{1.14}$$
$$\Delta s^2 < 0, \quad \text{time-like trajectory} \tag{1.15}$$
So in Figure 1.3, for instance, events B and C are separated by a time-like interval from the origin (or simply, B,C are time-like events), while A is space-like. To see if, e.g., BA is time- or space-like, one has to construct a light-cone centered on B or A.

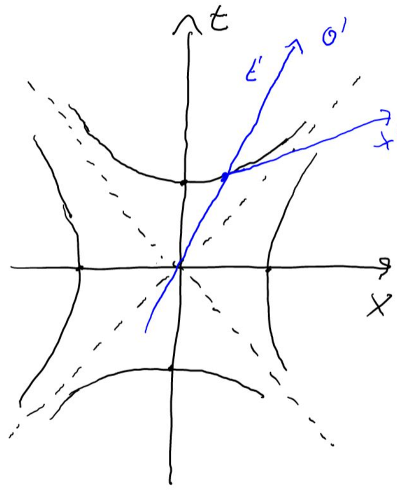

Figure 1.4: Invariant hyperbolae.

1.4 Light-cones and invariant hyperbolae

1. In the space t, x, y, the null lines define a light-cone at every point. With respect to that point, points inside the light cone are said to be separated by time-like intervals (future or past), points outside by space-like intervals, points on the cone by null- or light-intervals.

2. Invariant hyperbolae are defined by the equations of the type

$$-t^2 + x^2 = \pm a^2 \tag{1.16}$$

with the plus sign for space-like points, negative sign for time-like ones. Since such an equation is a space-time interval, it is invariant, i.e. the same for every inertial observer. (Figure 1.4).

3. Points on the invariant hyperbolae have all the same space-time interval wrt the origin. They are then analogous to circles in Euclidean space.

4. Invariant hyperbolae can be used to calibrate distances. If observer O has defined a certain time interval as unity, it can draw the hyperbola

$$-t^2 + x^2 = -1 \tag{1.17}$$

such that, indeed, for $x = 0$ the interval wrt the origin is $t = 1$. This hyperbola will intersect the time axis of $\bar{O}$ at some point t'. The interval from the origin to t' will then be again 1. In this way, O and $\bar{O}$ can define a common time unit. So two clocks, one carried by O and one carried by $\bar{O}$, will tick a unity of time in correspondence of two events connected by the hyperbola.

5. The hyperbola has another important property. The tangent to it at the point $\bar{t}_0$ of intersection of the time line of observer $\bar{O}$ gives the direction of the $\bar{x}$ axis of $\bar{O}$, i.e. its axis of simultaneity with $\bar{t}_0$.

1.5 Time dilation and space contraction

1. By drawing the invariant hyperbola, we immediately realize that observer O will see the time interval 1 of $\bar{O}$ as occurring at a time $t = 1 + \Delta t$ on its clock.

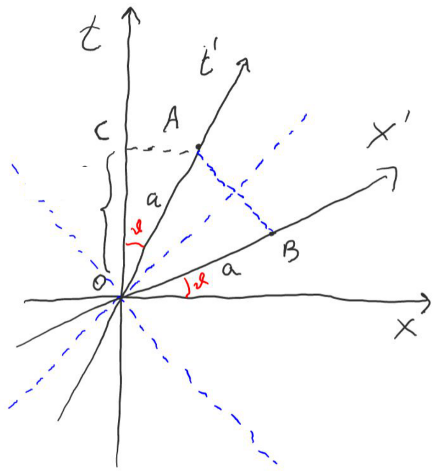

Figure 1.5: Invariance of space-time interval.

2. Simple geometry (see Fig. 1.6) shows that

$$t = \frac{1}{\sqrt{1-v^2}} \tag{1.18}$$

More in general, if $\Delta\bar{t}$ is a time interval between two events as measured by $\bar{O}$, the time interval between the same two events as measured by O will be dilated:

$$\Delta t = \frac{\Delta\bar{t}}{\sqrt{1-v^2}} \tag{1.19}$$

3. We define the proper time-interval between two time-like events (or simply *proper time* for short) as the time measured by a clock that moves with constant velocity through both events. This is evidently a directly measurable quantity since we can always set up an observer whose worldline connects the two events.

4. For a clock at rest at the origin, $\Delta\bar{x} = \Delta\bar{y} = \Delta\bar{z} = 0$ at all times, and one has that the proper time τ coincides with the space-time interval

$$\Delta\bar{s}^2 = -\Delta\bar{t}^2 = -\Delta\bar{\tau}^2 \tag{1.20}$$

This is then an invariant, and observer O will measure the same space-time interval

$$\Delta\tau^2 = -\Delta\bar{s}^2 = \Delta t^2 - \Delta x^2 \tag{1.21}$$

from which we derive again the relation between proper time and time measured by an observer moving with velocity v, namely

$$\Delta\tau = \Delta t\sqrt{1-v^2} \tag{1.22}$$

5. In an analogous manner, we can demonstrate the Lorentz contraction of distances (Fig. 1.7). Let us assume that observer $\bar{O}$ carries a rigid body of length $\bar{\ell} = OC'$ (the length has to be measured in the

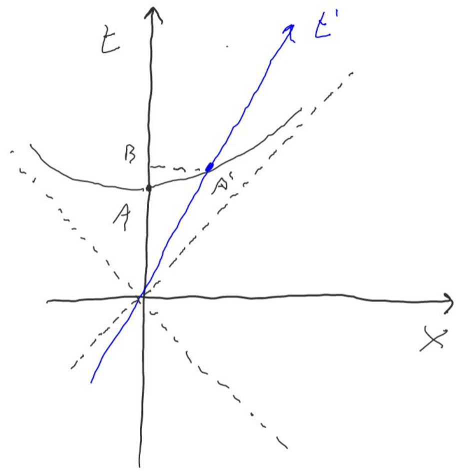

Figure 1.6: Time dilation.

observer's axis of simultaneity). The coordinates of C' can be found from the system

$$-t^2 + x^2 = \bar{\ell}^2 \tag{1.23}$$

$$t = xv \tag{1.24}$$

from which

$$C' = (\frac{v\bar{\ell}}{\sqrt{1-v^2}}, \frac{\bar{\ell}}{\sqrt{1-v^2}}) \tag{1.25}$$

From Figure 1.7 we see the rigid object of length $\bar{\ell}$ for $\bar{O}$ will appear to O (in its axis of simultaneity) of size

$$\ell = x_B - x_O = x_{C'} - vt_{C'} = \frac{\bar{\ell}}{\sqrt{1-v^2}} - v(\frac{v\bar{\ell}}{\sqrt{1-v^2}}) = \bar{\ell}\sqrt{1-v^2} \tag{1.26}$$

That is, the object will appear shorter for O than it is for $\bar{O}$. This is called *Lorentz contraction*. Of course, the same contraction will be measured by $\bar{O}$ for an object at rest with O.

1.6 Lorentz transformations

1. We mentioned before a transformation of coordinates from O to $\bar{O}$. We will now derive the full set of transformations.
2. We begin by assuming a *linear* transformation between x^α and $\bar{x}^\alpha$. The assumption of linearity is a crucial step and cannot be justified in a *a priori* manner, but just by simplicity. It is of course well supported by all the available observations.
3. We assume then that the relation between the O coordinates and the $\bar{O}$ coordinates is of this type:

$$\bar{t} = \alpha t + \beta x \tag{1.27}$$

$$\bar{x} = \gamma t + \sigma x \tag{1.28}$$

$$\bar{y} = y \tag{1.29}$$

$$\bar{z} = z \tag{1.30}$$

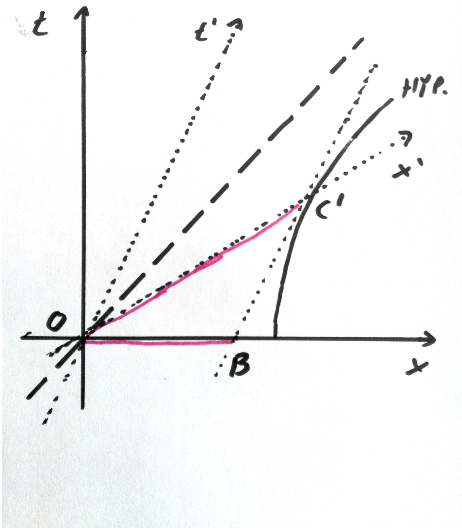

Figure 1.7: Space contraction.

and that $\alpha, \beta, \gamma, \delta$ are constants that only depend on a constant velocity v and that we are going now to determine.

4. We know already that the $\bar{t}$ axis ($\bar{x} = 0$) is given by the equation $vt - x = 0$ and that the $\bar{x}$ ($\bar{t} = 0$) axis is given by $t - vx = 0$. Then from the first one we have the relation

$$\gamma t + \sigma x = 0 \tag{1.31}$$

from which

$$-\frac{\gamma}{\sigma} = \frac{x}{t} = v \tag{1.32}$$

and from the second one

$$\frac{\beta}{\alpha} = -v \tag{1.33}$$

so that

$$\bar{t} = \alpha(t - vx) \tag{1.34}$$
$$\bar{x} = \sigma(-vt + x) \tag{1.35}$$

5. Now, referring to Figure 1.5, consider two points A, B connected by a light signal. In the $\bar{O}$ frame, point A has coordinates $(\bar{t}, \bar{x})$ equal to $(a, 0)_{\bar{O}}$, point B has coordinates $(0, a)_{\bar{O}}$. In the O frame, they have coordinates

$$A_O = \left(\frac{a}{\alpha(1-v^2)}, \frac{va}{\alpha(1-v^2)}\right) \tag{1.36}$$
$$B_O = \left(\frac{va}{\sigma(1-v^2)}, \frac{a}{\sigma(1-v^2)}\right) \tag{1.37}$$

In fact, let's consider for instance point B and apply the transformations 1.34-1.35:

$$0 = \alpha(t - vx) \tag{1.38}$$
$$a = \sigma(-vt + x) \tag{1.39}$$

from which indeed the x-coordinate of B_O is

$$x = \frac{a}{\sigma(1-v^2)} \tag{1.40}$$

while the t-coordinate is v times the x-coordinate. Similarly, one obtains the coordinates of A_O.

6. Now since A, B are separated by $\Delta s^2 = \Delta \bar{s}^2 = 0$, we can also write

$$\Delta s^2 = -(\frac{a}{1-v^2})^2(\frac{1}{\alpha} - \frac{v}{\sigma})^2 + (\frac{a}{1-v^2})^2(\frac{v}{\alpha} - \frac{1}{\sigma})^2 = 0 \tag{1.41}$$

Since this has to be valid for any v, we obtain

$$\alpha = \sigma \tag{1.42}$$

The transformations take then the form

$$\bar{t} = \alpha(t - vx) \tag{1.43}$$
$$\bar{x} = \alpha(-vt + x) \tag{1.44}$$

7. One last step is needed now. Since $\Delta s^2 = \Delta \bar{s}^2$, we have

$$-\Delta\bar{t}^2 + \Delta\bar{x}^2 = -\Delta t^2 + \Delta x^2 \tag{1.45}$$

from which

$$\alpha = \pm\frac{1}{\sqrt{1-v^2}} \equiv \pm\gamma \tag{1.46}$$

where γ is often called "relativistic factor": for $v \to 0$, $\gamma \to 1$, while for a relativistic velocity $v \to 1$, $\gamma \to \infty$. As before, we choose now the positive sign so that for $v = 0$ the Lorentz transformation is an identity.

8. The Lorentz transformation (LT) in one direction (boost with positive velocity v) is then

$$\bar{t} = \gamma(t - vx) \tag{1.47}$$
$$\bar{x} = \gamma(-vt + x) \tag{1.48}$$
$$\bar{y} = y \tag{1.49}$$
$$\bar{z} = z \tag{1.50}$$

Putting back c, the transformations remain the same, except the first one that becomes $\bar{t} = \gamma(t - \frac{vx}{c^2})$.

9. Since O is moving with velocity $-v$ wrt $\bar{O}$, the inverse transformation will be

$$t = \gamma(\bar{t} + v\bar{x}) \tag{1.51}$$
$$x = \gamma(v\bar{t} + \bar{x}) \tag{1.52}$$
$$y = \bar{y} \tag{1.53}$$
$$x = \bar{z} \tag{1.54}$$

10. The postulate of an absolute speed of light means that velocity cannot simply be added. The Lorentz transformation define then a new velocity composition law.

11. Imagine an observer $\bar{O}$ moving with velocity v wrt O; moreover, in the frame of $\bar{O}$, a particle is moving with constant velocity $\bar{w}$ as measured by $\bar{O}$ (Figure 1.8). Which velocity will O measure for the particle?

12. We have $w = \frac{\Delta\bar{x}}{\Delta\bar{t}}$ and we need to evaluate $w_O = \frac{\Delta x}{\Delta t}$. From the inverse LT we have

$$\Delta x = \gamma(v\Delta\bar{t} + \Delta\bar{x}) \tag{1.55}$$
$$\Delta t = \gamma(\Delta\bar{t} + v\Delta\bar{x}) \tag{1.56}$$

from which the composition law becomes

$$w_O = \frac{\Delta x}{\Delta t} = \frac{\gamma(v\Delta\bar{t} + \Delta\bar{x})}{\gamma(\Delta\bar{t} + v\Delta\bar{x})} = \frac{w + v}{1 + wv} \tag{1.57}$$

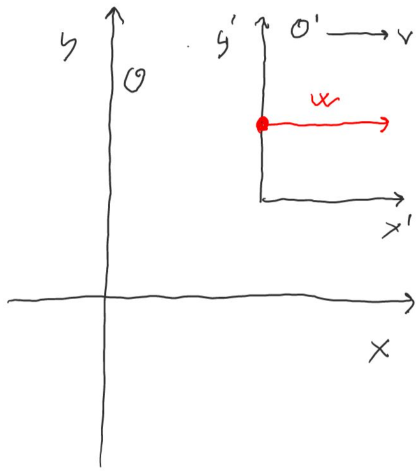

Figure 1.8: Composition of velocities (notice this is not the usual t, x space-time plane, it is a x, y plane).

13. If $w = 1$, then $w_O = 1$ as well, as it should, and viceversa (if $w_O = 1$, one could also have $v = 1$). In other words, if an observer moves with the speed of light, no velocity can be added. If $v, w \ll 1$, one goes back to the usual addition law, $w_O \approx v + w$.

14. The LT can be seen as a rotation in a pseudo-Euclidean plane. defining

$$v = \tanh u \tag{1.58}$$

we can write

$$\bar{t} = t \cosh u - x \sinh u \tag{1.59}$$
$$\bar{x} = -t \sinh u + x \cosh u \tag{1.60}$$

This is then formally similar to a rotation in Euclidean space, if the hyperbolic functions are replaced by the trigonometric ones (up to a sign).

1.7 Vectors in SR

1. We begin now to put the previous results in a form suitable for the mathematical developments of GR.

2. We define the prototypical coordinate vector between two points (events) in the frame O as

$$(\Delta\vec{x})_O \to (\Delta t, \Delta x, \Delta y, \Delta z) \equiv \{\Delta x^\alpha\} \tag{1.61}$$

In another frame $\bar{O}$ this will be denoted as

$$(\Delta\vec{x})_{\bar{O}} \to (\Delta\bar{t}, \Delta\bar{x}, \Delta\bar{y}, \Delta\bar{z}) \equiv \{\Delta x^{\bar{\alpha}}\} \tag{1.62}$$

Note that from now one, we put a bar over the indexes rather than over the variables!

3. An expression like
$$\Delta\bar{t} = \gamma(\Delta t - v\Delta x) \tag{1.63}$$
can now be written as
$$\Delta x^{\bar{0}} = \gamma(\Delta x^0 - v\Delta x^1 + 0\Delta x^2 + 0\Delta x^3) \tag{1.64}$$
$$= \sum_{\beta=0}^{3} \Lambda^{\bar{0}}{}_{\beta}\Delta x^{\beta} \tag{1.65}$$
where
$$\Lambda^{\bar{0}}{}_{\beta} = (\gamma, -v\gamma, 0, 0) \tag{1.66}$$

4. More in general, we write the LT as a matrix Λ such that
$$\Delta x^{\bar{\alpha}} = \sum_{\beta=0}^{3} \Lambda^{\bar{\alpha}}{}_{\beta}\Delta x^{\beta} \tag{1.67}$$
$$= \Lambda^{\bar{\alpha}}{}_{\beta}\Delta x^{\beta} \tag{1.68}$$
In the second line, the symbol of sum is understood whenever there are two identical indexes up and down. This is called *Einstein's convention.* It's a powerful trick to simplify many expressions. Notice that in order to preserve the matching of free indexes, the LT matrix has to be written with indexes up and down (mixed indexes).

5. The transformation can also be interpreted as the partial derivative of the new coordinates wrt the old ones:
$$\Lambda^{\bar{\alpha}}{}_{\beta} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}} \tag{1.69}$$
Since
$$\frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}}\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}} = \delta^{\bar{\alpha}}_{\bar{\beta}} \tag{1.70}$$
(please notice that one is not allowed to "simplify" ∂x^{β}!), we see that $\Lambda^{\beta}{}_{\bar{\alpha}}$ is the inverse of $\Lambda^{\bar{\alpha}}{}_{\beta}$.

6. The order of indexes is in general important, so one must make clear whether an index is the first or the second, e.g. leaving a space as in $\Lambda^{\bar{\alpha}}{}_{\beta}$ or putting a dot as in $\Lambda^{\bar{\alpha}}{}_{.\beta}$. The first index denotes the rows of the matrix, the second denotes the columns. The matrix product
$$\mathbf{C} = \mathbf{A}\cdot\mathbf{B} \tag{1.71}$$
means sum (rows of the first matrix)×(columns of the second matrix), i.e. $C^{\alpha}{}_{\gamma} = A^{\alpha}{}_{\beta}B^{\beta}{}_{\gamma}$. This can be written in any order, e.g. $C^{\alpha}{}_{\gamma} = B^{\beta}{}_{\gamma}A^{\alpha}{}_{\beta}$. But if I write $A^{\beta}{}_{\gamma}B^{\alpha}{}_{\beta}$, I am summing the columns of the first matrix times the rows of the second, that is, I am doing
$$\mathbf{C}' = \mathbf{B}\cdot\mathbf{A} \tag{1.72}$$
which of course is in general a different matrix. For a symmetric matrix, the order of the indexes is not important.

7. The LT (x-boost) is then represented by the *symmetric matrix*
$$\mathbf{\Lambda}_x = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.73}$$

8. The general form of the LT for a boost of speed v along the direction given by the unit vector (n_x, n_y, n_z) is instead the *symmetric matrix*

$$\mathbf{\Lambda}(v,\mathbf{n}) = \begin{pmatrix} \gamma & -\gamma v n_x & -\gamma v n_y & -\gamma v n_z \\ -\gamma v n_x & 1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z \\ -\gamma v n_y & (\gamma-1)n_y n_x & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z \\ -\gamma v n_z & (\gamma-1)n_x n_z & (\gamma-1)n_y n_z & 1+(\gamma-1)n_z^2 \end{pmatrix} \tag{1.74}$$

The inverse of Λ is simply

$$\Lambda^{-1} = \Lambda(-v, \mathbf{n}) \tag{1.75}$$

For $(n_x, n_y, n_z) = (1, 0, 0)$ we are of course back to the x-boost.

9. LT are more general than this. One can still add space and time translations and a spatial rotation. The combination of boosts, translations and rotations form a general LT group, the *Poincaré group*. It is a group because two general LT are another general LT and because there is an identity and an inverse operation. The only LT of interest in SR are however the boosts.

10. Note that in an expression like $\Lambda^{\bar{\alpha}}_{\ \beta}\Delta x^{\beta}$, the symbol that is summed over, β, is a "dummy" index (while $\bar{\alpha}$ is called a free index, ie. is not paired with another $\bar{\alpha}$ in the same term): we could have used any other symbol and still the expression would have had the same meaning. However, we should not have used $\bar{\alpha}$ or any other symbol already taken in the same term.

11. So the equations

$$\Delta x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\ \beta}\Delta x^{\beta} \tag{1.76}$$

$$\Delta x^{\bar{\beta}} = \Lambda^{\bar{\beta}}_{\ \gamma}\Delta x^{\gamma} \tag{1.77}$$

$$\Delta x^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\ \tau}\Delta x^{\tau} \tag{1.78}$$

have exactly the same meaning.

12. A simple rule will emerge more clearly later on but is useful to note already now: the free indexes must match on either side of every equation. An equation like $\Delta x^{\bar{\beta}} = \Lambda^{\bar{\alpha}}_{\ \tau}\Delta x^{\tau}$ does not make sense and should never appear.

13. We will use often also the 4D identity matrix or *Kronecker symbol*

$$\delta^{\alpha}_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.79}$$

By convention, the identity vector is always written with indexes up and down (mixed indexes). In this way we can write well-formed equations like

$$x^{\beta} = x^{\alpha}\delta^{\beta}_{\alpha} \tag{1.80}$$

which, although trivial, are often very useful to manipulate indexes. For the Kronecker symbol, like for all symmetric matrices, the order of index is not important.

14. Now we call *vector* (under a Lorentz transformation) any quadruplet

$$\vec{A}_O \rightarrow (A^0, A^1, A^2, A^3) = \{A^{\alpha}\} \tag{1.81}$$

that has the same transformation properties of x^{α}, i.e. such that

$$A^{\bar{\beta}} = \Lambda^{\bar{\beta}}_{\ \alpha}A^{\alpha} \tag{1.82}$$

Obviously linear combinations of vectors are other vectors, e.g.

$$A^{\alpha} = bx^{\alpha} + y^{\alpha} \tag{1.83}$$

where b is a constant.

15. We define now the *basis vectors.* In the frame O we have four special unitary (i.e., the norm is unity) vectors

$$\begin{aligned} \vec{e}_0 &\to (1,0,0,0) & (1.84)\\ \vec{e}_1 &\to (0,1,0,0) & (1.85)\\ \vec{e}_2 &\to (0,0,1,0) & (1.86)\\ \vec{e}_3 &\to (0,0,0,1) & (1.87) \end{aligned}$$

i.e.

$$(\vec{e}_\alpha)^\beta = \delta_\alpha^\beta \tag{1.88}$$

Notice the position of the index of $\vec{e}_\alpha$ as a subscript, not superscript. This set of basis vector is in the O frame. In the $\bar{O}$ frame we will have four different basis vectors $\vec{e}_{\bar{0}} \to (1,0,0,0), \vec{e}_{\bar{1}} \to (0,1,0,0)$ etc. Their components are the same, *but the vectors are different*! That is, they live in different spaces, one in O, one in $\bar{O}$. In other words, they form different *vector spaces.*

16. Every vector can be expressed in terms of the basis vector (all wrt a given frame). In fact, using the elementary rules for summing vectors, we can write

$$\vec{A} \to (A^0, A^1, A^2, A^3) = A^0\vec{e}_0 + A^1\vec{e}_1 + A^2\vec{e}_2 + A^3\vec{e}_3 = A^\alpha\vec{e}_\alpha \tag{1.89}$$

Notice that the components of the vectors $\vec{e}_\alpha$ can be indicated as $e_\alpha^{\ \beta}$. The equation above can then be written in explicit component form as

$$A^\beta = A^\alpha e_\alpha^{\ \beta} \tag{1.90}$$

which mathematically is totally trivial since $e_\alpha^{\ \beta} = \delta_\alpha^\beta$.

17. Now comes an important concept. A vector, for instance the position of an event in space-time, is an object that is independent of the frame. Then, we can write it equivalently with respect to different frames as

$$\vec{A} = A^\alpha\vec{e}_\alpha = A^{\bar{\alpha}}\vec{e}_{\bar{\alpha}} \tag{1.91}$$

But now we can also transform the frame $\bar{O}$ into O as follows

$$\vec{A} = A^\beta\vec{e}_\beta = A^{\bar{\alpha}}\vec{e}_{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\ \beta}A^\beta\vec{e}_{\bar{\alpha}} \tag{1.92}$$

By comparing the second and the last expression (notice that we deliberately chose the dummy index in the second expression to be β so to ease the comparison with the last expression), we deduce that

$$\vec{e}_\beta = \Lambda^{\bar{\alpha}}_{\ \beta}\vec{e}_{\bar{\alpha}} \tag{1.93}$$

This gives the law of transformation of the basis vectors (Figure 1.9).

18. As an example of LT of a vector, if we put $\vec{A}_O \to (5,0,0,2)$, the transformation induced by a x-boost gives, for the first component in the $\bar{O}$ frame

$$A^{\bar{0}} = \Lambda^{\bar{0}}_{\ 0}A^0 + \Lambda^{\bar{0}}_{\ 1}A^1 + \Lambda^{\bar{0}}_{\ 2}A^2 + \Lambda^{\bar{0}}_{\ 3}A^3 = 5\gamma \tag{1.94}$$

while the same transformation changes the first basis vectors as

$$\vec{e}_{\bar{0}} = \Lambda^{\alpha}_{\ \bar{0}}\vec{e}_\alpha = \gamma\vec{e}_0 + \gamma v\vec{e}_1 \tag{1.95}$$

All the other components and basis vectors can be derived in the same way.

19. The LT matrix $\Lambda^{\bar{\alpha}}_{\ \beta}(v)$ is a transformation that depends on v, i.e. on the relative velocity between the "upper index frame" $\bar{O}$ and the "lower index frame" O. Conversely, $\Lambda^{\alpha}_{\ \bar{\beta}}(-v)$ is the transformation between the "upper index frame" O and the "lower index frame" $\bar{O}$.

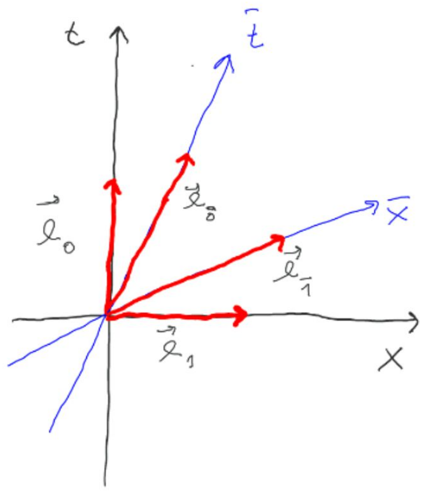

Figure 1.9: Basis vectors.

20. If we apply successively a transformation with velocity v and a transformation with velocity $-v$, we are back to the original frame. That is, from

$$\vec{e}_\alpha = \Lambda^{\bar{\beta}}{}_\alpha(v)\vec{e}_{\bar{\beta}}, \quad \vec{e}_{\bar{\beta}} = \Lambda^\nu_{\bar{\beta}}(-v)\vec{e}_\nu \tag{1.96}$$

we see that

$$\vec{e}_\alpha = \Lambda^{\bar{\beta}}{}_\alpha(v)\vec{e}_{\bar{\beta}} = \Lambda^{\bar{\beta}}{}_\alpha(v)\Lambda^\nu{}_{\bar{\beta}}(-v)\vec{e}_\nu \tag{1.97}$$

from which

$$\Lambda^{\bar{\beta}}{}_\alpha(v)\Lambda^\nu{}_{\bar{\beta}}(-v) = \delta^\nu_\alpha \tag{1.98}$$

i.e., as already mentioned, $\mathbf{\Lambda}(-v)$ is the inverse of $\mathbf{\Lambda}$.

21. Using this property, one can easily obtain the inverse of the vector transformation (1.82) is

$$A^\alpha = \Lambda^\alpha_{\bar{\beta}} A^{\bar{\beta}} \tag{1.99}$$

22. Basis vectors must remain unitary when transformed. This implies that the LT matrix Λ must also be unitary, i.e. its determinant is unity.

1.8 Four-velocity

1. As already said, every linear combination of vectors (in the same frame) is another vector in that frame, so for instance

$$\Delta z^\alpha = c\Delta x^\alpha + b\Delta y^\alpha \tag{1.100}$$

is a vector.

2. Let us the define the *particle frame* (or rest-frame, RF) in a point P. This is the frame $\bar{O}$ that is at rest with a particle in a point (event) P. If the particle has constant velocity, the particle frame is always the same. If the particle velocity changes in time, at every point one should associate a different particle frame. This is called *momentarily comoving rest-frame* (MCRF).

3. A very important vector is the *four-velocity*

$$U^\alpha \equiv \frac{dx^\alpha}{d\tau} \tag{1.101}$$

where τ is the proper time in the particle frame, $d\tau^2 = -ds^2$. This vector is the tangent to a world line in the particle frame and has length equal to one unity of time in that frame. In fact, in that frame at rest with the particle, $dx^1 = dx^2 = dx^3 = 0$ so that $dx^\alpha \to (dt, 0, 0, 0)$ and $ds^2 = -dt^2$, so

$$U^\alpha \to (1, 0, 0, 0) \tag{1.102}$$

Therefore, in the particle frame,

$$\vec{U} = \vec{e}_0 \tag{1.103}$$

4. The *four-momentum* is defined simply as

$$\vec{p} = m\vec{U} \tag{1.104}$$

5. In the RF, then, $\vec{p} = m\vec{e}_{\bar{0}} \to (m, 0, 0, 0)$. In the O frame, we have

$$U^\alpha = \Lambda^\alpha_{\ \bar{\beta}} (\vec{e}_{\bar{0}})^{\bar{\beta}} \tag{1.105}$$

where

$$\Lambda^\alpha_{\ \bar{\beta}} \to \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.106}$$

This gives

$$\begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ v \\ 0 \\ 0 \end{pmatrix} \tag{1.107}$$

6. For small velocities, obviously, we have for the space part, $\mathbf{U} \to (v, 0, 0)$ and $\mathbf{p} \to (mv, 0, 0)$.

7. In general, for small v,

$$p^0 = m\gamma = \frac{m}{\sqrt{1 - v^2}} \approx m + \frac{1}{2}mv^2 \tag{1.108}$$

so we see that for a generic observer moving with small velocity $-v$ with respect to the particle, the 0-component of the 4-momentum is a constant plus the kinetic energy. The constant is called rest energy E_0. If we put back the constant c, we get $E_0 = mc^2$.

8. The general expression for the 4-momentum is then

$$\vec{p} \to (E, p^1, p^2, p^3) = (m\gamma, mv_x\gamma, mv_y\gamma, mv_z\gamma) \tag{1.109}$$

9. The frame in which the space part of the momentum vanishes, is the particle rest frame. If we have several particles, we can define a center of mass (CM) rest frame, defined as that frame in which

$$\sum \vec{p}_i \to (E_{total}, 0, 0, 0) \tag{1.110}$$

1.9 Operations with vectors

1. We need now to learn how to operate with vectors. In analogy with the standard scalar product, we define the scalar product of 4-vectors as

$$\vec{A}^2 = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2 \tag{1.111}$$

We call the result of such an operation a *scalar*.

2. Since $\vec{A}$ is defined to be a quantity that transforms as $\Delta\vec{x}$, and since the scalar product

$$\Delta s^2 = (\Delta\vec{x})^2 \tag{1.112}$$

is invariant, it follows that $(\vec{A})^2$ is invariant as well under a LT. A scalar is a quantity that is invariant under a group of transformations.

3. We can immediately see that the 4-velocity of a time-like world line (i.e. world line such that all its time interval are time-like)

$$\vec{U}\cdot\vec{U} = -1 \tag{1.113}$$

From now on, all 4-velocities will always refer to particles moving along time-like trajectories.

4. Just as for Δs^2, we say that $(\vec{A})^2$ is null, space-like or timelike if it is 0, positive or negative.

5. We can form a scalar product also with two different vectors

$$\vec{A}\cdot\vec{B} = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \tag{1.114}$$

That this is also invariant, can be demonstrated in this way. Let us form the scalar product

$$(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = (\vec{A})^2 + (\vec{B})^2 + 2\vec{A}\cdot\vec{B} \tag{1.115}$$

Now, the term on the lhs is invariant because is the scalar product of $\vec{C} = \vec{A}+\vec{B}$; the first two terms on the rhs are also obviously invariant; therefore, the last term must be be invariant as well.

6. Two 4-vectors are called orthogonal if

$$\vec{A}\cdot\vec{B} = 0 \tag{1.116}$$

Notice that in a pseudo-Euclidean space, orthogonality has nothing to do with angles of 90^0! Two vectors along null lines are always orthogonal, even if they appear “parallel” on the t, x plane. In fact, a null vector is orthogonal to itself.

7. We now introduce one of the most fundamental concept in SR and, even more, in GR. Let us write down all the scalar products among the basis vectors

$$\vec{e}_0\cdot\vec{e}_0 = -1 \tag{1.117}$$
$$\vec{e}_1\cdot\vec{e}_1 = \vec{e}_2\cdot\vec{e}_2 = \vec{e}_3\cdot\vec{e}_3 = 1 \tag{1.118}$$
$$\vec{e}_1\cdot\vec{e}_2 = 0 \tag{1.119}$$

etc. That is

$$\vec{e}_\alpha\cdot\vec{e}_\beta = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.120}$$

The matrix $\eta_{\alpha\beta}$ is called *Minkowski metric.*

8. Using the Minkowski metric the space-time interval can be written as

$$\Delta s^2 = (\Delta\vec{x})^2 = \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta \tag{1.121}$$

1.10 Four-acceleration

1. The vector $d\vec{x}$ is clearly tangent to a world line. As we already mentioned, therefore the 4-velocity vector $\vec{U}$ is also tangent to the world line, with magnitude -1.

2. Similarly, we can define the 4-acceleration
$$\vec{a} = \frac{d\vec{U}}{d\tau} \tag{1.122}$$
Since $\vec{U} \cdot \vec{U} = -1$, we have
$$\frac{d}{d\tau}(\vec{U} \cdot \vec{U}) = 2\vec{U} \cdot \frac{d\vec{U}}{d\tau} = 0 \tag{1.123}$$
This shows that $\vec{a}$ is always orthogonal to $\vec{U}$. In the MCRF, one has therefore
$$\vec{a} \to (0, a^1, a^2, a^3) \tag{1.124}$$

1.11 Energy and momentum

1. The scalar product of the momentum with itself is
$$\vec{p} \cdot \vec{p} = m^2 \vec{U} \cdot \vec{U} = -m^2 \tag{1.125}$$
so
$$-E^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -m^2 \tag{1.126}$$
and therefore
$$E^2 = m^2 + \boldsymbol{p}^2 \tag{1.127}$$
Restoring c, this relation would be
$$E^2 = m^2 c^4 + \boldsymbol{p}^2 c^2 \tag{1.128}$$

2. Consider now an observer $\bar{O}$ in arbitrary motion with respect to a particle. In the $\bar{O}$ frame, its own velocity vector is $\vec{U}_{obs} = \vec{e}_{\bar{0}}$. Therefore, since $\vec{p}_{\bar{O}} \to (\bar{E}, \bar{\mathbf{p}})$, we have
$$\vec{p} \cdot \vec{U}_{obs} = -\bar{E} \tag{1.129}$$
This expression gives therefore the energy of a particle with momentum $\vec{p}$ as seen by an observer with 4-velocity $\vec{U}_{obs}$. This is an example of invariant expression: if we move to another frame $\bar{\bar{O}}$, all elements of $\vec{p}$ and $\vec{U}$ will change, but their scalar product will remain $-\bar{E}$.

3. Notice that the energy an observer measures is not an invariant: even in classical physics, the kinetic energy of a particle is zero in the particle's rest frame, and positive in any other frame. What is invariant is the product $\vec{p} \cdot \vec{U}$. Every inertial observer will agree that the energy measured by an observer with velocity v wrt to a particle with momentum $\vec{p}$ as seen by that observer, is $-\vec{p} \cdot \vec{U}$.

4. On null lines the proper time $d\tau$ vanishes so we cannot define a 4-velocity. Therefore we cannot define a rest frame for a photon. We can of course still define vectors that are tangent to the null line (any vector proportional to an interval $\Delta\vec{x}$ taken on the null line).

5. Therefore, we can define a momentum as a null vector tangent to the null line. In fact, using the definition
$$\vec{p} \cdot \vec{p} = -E^2 + (\mathbf{p})^2 = 0 \tag{1.130}$$
we see that the 4-momentum of a photon is a vector with elements $(E, \boldsymbol{p})$ such that
$$E^2 = \boldsymbol{p}^2 \tag{1.131}$$
This shows that null lines are the trajectories of particles of zero mass.

1.12 Doppler shift

1. We know that the energy of a photon depends on its frequency as

$$E = h\nu \tag{1.132}$$

where h is Planck's constant.

2. Imagine now an observer $\bar{O}$ moving with velocity v along x emits a photon with energy $\bar{E} = h\bar{\nu}$. How will this photon be seen by O?

3. We need to transform the photon momentum. In particular, its component 0 will transform as

$$\bar{E} = \Lambda^{\bar{0}}{}_0 E^0 + \Lambda^{\bar{0}}{}_1 p^1 = \gamma E^0 - \gamma v E^0 \tag{1.133}$$

since $E = p^1$ (photon momentum is a null vector).

4. Then since $E^0 = h\nu$ and $\bar{E} = h\bar{\nu}$, we have

$$\frac{\bar{\nu}}{\nu} = \sqrt{\frac{1-v}{1+v}} \tag{1.134}$$

i.e. $\bar{\nu} < \nu$: the frequency decreases, so the wavelength increases. This is then called *redshift*.

5. For small v, we obtain the classical Doppler shift

$$\frac{\bar{\nu}}{\nu} \approx 1 - v \tag{1.135}$$

1.13 Rindler coordinates

1. The time-like invariant hyperbola $-t^2+x^2 = X^2$, with parameter X, represents a possible physical motion. Obviously, since it is not a straight line, it is not a constant speed motion. One see that it is the motion of a particle coming from $x = +\infty$, initially with the speed of light, then reaching the closest point $x = X$ to the origin at $t = 0$, standing still for a moment, and then moving back to infinity, reaching again asymptotically c.

2. This is actually the motion of a uniformly accelerated particle, with acceleration $1/X$. One can build a family of such hyperbolae with different X, all lying in the quadrant I, defined by $x > 0$, $|t| < |x|$. These hyperbolae define a transformation of the Minkowski space called Rindler coordinates extending to this quadrant, called Rindler wedge (see Fig. 1.10).

3. The proper distance is the spatial distance measured on the axis of simultaneity of the observers that are instantaneously at rest with the hyperbolic motion, i.e. along the straight tilted lines in Fig. 1.10. An important property of the Rindler hyperbolae is that the proper distance between any two Rindler observers remains constant.

4. This means that if one wants to build a laboratory that remains rigid under acceleration, i.e. such that all distances measured within the laboratory are constant, one must realize a (sector of) Rindler wedge, i.e. different parts of the laboratory must move with different acceleration in order for the proper distances not to change. For a 1-dimensional laboratory extended along x and moving towards $+\infty$, this means the trailing part must be more accelerated than the leading part.

5. The transformation from x, t in Minkowski coordinates to X, T in Rindler coordinates is

$$x = X \cosh gT \tag{1.136}$$
$$t = X \sinh gT \tag{1.137}$$

(y, z are unaffected) where g is a constant that we will identify later on as the acceleration. Then one sees that lines of constant X correspond to Rindler hyperbolae $-t^2 + x^2 = X^2$. Lines of constant T are straight tilted lines in the Rindler wedge. The transformation covers only the Rindler wedge, not the entire Cartesian plane as the original coordinates. The Rindler wedge is an example of a *geodesically incomplete* manifold (geodesics will be discussed later; for now, they are just time-like straight lines in Minkowski space).

6. The 4-velocity U^μ is by definition a vector of norm -1 tangent to the hyperbola. Therefore $dx/dt = U^1/U^0 = x^{-1}\sqrt{x^2 - X^2}$ and $-(U^0)^2 + (U^1)^2 = -1$. From these two conditions, we see that

$$U^\mu \to \{\frac{x}{X}, \frac{\sqrt{x^2 - X^2}}{X}, 0, 0\} \tag{1.138}$$

In order to obtain the acceleration, we need to differentiate U^μ wrt τ, but the expression above gives the velocity as a function of x, not τ. However, we know that $U^1 = dx/d\tau$, so we have

$$\tau = \int \frac{X}{\sqrt{x^2 - X^2}} dx = X \text{ArcCoth} \frac{x}{\sqrt{x^2 - X^2}} \tag{1.139}$$

and $x = X \cosh(\tau/X)$ (putting $\tau = 0$ when $x = X$), so we can replace x with τ and obtain

$$U^\mu \to \{\cosh(\tau/X), \sinh(\tau/X), 0, 0\} \tag{1.140}$$

This is the 4-velocity of the MCRF. Notice that, replacing in τ the Rindler coordinates, we see that $\tau = XgT$. On a given hyperbola, in which $X = const$, we can choose $g = 1/X$ so that $\tau = T$. The proper time of a particular Rindler observer, the one with with $X = 1/g$, is then made to coincide with the coordinate time T for all Rindler observers.

7. The four-acceleration as measured in an inertial frame is then

$$a^\mu = \frac{dU^\mu}{d\tau} \to \{\frac{1}{X} \sinh \frac{\tau}{X}, \frac{1}{X} \cosh \frac{\tau}{X}, 0, 0\} \tag{1.141}$$

and therefore

$$a_\mu a^\mu = \frac{1}{X^2} \tag{1.142}$$

so that the four-acceleration has norm $1/X$. The points such that $X = 0$ (i.e. the null-lines $t = \pm x$) cannot be reached because they require an infinite acceleration.

8. Since $U^\mu \to \{\gamma, \gamma \mathbf{v}\}$, acceleration as measured by an IF can be written in general as

$$a^\mu = \frac{dU^\mu}{d\tau} = \{\gamma\dot{\gamma}, \gamma^2 \mathbf{a} + \gamma\dot{\gamma}\mathbf{v}\} \tag{1.143}$$

$$= \gamma^4 \{\mathbf{a} \cdot \mathbf{v}, \mathbf{a} + \mathbf{v} \times (\mathbf{v} \times \mathbf{a})\} \tag{1.144}$$

where $\mathbf{a} = d\mathbf{v}/dt$ and a dot stands for d/dt. (We used the relations $\dot{\gamma} = \gamma^3 \mathbf{v} \cdot \mathbf{a}$ and the triple vector product rule $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$).

9. Since in an instantaneously comoving frame $\mathbf{v} = 0$, it follows $a^\mu = \{0, \mathbf{a}\}$ (proper acceleration), and from (1.142) we see then that the 3D proper acceleration is $1/X$. So Rindler particles with $X = const$ are indeed particles of constant proper acceleration.

10. Rindler observers cannot detect any signal coming from the upper quadrant II, i.e. $t > 0$, $|t| > |x|$: this is an example of a horizon. Similarly, observers in the bottom quadrant IV, i.e. $t < 0$, $|t| > |x|$ will not detect any signal from Rindler particles.

11. In Rindler coordinates, the Minkowski metric becomes

$$ds^2 = -(gX)^2 dT^2 + dX^2 + dY^2 + dZ^2 \tag{1.145}$$

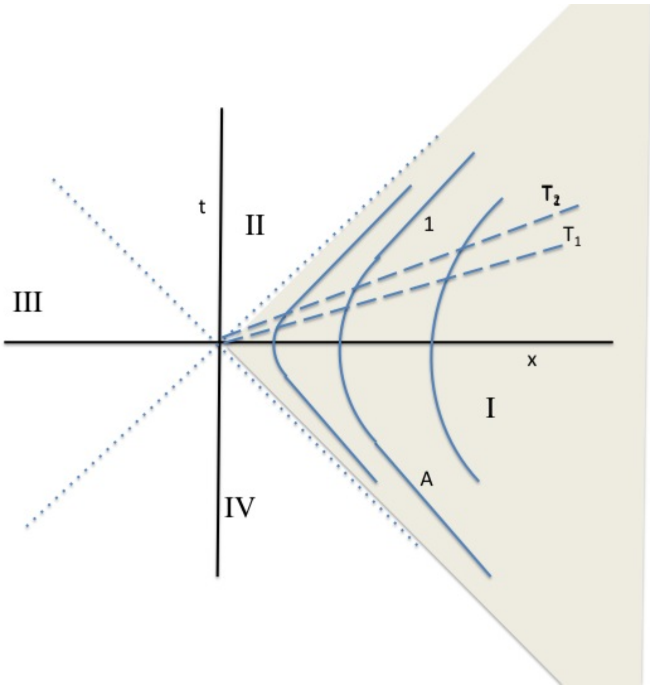

Figure 1.10: Rindler hyperbolae.

Chapter 2

Tensors

2.1 Metric tensor

1. We have seen that
$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \tag{2.1}$$
The quantity $\eta_{\alpha\beta}$ we call *metric tensor.*

2. Decomposing two vectors into their components, we have
$$\vec{A} = A^\alpha \vec{e}_\alpha, \quad \vec{B} = B^\alpha \vec{e}_\alpha \tag{2.2}$$
and therefore
$$\vec{A} \cdot \vec{B} = (A^\alpha B^\beta)(\vec{e}_\alpha \cdot \vec{e}_\beta) = A^\alpha B^\beta \eta_{\alpha\beta} \tag{2.3}$$
(notice that we changed the second index to β to distinguish from the first α). Therefore the metric performs the scalar product. One can see it as a quantity such that, given two vectors, produces a scalar.

3. More exactly, we say that $\eta_{\alpha\beta}$ is a $(0,2)$ tensor.

2.2 General definition of tensors

1. A tensor of type $\begin{pmatrix} 0 \\ N \end{pmatrix}$ or $(0, N)$, is a function of N vectors into the real numbers (i.e. it takes N vectors and produces a real number), linear in each of its arguments.

2. An ordinary function $f(t, x, y, z)$ is then a $(0, 0)$ tensor.

3. *Linearity* means that if we have a tensor $g(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}$, then
$$(\alpha\vec{A}) \cdot \vec{B} = \alpha(\vec{A} \cdot \vec{B}) \tag{2.4}$$
$$(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C} \tag{2.5}$$

4. We will use the notation $f(\vec{A}, \vec{B})$ to express the action of the tensor on the vectors $\vec{A}, \vec{B}$. Eg, for the metric, we have
$$g(\vec{A}, \vec{B}) \coloneqq \vec{A} \cdot \vec{B} \tag{2.6}$$

5. Notice that we made no reference to components. This is then a frame-independent definition.

6. The components of a tensor *in a given frame* can be obtained by inserting the basis vectors of that frame. For instance, for the metric,
$$g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \tag{2.7}$$
This extends to all tensors.

7. Now we introduce the $(0,1)$ tensors. They are also called one-forms, or covariant vectors (the usual vectors we introduced earlier are also called contravariant tensors or $(1,0)$ tensors).

8. A one-form is denoted as $\tilde{p}(\vec{A})$: it takes a vector and gives a scalar (a real number).

9. Because of linearity,

$$\tilde{s} = \tilde{p} + \tilde{q} \tag{2.8}$$
$$\tilde{r} = \alpha\tilde{p} \tag{2.9}$$

are also one-forms. Therefore, one-forms form a vector space, called *dual vector space*. This vector space is completely separated from the standard vector space. An operation like $\vec{A} + \tilde{p}$ makes no sense.

10. The components of a one-form in a given frame is given as usual through the basis vectors

$$p_\alpha := \tilde{p}(\vec{e}_\alpha) \tag{2.10}$$

Components written with a lower index are components of a one-form; components with an upper index are components of a vector.

11. Because of linearity, we can write

$$\tilde{p}(\vec{A}) = \tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha) = A^\alpha p_\alpha \tag{2.11}$$

which is a scalar; this is called a contraction of $\vec{A}$ with $\tilde{p}$.

12. The contraction is just a ordinary sum

$$A^\alpha p_\alpha = A^0 p_0 + A^1 p_1 + A^2 p_2 + A^3 p_3 \tag{2.12}$$

Notice that there is no minus sign; that is, no need of a metric. The contraction is an entity that comes before the introduction of a metric.

13. Let us see now how one-forms transform. We have

$$p_{\bar\beta} = \tilde{p}(\vec{e}_{\bar\beta}) = \tilde{p}(\Lambda^\alpha{}_{\bar\beta}\vec{e}_\alpha) = \Lambda^\alpha{}_{\bar\beta}\tilde{p}(\vec{e}_\alpha) = \Lambda^\alpha{}_{\bar\beta} p_\alpha \tag{2.13}$$

i.e. a similar transformation as for the basis vectors, $\vec{e}_{\bar\beta} = \Lambda^\alpha{}_{\bar\beta}\vec{e}_\alpha$. So, one-forms transform like basis vectors, i.e. in the opposite way (notice the position of barred and unbarred indexes) of vectors.

14. We have seen that $\Lambda^\alpha{}_{\bar\beta}$ is the LT from $\bar{O}$ to O, inverse of a transformation from O to $\bar{O}$. This shows that any contraction like

$$A^\alpha p_\alpha = A^{\bar\alpha} p_{\bar\alpha} \tag{2.14}$$

is automatically invariant under LT.

15. To recap: something that transforms like the basis vectors, is said to be co-variant (lower indexes); something that transforms opposite, is called contra-variant (upper indexes).

16. Just like for vectors, we can now introduce the basis 1-forms, i.e. the basis of the dual vector space. Let us denote them as $\tilde{\omega}^\alpha$. We can then decompose a 1-form into components along the basis one-forms:

$$\tilde{p} = p_\alpha \tilde{\omega}^\alpha \tag{2.15}$$

where $\tilde{\omega}^\alpha$ are four different 1-forms.

17. We can now write

$$\tilde{p}(\vec{A}) = p_\alpha \tilde{\omega}^\alpha(\vec{A}) = p_\alpha \tilde{\omega}^\alpha(A^\beta \vec{e}_\beta) = p_\alpha A^\beta \tilde{\omega}^\alpha(\vec{e}_\beta) \tag{2.16}$$

On the other hand, we know that

$$\tilde{p}(\vec{A}) = p_\alpha A^\alpha \tag{2.17}$$

and therefore

$$A^\alpha = A^\beta \tilde{\omega}^\alpha(\vec{e}_\beta) \tag{2.18}$$

from which we see that

$$\delta^\alpha_\beta = \tilde{\omega}^\alpha(\vec{e}_\beta) \tag{2.19}$$

This relation, finally, defines the components of the dual basis vectors in terms of the basis vectors. Then one has finally in the same frame O of the basis vectors the components

$$\tilde{\omega}^0 \quad \to_O \quad (1,0,0,0) \tag{2.20}$$

$$\tilde{\omega}^1 \quad \to_O \quad (0,1,0,0) \tag{2.21}$$

etc.

18. The 1-form basis vectors transform as the vector components, i.e.

$$\tilde{\omega}^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}_{\ \beta}\tilde{\omega}^\beta \tag{2.22}$$

19. Now we show that the gradient of a function is the prototypical 1-form, just like Δx^α is the prototypical vector. Let us consider a scalar field, ie. a function $\phi(t,x,y,z)$ defined in the entire space-time. Let us also consider a particular world-line parametrized by its proper time τ, with tangent vector

$$\vec{U} \to (\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}) \tag{2.23}$$

20. Consider now the variation of ϕ

$$\frac{d\phi}{d\tau} = \frac{\partial\phi}{\partial t}\frac{dt}{d\tau} + \frac{\partial\phi}{\partial x}\frac{dx}{d\tau} + \frac{\partial\phi}{\partial y}\frac{dy}{d\tau} + \frac{\partial\phi}{\partial z}\frac{dz}{d\tau} \tag{2.24}$$

$$= \frac{\partial\phi}{\partial x^\alpha}U^\alpha \equiv (\tilde{d}\phi)_\alpha U^\alpha \tag{2.25}$$

We see then that $d\phi/d\tau$, ie the derivative of ϕ along a worldline, is a scalar that can be written as the contraction between the gradient of ϕ, denoted as $\tilde{d}\phi$, and the vector $\vec{U}$. Therefore $(\tilde{d}\phi)_\alpha$ are the components of a 1-form:

$$\tilde{d}\phi \to (\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \tag{2.26}$$

and

$$\frac{d\phi}{d\tau} = \tilde{d}\phi(\vec{U}) \tag{2.27}$$

21. One can also directly check that $\tilde{d}\phi$ transforms as a 1-form. In fact,

$$\frac{\partial\phi}{\partial x^{\bar{\alpha}}} = \frac{\partial\phi}{\partial x^\beta}\frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} = \Lambda^\beta_{\ \bar{\alpha}}\frac{\partial\phi}{\partial x^\beta} \tag{2.28}$$

(see eq. 1.69) so

$$(\tilde{d}\phi)_{\bar{\alpha}} = \Lambda^\beta_{\ \bar{\alpha}}(\tilde{d}\phi)_\beta \tag{2.29}$$

22. For the derivatives, we introduce now a handy notation:

$$\frac{\partial\phi}{\partial x} := \phi_{,x} \tag{2.30}$$

$$\frac{\partial\phi}{\partial x^\alpha} := \phi_{,\alpha} \tag{2.31}$$

Also,

$$x^\alpha_{\ ,\beta} = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta \tag{2.32}$$

Notice that $x^{\alpha}_{\ ,\beta}$ are the components of the 1-form $\tilde{d}x^{\alpha}$. This shows that

$$\tilde{d}x^{\alpha} = \tilde{\omega}^{\alpha} \tag{2.33}$$

For any function, we write the differential as

$$\tilde{d}f = \frac{\partial f}{\partial x^{\alpha}} \tilde{d}x^{\alpha} \tag{2.34}$$

23. Metric is then a $(0, 2)$ tensor. The component of the metric can be obtained by multiplication of two vectors

$$\eta_{\alpha\beta} = \vec{e}_{\alpha} \cdot \vec{e}_{\beta} \tag{2.35}$$

Now however we define the metric as a $(0, 2)$ tensor by introducing a new operation, called outer product $\otimes$, between other tensors, in particular the basis $(0, 1)$ forms (i.e. 1-forms). Calling g the metric we can write

$$g = g_{\alpha\beta}\tilde{\omega}^{\alpha} \otimes \tilde{\omega}^{\beta} = g_{\alpha\beta}\tilde{d}x^{\alpha} \otimes \tilde{d}x^{\beta} \tag{2.36}$$

The result of this operation is clearly a scalar: ds^2. This last expression is indeed the space-time interval ds^2 written in a different way: as the linear combination of the basis $\tilde{d}x^{\alpha} \otimes \tilde{d}x^{\beta}$ of the $(0, 2)$ tensor space, with coefficients given by the components $g_{\alpha\beta}$.

24. Again, the important point is that this definition does not refer to components: it's valid for every frame. In general, a new tensor can always be obtained as a similar product

$$f = \tilde{p} \otimes \tilde{q} \tag{2.37}$$

This expression means: the result of the operation $f(\vec{A}, \vec{B})$ is a number given by

$$\tilde{p}(\vec{A}) \cdot \tilde{q}(\vec{B}) \tag{2.38}$$

where here the $\cdot$ is an ordinary multiplication. In components, this is

$$p_{\alpha}A^{\alpha}q_{\beta}B^{\beta} \tag{2.39}$$

25. The order is important: this is different from

$$\tilde{p}(\vec{B}) \cdot \tilde{q}(\vec{A}) \tag{2.40}$$

That is, the operation $\otimes$ is not commutative. Indeed, $p_{\alpha}A^{\alpha}q_{\beta}B^{\beta} \neq p_{\alpha}B^{\alpha}q_{\beta}A^{\beta}$.

26. Although given $\tilde{p}, \tilde{q}$ we can always construct a $(0, 2)$ tensor, not every $(0, 2)$ tensor can be written as a product of two 1-forms, just as not every matrix can be written as the product of two vectors. However, as we show soon, every $(0, 2)$ tensor can be written as the sum of outer products.

27. The outer product can be extended to any number of 1-forms

$$f = \tilde{p} \otimes \tilde{q} \otimes \tilde{t} \tag{2.41}$$

gives

$$f(\vec{A}, \vec{B}, \vec{C}) = \tilde{p}(\vec{A}) \cdot \tilde{q}(\vec{B}) \cdot \tilde{t}(\vec{C}) \tag{2.42}$$

28. The components of a general $(0, 2)$ tensor in a particular frame are as usual obtained by inserting the basis vectors:

$$f_{\alpha\beta} = f(\vec{e}_{\alpha}, \vec{e}_{\beta}) \tag{2.43}$$

so that in general, again using the linearity of the tensor in its arguments, and the fact that $\vec{A} = A^{\alpha}\vec{e}_{\alpha}$, one has

$$f(\vec{A}, \vec{B}) = A^{\alpha}B^{\beta}f_{\alpha\beta} \tag{2.44}$$

29. To find the basis for $(0,2)$ tensors, we proceed as follows. We want to find the $(0,2)$ tensors $\tilde{\omega}^{\alpha\beta}$ such that every tensor can be decomposed as

$$f = f_{\alpha\beta}\tilde{\omega}^{\alpha\beta} \tag{2.45}$$

Then we have

$$f_{\mu\nu} = f(\vec{e}_\mu, \vec{e}_\nu) = f_{\alpha\beta}\tilde{\omega}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu) \tag{2.46}$$

So, comparing with the expression

$$f_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu f_{\alpha\beta} \tag{2.47}$$

we obtain

$$\tilde{\omega}^{\alpha\beta}(\vec{e}_\mu, \vec{e}_\nu) = \delta^\alpha_\mu \delta^\beta_\nu \tag{2.48}$$

Finally, since $\delta^\alpha_\mu = \tilde{\omega}^\alpha(\vec{e}_\mu)$, we have the set of basis $(0,2)$ tensor:

$$\tilde{\omega}^{\alpha\beta} = \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \tag{2.49}$$

Every $(0,2)$ tensor can then be explicitly decomposed as

$$f = f_{\alpha\beta}\tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \tag{2.50}$$

30. We say that a tensor is symmetric if

$$f(\vec{A}, \vec{B}) = f(\vec{B}, \vec{A}) \tag{2.51}$$

from which it derives that the components are also symmetric, $f_{\alpha\beta} = f_{\beta\alpha}$. That is, the matrices of the components are equal to their transpose.

31. Given any $(0,2)$ tensor we can define a new symmetric tensor as

$$h_{(S)}(\vec{A}, \vec{B}) = \frac{1}{2}[h(\vec{A}, \vec{B}) + h(\vec{B}, \vec{A})] \tag{2.52}$$

or, componentwise,

$$h_{(S)\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) \equiv h_{(\alpha\beta)} \tag{2.53}$$

32. Conversely, we say a tensor is antisymmetric if

$$f(\vec{A}, \vec{B}) = -f(\vec{B}, \vec{A}) \tag{2.54}$$

i.e. $f_{\alpha\beta} = -f_{\beta\alpha}$, and we can obtain an antisymmetrization as follows,

$$h_{(A)}(\vec{A}, \vec{B}) = \frac{1}{2}[h(\vec{A}, \vec{B}) - h(\vec{B}, \vec{A})] \tag{2.55}$$

or, componentwise,

$$h_{(A)\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) \equiv h_{[\alpha\beta]} \tag{2.56}$$

33. Every tensor can be decomposed into a symmetric and an antisymmetric part

$$h_{\alpha\beta} = h_{(\alpha\beta)} + h_{[\alpha\beta]} \tag{2.57}$$

The metric tensor is symmetric since

$$\eta_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta = \vec{e}_\beta \cdot \vec{e}_\alpha \tag{2.58}$$

2.3 Relation between vectors and 1-forms

1. We have seen that
$$\vec{A} \cdot \vec{B} = \eta_{\alpha\beta} A^{\alpha} B^{\beta} \tag{2.59}$$
Now we can define an entity with components
$$V_{\beta} = \eta_{\alpha\beta} A^{\alpha} \tag{2.60}$$
that acts just like a 1-form: it takes a vector and produces a scalar which is obviously linear in the vector. In fact, we have just defined a 1-form, which we denote as
$$g(\vec{A},\ ) \equiv \tilde{V}(\ ) \tag{2.61}$$
where the empty slot reminds us that this 1-form has an argument given by a vector.

2. So we can write
$$\tilde{V}(\vec{B}) = g(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} \tag{2.62}$$
Because of symmetry, we could have also defined the 1-form as $g(\ , \vec{B})$.

3. We know already that $\eta_{\alpha\beta} A^{\alpha}$ are the components of $g(\vec{A},\ )$. We could have obtained the same result as follows:
$$V_{\beta} = \tilde{V}(\vec{e}_{\beta}) = g(\vec{A}, \vec{e}_{\beta}) = g(A^{\alpha}\vec{e}_{\alpha}, \vec{e}_{\beta}) = A^{\alpha} g(\vec{e}_{\alpha}, \vec{e}_{\beta}) \tag{2.63}$$
$$= \eta_{\alpha\beta} A^{\alpha} \tag{2.64}$$
The expression
$$A_{\beta} = \eta_{\alpha\beta} A^{\alpha} \tag{2.65}$$
in which we use the same symbol for the components of the 1-form A_{β} and of the vector A^{α}, is often described as "the metric lowers the indexes".

4. This shows the importance of the metric: it is that tensor $(0, 2)$ that uniquely maps vectors into 1-forms. In fact, one could have skipped the entire definition of 1-forms and immediately introduce just vectors and the metric. This is indeed the approach of many textbooks to GR, especially the old ones. However, it is important from a mathematical point of view to realize that vectors and 1-forms are independent of frames and of the existence of a special tensor, the metric, that arises when we define distances over a manifold.

5. So we have
$$V_0 = -V^0, \quad V_1 = V^1, \quad V_2 = V^2, \quad V_3 = V^3 \tag{2.66}$$
If $\vec{V} \to (a, b, c, d)$ in a frame, then $\tilde{V} \to (-a, b, c, d)$ in the same frame.

6. One could have started instead by defining first 1-forms and then vectors. Then one would have written
$$V^{\alpha} = \eta^{\alpha\beta} V_{\beta} \tag{2.67}$$
where the $(2, 0)$ tensor $\eta^{\alpha\beta}$ is just the inverse of $\eta_{\alpha\beta}$ (and has identical components in a Cartesian system of coordinates). So we have
$$\eta_{\alpha\beta}\eta^{\beta\mu} = \delta^{\mu}_{\alpha} \tag{2.68}$$

7. In an Euclidian space, the metric is δ_{ij}, so the components of 1-forms and vectors are identical. Because of this, normally the definition of 1-forms etc are less important.

8. Using the metric we can now easily transform 1-forms into vectors and viceversa. For instance the 1-form gradient
$$\tilde{d}\phi \to (\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \tag{2.69}$$
is mapped into a vector gradient
$$\vec{d}\phi \to (-\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \tag{2.70}$$

9. Just like vectors, we define the scalar product of 1-forms. The definition is such the scalar product of 1-forms is the same as the scalar product of the corresponding vector:
$$\tilde{p}^2 = \vec{p}^2 = \eta_{\alpha\beta} p^\alpha p^\beta \tag{2.71}$$
This can be written as
$$\begin{aligned} \tilde{p}^2 &= \eta_{\alpha\beta}(\eta^{\alpha\mu} p_\mu)(\eta^{\beta\nu} p_\nu) = \eta^{\alpha\mu} p_\mu p_\alpha & (2.72)\\ &= -(p_0)^2 + (p_1)^2 + (p_2)^2 + (p_3)^2 & (2.73) \end{aligned}$$
which is then invariant. Similarly, we say that 1-forms are null, time-like or space-like.
10. Obviously we have also scalar product among different 1-forms
$$\tilde{p} \cdot \tilde{q} \tag{2.74}$$
which is also an invariant under LT.
11. Vectors can be understood now as $(1,0)$ tensors, that take 1-forms and generate scalars via a contraction.
12. One introduces then the general (M,N) tensor, a linear function of M 1-forms and N vectors into the real numbers.
13. The components, as usual, are given when the M 1-forms and the N vectors are the respective basis vectors.
14. E.g., if R is a $(1,1)$ tensor, its components are
$$R(\tilde{\omega}^\alpha, \vec{e}_\beta) = R^\alpha_\beta \tag{2.75}$$
A LT gives the new components
$$R^{\bar{\alpha}}_{\bar{\beta}} = R(\tilde{\omega}^{\bar{\alpha}}, \vec{e}_{\bar{\beta}}) = \Lambda^{\bar{\alpha}}_{\ \mu} \Lambda^{\nu}_{\ \bar{\beta}} R^\mu_\nu \tag{2.76}$$
15. We have seen that the metric raises and lowers the indexes. This is true for every tensor:
$$T^\alpha_{\ \beta\gamma} = \eta_{\beta\mu} T^{\alpha\mu}_{\ \ \gamma} \tag{2.77}$$
etc. That is, in this case, the metric transformed a $(1,2)$ tensor into a $(2,1)$ tensor. The order of the index is in general important; $T^\alpha_{\ \beta\gamma} \neq T^{\alpha\beta}_{\ \ \gamma}$.
16. The transformation rules are a direct generalization of what we have already seen. For instance,
$$T^{\bar{\alpha}}_{\ \bar{\beta}\bar{\gamma}} = \Lambda^{\bar{\alpha}}_{\ \mu} \Lambda^{\nu}_{\ \bar{\beta}} \Lambda^{\tau}_{\ \bar{\gamma}} T^\mu_{\ \nu\tau} \tag{2.78}$$
17. The basis are obtained, as for the $(0,2)$ tensor, as outer products of N basis vectors and M basis 1-forms:
$$T = T^\alpha_{\ \beta\gamma} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \tag{2.79}$$

2.4 Differentiation of tensors

1. We have seen that if f is a function, or $(0,0)$ tensor (i.e. a scalar), its gradient $\tilde{d}f$ is a $(0,1)$ tensor (1-form). That is, differentiation produces a tensor of higher (covariant) order. We now apply this rule to other tensors.
2. Imagine we have a $(1,1)$ tensor
$$T = T^\alpha_\beta \vec{e}_\alpha \otimes \tilde{\omega}^\beta \tag{2.80}$$
If we differentiate it we obtain another $(1,1)$ tensor
$$\frac{dT}{d\tau} = \frac{dT^\alpha_\beta}{d\tau} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \tag{2.81}$$
Notice that we used the crucial information that in SR the basis do not depend on position! This will change in GR.

3. Now we write
$$\frac{dT^\alpha_\beta}{d\tau} = T^\alpha_{\beta,\gamma} U^\gamma \tag{2.82}$$
and so
$$\frac{dT}{d\tau} = \frac{dT^\alpha_\beta}{d\tau} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \tag{2.83}$$
$$= (T^\alpha_{\beta,\gamma} U^\gamma) \vec{e}_\alpha \otimes \tilde{\omega}^\beta \tag{2.84}$$
Now we can define a $(1,2)$ tensor of components $T^\alpha_{\beta,\gamma}$, called gradient of T
$$\nabla T = T^\alpha_{\beta,\gamma} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \tag{2.85}$$
This is indeed a $(1,2)$ tensor since when it takes as last argument a vector $\vec{U}$ produces a $(1,1)$ tensor:
$$\frac{dT}{d\tau} = \nabla T(\vec{U}) \tag{2.86}$$
since
$$\nabla T(\vec{U}) = T^\alpha_{\beta,\gamma} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma(\vec{U}) = T^\alpha_{\beta,\gamma} \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma(U^\tau \vec{e}_\tau) \tag{2.87}$$
$$= T^\alpha_{\beta,\gamma} U^\tau \delta^\gamma_\tau \vec{e}_\alpha \otimes \tilde{\omega}^\beta = T^\alpha_{\beta,\gamma} U^\gamma \vec{e}_\alpha \otimes \tilde{\omega}^\beta \tag{2.88}$$
This tensor, obtained by contracting the gradient ∇T (a $(1,2)$ tensor) with the vector $\vec{U}$, equals indeed the $(1,1)$ tensor (2.84). In components, all this simply shows that the derivative of T^α_β can be written in terms of the component of higher-rank tensor,
$$\frac{dT^\alpha_\beta}{d\tau} = T^\alpha_{\beta,\gamma} U^\gamma \tag{2.89}$$

4. As a shorthand, we write
$$\nabla_\gamma T^\alpha_\beta = T^\alpha_{\beta,\gamma} \tag{2.90}$$

Chapter 3

The energy-momentum tensor

3.1 Density

1. A *field* is a quantity that is continuously defined in every point of the space. For instance, the density of a substance can be defined as a function $\rho(x, y, z)$. If it varies in time, then we write $\rho(t, x, y, z)$.

2. A substance is composed of molecules, atoms or particles. We can approximate it as a field if we average over a small element of volume.

3. A *fluid* is a substance that has small anti-slipping forces: that is, the little volume elements are free to move with respect to one another. A rock therefore is not a fluid. A *perfect fluid* has zero anti-slipping forces.

4. We call *dust* a collection of particles all *at rest* in some Lorentz frame (a MCRF).

5. We denote with n the number density (number of particle inside an element of volume divided by the volume). For dust, n can depend on space but, for the MCRF, not on time.

6. How does the density change in another frame? If in O an element has volume

$$V_O = \Delta x \Delta y \Delta z \tag{3.1}$$

in $\bar{O}$ (a frame x-boosted with velocity v), the volume will be contracted along x

$$V_{\bar{O}} = \frac{\Delta x}{\gamma} \Delta y \Delta z = \frac{V_O}{\gamma} \tag{3.2}$$

so that, since the number of particles in the volume element remains evidently the same, the density will be

$$n_{\bar{O}} = \frac{N}{V_{\bar{O}}} = n\gamma \tag{3.3}$$

3.2 Flux across a surface

1. The flux across a surface is the number of particles crossing a unit area in a unit time.

2. In the rest frame there is zero velocity, so zero flux.

3. In a frame moving with velocity $-v$, or if the fluid is moving with velocity v with respect to a frame $\bar{O}$, the total number of particles crossing an area ΔA in $\Delta \bar{t}$ is (see Fig. 3.1)

$$\Delta N = n\gamma v \Delta \bar{t} \Delta A \tag{3.4}$$

so the flux f_x along x is

$$f_x = \frac{\Delta N}{\Delta A \Delta \bar{t}} = n\gamma v \tag{3.5}$$

That is, the flux is the density times the velocity. Notice that even if the fluid is moving with a velocity $\vec{v}$ not directed along x, only the x component v_x counts for as concerns the flux along x. So the expression above can be written as

$$\boldsymbol{f} = n\gamma\boldsymbol{v} \tag{3.6}$$

4. We define now a number-flux 4-vector as

$$\vec{N} = n\vec{U} \tag{3.7}$$

where as we know $\vec{U} \to_{\bar{O}} (\gamma, \gamma\mathbf{v})$. So we see that

$$\vec{N} \to_{\bar{O}} (n\gamma, n\gamma\mathbf{v}) \tag{3.8}$$

This vector unifies number density and flux: these two quantities were a scalar and a 3D vector in Newtonian mechanics, but are now components of the same 4-vector. We can think of the number density as the flux of particles through a space-like surface, i.e. through a surface orthogonal to the time direction. So just like the flux is the number of particles in the 'volume' $\Delta A \Delta t$, the number density is the number of particle in the volume $\Delta x \Delta y \Delta z$.

5. Then we have

$$\vec{N} \cdot \vec{N} = -n^2 \tag{3.9}$$

as one can explicitly confirm by taking Eq. (3.8). Since this is an invariant equation, this defines the *rest density* n in the fluid rest frame as an invariant quantity (i.e., a scalar).

6. We can now generalize this. A 3D surface is defined as

$$\phi(t, x, y, z) = const \tag{3.10}$$

The normal to the surface is the gradient 1-form

$$\tilde{d}\phi \tag{3.11}$$

(let us remember that its dual vector, $\vec{d}\phi$, is orthogonal to the surface). The unit normal is then

$$\tilde{n} = \frac{\tilde{d}\phi}{|\tilde{d}\phi|} \tag{3.12}$$

where $|\tilde{d}\phi| = |\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}|^{1/2}$. So for instance the unit normal across x is

$$\tilde{d}x \to_{\bar{O}} (0, 1, 0, 0) \tag{3.13}$$

if $\bar{O}$ is the observer at rest with the fluid. Now (all in the $\bar{O}$ frame)

$$\tilde{d}x(\vec{N}) = \langle \tilde{d}x, \vec{N} \rangle = N^{\bar{\alpha}}(\tilde{d}x)_{\bar{\alpha}} = N^x \tag{3.14}$$

(where we introduce the alternative notation $\langle\ ,\ \rangle$ for the scalar product) i.e. the flux across a x-surface (we call a x-surface, a surface orthogonal to x). So in general we see that the flux across any surface $\phi = const$ with normal $\tilde{n}$ is given by

$$\langle \tilde{n}, \vec{N} \rangle = n^\alpha N_\alpha \tag{3.15}$$

3.3 Dust energy-momentum tensor

1. In order to describe fully a system of particles, we should consider also energy and momentum, not just density and velocity.

2. For dust, the particles are supposed not to interact with each other, so their energy is just the sum of the rest energy of each particle. Therefore the energy density is

$$\rho = nm \tag{3.16}$$

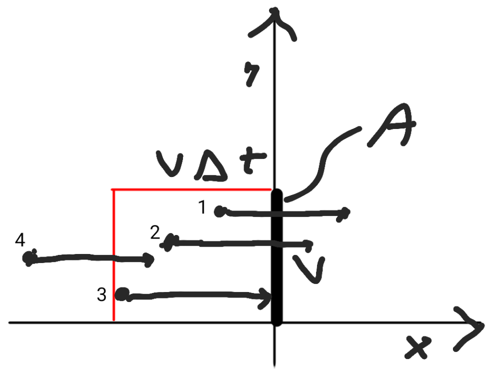

Figure 3.1: Flux across a surface A. Particles inside the box (for instance, particles 1,2,3) will cross the surface in Δt, particles outside (e.g., 4) will not.

3. In a frame with velocity v wrt the MCRF, we have for each particle a number density $n\gamma$ and an energy $m\gamma$. Therefore the energy density in this frame will be

$$\rho_{\bar{O}} = (n\gamma)(m\gamma) = nm\gamma^2 \tag{3.17}$$

The fact that there are two factors of γ, and therefore two factors of $\Lambda^{\bar{O}}_{O}$, makes us think that this quantity is not part of a vector but of a tensor, which transforms as

$$T^{\bar{\alpha}\bar{\beta}} = \Lambda^{\bar{\alpha}}_{\mu}\Lambda^{\bar{\beta}}_{\nu}T^{\mu\nu} \tag{3.18}$$

4. We proceed now then to identify and construct such a energy (or stress)-momentum tensor T with components $T^{\alpha\beta}$. We define

$$T^{\alpha\beta} = \text{flux of } \alpha - \text{momentum across a surface of constant } x^{\beta} \tag{3.19}$$

(α-momentum means the component α of momentum, i.e. p^{α}, while sometimes we refer to a x^{β}-surface or β-surface to mean a surface orthogonal to the x^{β} direction).

5. So for instance T^{00} is the flux of the 0-component of momentum across a surface at $t = const$. That is, it's the energy density.

6. Similarly, T^{0i} is the flux of p^0 (energy) across a x^i-surface. T^{i0} is the flux of p^i across $t = const$, that is, the density of momentum (momentum divided volume).

7. Finally, T^{ij} is the flux of i-momentum across a x^j-surface.

8. Let's evaluate T in the dust MCRF. As we have seen already, $(T^{00})_{MCRF} = \rho$, the number density. In the MCRF frame, there is no flux of particles in any direction, so

$$T^{ij} = T^{i0} = T^{0i} = 0 \tag{3.20}$$

(this equation is badly formed: it just means that all components vanish).

9. Now one can easily verify that the tensor
$$T = \vec{p} \otimes \vec{N} \tag{3.21}$$
has exactly the same component as the dust energy-momentum tensor, since
$$\vec{p} \quad \rightarrow \quad (E, 0, 0, 0) \tag{3.22}$$
$$\vec{N} = n\vec{U} \quad \rightarrow \quad (n, 0, 0, 0) \tag{3.23}$$
Componentwise, we can write
$$T^{\alpha\beta} = p^\alpha N^\beta \tag{3.24}$$
Since this is a well formed covariant equation (that is, with tensors of the same type on both sides), if it is valid in one frame, it remains valid in all frames. But we have shown that it is indeed valid in the MCRF frame so we have found a general, covariant expression for the energy-momentum tensor.

10. We can also write
$$T = \vec{p} \otimes \vec{N} = mn\vec{U} \otimes \vec{U} = \rho\vec{U} \otimes \vec{U} \tag{3.25}$$
or
$$T^{\alpha\beta} = \rho U^\alpha U^\beta \tag{3.26}$$

11. So in a frame $\bar{O}$ moving with velocity v one has
$$T^{\bar{0}\bar{0}} = \rho\gamma^2, \quad T^{\bar{i}\bar{0}} = \rho v^i \gamma^2 \tag{3.27}$$
$$T^{\bar{0}\bar{i}} = \rho v^i \gamma^2, \quad T^{\bar{i}\bar{j}} = \rho v^i v^j \gamma^2 \tag{3.28}$$
Notice that $T^{\alpha\beta} = T^{\beta\alpha}$, i.e. it is a symmetric tensor. Although we have shown this only in a particular case, this result is actually true for every energy-momentum tensor.

12. We have already seen in Eq. (3.9) that n is a scalar, and therefore also $\rho = nm$ is. This can be seen now in another way by simply forming the scalar
$$T^{\alpha\beta} U_\alpha U_\beta = \rho \tag{3.29}$$
This is then the invariant general expression for the energy density in a frame with 4-velocity U^α.

3.4 General fluids

1. To describe a general fluid, we need to 1) include the random motion of particles, i.e their kinetic energy, and b) include their interaction, i.e. their potential energy.

2. Since now the fluid is now in arbitrary motion, there is no single MCRF such that the fluid is at rest. However, we can still define a MCRF at rest with every single infinitesimal element of fluid. So from now on we assume that all the scalar quantities associated with a fluid element (number density, energy density, temperature etc), are defined in the associated MCRF.

3. Focusing now on a single element, we can say again that T^{00} is the energy density, except now this ρ is meant to be also time-dependent quantity (it was in general space-dependent also for dust). Also, again T^{0i} and T^{i0} vanish, since by definition the element is at rest with the MCRF, so there is no net momentum (each particle has momentum, but the total one must be zero) and no net change of energy. This however is only true if particles do not exchange energy, e.g. through heat conduction, i.e. without the particle themselves moving on average.

4. The remaining part, T^{ij}, is also called the stress tensor. As we have seen, it is defined as
$$\text{flux} = \frac{\Delta\text{momentum}}{\Delta A \Delta t} = \frac{F}{\Delta A} \tag{3.30}$$
or more in general, T^{ij} is given by
$$\frac{F^i}{\Delta A^j} = \frac{\text{force along } i}{\text{Area} \perp \text{to } j} \tag{3.31}$$
so T^{ij} gives the forces acting on adjacent fluid elements.

5. Now T^{ij} for $i \neq j$ represent forces directed parallel to the side of the element. If T^{ij} were different from T^{ji},every fluid element would be subject to a torque and would start rotating on itself. Since this certainly does not happen, fluids must have $T^{ij} = T^{ji}$. In fact one can show in very general terms that $T^{\alpha\beta}$ must be a symmetric tensor. The full demonstration is however very complex. All the energy-momentum tensor (EMT) that are ever encountered in physical applications are symmetric.

6. We call the forces directed parallel to the sides of fluid elements, *viscosity*. In a perfect fluid, one assumes no heat conduction nor viscosity. Then the general EMT for a perfect fluid has $T^{0i} = T^{i0} = 0$ (in the MCRF) and, moreover, T^{ij} is a diagonal matrix (in all frames).

7. We know from thermodynamics that the pressure in a fluid is isotropic, that is, in a small fluid element,

$$\frac{F}{\Delta A} = p \tag{3.32}$$

however one chooses the small area ΔA. Therefore the stress tensor is

$$T^{ij} = p\delta^{ij} \tag{3.33}$$

8. Finally, we can say that $T^{\alpha\beta}$ for an element of perfect fluid in the MCRF is

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{3.34}$$

This can be written in components as

$$T^{\alpha\beta} = (\rho + p)U^\alpha U^\beta + p\eta^{\alpha\beta} \tag{3.35}$$

and in a component-free way as

$$T = (\rho + p)\vec{U} \otimes \vec{U} + pg^{-1} \tag{3.36}$$

Dust is obtained for $p = 0$, therefore dust is called also pressureless perfect fluid. Notice that once again

$$T^{\alpha\beta}U_\alpha U_\beta = \rho \tag{3.37}$$

while

$$T^{\alpha\beta}\eta_{\alpha\beta} = -\rho + 3p \tag{3.38}$$

so

$$p = \frac{1}{3}\left(T^{\alpha\beta}\eta_{\alpha\beta} + T^{\alpha\beta}U_\alpha U_\beta\right) = \frac{1}{3}T^{\alpha\beta}(\eta_{\alpha\beta} + U_\alpha U_\beta) \tag{3.39}$$

This shows that ρ, p are scalar quantities.

3.5 Conservation laws

1. The flux vector $\vec{N}$ is a conserved quantity. This means that

$$N^\alpha_{\ ,\alpha} = (nU^\alpha)_{,\alpha} = 0 \tag{3.40}$$

2. In fact, let us consider particles moving only along x in or out of a volume element R across an area A as in Fig. (3.2). One sees that

$$A(N^x(0) - N^x(x))\Delta t = -A\Delta t\Delta N^x \tag{3.41}$$

is the number of particles flowing in R, minus those flowing out, in the time interval $\Delta t = t_2 - t_1$. But this number is also equal to the change in density in the same time interval, so

$$V(N^0(t_2) - N^0(t_1)) = V\Delta N^0 \tag{3.42}$$

Therefore

$$\Delta N^0 = -\frac{A}{V}\Delta t \Delta N^x \tag{3.43}$$

from which

$$\frac{\Delta N^0}{\Delta t} = -\frac{\Delta N^x}{\Delta x} \tag{3.44}$$

If we apply this to all three directions, we obtain

$$\frac{\Delta N^0}{\Delta t} = -\frac{\Delta N^x}{\Delta x} - \frac{\Delta N^y}{\Delta y} - \frac{\Delta N^z}{\Delta z} \tag{3.45}$$

which in the limit of small interval, is exactly Eq. (3.40).

3. Now we show that also $T^{\mu\nu}$ is a conserved quantity when we consider its SR limit. Let us assume the perfect fluid form

$$T^{\alpha\beta} = (\rho + p)U^\alpha U^\beta + p\eta^{\alpha\beta} \tag{3.46}$$

Then we have

$$\begin{aligned} T^{\alpha\beta}_{\ ,\beta} &= [(\rho+p)U^\alpha U^\beta]_{,\beta} + p_{,\beta}\eta^{\alpha\beta} && (3.47)\\ &= [\frac{(\rho+p)}{n}U^\alpha n U^\beta]_{,\beta} + p_{,\beta}\eta^{\alpha\beta} && (3.48)\\ &= \frac{(\rho+p)}{n}U^\alpha (nU^\beta)_{,\beta} + [\frac{(\rho+p)}{n}U^\alpha]_{,\beta} n U^\beta + p_{,\beta}\eta^{\alpha\beta} && (3.49)\\ &= [\frac{(\rho+p)}{n}U^\alpha]_{,\beta} n U^\beta + p_{,\beta}\eta^{\alpha\beta} && (3.50) \end{aligned}$$

(where we used $nU^\beta_{\ ,\beta} = 0$). Then we have, for $\alpha = i$

$$T^{i\beta}_{\ ,\beta} = [\tfrac{(\rho+p)}{n}]_{,\beta} n U^i U^\beta + \quad (\rho+p)U^i_{\ ,\beta}U^\beta + p_{,\beta}\eta^{i\beta} \tag{3.51}$$

The first term on the rhs vanishes because in the RF, $U^i = 0$ (but $U^i_{,\beta} \neq 0$ since the RF is defined assuming the velocity is the same as the volume element, but not the acceleration) and finally we obtain the SR version of the Euler equation

$$(\rho + p)a^i + p^{,i} = 0 \tag{3.52}$$

where

$$a^i \equiv U^i_{,\beta}U^\beta \tag{3.53}$$

4. The acceleration for small velocities ($d\tau \approx dt$) can be written as

$$a^i = U^i_{,0}U^0 + U^i_{,j}U^j = \dot{v}^i + (v^i_{,j})v^j \tag{3.54}$$

or in vector form as

$$\boldsymbol{a} = \dot{\boldsymbol{v}} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \equiv \frac{d\boldsymbol{v}}{dt} \tag{3.55}$$

where clearly

$$\boldsymbol{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \tag{3.56}$$

and

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \boldsymbol{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \boldsymbol{v}}{\partial z}\frac{dz}{dt} \tag{3.57}$$

5. We have got then a SR version of the *Euler equation*: the acceleration acting on an element of fluid is equal to minus the gradient of the pressure

$$(\rho + p)\boldsymbol{a} = -\boldsymbol{\nabla} p \tag{3.58}$$

The term p on the lhs is due to SR effects and in fact it is negligible for non-relativistic particles, for which the kinetic energy, and therefore the pressure, is much smaller than the rest energy. In some cases, e.g. stars with high-density and high-pressure cores, as neutron stars, the p terms is very important.

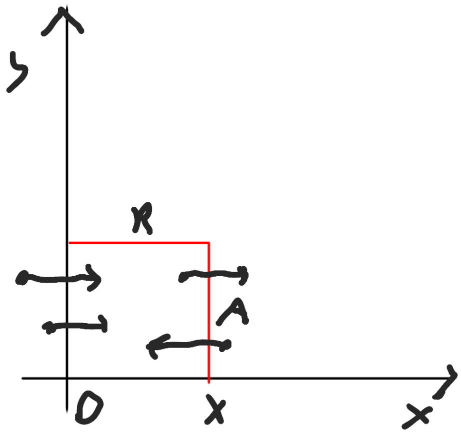

Figure 3.2: Conservation of flux: the number of particles inside the box changes when particles enter or leave it.

6. Multiplying by U_α and summing over α we obtain instead

$$U_\alpha T^{\alpha\beta}_{\ ,\beta} = U_\alpha[(\rho+p)U^\alpha U^\beta]_{,\beta} + p_{,\beta}U^\beta \tag{3.59}$$

$$= U_\alpha[\frac{(\rho+p)}{n}U^\alpha n U^\beta]_{,\beta} + p_{,\beta}U^\beta \tag{3.60}$$

$$= -\frac{(\rho+p)}{n}(nU^\beta)_{,\beta} + U_\alpha[\frac{(\rho+p)}{n}U^\alpha]_{,\beta} nU^\beta + p_{,\beta}U^\beta \tag{3.61}$$

$$= U_\alpha[\frac{(\rho+p)}{n}U^\alpha]_{,\beta} nU^\beta + p_{,\beta}U^\beta \tag{3.62}$$

$$= -[\frac{(\rho+p)}{n}]_{,\beta} nU^\beta + U_\alpha(\rho+p)U^\alpha_{,\beta}U^\beta + p_{,\beta}U^\beta \tag{3.63}$$

$$= -(\rho_{,0}+p_{,0})U^0 + [\frac{(\rho+p)n_{,0}}{n^2}]nU^0 + p_{,0}U^0 \tag{3.64}$$

$$= [-\rho_{,0} + \frac{(\rho+p)n_{,0}}{n}]U^0 \tag{3.65}$$

(we used $(U_\alpha U^\alpha)_{,\beta} = 0$, from which $U_\alpha U^\alpha_{,\beta} = 0$) which gives the energy conservation equation

$$\frac{d\rho}{d\tau} = \frac{(\rho+p)}{n}\frac{dn}{d\tau} \tag{3.66}$$

which embodies the conservation of energy for perfect fluids: the increase of ρ is proportional to the increase of particle density $dn/d\tau$. For small velocities $d\tau \approx dt$ and $p \ll \rho$, so finally

$$\frac{d\rho}{dt} = \frac{\rho}{n}\frac{dn}{dt} = m\frac{dn}{dt} \tag{3.67}$$

3.6 Gauss' theorem

1. We need now to recall an important mathematical result, Gauss' theorem. In 4D, it says that

$$\int_V V^\alpha_{,\alpha} d^4x = \oint V^\alpha n_\alpha d^3s \tag{3.68}$$

where the second integral runs over the surface of the volume V of the first integral, and $\vec{n}$ is the unit vector orthogonal to the surface, outward pointing (it is not the number density!). This is essentially a generalization of the standard 1D integral

$$\int \frac{df}{dx} dx = f|_b^a \tag{3.69}$$

where a, b are the boundaries ("surface") of the domain of integration.

2. The operation

$$V^\alpha_{,\alpha} \tag{3.70}$$

is called divergence of V, or div(V), in analogy to the 3D case.

3. Applying it to a conserved quantity like $\vec{N}$, we see that

$$0 = \int_V N^\alpha_{,\alpha} d^4x = \oint N^\alpha n_\alpha d^3s \tag{3.71}$$

and therefore

$$\oint N^0 n_0 d^3s = -\oint N^i n_i d^3s \tag{3.72}$$

This is the integral form of the conservation equation. It expresses once again the intuitive property that the number of particles inside a surface, N^0, changes by a quantity that is minus the total flux N^i leaving that surface.

Chapter 4

Curved spaces

4.1 General coordinate transformations

1. Let us consider a general coordinate transformation (GCT) in the x, y plane:

$$\xi = \xi(x, y) \tag{4.1}$$
$$\eta = \eta(x, y) \tag{4.2}$$

   from which the variation

$$\Delta\xi = \frac{\partial\xi}{\partial x}\Delta x + \frac{\partial\xi}{\partial y}\Delta y \tag{4.3}$$
$$\Delta\eta = \frac{\partial\eta}{\partial x}\Delta x + \frac{\partial\eta}{\partial y}\Delta y \tag{4.4}$$

   or

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = J \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \tag{4.5}$$

   where J is the Jacobian matrix

$$J = \begin{pmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{pmatrix} \tag{4.6}$$

2. This transformation is a "good one" only when distinct points in the x, y plane are mapped into distinct points in the ξ, η plane, and same points into same points. That is, when $\Delta x = \Delta y = 0$ then we wish that also $\Delta\xi = \Delta\eta = 0$. From linear algebra we know that the general condition for this to happen is that the Jacobian matrix is non singular, i.e. that

$$\det J \neq 0 \tag{4.7}$$

3. In component notation, we can write

$$\Delta\xi_i = J_{ij}\Delta x_j \tag{4.8}$$
$$\Delta x_i = (J^{-1})_{ij}\Delta\xi_j \tag{4.9}$$

   where $\boldsymbol{\xi} = \{\xi, \eta\}$ and $\boldsymbol{x} = \{x, y\}$ and ξ_i, x_i are their components.

4. Let us define then the matrix of transformations (not necessarily a Lorentz transformation, now the transformation will in general depend on space and time!)

$$\frac{\partial x^{\bar\alpha}}{\partial x^\beta} = \Lambda^{\bar\alpha}_{\beta} = J \tag{4.10}$$

   Now we can define again vectors as those objects that transform according to the law

$$V^{\bar\alpha} = \Lambda^{\bar\alpha}_{\beta} V^\beta \tag{4.11}$$

   (we are still in 2D so for now $\alpha, \beta = 1, 2$).

5. Note that
$$\frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}}\frac{\partial x^{\beta}}{\partial x^{\gamma'}} = \delta^{\bar{\alpha}}_{\gamma'} \tag{4.12}$$
That is, if
$$J \to \{\frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}}\} \tag{4.13}$$
then
$$J^{-1} \to \{\frac{\partial x^{\beta}}{\partial x^{\bar{\alpha}}}\} \tag{4.14}$$
But note that it is in general not true that exchanging numerator and denominator in a partial derivative one has the inverse,
$$\frac{\partial f}{\partial x} \neq \left(\frac{\partial x}{\partial f}\right)^{-1} \tag{4.15}$$

6. More explicitly,
$$\begin{pmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{pmatrix}\begin{pmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.16}$$
where we used the fact that
$$\frac{\partial \xi}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial \xi}{\partial y}\frac{\partial y}{\partial \eta} = \frac{\partial \xi}{\partial \eta} = 0 \tag{4.17}$$
since ξ, η are the two independent coordinates (and similarly for $\partial\eta/\partial\xi = 0$).

7. We can define vectors also starting from 1-forms. That is, let us take a generic scalar function (or field) $\phi(\xi, \eta)$ and take its gradient $\tilde{d}\phi$, of components
$$\tilde{d}\phi \to (\frac{\partial \phi}{\partial \xi}, \frac{\partial \phi}{\partial \eta}) \tag{4.18}$$
The transformation rule for $\tilde{d}\phi$ can be obtained immediately as a consequence of the chain rule for partial derivatives
$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \eta} \tag{4.19}$$
and similarly for $\phi_{,\eta}$. That is, we have the usual co-variant transformation law
$$\phi_{,\bar{\alpha}} = \Lambda^{\beta}_{\ \bar{\alpha}}\phi_{,\beta} \tag{4.20}$$
where
$$\Lambda^{\beta}_{\ \bar{\alpha}} = \frac{\partial x^{\beta}}{\partial \xi^{\bar{\alpha}}} \tag{4.21}$$

8. The entire set of relation and transformation laws we have seen before is then again valid now. Since $d\phi/d\tau$ is a scalar, and $\tilde{d}\phi$ is a 1-form, the 4-velocity $U \to \{\frac{dx^{\alpha}}{d\tau}\}$ must be a vector that transforms in the opposite way as 1-forms.

9. This shows that one can define 1-forms, vectors, tensors etc under a general coordinate transformation, using exactly the same rules and notation that we have already employed for SR.

4.2 Tangent manifold

1. A *curve* is a mapping of an interval of a real line into a path in the space-time.

2. Given a parameter s in an interval (a, b) along the path that identifies uniquely every point (eg, the proper time, or any monotonous function of it), a curve in 2D can be defined as
$$\xi = f(s), \quad \eta = g(s) \tag{4.22}$$

3. If we change to a new parameter s', we have to think of the resulting curve as a new curve, even if the path is the same.

4. Now, for every scalar field ϕ we can differentiate and obtain

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial\xi}\frac{d\xi}{ds} + \frac{\partial\phi}{\partial\eta}\frac{d\eta}{ds} = \langle\tilde{d}\phi, \vec{V}\rangle \tag{4.23}$$

where

$$\vec{V} \to (\frac{d\xi}{ds}, \frac{d\eta}{ds}) \tag{4.24}$$

is the tangent vector to the curve. As one can see, a path has a unique tangent, but a curve will have infinite tangents, depending on the parameter s.

5. A vector can always be defined directly in terms of tangent to curves. I.e., a vector can be defined as that function that gives $\frac{d\phi}{ds}$ when it takes $\tilde{d}\phi$ as argument. The vector space defined as the collection of all the tangent vectors in a point is called the tangent manifold.

4.3 Polar coordinates

1. As a particular example of general coordinate transformations, we take now the case of polar coordinates r, θ

$$r = (x^2 + y^2)^{1/2} \tag{4.25}$$

$$\theta = \arctan\frac{y}{x} \tag{4.26}$$

and

$$x = r\cos\theta \tag{4.27}$$

$$y = r\sin\theta \tag{4.28}$$

By differentiation,

$$\Delta r = \cos\theta\Delta x + \sin\theta\Delta y \tag{4.29}$$

$$\Delta\theta = -\frac{1}{r}\sin\theta\Delta x + \frac{1}{r}\cos\theta\Delta y \tag{4.30}$$

which, compared to the chain differential rule for a generic function $f(x, y)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

shows that

$$\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta \tag{4.31}$$

etc.

2. If we say that the barred frame is (r, θ) and the unbarred one is (x, y), the transformation matrix is now

$$\Lambda^{\bar{\alpha}}_{\beta} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \tag{4.32}$$

and its inverse by

$$\Lambda^{\alpha}_{\bar{\beta}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\beta}}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \tag{4.33}$$

Notice that, contrary to the LT, a general transformation need not be symmetric. Clearly

$$\Lambda^{\bar{\alpha}}_{\beta}\Lambda^{\beta}_{\bar{\gamma}} = \delta^{\bar{\alpha}}_{\bar{\gamma}} \tag{4.34}$$

Let us stress once again that

$$\frac{\partial x}{\partial r} \neq \left(\frac{\partial r}{\partial x}\right)^{-1} \tag{4.35}$$

3. With this metric, we can apply all the rules we learned previously. For instance, the basis vectors transform as

$$\vec{e}_{\bar{\alpha}} = \Lambda^{\beta}{}_{\bar{\alpha}} \vec{e}_{\beta} \tag{4.36}$$

This means that the basis vectors in the r, θ frame are (with new, but obvious notation for the indexes)

$$\vec{e}_r = \Lambda^x{}_r \vec{e}_x + \Lambda^y{}_r \vec{e}_y = \cos\theta \vec{e}_x + \sin\theta \vec{e}_y \tag{4.37}$$

$$\vec{e}_\theta = \Lambda^x{}_\theta \vec{e}_x + \Lambda^y{}_\theta \vec{e}_y = -r\sin\theta \vec{e}_x + r\cos\theta \vec{e}_y \tag{4.38}$$

Since we know the basis vectors of the original frame, $\vec{e}_x \to (1,0), \vec{e}_y \to (0,1)$, we obtain immediately that the basis vectors of the new frame (as seen in the old frame) are

$$\vec{e}_r \to (\cos\theta, \sin\theta) \tag{4.39}$$

$$\vec{e}_\theta \to (-r\sin\theta, r\cos\theta) \tag{4.40}$$

One can easily convince themselves that these two vectors, represented in the (x, y) plane, are indeed pointing one along the radius and the other tangent to the circumference, and orthogonal to each other, see Fig. (4.1).

4. Similarly, an observer using the r, θ frame will define basis vectors $\vec{e}_r \to (1,0)$ and $\vec{e}_\theta \to (0,1)$ and will use the inverse transformation to evaluate the old x, y basis in terms of the new frame basis

$$\vec{e}_x = \Lambda^r{}_x \vec{e}_r + \Lambda^\theta{}_x \vec{e}_\theta = \cos\theta \vec{e}_r - \frac{1}{r}\sin\theta \vec{e}_\theta \tag{4.41}$$

$$\vec{e}_y = \sin\theta \vec{e}_r + \frac{1}{r}\cos\theta \vec{e}_\theta \tag{4.42}$$

These transformations express the basis of the old frame as a linear combination of the basis of the new frame

5. Similarly, the basis 1-forms of the original (x, y) frame are $\tilde{d}x, \tilde{d}y$, and those of the new frame will be

$$\tilde{d}\theta = \frac{\partial\theta}{\partial x}\tilde{d}x + \frac{\partial\theta}{\partial y}\tilde{d}y = -\frac{1}{r}\sin\theta \tilde{d}x + \frac{1}{r}\cos\theta \tilde{d}y \tag{4.43}$$

$$\tilde{d}r = \cos\theta \tilde{d}x + \sin\theta \tilde{d}y \tag{4.44}$$

6. One important point to remark is that now the basis vectors/1-forms in the new frame depend on the position, contrary to the old frame; e.g.

$$|\vec{e}_\theta|^2 = r^2 \tag{4.45}$$

7. The metric tensor in the old x, y frame has components

$$g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta} \tag{4.46}$$

In the new frame, the components will be (barred symbols run over r, θ)

$$g(\vec{e}_{\bar{\alpha}}, \vec{e}_{\bar{\beta}}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{4.47}$$

8. A small interval vector can be expanded into the new frame basis vectors as

$$\vec{d\ell} = dr\vec{e}_r + d\theta\vec{e}_\theta \tag{4.48}$$

The length of this vector, i.e. the interval ds, is then

$$\vec{d\ell} \cdot \vec{d\ell} = ds^2 = |dr\vec{e}_r + d\theta\vec{e}_\theta|^2 = dr^2 + r^2 d\theta^2 + 0 \times (2dr d\theta) \tag{4.49}$$

where we can identify the various entries of the metric tensor $g(\vec{e}_{\bar{\alpha}}, \vec{e}_{\bar{\beta}})$: we have in fact $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{r\theta} = 0$.

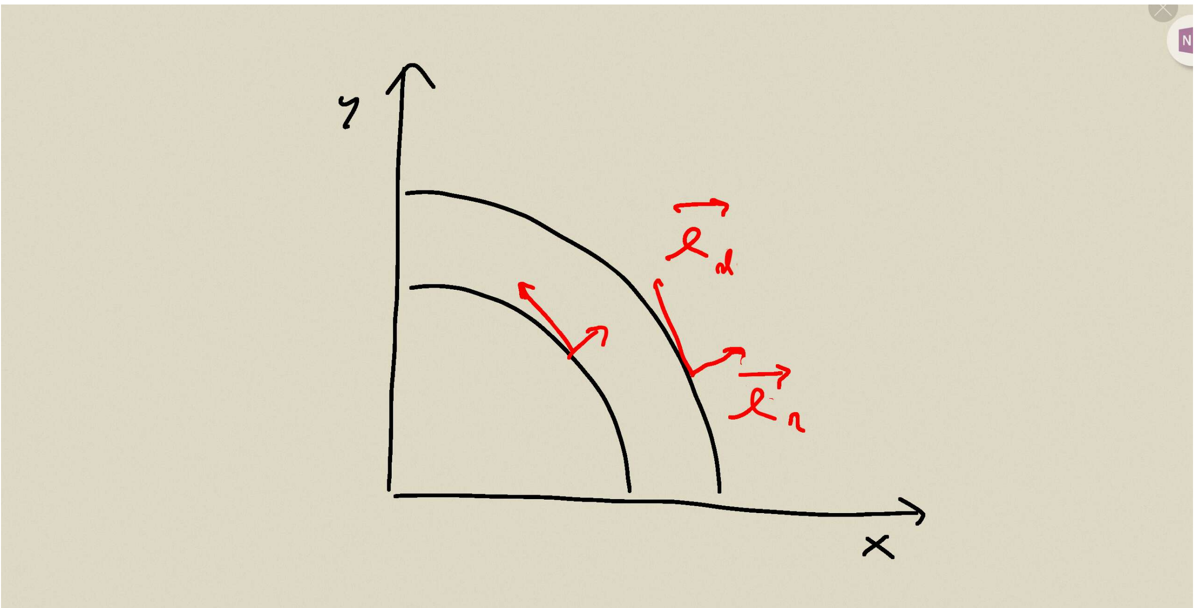

Figure 4.1: Basis in polar coordinates.

9. The metric inverse, whose components are denoted as $g^{\mu\nu}$, is clearly

$$g^{-1} \to \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \tag{4.50}$$

10. The metric tensor is still the quantity that maps vectors into 1-forms, or, in other words, raises and lowers the indexes. Now however the difference is not just a sign change.

11. For instance, if $\tilde{d}\phi \to (\phi_{,r}, \phi_{,\theta})$ is the gradient of a function, the component of the dual 1-form are obtained by

$$(\vec{d}\phi)^{\alpha} = g^{\alpha\beta}\phi_{,\beta} \tag{4.51}$$

and are therefore $(\phi_{,r}, r^{-2}\phi_{,\theta})$.

4.4 Derivatives of basis vectors

1. The basis vector in the Cartesian frame are constant, i.e. the same at every point,

$$\frac{\partial \vec{e}_{x,y}}{\partial x} = 0 \tag{4.52}$$

and therefore also

$$\frac{\partial \vec{e}_{x,y}}{\partial r} = 0 \tag{4.53}$$

and the same for derivative wrt y, θ. The basis vectors in the Polar frame are not:

$$\frac{\partial \vec{e}_r}{\partial \theta} = \frac{\partial}{\partial \theta}(\cos\theta \vec{e}_x + \sin\theta \vec{e}_y) \tag{4.54}$$

and similarly for $\frac{\partial \vec{e}_r}{\partial r}, \frac{\partial \vec{e}_\theta}{\partial r}, \frac{\partial \vec{e}_\theta}{\partial \theta}$.

2. However, if we evaluate the x, y basis in terms of the r, θ basis using the components that appear in Eq. (4.42) we obtain that $\vec{e}_x$ does depend on r, θ, in contradiction with something we know is certainly true, namely Eq. (4.53)!

3. This teaches us that the derivative of a vector is not in general given by the derivative of its components, as we have seen in SR (see Sec. 2.4). So this is how we should proceed now:

$$\frac{\partial \vec{e}_r}{\partial r} = \cos\theta \frac{\partial \vec{e}_x}{\partial r} + \sin\theta \frac{\partial \vec{e}_y}{\partial r} = 0 \tag{4.55}$$

$$\frac{\partial \vec{e}_r}{\partial \theta} = -\sin\theta \vec{e}_x + \cos\theta \vec{e}_y = \frac{1}{r}\vec{e}_\theta \tag{4.56}$$

and similarly one gets

$$\frac{\partial \vec{e}_\theta}{\partial r} = \frac{1}{r}\vec{e}_\theta, \quad \frac{\partial \vec{e}_\theta}{\partial \theta} = -r\vec{e}_r \tag{4.57}$$

4. Now we can do the differentiation properly:

$$\frac{\partial \vec{e}_x}{\partial \theta} = \frac{\partial}{\partial \theta}[\cos\theta \vec{e}_r - \frac{1}{r}\sin\theta \vec{e}_\theta] = 0 \tag{4.58}$$

as it should. That is, one must take into account the fact that the basis vectors might depend on position.

5. So the derivative of $\vec{V} \to (V^r, V^\theta)$ in general is

$$\frac{\partial \vec{V}}{\partial r} = \frac{\partial}{\partial r}(V^r\vec{e}_r + V^\theta\vec{e}_\theta) = \frac{\partial V^r}{\partial r}\vec{e}_r + V^r\frac{\partial \vec{e}_r}{\partial r} + \frac{\partial V^\theta}{\partial r}\vec{e}_\theta + V^\theta\frac{\partial \vec{e}_\theta}{\partial r}) \tag{4.59}$$

and the same for θ; that is, in general for the coordinate x^β,

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}(V^\alpha\vec{e}_\alpha) = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha\frac{\partial \vec{e}_\alpha}{\partial x^\beta} \tag{4.60}$$

6. We now *assume* that the expression

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} \tag{4.61}$$

can be written in terms of the basis vectors $\vec{e}_\alpha$ as

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu \tag{4.62}$$

where the $\Gamma^\mu{}_{\alpha\beta}$ are called *Christoffel's symbols.* They represent the μ components of $\vec{e}_{\alpha,\beta}$.

7. Let us evaluate the Christoffel symbols for the polar coordinate case. For instance, since

$$\frac{\partial \vec{e}_r}{\partial \theta} = \frac{1}{r}\vec{e}_\theta + 0\,\vec{e}_r \tag{4.63}$$

we see that $\Gamma^r{}_{r\theta} = 0$ and $\Gamma^\theta{}_{r\theta} = r^{-1}$. Also, we find

$$\Gamma^\mu{}_{rr} = 0, \quad \Gamma^r{}_{\theta r} = 0, \quad \Gamma^\theta{}_{\theta r} = \frac{1}{r} \tag{4.64}$$

8. We can now go back to Eq. (4.60) and put

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = V^\alpha_{,\beta}\vec{e}_\alpha + V^\alpha\Gamma^\mu{}_{\alpha\beta}\vec{e}_\mu \tag{4.65}$$

$$= V^\alpha_{,\beta}\vec{e}_\alpha + V^\mu\Gamma^\alpha{}_{\mu\beta}\vec{e}_\alpha = (V^\alpha_{,\beta} + V^\mu\Gamma^\alpha{}_{\mu\beta})\vec{e}_\alpha \tag{4.66}$$

(in the second line we interchanged the dummy indexes μ, α). This means that the $(1,1)$ tensor $\frac{\partial \vec{V}}{\partial x^\beta}$ has components

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + V^\mu\Gamma^\alpha{}_{\mu\beta} \tag{4.67}$$

This new operation, denoted by a ";", is called *covariant derivative.* We also write

$$\nabla_\beta(\vec{V})^\alpha = V^\alpha_{;\beta} \tag{4.68}$$

In Cartesian coordinates, of course, the Christoffel symbols vanish and the covariant derivative is identical to the ordinary one.

9. For a scalar, i.e. a quantity that is independent of the basis, the covariant derivative is always identical to the standard one, in any coordinate frame:

$$\phi_{,\alpha} = \phi_{;\alpha} \tag{4.69}$$

4.5 Divergence and Laplacian

1. We define the *covariant divergence* of a vector as $V^{\alpha}_{;\alpha}$. This can be written in several equivalent ways

$$\vec{\nabla} \cdot \vec{V} \equiv \nabla_{\alpha} V^{\alpha} \equiv V^{\alpha}_{;\alpha} \equiv \delta^{\alpha}_{\beta} V^{\beta}_{;\alpha} \tag{4.70}$$

that is,

$$V^{\alpha}_{;\alpha} = V^{\alpha}_{,\alpha} + \Gamma^{\alpha}_{\mu\alpha} V^{\mu} \tag{4.71}$$

Notice that this is a scalar quantity, i.e. has no free indexes.

2. In polar coordinates, for instance, we have ($\alpha = r, \theta$)

$$\Gamma^{\alpha}_{r\alpha} = \Gamma^{r}_{rr} + \Gamma^{\theta}_{r\theta} = \frac{1}{r}, \quad \Gamma^{\alpha}_{\theta\alpha} = 0 \tag{4.72}$$

so

$$V^{\alpha}_{;\alpha} = \frac{\partial V^r}{\partial r} + \frac{\partial V^{\theta}}{\partial \theta} + \frac{1}{r} V^r = \frac{1}{r}\frac{\partial}{\partial r}(r V^r) + \frac{\partial}{\partial \theta} V^{\theta} \tag{4.73}$$

which is indeed the formula for the divergence in polar coordinates.

3. The Laplacian of a scalar field is defined as

$$\vec{\nabla} \cdot \vec{\nabla} \phi \tag{4.74}$$

The second gradient $\vec{\nabla}\phi$ is the vector associated to the gradient 1-form $\tilde{d}\phi$. That is

$$\tilde{d}\phi \rightarrow (\phi_{,r}, \phi_{,\theta}) \tag{4.75}$$

$$\vec{\nabla}\phi \rightarrow (\phi^{,r} \equiv \phi_{,r}, \phi^{,\theta} \equiv \frac{1}{r^2}\phi_{,\theta}) \tag{4.76}$$

In fact, the component of $\vec{\nabla}\phi$ can be obtained as

$$(\vec{\nabla}\phi)^{\alpha} = g^{\alpha\beta}(\tilde{d}\phi)_{\beta} \tag{4.77}$$

Then we have that Eq. (4.74) is the divergence of the vector $\vec{\nabla}\phi$, so using (4.73) it follows

$$\vec{\nabla} \cdot \vec{\nabla}\phi = \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}(r(\phi_{,r})) + \frac{\partial}{\partial \theta}(\frac{1}{r^2}\phi_{,\theta}) \tag{4.78}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\phi_{,r}) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\phi_{,\theta} \tag{4.79}$$

which is the Laplacian in polar coordinates. Once again, the result is a scalar quantity. In component form we write it simply as

$$\vec{\nabla} \cdot \vec{\nabla}\phi \equiv \phi^{;\alpha}_{;\alpha} \equiv \Box\Phi \tag{4.80}$$

where we introduced the *box* operator, called *D'Alambertian.* Notice that although $\phi_{,\alpha} = \phi_{;\alpha}$, for the Laplacian the covariant derivative is not the same as the ordinary derivative, since the second derivative acts on $\phi_{;\alpha}$, that is a vector. Since the result is a scalar, it must remain the same in every frame, so indeed one can verify that

$$\phi^{;\alpha}_{;\alpha} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \tag{4.81}$$

4.6 Derivative of general tensors

1. We know that the covariant derivative of a scalar coincides with the ordinary one. Then let us assume a scalar obtained as a contraction $\phi = p_\alpha V^\alpha$. The covariant derivative is *assumed* to obey the Leibniz product rule

$$\phi_{;\beta} = p_{\alpha;\beta} V^\alpha + p_\alpha V^\alpha_{;\beta} \tag{4.82}$$

On the other hand, since this is equal to the ordinary derivative, we have

$$\begin{aligned} \phi_{;\beta} &= \phi_{,\beta} = p_{\alpha,\beta} V^\alpha + p_\alpha V^\alpha_{,\beta} & (4.83)\\ &= p_{\alpha,\beta} V^\alpha + p_\alpha (V^\alpha_{;\beta} - \Gamma^\alpha{}_{\mu\beta} V^\mu) & (4.84)\\ &= (p_{\alpha,\beta} - p_\mu \Gamma^\mu{}_{\alpha\beta}) V^\alpha + p_\alpha V^\alpha_{;\beta} & (4.85) \end{aligned}$$

where in the last line, second term inside parentheses, we interchanged the dummy indexes α, μ. Now the term on the lhs is a vector, and the last term on the rhs is also a vector. So the term in the middle must be a vector as well. That is, comparing with (4.82), we find the covariant derivative of a 1-form

$$p_{\alpha;\beta} = p_{\alpha,\beta} - p_\mu \Gamma^\mu{}_{\alpha\beta} \tag{4.86}$$

2. In a similar way, one can find the general rule for every tensor. We only need to write down the one for the various forms of tensors:

$$T_{\mu\nu;\beta} = T_{\mu\nu,\beta} - T_{\alpha\nu} \Gamma^\alpha{}_{\mu\beta} - T_{\mu\alpha} \Gamma^\alpha{}_{\nu\beta} \tag{4.87}$$

$$T^\mu{}_{\nu;\beta} = T^\mu{}_{\nu,\beta} - T^\mu{}_\alpha \Gamma^\alpha{}_{\nu\beta} + T^\alpha{}_\nu \Gamma^\mu{}_{\alpha\beta} \tag{4.88}$$

$$T^{\mu\nu}{}_{;\beta} = T^{\mu\nu}{}_{,\beta} + T^{\mu\alpha} \Gamma^\nu{}_{\alpha\beta} + T^{\nu\alpha} \Gamma^\mu{}_{\alpha\beta} \tag{4.89}$$

3. Let us apply these rules to the metric tensor defined in terms of the basis vectors, $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$ now. We have

$$(\vec{e}_\mu \cdot \vec{e}_\nu)_{;\beta} = (\vec{e}_\mu \cdot \vec{e}_\nu)_{,\beta} - \Gamma^\alpha{}_{\mu\beta} \vec{e}_\alpha \cdot \vec{e}_\nu - \Gamma^\alpha{}_{\nu\beta} \vec{e}_\mu \cdot \vec{e}_\alpha \tag{4.90}$$

But we defined the Christoffel symbols as

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu{}_{\alpha\beta} \vec{e}_\mu \tag{4.91}$$

so

$$(\vec{e}_\mu \cdot \vec{e}_\nu)_{;\beta} = (\vec{e}_\mu \cdot \vec{e}_\nu)_{,\beta} - \frac{\partial \vec{e}_\mu}{\partial x^\beta} \cdot \vec{e}_\nu - \vec{e}_\mu \cdot \frac{\partial \vec{e}_\nu}{\partial x^\beta} = 0 \tag{4.92}$$

So, we have shown that

$$g_{\mu\nu;\gamma} = 0 \tag{4.93}$$

that is,

$$g_{\alpha\beta,\mu} = \Gamma^\nu{}_{\alpha\mu} g_{\nu\beta} + \Gamma^\nu{}_{\beta\mu} g_{\alpha\nu} \tag{4.94}$$

4. However, we have assumed erlier that $\frac{\partial \vec{e}_\alpha}{\partial x^\beta}$ can be expanded as in Eq. (4.62), and we have also assumed Eq. (4.82), and one can obtain a non vanishing $g_{\mu\nu;\gamma}$ if these assumption are lifted. The choice that gives $g_{\mu\nu;\gamma} = 0$, is called *metric-compatibility* (the Christoffel symbols are said to be metric-compatible). In fact, we could have started from postulating metric-compatibility and derive the other conditions.

5. This choice simplifies enormously our calculations and so far seems satisfied by observations. Non-metric-compatible theories of gravity have however been proposed many times. From now on, we will always assume metric-compatibility.

6. As a consequence

$$V_{\alpha;\beta} = (g_{\alpha\mu} V^\mu)_{;\beta} = g_{\alpha\mu} V^\mu_{;\beta} \tag{4.95}$$

so a vector and a 1-form that are dual, remain dual even after differentiation. This is a very useful property. For instance, if a vector is covariantly conserved, then so is its dual.

7. If we apply this to polar coordinates, we see that, indeed, using (4.64)

$$g_{\theta\theta;r} = (r^2)_{,r} - \frac{1}{r} r^2 - \frac{1}{r} r^2 = 0 \tag{4.96}$$

4.7 Christoffel symbols as function of the metric

1. The metric-compatibility is a very important assumption also because it allows to express the Christoffel symbols in function of the metric.

2. We need however first another assumption, that the Christoffel symbols are symmetric in the lower indices
$$\Gamma^{\mu}_{\ \alpha\beta} = \Gamma^{\mu}_{\ \beta\alpha} \tag{4.97}$$
This is called *torsion-free* assumption. This condition is equivalent to imposing that covariant derivatives of scalar functions commute
$$\phi_{;\alpha\beta} = \phi_{;\beta\alpha} \tag{4.98}$$
that is
$$\phi_{,\alpha\beta} - \phi_{,\mu}\Gamma^{\mu}_{\ \alpha\beta} = \phi_{,\beta\alpha} - \phi_{,\mu}\Gamma^{\mu}_{\ \beta\alpha} \tag{4.99}$$
from which
$$\phi_{,\mu}(\Gamma^{\mu}_{\ \alpha\beta} - \Gamma^{\mu}_{\ \beta\alpha}) = 0 \tag{4.100}$$
for any scalar function ϕ. Again, one can build a different theory of space-time by violating this condition.

3. The commutation holds only for scalar functions. For vector and tensors in general
$$V^{\alpha}_{;\mu\nu} \neq V^{\alpha}_{;\nu\mu} \tag{4.101}$$

4. To find an explicit expression of the Christoffel symbols, we rewrite Eq. (4.94) three times with a permutation of the indices:
$$g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\ \alpha\mu}g_{\nu\beta} + \Gamma^{\nu}_{\ \beta\mu}g_{\alpha\nu} \tag{4.102}$$
$$g_{\alpha\mu,\beta} = \Gamma^{\nu}_{\ \alpha\beta}g_{\nu\mu} + \Gamma^{\nu}_{\ \mu\beta}g_{\alpha\nu} \tag{4.103}$$
$$g_{\beta\mu,\alpha} = \Gamma^{\nu}_{\ \beta\alpha}g_{\nu\mu} + \Gamma^{\nu}_{\ \mu\alpha}g_{\beta\nu} \tag{4.104}$$
and add the first two expressions and subtract the third one. We obtain
$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2g_{\alpha\nu}\Gamma^{\nu}_{\ \beta\mu} \tag{4.105}$$
and finally, multiplying by $\frac{1}{2}g^{\alpha\gamma}$ on both sides
$$\Gamma^{\gamma}_{\ \beta\mu} = \frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\mu\beta,\alpha}) \tag{4.106}$$
This is one of the most important equation in GR. It is also called Levi-Civita connection. It allows to find the Christoffel symbols given any metric. Notice that indeed there is symmetry between β, μ.

5. The Christoffel symbol is not a tensor, which means that, for instance, $\Gamma^{\mu}_{\ \alpha\beta} = 0$ is an equation that is not valid in every frame. The transformation law is indeed not in tensorial form:
$$\Gamma^{\bar{\gamma}}_{\ \bar{\alpha}\bar{\beta}} = \frac{\partial x^{\bar{\gamma}}}{\partial x^{\gamma}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\Gamma^{\gamma}_{\ \alpha\beta} + \frac{\partial^2 x^{\gamma}}{\partial x^{\bar{\alpha}}\partial x^{\bar{\beta}}}\frac{\partial x^{\bar{\gamma}}}{\partial x^{\gamma}} \tag{4.107}$$
This transformation law can be obtained directly from Eq. (4.106) but it applies also to non-torsion-free connections. The torsion is defined as $T^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} = \Gamma^{\bar{\gamma}}_{\ \bar{\alpha}\bar{\beta}} - \Gamma^{\bar{\gamma}}_{\ \bar{\beta}\bar{\alpha}}$, and one can immediately check that this is instead a tensor. Also, if one transforms only the γ index and one of the other indexes, so that for instance $\bar{\alpha} = \alpha$, then $\partial^2 x^{\gamma}/\partial x^{\alpha}\partial x^{\bar{\beta}} = \partial\delta^{\gamma}_{\alpha}/\partial x^{\bar{\beta}} = 0$ and $\Gamma^{\bar{\gamma}}_{\ \alpha\bar{\beta}}$ transforms as a $(1,1)$ tensor with respect to $\bar{\gamma}, \bar{\beta}$.

6. The expressions that we will build with the use of the Christoffel symbols, in particular the covariant derivatives, will always be tensor expressions.

7. The Levi-Civita connection is the simplest possible connection, and it is fully determined by the metric. More general connections (with non-zero torsion or non-metricity), will not be fully determined by the metric. This could in principle be used to add non-gravitational forces into GR.

Chapter 5

Einstein's equations

5.1 Manifolds

1. A *manifold* is a continuum space that locally looks Euclidian, i.e. it can approximated by a Minkowski metric around every point. A *differential manifold* is a manifold on which scalar functions can be defined that are continuous and differentiable everywhere. That is, a differential manifold is a *smooth (hyper)surface*. We will only use differential manifolds in the following (just manifold for short).

2. A surface that contains singularities, spikes, etc, is not a manifold.

3. In a slightly more abstract way, a manifold is any set of points that can be continuously parametrized. The number of independent parameters is equal to the dimension of the manifold. The ordinary Euclidian space, for instance, is a 3D manifold because we can identify every point with three parameters, x, y, z. The space of permutations of a sequence of numbers is not a manifold because the order or permutations is not a continuous parameter. The surface of a cone is not a manifold because the vertex is not differentiable.

4. Since a D-dimensional manifold can be continuously parametrized by D parameters, every manifold can be mapped into a D -dimensional Euclidian space, that is, every point of the manifold can be identified with (or mapped into) a point in the Euclidian space.

5. Since we can differentiate scalar functions, we can define vectors, 1-forms, tensors etc, without the need of introducing a metric.

6. Introducing the metric, we say that a differential manifold endowed with a metric is a *Riemannian space*. If the metric has Minkowskian *signature* (i.e., the eigenvalues of the diagonalized metric have the same signs as the Minkowski metric; the signature it's an invariant quantity) instead of Cartesian, we say sometimes it's a pseudo-Riemannian manifold.

7. *Coordinate transformations* are defined as

$$\Lambda^{\bar{\alpha}}_{\ \beta} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}} \tag{5.1}$$

   Since second derivatives commute, we have that

$$\frac{\partial \Lambda^{\bar{\alpha}}_{\ \beta}}{\partial x^{\gamma}} = \frac{\partial \Lambda^{\bar{\alpha}}_{\ \gamma}}{\partial x^{\beta}} \tag{5.2}$$

   Only matrices that obey this condition are coordinate transformations. Matrices that do not obey this, are called just transformations.

8. A theorem says that for every non-singular symmetric tensor there is a transformation (not, in general, a coordinate transformation) that transforms the metric in the vicinity of a point into a diagonal one with ± 1 on the diagonal. This sequence of ± 1 is a invariant characteristic of every metric tensor.

9. The trace of this diagonal metric is called *signature*. Moreover, a *coordinate transformation* exists such that around a point P

$$g_{\alpha\beta}(P) = \eta_{\mu\nu} \tag{5.3}$$
$$g_{\alpha\beta,\gamma}(P) = 0 \tag{5.4}$$
$$g_{\alpha\beta,\gamma\mu}(P) \neq 0 \tag{5.5}$$

This means that it always exist a *local inertial frame* (LIF), i.e. a frame such that the metric is locally (i.e., up to first derivatives, which turn out to express the forces) Minkowskian. But, in general, there is no coordinate transformation that makes $g_{\mu\nu}$ Minkowskian *everywhere*. The second derivatives express the *tidal effects*.

10. In particular, this means that locally we can always set up a locally inertial observer such that $\Gamma^{\mu}{}_{\alpha\beta} = 0$, but, in general,

$$\Gamma^{\mu}{}_{\alpha\beta,\gamma} \neq 0 \tag{5.6}$$

11. Finding for a point P a general coordinate transformation $\Lambda^{\bar{\alpha}}_{\gamma} = \partial x^{\bar{\alpha}}/\partial x^{\gamma}|_P$ (formed by 16 entries) that defines a LIF means solving Eqs. (5.3-5.4). However a LIF can be determined only up to the 6 free parameters of the Lorentz group, namely three velocities v^i and three rotations, so there are only 10 unknowns in $\Lambda^{\bar{\alpha}}_{\gamma}$. They are fixed by the condition $g_{\alpha\beta} = 0$, which indeed imposes 10 equations (the independent entries of the symmetric matrix $g_{\alpha\beta}$). For the second condition, $g_{\alpha\beta,\gamma} = 0$, one has 40 equations for the 40 unknowns (4×10) given by $\Lambda^{\bar{\alpha}}_{\beta,\gamma} = \partial^2 x^{\bar{\alpha}}/\partial x^{\gamma}\partial x^{\beta}$. If now we want to impose a further condition, i.e. $g_{\alpha\beta,\gamma\mu}(P) = 0$, we see that we would get 100 equations for $\Lambda^{\bar{\alpha}}_{\beta,\gamma\mu} = \partial^3 x^{\bar{\alpha}}/\partial x^{\gamma}\partial x^{\beta}\partial x^{\mu}$, which however consist of only 80 unknowns due to the symmetry between permutations of γ, β, μ (a fully symmetric matrix of dimension D with m indexes has $(D+m-1)!/(D-1)!m!$ independent entries). It is therefore in general impossible to find a frame in which the second derivatives of the metric vanish. These derivatives represent an intrinsic property of the manifold that cannot be absorbed by a coordinate transformation. They can be put in relation to the general curvature tensor, which, as we will see later, has exactly 20 independent entries.

5.2 Lengths and volumes in manifolds

1. The infinitesimal interval of length in a manifold is

$$ds^2 = (\vec{dx}) \cdot (\vec{dx}) = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{5.7}$$

A finite interval along a certain curve parametrized by λ will be therefore

$$\ell = \int ds = \int |g_{\alpha\beta} dx^{\alpha} dx^{\beta}|^{1/2} \tag{5.8}$$
$$= \int |g_{\alpha\beta} \frac{dx^{\alpha}}{d\lambda} \frac{dx^{\beta}}{d\lambda}|^{1/2} d\lambda \tag{5.9}$$
$$= \int |\vec{V} \cdot \vec{V}|^{1/2} d\lambda \tag{5.10}$$

if $\vec{V} \to \{\frac{dx^{\alpha}}{d\lambda}\}$ is a vector tangent to the curve.

2. The infinitesimal volume in flat space with Cartesian coordinates is

$$d^4x = dx^0 dx^1 dx^2 dx^3 \tag{5.11}$$

Under a general coordinate transformation, this becomes

$$d^4x = |J^{-1}| d^{\bar{4}}x = |J^{-1}| dx^{\bar{0}} dx^{\bar{1}} dx^{\bar{2}} dx^{\bar{3}} \tag{5.12}$$

where $|...|$ means determinant, and

$$J^{-1} \to \frac{\partial x^{\alpha}}{\partial x^{\bar{\beta}}} = \Lambda^{\alpha}_{\bar{\beta}} \tag{5.13}$$

The volume, therefore, is not a scalar quantity, because it changes under a coordinate transformation, nor evidently is a tensor. When a quantity acquires a power of the determinant of the Jacobian under a transformation, it is called a "tensor density", in this case a scalar density.

3. Now, we know that the metric transforms as

$$g_{\bar{\alpha}\bar{\beta}} = \Lambda^{\mu}_{\ \bar{\alpha}}\Lambda^{\nu}_{\ \bar{\beta}}g_{\mu\nu} \tag{5.14}$$

so that its determinant is

$$-|\bar{g}| = -|\Lambda|^2|g| \tag{5.15}$$

(notice therefore that $|g|$ is not a scalar but a scalar density). Then we see that

$$\sqrt{-|g|}d^4x = \sqrt{-|\bar{g}|}d^{\bar{4}}x \tag{5.16}$$

that is, $d^4x\sqrt{-g}$ is the scalar quantity that defines the volume element, also called *proper volume*. (The minus sign is needed because the determinant of g is always negative).

4. Derivative of the inverse of a matrix:

$$M^{-1}_{,\alpha} = -M^{-1}M_{,\alpha}M^{-1} \tag{5.17}$$

or, componentwise,

$$(M^{-1})_{\alpha\beta,\gamma} = -(M^{-1})_{\alpha\mu}M^{\mu\nu}_{,\gamma}(M^{-1})_{\nu\beta} \tag{5.18}$$

5. Derivative of metric determinant. We know that the inverse of a matrix is the transpose of the co-factor matrix $C = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor matrix (the matrix obtained by replacing every element ij by the determinant of the matrix obtained by excluding the row and column of ij) divided by the determinant,

$$g^{-1} = \frac{C}{|g|}$$

So we have

$$\frac{\partial}{\partial g^{\mu\nu}}|g|g^{-1} = \frac{\partial C}{\partial g^{\mu\nu}} = 0 \tag{5.19}$$

because $C^{\mu\nu}$ does not depend on $g^{\mu\nu}$ (its entries are the determinant formed by excluding the μ-th row and ν-th column). Then we find

$$\frac{\partial}{\partial g^{\mu\nu}}|g|g^{-1} = g^{-1}\frac{\partial}{\partial g^{\mu\nu}}|g| - |g|g^{-1}\frac{\partial g}{\partial g^{\mu\nu}}g^{-1} = 0 \tag{5.20}$$

or, in explicit index form

$$\frac{\partial}{\partial g^{\mu\nu}}|g| = |g|\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}}g^{\alpha\beta} \tag{5.21}$$

Multiplying by $\frac{\partial g^{\mu\nu}}{\partial x^{\gamma}}$ this can be written as an ordinary derivative of the determinant,

$$|g|_{,\gamma} = |g|g_{\alpha\beta,\gamma}g^{\alpha\beta} \tag{5.22}$$

This is a very useful formula. Be careful: the *covariant* derivative of $|g|$, in contrast, vanishes, because $g_{\alpha\beta;\gamma} = 0$.

6. Notice that

$$\frac{\partial g_{\alpha\beta}}{\partial g_{\mu\nu}} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} \tag{5.23}$$

$$\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}} = -g_{\alpha\mu}g_{\beta\nu} \tag{5.24}$$

(in the second line we applied Eq. (5.18)). Therefore

$$\frac{\partial}{\partial g^{\mu\nu}}|g| = |g|\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}}g^{\alpha\beta} = -|g|g_{\mu\nu} \tag{5.25}$$

7. As a consequence,

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} + g_{\beta\alpha,\mu} - g_{\mu\alpha,\beta}) \tag{5.26}$$
$$= \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\alpha} - g_{\mu\alpha,\beta}) + \frac{1}{2}g^{\alpha\beta}g_{\beta\alpha,\mu} \tag{5.27}$$

but the first term is zero because is the trace of the product of a symmetric and an antisymmetric term. Then

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta,\mu} = \frac{1}{2}\frac{g_{,\mu}}{g} = \frac{(\sqrt{-g})_{,\mu}}{\sqrt{-g}} = \frac{1}{2}[\log(-g)]_{,\mu} \tag{5.28}$$

8. The equation for the divergence (4.71) simplifies then in this way

$$V^{\alpha}_{;\alpha} = V^{\alpha}_{,\alpha} + V^{\mu}\Gamma^{\nu}_{\nu\alpha} = V^{\alpha}_{,\alpha} + V^{\alpha}\frac{(\sqrt{-g})_{,\alpha}}{\sqrt{-g}} \tag{5.29}$$
$$= \frac{(V^{\alpha}\sqrt{-g})_{,\alpha}}{\sqrt{-g}} \tag{5.30}$$

9. This implies that, by applying Gauss' law,

$$\int V^{\alpha}_{;\alpha}\sqrt{-g}d^4x = \int (V^{\alpha}\sqrt{-g})_{,\alpha}d^4x = \oint V^{\alpha}n_{\alpha}\sqrt{-g}d^3s \tag{5.31}$$

This form of Gauss' law is very convenient.

10. Often in GR one applies the previous equation to fields that vanish at infinity, e.g. far from any mass. Then the boundary integral vanishes at infinity. Then one has

$$\int d^4x\sqrt{-g}AB^{\alpha}{}_{;\alpha} = \int d^4x\sqrt{-g}[(AB^{\alpha})_{;\alpha} - A_{;\alpha}B^{\alpha}] = -\int d^4x\sqrt{-g}A_{;\alpha}B^{\alpha} \tag{5.32}$$

which is similar to an ordinary integration by parts.

11. If we apply the divergence to a vector defined as $V^{\mu} \equiv \phi^{;\mu} = \phi^{,\mu}$, i.e. as a derivative of a scalar, we find the very useful relation valid in every coordinate frame

$$\Box\phi = \phi^{;\mu}_{;\mu} = \frac{(\phi^{,\alpha}\sqrt{-g})_{,\alpha}}{\sqrt{-g}} = \frac{1}{\sqrt{-g}}\partial_{\alpha}(\sqrt{-g}g^{\alpha\beta}\partial_{\beta}\phi) \tag{5.33}$$

5.3 Parallel transport

1. A vector is said to be parallel-transported (same direction and same length) along a curve parametrized by λ when its components remain constant in a locally inertial frame around point P (see Fig. 5.1).

2. That is,

$$\frac{dV^{\alpha}}{d\lambda} = \frac{d\tau}{d\lambda}U^{\beta}V^{\alpha}_{,\beta} = 0 \tag{5.34}$$

or simply

$$U^{\beta}V^{\alpha}_{,\beta} = 0 \tag{5.35}$$

Now, in order for this equation to be valid in every frame, we upgrade it to

$$U^{\beta}V^{\alpha}_{;\beta} = 0 \tag{5.36}$$

3. If we choose $\vec{V} = \vec{U}$, we are parallel-transporting (PT) the tangent vector to the curve. This imposes a condition on the curve itself, that is, it identifies those curves along which the tangent vectors are PT:

these will be called *auto-parallels* (when we only refer to the parallel-transport property) or *geodesics* (when we include also the property of extremality, see below). Then we have

$$U^\beta U^\alpha_{;\beta} = U^\beta U^\alpha_{,\beta} + U^\beta \Gamma^\alpha_{\nu\beta} U^\nu \tag{5.37}$$

$$= \frac{dx^\beta}{d\tau}\frac{\partial}{\partial x^\beta}\frac{dx^\alpha}{d\tau} + \frac{dx^\beta}{d\tau}\Gamma^\alpha_{\nu\beta}\frac{dx^\nu}{d\tau} \tag{5.38}$$

$$= \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\nu\beta}\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{5.39}$$

This is called *geodesic equation* (they are actually four equations). Along the curve $x^\mu(\tau)$ that is a solution of this equation, the tangent vector is PT.

4. The length of a tangent vector is

$$|\frac{dx^\mu}{d\lambda}| = \left(g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\alpha}{d\lambda}\right)^{1/2} = |\frac{d\tau}{d\lambda}| \tag{5.40}$$

so we see that it does not change along the curve parametrized by λ only if τ and λ are linearly related. Parameters that are linearly related, i.e. $\lambda = a\tau + b$ are called *affine parameters*.

5. For a light ray, we cannot use neither $\vec{U}$ nor the proper time τ. We use then the momentum $\vec{p}$ and any monotonic parametrization λ of the null path. One can always use one of the geodesic equations to express λ as a function of a coordinate.

6. The geodesic equation describes the dynamics (acceleration) of a particle along the geodesic path

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\nu\beta}\frac{dx^\beta}{d\tau}\frac{dx^\nu}{d\tau} \tag{5.41}$$

If there are non-gravitational forces, then a particle will deviate from the geodesic and if the forces are described by a potential $V(x^\mu)$ then a term $-\partial V/\partial x_\alpha$ is to be added at the rhs.

7. Now we show an important property of geodesics: they are the paths of extremal distance (interval) between two points (also called stationary paths), defined as (for time-like paths, so $d\tau = \sqrt{-ds^2}$)

$$\tau = \int d\tau = \int |g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}|^{1/2} d\tau \tag{5.42}$$

Now, if we distort a little the path from a geodesic, we will have a small perturbation $\tau + \delta\tau$. If the geodesic is extremal, then $\delta\tau = 0$. That is what we are going to show. First notice that for an arbitrary function $f(\tau)$

$$\delta\int\sqrt{-f}d\tau = \int(\delta\sqrt{-f})d\tau \tag{5.43}$$

$$= -\frac{1}{2}\int(-f)^{-1/2}\delta f d\tau \tag{5.44}$$

Since in our case

$$f = g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = U^\alpha U_\alpha = -1 \tag{5.45}$$

we have

$$\delta\tau = -\frac{1}{2}\int(\delta f)d\tau \tag{5.46}$$

that is, extremizing $\sqrt{f}$ is the same as extremizing f. Then our problem is to show that

$$\delta I = \delta[\int g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}d\tau] = 0 \tag{5.47}$$

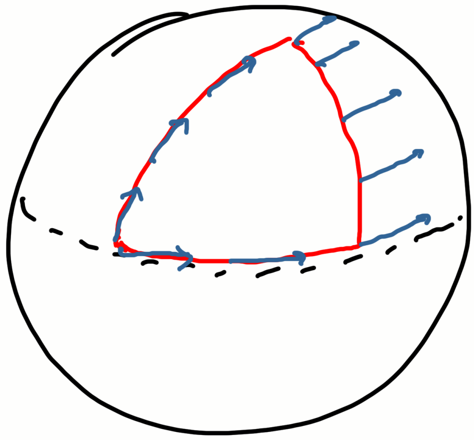

Figure 5.1: Parallel transport on a sphere.

8. The variation δ means that we have to perturb the coordinates

$$x^\mu \to x^\mu + \delta x^\mu \tag{5.48}$$

and consequently,

$$g_{\mu\nu} \to g_{\mu\nu} + (g_{\mu\nu,\sigma})\delta x^\sigma \tag{5.49}$$

(notice it's a normal derivative, we are doing a simple Taylor expansion of the functions $g_{\mu\nu}$). Then we have

$$\delta I = \int [\delta g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{d\delta x^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\delta x^\beta}{d\tau}]d\tau \tag{5.50}$$

$$= \int [g_{\alpha\beta,\sigma}\delta x^\sigma\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{d\delta x^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\delta x^\beta}{d\tau}]d\tau \tag{5.51}$$

The second term can be integrated by parts:

$$\int d\tau g_{\alpha\beta}\frac{d\delta x^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = -\int d\tau (g_{\alpha\beta,\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{d^2x^\beta}{d\tau^2})\delta x^\alpha \tag{5.52}$$

and similarly for the third term. Collecting, we have

$$\delta I = -2\int [g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}]\delta x^\sigma d\tau = 0 \tag{5.53}$$

Now a crucial step: we want this equation to be satisfied for every variation δx^σ. This means the expression within square brackets must vanish:

$$g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{5.54}$$

Finally, multiplying by $g^{\sigma\tau}$, we obtain the geodesic equation

$$\frac{d^2x^\tau}{d\tau^2} + \Gamma^\tau{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{5.55}$$

This shows that the geodesic equation produces paths of shortest (actually, extremal) interval. The geodesics generalize the concept of "straight line" on a curved manifold.

9. Although one can PT a tangent vector defined for any parameter λ, only for a special class of λ it happens that the geodesic is also the shortest path, namely, only when they are related to τ by an *affine transformation*, i.e. $\lambda = a\tau + b$. In this case in fact Eq. (5.55) coincides with Eq. (5.39). Therefore τ and all the affinely related λ's are called *affine parameters*. From now on we only consider such parameters for time-like geodesics.

10. A "straight line" on a curved manifold is then defined as that line on which a tangent vector remains parallel to itself.

11. The connection with physics is given by extending to the geodesic the Galilean principle: a particle on which no external force act will follow a geodesic. The effect of the curved manifold is, in this sense, not an external force, it's just the geometry of space-time. In GR, we will see that gravity is what causes the manifold to be curved. So, a particle under the sole action of gravity follows a geodesic. This is called *free-fall.*

12. Summarizing: a tangent vector is PT if it moves along a geodesic parametrized by λ. If $\tau = \lambda$, then the particle whose 4-velocity is $\vec{U}$ is in free-fall. The Christoffel symbols express the effect of gravity (geometry) on the particle: they are the "gravitational force" acting on the particle.

13. A vector orthogonal to the tangent vector, is also PT along the same geodesic. Therefore, since every vector on a point P can be decomposed into components orthogonal and parallel to the tangent vector, it will also be PT. In other words, along a geodesic with tangent vector $\vec{U}$, a vector $\vec{V}$ defined at one point P is PT to another point P' if
$$U^\alpha V^\beta_{;\alpha} = 0 \tag{5.56}$$
between P and P'.

14. Let us define now the operator of directional covariant derivative of a vector $\vec{V}$ along a curve with tangent vector $\vec{U}$:
$$\nabla_{\vec{U}} \vec{V} \to \frac{D}{d\tau} V^\alpha = \frac{dx^\mu}{d\tau} \nabla_\mu V^\alpha = U^\mu \nabla_\mu V^\alpha \tag{5.57}$$
(where ∇_μ is an alternative symbol for the covariant derivative, so $\nabla_\mu V^\nu \equiv V^\nu_{;\mu}$): PT means then $\nabla_{\vec{U}} \vec{V} = 0$. Now, since
$$\nabla_{\vec{U}} g_{\alpha\beta} = g_{\alpha\beta;\gamma} U^\gamma = 0 \tag{5.58}$$
it follows also that
$$\nabla_{\vec{U}} (g_{\alpha\beta} A^\alpha B^\beta) = g_{\alpha\beta}[(\nabla_{\vec{U}} A^\alpha) B^\beta + A^\alpha (\nabla_{\vec{U}} B^\beta)] = 0 \tag{5.59}$$
This shows that (for metric-compatible theories) the norm of vectors does not vary when transported along the geodesic. Therefore, a vector remains time-, null-, or space-like when PT.

5.4 The Riemann tensor

1. Let us consider the curved parallelogram in Fig. 5.2. A vector in A$=(a,b)$ is PT to B along $x^2 = const$, i.e. along the basis vector $\vec{e}_1 \to \{1,0\}$ if
$$\nabla_{\vec{e}_1} \vec{V} = 0 \quad \to (\vec{e}_1)^\beta (V^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} V^\mu) = 0 \to \quad \frac{\partial V^\alpha}{\partial x^1} = -\Gamma^\alpha_{\mu 1} V^\mu \tag{5.60}$$
and similarly if PT along $\vec{e}_2$. So, at B$=(a+\delta a, b)$, it has components
$$V^\alpha(B) = V^\alpha(A_{init}) - \int_{x^1=a}^{x^1=a+\delta a} \Gamma^\alpha_{\mu 1} V^\mu dx^1|_{x^2=b} = V^\alpha(A_{init}) - \delta a (\Gamma^\alpha_{\mu 1} V^\mu dx^1)|_{x^2=b} \tag{5.61}$$
From B to C$=(a+\delta a, b+\delta b)$ and D$=(a, b+\delta b)$ we have similarly
$$V^\alpha(C) = V^\alpha(B) - \int_{x^2=b}^{x^2=b+\delta b} \Gamma^\alpha_{\mu 2} V^\mu dx^2|_{x^1=a+\delta a} = V^\alpha(B) - \delta b (\Gamma^\alpha_{\mu 2} V^\mu dx^2)|_{x^1=a+\delta a} \tag{5.62}$$

$$V^\alpha(D) = V^\alpha(C) + \int_{x^1=a}^{x^1=a+\delta a} \Gamma^\alpha_{\mu 1} V^\mu dx^1|_{x^2=b+\delta b} = V^\alpha(C) + \delta a \Gamma^\alpha_{\mu 1} V^\mu dx^1|_{x^2=b+\delta b} \tag{5.63}$$

(notice the sign inversion because we move from $a + \delta a$ to a) and finally

$$V^\alpha(A_{final}) = V^\alpha(D) + \int_{x^2=b}^{x^2=b+\delta b} \Gamma^\alpha_{\mu 2} V^\mu dx^2|_{x^1=a} = V^\alpha(D) + \delta b(\Gamma^\alpha_{\mu 2} V^\mu dx^2)|_{x^1=a} \tag{5.64}$$

Now, this means

$$\begin{aligned}
\delta V^\alpha &= V^\alpha(A_{final}) - V^\alpha(A_{init}) && (5.65)\\
&= \delta b(\Gamma^\alpha_{\mu 2} V^\mu dx^2)|_{x^1=a} + \delta a \Gamma^\alpha_{\mu 1} V^\mu dx^1|_{x^2=b+\delta b} && (5.66)\\
&- \delta b(\Gamma^\alpha_{\mu 2} V^\mu dx^2)|_{x^1=a+\delta a} - \delta a(\Gamma^\alpha_{\mu 1} V^\mu dx^1)|_{x^2=b} && (5.67)\\
&= \delta b[(\Gamma^\alpha_{\mu 2} V^\mu dx^2)|_{x^1=a} - (\Gamma^\alpha_{\mu 2} V^\mu dx^2)|_{x^1=a+\delta a}] && (5.68)\\
&+ \delta a[\Gamma^\alpha_{\mu 1} V^\mu dx^1|_{x^2=b+\delta b} - (\Gamma^\alpha_{\mu 1} V^\mu dx^1)|_{x^2=b}] && (5.69)\\
&= -\delta b \delta a(\Gamma^\alpha_{\mu 2} V^\mu)_{,1} dx^2 + \delta a \delta b(\Gamma^\alpha_{\mu 1} V^\mu)_{,2} dx^1 && (5.70)\\
&= \delta a \delta b[(-\Gamma^\alpha_{\mu 2} V^\mu)_{,1} + (\Gamma^\alpha_{\mu 1} V^\mu)_{,2}] && (5.71)
\end{aligned}$$

Moreover, from Eq. (5.60), $V^\alpha_{,1} = -\Gamma^\alpha_{\mu 1} V^\mu$ and therefore

$$\begin{aligned}
\delta V^\alpha &= \delta a \delta b[\Gamma^\alpha_{\mu 1,2} V^\mu - \Gamma^\alpha_{\mu 2,1} V^\mu + \Gamma^\alpha_{\mu 1} V^\mu_{,2} - \Gamma^\alpha_{\mu 2} V^\mu_{,1}] && (5.72)\\
&= \delta a \delta b[\Gamma^\alpha_{\mu 1,2} V^\mu - \Gamma^\alpha_{\mu 2,1} V^\mu + \Gamma^\alpha_{1\sigma} V^\sigma_{,2} - \Gamma^\alpha_{\sigma 2} V^\sigma_{,1}] && (5.73)\\
&= \delta a \delta b[\Gamma^\alpha_{\mu 1,2} V^\mu - \Gamma^\alpha_{\mu 2,1} V^\mu + \Gamma^\alpha_{1\tau}(-\Gamma^\tau_{\mu 2} V^\mu) - \Gamma^\alpha_{\tau 2}(-\Gamma^\tau_{\mu 1} V^\mu)] && (5.74)\\
&= \delta a \delta b[\Gamma^\alpha_{\mu 1,2} - \Gamma^\alpha_{\mu 2,1} - \Gamma^\alpha_{1\tau}\Gamma^\tau_{\mu 2} + \Gamma^\alpha_{\tau 2}\Gamma^\tau_{\mu 1}] V^\mu && (5.75)
\end{aligned}$$

The generalization to paths along coordinates $x^\sigma x^\lambda$ is an obvious one:

$$\begin{aligned}
\delta V^\alpha &= \delta x^\sigma \delta x^\lambda [\Gamma^\alpha_{\mu\sigma,\lambda} - \Gamma^\alpha_{\mu\lambda,\sigma} - \Gamma^\alpha_{\sigma\tau}\Gamma^\tau_{\mu\lambda} + \Gamma^\alpha_{\tau\lambda}\Gamma^\tau_{\mu\sigma}] V^\mu && (5.76)\\
&= \delta x^\sigma \delta x^\lambda V^\mu R^\alpha_{\mu\lambda\sigma} && (5.77)
\end{aligned}$$

This defines the *Riemann tensor* (RT)

$$\begin{aligned}
R^\alpha_{\beta\mu\nu} &= \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\nu\beta} && (5.78)\\
&= \Gamma^\alpha_{\beta\nu;\mu} - \Gamma^\alpha_{\beta\mu;\nu} && (5.79)
\end{aligned}$$

The last line has to be understood only in the sense that Γ is treated as a vector with index α.

2. The Riemann tensor therefore expresses how the components of the vector V^α vary when it is PT along a closed circuit. Although the vector is PT, its components change if the manifold is not flat. The Riemann tensor is therefore an *intrinsic* (because obtained on the manifold itself, without making reference to an external higher dimensional space) measure of curvature.

5.5 Properties of the Riemann tensor

1. In a local inertial frame (LIF), i.e. when the Christoffel symbols vanish but not their derivative, one has

$$R^\alpha_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(g_{\sigma\nu,\beta\mu} - g_{\beta\nu,\sigma\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu}) \tag{5.80}$$

and

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\beta\nu,\alpha\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu}) \tag{5.81}$$

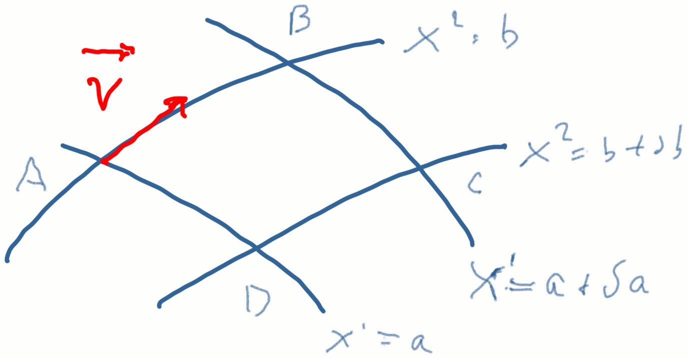

Figure 5.2: PT of a vector around a closed loop.

2. From these expressions we can derive some properties of the RT:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta} \tag{5.82}$$

i.e., antisymmetry under exchange of indexes within the first and second pair, symmetry under exchange of indexes among the pairs.

3. Cyclic permutation among the last three indexes:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \tag{5.83}$$

4. Because of these symmetries, of the $4^4 = 256$ components of the RT, only 20 are independent. (Think of the RT as a R_{AB} matrix; because of antisymmetry within the pairs A and B, only 6 values of A and 6 of B are independent, so R_{AB} can be written as a $6{\times}6$ matrix; this matrix is symmetric, so only $6 \cdot 7/2 = 21$ entries are independent; the permutation condition adds yet another independent constraint, so we are left with 20.)

5. That's exactly the number of components of $g_{\alpha\beta,\mu\nu}$ that remain even after exploiting the freedom allowed by a general coordinate transformation: the RT is then a (actually the only one) covariant way to combine the second derivatives of the metric.

6. Sufficient and necessary condition for a flat manifold is

$$R^{\alpha}{}_{\beta\mu\nu} = 0 \tag{5.84}$$

Since this is a covariant equation, flatness is an invariant property of a manifold: it depends on its intrinsic properties, not on the choice of coordinates. For instance, a Minkowskian metric, whether in Cartesian form, in spherical coordinates, or in Rindler coordinates (see Eq. 1.145), has zero Riemann tensor.

7. Notice that a manifold has also global properties that are not described by curvature. For instance, curvature does not tell us if the manifold has multiple connectdness, that is, is closed like a torus or with holes, etc. These properties are called topological and are not described by General Relativity.

5.6 Commutation of covariant derivatives and geodesic deviation.

1. As we mentioned, covariant derivatives do not commute in general. So we have

$$(V^{\mu}_{;\beta})_{;\alpha} = (V^{\mu}_{;\beta})_{,\alpha} + (V^{\nu}_{;\beta})\Gamma^{\mu}_{\nu\alpha} - V^{\mu}_{;\nu}\Gamma^{\nu}_{\beta\alpha} \tag{5.85}$$

In a LIF, this simplifies to

$$(V^{\mu}_{;\beta})_{;\alpha} = V^{\mu}_{,\alpha\beta} + \Gamma^{\mu}_{\nu\beta,\alpha}V^{\nu} \tag{5.86}$$

Exchanging α, β and subtracting the two expressions, we obtain the commutator

$$[\nabla_\alpha, \nabla_\beta]V^\mu = \nabla_\alpha\nabla_\beta V^\mu - \nabla_\beta\nabla_\alpha V^\mu = (\Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta})V^\nu \tag{5.87}$$

The term in parentheses coincides with the RT in the LIF. So we draw the conclusion that

$$[\nabla_\alpha, \nabla_\beta]V^\mu = R^{\mu}_{\ \nu\alpha\beta}V^\nu \tag{5.88}$$

I.e., the RT expresses the non-commutativity of covariant derivatives. (For a covariant vector, the equation is $[\nabla_\alpha, \nabla_\beta]V_\mu = -R^{\nu}_{\ \mu\alpha\beta}V_\nu$.)

2. Similarly, we show below that the RT expresses the *geodesic deviation*. Let us define a family of geodesic curves $x^\mu = x^\mu(p, \lambda)$, in which λ runs along each curve, while p distinguishes one geodesic from another one (such sets of non-intersecting curves that fill a manifold is called a *congruence*). We want to find how a separation vector $\xi^\mu = \partial x^\mu/\partial p$ between nearby geodesics, with tangent vector $\vec{V} = \partial x^\mu/\partial\lambda$, changes when the parameter λ changes:

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha = R^{\alpha}_{\ \mu\nu\beta}V^\mu V^\nu \xi^\beta \tag{5.89}$$

3. Proof of Eq. (5.89). Following Uzan-DeRuelle, *Relativity in Modern Physics*, let us first derive the following identity:

$$\xi^\alpha\nabla_\alpha V^\beta = \frac{\partial x^\alpha}{\partial p}(V^\beta_{,\alpha} + \Gamma^{\beta}_{\mu\alpha}V^\mu) = \frac{\partial x^\alpha}{\partial p}(V^\beta_{,\alpha} + \Gamma^{\beta}_{\mu\alpha}V^\mu) \tag{5.90}$$

$$= \frac{\partial x^\alpha}{\partial p}\left(\frac{\partial}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\lambda} + \Gamma^{\beta}_{\mu\alpha}\frac{\partial x^\mu}{\partial\lambda}\right) = \frac{\partial}{\partial p}\frac{\partial x^\beta}{\partial\lambda} + \Gamma^{\beta}_{\mu\alpha}\frac{\partial x^\alpha}{\partial p}\frac{\partial x^\mu}{\partial\lambda} \tag{5.91}$$

$$= \frac{\partial}{\partial\lambda}\frac{\partial x^\beta}{\partial p} + \Gamma^{\beta}_{\alpha\mu}\frac{\partial x^\mu}{\partial p}\frac{\partial x^\alpha}{\partial\lambda} = \frac{\partial x^\alpha}{\partial\lambda}\left(\frac{\partial}{\partial x^\alpha}\frac{\partial x^\beta}{\partial p} + \Gamma^{\beta}_{\mu\alpha}\frac{\partial x^\mu}{\partial p}\right) \tag{5.92}$$

$$= V^\alpha\nabla_\alpha\xi^\beta \tag{5.93}$$

Then we have

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha = V^\mu\nabla_\mu(V^\nu\nabla_\nu\xi^\alpha) = V^\mu\nabla_\mu(\xi^\nu\nabla_\nu V^\alpha) \tag{5.94}$$

$$= V^\mu\xi^\nu\nabla_\mu\nabla_\nu V^\alpha + V^\mu(\nabla_\mu\xi^\nu)(\nabla_\nu V^\alpha) \tag{5.95}$$

$$= V^\mu\xi^\nu(\nabla_\nu\nabla_\mu V^\alpha + R^{\alpha}_{\ \beta\mu\nu}V^\beta) + V^\mu(\nabla_\mu\xi^\nu)(\nabla_\nu V^\alpha) \tag{5.96}$$

But is also true that, since $V^\nu\nabla_\nu V^\alpha = 0$ for auto-parallels, then

$$0 = \xi^\mu\nabla_\mu(V^\nu\nabla_\nu V^\alpha) = \xi^\mu V^\nu(\nabla_\mu\nabla_\nu V^\alpha) + \xi^\mu(\nabla_\mu V^\nu)(\nabla_\nu V^\alpha) \tag{5.97}$$

$$= \xi^\mu V^\nu(\nabla_\mu\nabla_\nu V^\alpha) + V^\mu(\nabla_\mu\xi^\nu)(\nabla_\nu V^\alpha) \tag{5.98}$$

so that $V^\mu(\nabla_\mu\xi^\nu)(\nabla_\nu V^\alpha) = -\xi^\mu V^\nu(\nabla_\mu\nabla_\nu V^\alpha)$. Finally, inserting this in Eq. (5.96), we obtain

$$\nabla_{\vec{V}}\nabla_{\vec{V}}\xi^\alpha = V^\mu\xi^\nu(\nabla_\nu\nabla_\mu V^\alpha + R^{\alpha}_{\ \beta\mu\nu}V^\beta) - \xi^\mu V^\nu(\nabla_\mu\nabla_\nu V^\alpha) = R^{\alpha}_{\ \mu\nu\beta}V^\mu V^\nu\xi^\beta \tag{5.99}$$

(after some reshuffling of dummy indexes).

5.7 Bianchi identities, Ricci tensor, Ricci scalar, Weyl tensor

1. Putting ourselves again in the LIF, where the Γs vanish but their derivatives do not, we can evaluate

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}) \tag{5.100}$$

Now using the symmetry $g_{\alpha\beta} = g_{\beta\alpha}$ one can show that

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0 \tag{5.101}$$

In general, this relation becomes

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \tag{5.102}$$

These equations are known as the *Bianchi identities*. Notice that they are obtained through a cyclic permutation of the last three indexes.

2. The contraction of Riemann gives the Ricci tensor

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} \tag{5.103}$$

The other possible contractions are either zero ($R^{\mu}{}_{\mu\alpha\beta} = 0$) or equivalent to Ricci ($R^{\mu}{}_{\alpha\beta\mu} = -R_{\alpha\beta}$).

3. Since

$$R_{\beta\alpha} = g^{\mu\nu} R_{\nu\beta\mu\alpha} = g^{\mu\nu} R_{\mu\alpha\nu\beta} = R_{\alpha\beta} \tag{5.104}$$

it follows that the Ricci tensor is symmetric.

4. We can write explicitly

$$R_{\beta\nu} = \Gamma^{\alpha}{}_{\beta\nu,\alpha} - \Gamma^{\alpha}{}_{\beta\alpha,\nu} - \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\beta\alpha} + \Gamma^{\alpha}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} \tag{5.105}$$
$$= \Gamma^{\alpha}{}_{\beta\nu;\alpha} - \Gamma^{\alpha}{}_{\beta\alpha;\nu} \tag{5.106}$$

5. Finally, contracting the Ricci tensor (i.e., taking the trace of the mixed tensor) we get the Ricci or curvature scalar

$$g^{\alpha\beta} R_{\alpha\beta} = R \tag{5.107}$$

6. The Bianchi identities impose some constraints on $R_{\alpha\beta}$. In fact, by contracting once

$$g^{\alpha\mu}\left(R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}\right) = R_{\beta\nu;\lambda} + (-R_{\beta\lambda;\nu}) + R^{\mu}{}_{\beta\nu\lambda;\mu} \tag{5.108}$$

(we used $R_{\alpha\beta\lambda\mu;\nu} = -R_{\alpha\beta\mu\lambda;\nu}$) and by contracting again

$$g^{\beta\nu}(R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu}) = R_{;\lambda} - R^{\nu}{}_{\lambda;\nu} - R^{\nu}{}_{\lambda;\nu} \tag{5.109}$$
$$= (-2R^{\nu}{}_{\lambda} + \delta^{\nu}_{\lambda} R)_{;\nu} = 0 \tag{5.110}$$

Now, if we define Einstein's tensor

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \tag{5.111}$$

or, in mixed form,

$$G^{\alpha}_{\beta} \equiv R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} R \tag{5.112}$$

we see that it is covariantly conserved, i.e.

$$G^{\alpha\beta}_{;\beta} = 0 \tag{5.113}$$

These equations are also called Bianchi identities. It can be shown that $G^{\alpha\beta}$ is the *only* combination of the RT that is conserved. It is actually possible to show that it is the only combination of second derivatives of the metric that is a tensor and that is conserved. If one considers also terms of less-than-second derivatives, then a term $\Lambda g_{\mu\nu}$ can also be added (more about this later on).

7. Notice that for a manifold to be flat, all components of the Riemann tensor must vanish; it is not sufficient that the Ricci tensor vanishes.

8. Another tensor is often defined out of the Riemann tensor, called Weyl tensor:

$$C_{\alpha\beta\mu\nu} \equiv R_{\alpha\beta\mu\nu} + \frac{1}{2}(R_{\alpha\nu}g_{\beta\mu} - R_{\alpha\mu}g_{\beta\nu} + R_{\beta\mu}g_{\alpha\nu} - R_{\beta\nu}g_{\alpha\mu}) \quad (5.114)$$

$$+ \frac{1}{6}R(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}) \quad (5.115)$$

This tensor has the same symmetries of the Riemann tensor but also other interesting properties. First, it is traceless under contraction of any two indexes. Secondly, its components do not vanish in general in empty space (where $R_{\mu\nu} = 0$, a condition called *Ricci-flatness*). Third, because of this reason, it governs the propagation of gravitational waves in empty space. Fourth, it is invariant under a *conformal transformation*, that is, it is the same for a metric $g_{\mu\nu}$ and for a conformally rescaled metric $g'_{\mu\nu} = \phi g_{\mu\nu}$, where ϕ is any function of coordinates (it's called conformal because angles do not change under it). As a consequence, if the Weyl tensor vanishes for a given metric, then there exists a coordinate transformation such that the metric is everywhere proportional to the Minkowski metric, that is, $g_{\mu\nu} = \phi\eta_{\mu\nu}$. Such a metric is called *conformally flat* (see e.g. the metric in Eq. 8.16). Finally, in dimensions less than four, the Weyl tensor vanishes identically. We will not make any further use of this tensor.

5.8 Einstein tensor

1. We have now all the ingredients to build Einstein's equations. We need now a basic postulate. We know that in flat space free particles move on straight lines, i.e. according to

$$U^\mu U^\nu_{,\mu} = 0 \quad (5.116)$$

Now we assume this extends to curved manifolds, i.e. free-falling particle (i.e. particles subject only to gravity) always move along geodesics,

$$U^\mu U^\nu_{;\mu} = 0 \quad (5.117)$$

2. This is one example of a more general principle, called strong equivalence principle (SEP): Every non-gravitational law of physics that in SR can be expressed as a tensor equation, has the same form in a local inertial frame in GR. In other words, Minkowski metric maps into the general metric, SR tensors map into GR tensors, and derivatives map into covariant derivatives.

3. The equivalence principle represents the famous Einstein's "elevator gedanken experiment". That is, on an elevator falling on a gravitational field, all non-gravitational experiments (eg, playing with billiard balls) perform the same way as when at rest. This is due to the fact that the gravitational field act in the same way on everything, regardless of the mass and of their composition; if we fall on a electromagnetic potential, instead, particles of opposite charge would act differently. If the "elevator" is very large or the observations are carried out for a long time, however, we would see tidal effects (for instance, two balls will be seen to approach each other as they follow radial, not parallel, free-falling trajectories converging towards the centre of the Earth): this corresponds to the non-vanishing second-derivatives of the metric.

4. Notice that in principle one could imagine that particles move along lines that obey a different equation, eg

$$U^\mu U^\nu_{;\mu} = R^{;\nu} \quad (5.118)$$

which would still reduce to (5.116) in flat space (not in a LIF however!), but the SEP excludes this possibility. Of course, ultimately, only experiments can say which equation is right.

5. From the SEP it follows that the flux-density vector and the EMT are covariantly conserved:

$$N^\alpha_{;\alpha} = 0 \quad (5.119)$$

$$T^{\mu\nu}_{;\mu} = 0 \quad (5.120)$$

6. Also other expressions we found in SR are automatically extended in GR replacing a comma with a semicolon and $\eta_{\mu\nu}$ with $g_{\mu\nu}$, for instance the definition of the energy density and pressure in a frame with velocity $\vec{U}$,

$$T^{\alpha\beta}U_\alpha U_\beta = \rho \tag{5.121}$$

and

$$p = \frac{1}{3}T^{\alpha\beta}(g_{\alpha\beta} + U_\alpha U_\beta) \tag{5.122}$$

5.9 Weak field limit

1. If the new conservation equation is correct, we should recover Newton's gravity in the limit of weak fields and slow velocities in which it is normally tested.

2. We have seen that the geodesic extremizes $\int d\tau$ or equivalently the space-time interval $\int ds$. But in presence of a gravitational potential Φ (e.g. for a point mass $\Phi = -GM/r$) a non-relativistic particle of mass m moves along trajectories that extremize the Lagrangian $m\int dt(\frac{1}{2}v^2 - \Phi)$. This is v in the usual units; to convert to our $c = 1$ units, this Lagrangian becomes

$$m\int dt(\frac{1}{2}(\frac{v}{c})^2 - \frac{\Phi}{c^2}) \rightarrow m\int dt(\frac{1}{2}v^2 - \Phi) \tag{5.123}$$

where now Φ includes the $1/c^2$ factor, so that for a point mass

$$\Phi = -\frac{GM}{rc^2} \tag{5.124}$$

For the Earth, for instance, $\Phi \approx 10^{-9}$, so much smaller than unity. From now on we revert to $c = 1$ units.

3. To convert the Langrangian into a new interval ds, we add a constant (which does not change the dynamics) and write

$$m\int dt(-1 - \Phi + \frac{1}{2}v^2) \quad \approx \quad m\int dt(-1 - 2\Phi + v^2)^{1/2} \equiv m\int ds \tag{5.125}$$

Then we see that, putting $v = dr/dt$, the new interval is

$$ds^2 = dt^2(-1 - 2\Phi + v^2) \tag{5.126}$$

$$= -dt^2(1 + 2\Phi) + dr^2 \tag{5.127}$$

4. If we had considered higher order terms in the velocity, we would have obtained a slight generalization, namely the weak field interval

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \tag{5.128}$$

with $\Phi = \Phi(t, x, y, z)$ a field over the entire space-time. Our goal now is to insert this metric

$$g_{\mu\nu} = \begin{pmatrix} -1-2\Phi & 0 & 0 & 0 \\ 0 & 1-2\Phi & 0 & 0 \\ 0 & 0 & 1-2\Phi & 0 \\ 0 & 0 & 0 & 1-2\Phi \end{pmatrix} \tag{5.129}$$

and

$$g^{\mu\nu} = \begin{pmatrix} (-1-2\Phi)^{-1} & 0 & 0 & 0 \\ 0 & (1-2\Phi)^{-1} & 0 & 0 \\ 0 & 0 & (1-2\Phi)^{-1} & 0 \\ 0 & 0 & 0 & (1-2\Phi)^{-1} \end{pmatrix} \tag{5.130}$$

$$\approx \begin{pmatrix} -1+2\Phi & 0 & 0 & 0 \\ 0 & 1+2\Phi & 0 & 0 \\ 0 & 0 & 1+2\Phi & 0 \\ 0 & 0 & 0 & 1+2\Phi \end{pmatrix} \tag{5.131}$$

into the geodesic equation and recover Newton's law. These expressions are only valid when $\Phi \ll 1$ (which is very well verified in all astrophysical bodies except black holes).

5. Since for now we confine ourselves to small velocities, we can actually put $g_{ii}, g^{ii} = 1$ in the following calculations.

6. We now show that this metric indeed recovers the standard Newtonian equations. For a massive particle, the geodesic equation can also be written with the four momentum $\vec{p}$ as

$$p^\alpha p^\beta_{;\alpha} = 0 \tag{5.132}$$

For $\beta = 0$ this is

$$p^\alpha p^0_{,\alpha} + \Gamma^0_{\alpha\beta} p^\alpha p^\beta = 0 \tag{5.133}$$

For small velocities, we can assume $p^0 \gg p^i$; moreover we have

$$p^\alpha \partial_\alpha = mU^\alpha \frac{\partial}{\partial x^\alpha} = m\frac{d}{d\tau} \tag{5.134}$$

so we obtain

$$m\frac{d}{d\tau}p^0 + \Gamma^0_{00} p^0 p^0 = 0 \tag{5.135}$$

Now, direct calculation shows that at first order

$$\Gamma^0_{00} = \frac{1}{2}g^{00}(g_{00,0} + g_{00,0} - g_{00,0}) = \frac{1}{2}g^{00}g_{00,0} = \dot{\Phi} \tag{5.136}$$

so finally, since $p^0 = m$ for small velocities (but $dp^0/d\tau \neq 0$), we get

$$\frac{dp^0}{d\tau} = -m\dot{\Phi} \tag{5.137}$$

7. So we see that in weak field GR the energy p^0 is conserved only if $\dot{\Phi} = 0$ (which is always the case in ordinary experiments).

8. A similar calculation for $\beta = i$, using now

$$\Gamma^i_{00} \approx \Phi^{,i} \tag{5.138}$$

(notice that $\Phi^{,i} = \Phi_{,i}$ at first order), gives

$$\frac{dp^i}{d\tau} = -m\Phi^{,i} \tag{5.139}$$

which as expected just says that the force acting on a particle equals minus the gradient of the potential. (Also, notice that $d\tau \approx dt$ for small velocities). This confirms that particles indeed move along geodesics.

5.10 Conserved quantities

1. Let us consider again the geodesic equation, written now with one index down,

$$p^\alpha p_{\beta;\alpha} = 0 \tag{5.140}$$

which is also

$$p^\alpha \partial_\alpha p_\beta - \Gamma^\gamma_{\alpha\beta} p^\alpha p_\gamma = m\frac{dp_\beta}{d\tau} - \Gamma^\gamma_{\alpha\beta} p^\alpha p_\gamma = 0 \tag{5.141}$$

Now we have that

$$\Gamma^\gamma_{\alpha\beta} p^\alpha p_\gamma = \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu})p^\nu p^\alpha \tag{5.142}$$

$$= \frac{1}{2}g_{\nu\alpha,\beta}p^\nu p^\alpha \tag{5.143}$$

(in fact, the first and third term, which together form an antisymmetric matrix in $\nu\alpha$, vanish when multiplied by the symmetric matrix $p^\nu p^\alpha$). Let us focus now on a particular coordinate $\hat{\beta}$. Therefore we have

$$m\frac{dp_{\hat{\beta}}}{d\tau} = \Gamma^{\gamma}_{\alpha\hat{\beta}} p^\alpha p_\gamma = \frac{1}{2} g_{\nu\alpha,\hat{\beta}} p^\nu p^\alpha \tag{5.144}$$

This shows that $p_{\hat{\beta}}$ is constant (conserved) if and only if

$$g_{\nu\alpha,\hat{\beta}} = 0 \tag{5.145}$$

that is, if the metric does not depend on the coordinate $\hat{\beta}$. In general, $p_{\hat{\beta}} = const$ does not imply $p^{\hat{\beta}} = const$.

2. So for instance, if the metric does not depend on time (stationary metric), then the energy, $p_0 = -E$, is conserved. If, in spherical coordinates, the metric does not depend on the azimuthal angle ϕ, the angular momentum $p_\phi = g_{\phi\phi} p^\phi = r^2 m d\phi/d\tau$ is conserved (notice that p^ϕ is not).

3. From the equation

$$p_\alpha p^\alpha = -m^2 = g^{\alpha\beta} p_\alpha p_\beta \tag{5.146}$$

in the weak field metric $\Phi \ll 1$ and in the small velocity limit $p^0 \gg p^i$, we find that at the lowest order

$$-m^2 = g^{\alpha\beta} p_\alpha p_\beta = (-1 + 2\Phi) p_0^2 + p^2 \tag{5.147}$$

(where $p = |\boldsymbol{p}|$ is the spatial momentum amplitude) from which (choosing the minus sign since $p_0 = -E$)

$$p_0 = -m\sqrt{\frac{1+v^2}{1-2\Phi}} \approx -m(1 + \frac{1}{2}v^2 + \Phi) \tag{5.148}$$

So, the energy of a massive particle for small velocities and small Φ (the so-called *Newtonian limit*) is

$$-p_0 = E \approx m + \frac{1}{2}mv^2 + m\Phi = m + \frac{p^2}{2m} + m\Phi \tag{5.149}$$

Terms like Φv^2 have been discarded because very small in this approximation. This recover the standard Netwonian expression (a parte the rest mass), i.e. kinetic energy plus potential energy. As we have seen, this energy is conserved only if Φ does not depend on time.

5.11 Killing vectors

1. The momentum conservation rule can be generalized to conservation along a set of vectors $\vec{\xi}$ known as Killing vectors or Killing fields. Suppose

$$p^\alpha \xi_\alpha = const \tag{5.150}$$

Then

$$\begin{aligned}
(p^\alpha \xi_\alpha)_{;\mu} &= p^\alpha_{;\mu}\xi_\alpha + p^\alpha \xi_{\alpha;\mu} && (5.151)\\
&= p^\alpha_{,\mu}\xi_\alpha + p^\alpha \xi_{\alpha,\mu} + \xi_\alpha \Gamma^\alpha_{\nu\mu} p^\nu - \xi_\nu \Gamma^\nu_{\alpha\mu} p^\alpha && (5.152)\\
&= p^\alpha_{,\mu}\xi_\alpha + p^\alpha \xi_{\alpha,\mu} = (p^\alpha \xi_\alpha)_{,\mu} = 0 && (5.153)
\end{aligned}$$

Now, this implies

$$p^\mu (p^\alpha \xi_\alpha)_{;\mu} = p^\mu p^\alpha_{;\mu} \xi_\alpha + p^\mu p^\alpha \xi_{\alpha;\mu} = 0 \tag{5.154}$$

and since the first term vanishes (because of the geodetic condition), the second must as well. Now $p^\mu p^\alpha$ is a symmetric matrix, so it vanishes when contracted with an antisymmetric matrix, so the condition for Eq. (5.150) to be verified is that

$$\xi_{\alpha;\mu} + \xi_{\mu;\alpha} = 0 \tag{5.155}$$

2. These conditions define the Killing vector field $\vec{\xi}$. This means that to find the Killing vectors in a given metric one should solve these differential equations, which in general is a very non-trivial task. The projection of momentum along a Killing vector is conserved.

3. When the metric does not depend on coordinate $\hat{\alpha}$, the Killing vector is $\xi^\mu = \delta^\mu_{\hat{\alpha}}$. In fact, $\xi_\mu = g_{\mu\nu}\xi^\nu_{\hat{\alpha}} = g_{\mu\hat{\alpha}}$ and $\xi_\mu p^\mu = p_{\hat{\alpha}}$ which is indeed constant. Considering $\xi_\mu = g_{\mu\hat{\alpha}}$ as a vector, and therefore applying the covariant derivative of vectors, we have infact that the antisymmetry condition is satisfied

$$g_{\mu\hat{\alpha};\nu} + g_{\nu\hat{\alpha};\mu} = g_{\mu\hat{\alpha},\nu} - \Gamma^\tau_{\mu\nu} g_{\tau\hat{\alpha}} + g_{\nu\hat{\alpha},\mu} - \Gamma^\tau_{\nu\mu} g_{\tau\hat{\alpha}} = g_{\mu\hat{\alpha},\nu} + g_{\nu\hat{\alpha},\mu} - 2\Gamma^\tau_{\nu\mu} g_{\tau\hat{\alpha}} \tag{5.156}$$

$$= g_{\mu\hat{\alpha},\nu} + g_{\nu\hat{\alpha},\mu} - g_{\tau\hat{\alpha}} g^{\tau\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \tag{5.157}$$

$$= g_{\mu\hat{\alpha},\nu} + g_{\nu\hat{\alpha},\mu} - \delta^\beta_{\hat{\alpha}}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \tag{5.158}$$

$$= g_{\mu\hat{\alpha},\nu} + g_{\nu\hat{\alpha},\mu} - g_{\mu\hat{\alpha},\nu} - g_{\nu\hat{\alpha},\mu} + g_{\mu\nu,\hat{\alpha}} \tag{5.159}$$

$$= g_{\mu\nu,\hat{\alpha}} \tag{5.160}$$

which indeed vanishes if $g_{\mu\nu,\hat{\alpha}} = 0$.

5.12 Einstein equations

1. We know that the Newtonian gravitational equations are the Poisson equation and the gravitational potential equation

$$\nabla^2_{(3)}\Phi = 4\pi G\rho \tag{5.161}$$

$$\Phi = -\frac{GM}{r} \tag{5.162}$$

We now look for the simplest set of equations involving the metric and the EMT such that this limit is reproduced. These will be the GR equations.

2. Since the Poisson equation contains two derivatives, we expect that also our GR equations are (at least) second order. Moreover, since ρ appears linearly (in the second equation, $M = 4\pi\rho R^3/3$ for a homogeneous sphere), and ρ is part of $T^{\mu\nu}$, we expect the equations to be linear in $T^{\mu\nu}$. So, since $T^{\mu\nu}$ is covariantly conserved, the metric should appear in a form that contains second order derivatives and that is conserved. There is then only a possibility left:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{5.163}$$

These are Einstein's equations.

3. Einstein equations are generally covariant, that is, they remain formally identical under a general transformation of coordinates, since the tensors on both sides transform in the same way. This property is also called *diffeomorphism invariance*, i.e. invariance under a differentiable transformation of coordinates. This property crucially requires that the metric $g_{\mu\nu}$ be a dynamical quantity, not a fixed one. Standard field theory is defined on a fixed Minkowski background and therefore is not diffeomorphism invariant.

4. In reality, we could add a term linear in the metric, $\Lambda g_{\mu\nu}$, which is also conserved. For now we do not consider this "cosmological constant" term.

5. If we allow for more derivatives, then the equations can become much more complicated. For instance, one can show that the tensor

$$E_{\mu\nu} \equiv \frac{df(R)}{dR}R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left(\frac{df(R)}{dR}\right)_{;\mu\nu} + g_{\mu\nu}g^{\alpha\beta}\left(\frac{df(R)}{dR}\right)_{;\alpha\beta} \tag{5.164}$$

where $f(R)$ is any function of R, is conserved ($E^{\mu\nu}_{;\mu} = 0$), and therefore could be added to $G_{\mu\nu}$. These "modified gravity" theories, although possible in principle, have received so far no experimental support.

6. In order to find κ, the only free parameter, we compare Einstein's equations with the Newtonian limit. First we choose units in which
$$1 = \frac{G}{c^2} = 7.45 \times 10^{-28} \frac{\mathrm{m}}{\mathrm{kg}} \tag{5.165}$$
so the mass can be expressed in meters. For instance
$$M_{\odot} \approx 1500\mathrm{m} \tag{5.166}$$
The correct dimensionality of the equations can be checked by noting that now mass, length and time are all in length units, e.g. meters. That is
$$1\mathrm{s} = 3 \cdot 10^8 \mathrm{m} \tag{5.167}$$
$$1\mathrm{kg} = 7.45 \cdot 10^{-28} \mathrm{m} \tag{5.168}$$

7. We now set $\kappa = 8\pi G/c^4 \to 8\pi$, and show that this is the correct choice by comparing the weak field limit to the Poisson equation.

8. Using units such that $c = h = 1$ (so-called *natural units*), one finds that G has dimensions of mass^{-2}. We can write therefore
$$G = M_P^{-2} \tag{5.169}$$
where $M_P \approx 10^{19}$ GeV is called Planck mass.

5.13 Linearized equations

1. Let us write
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{5.170}$$
(we can say that η is the *background metric,* and h is the *perturbed metric*). Since we are looking for the Newtonian limit, we assume the metric will always be close to Minkowski, that is
$$h_{\mu\nu} \ll \eta_{\mu\nu} \tag{5.171}$$
or $|h_{\mu\nu}| \ll 1$. Notice that one should write
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{5.172}$$
so that $g_{\mu\nu}g^{\nu\sigma} = \delta_\mu^\sigma$ at first order. The indexes of $h_{\mu\nu}$ can be raised/lowered simply with $\eta_{\mu\nu}$, since using the full $g_{\mu\nu}$ would introduce a second order correction. To keep track of the order, it is often convenient to add a factor ε to every perturbed quantity, and keep only $O(\varepsilon)$ terms. By the same reason, covariant derivatives become ordinary derivatives when linearizing.

2. Let us perform now a Lorentz transformation $\Lambda_\beta^{\bar{\alpha}}$. Schematically, we have
$$\bar{g} = \Lambda\Lambda g = \Lambda\Lambda\eta + \Lambda\Lambda h \tag{5.173}$$
But $\eta_{\mu\nu}$ is invariant under a Lorentz transformation, so
$$\bar{g} = \eta + \bar{h} = \eta + \Lambda\Lambda h \tag{5.174}$$
that is, although h is not a tensor under a general transformation (g is), it transforms as a tensor under LT.

3. How does h transform then under a general coordinate transformation? Since h is supposed to be a small quantity, it is enough we consider "small transformations", i.e. transformations that deviate from identity just by a small quantity ξ^α (these are called *gauge transformations*: they leave the background $\eta_{\alpha\beta}$ unchanged, but change the perturbation $h_{\alpha\beta}$)
$$x^{\bar{\alpha}} = x^\alpha + \xi^\alpha(x^\beta) \tag{5.175}$$

Then we have

$$\Lambda^{\bar{\alpha}}_{\beta} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \xi^{\alpha}_{,\beta} \tag{5.176}$$

and the inverse

$$\Lambda^{\alpha}_{\bar{\beta}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\beta}}} = \delta^{\alpha}_{\beta} - \xi^{\alpha}_{,\beta} \tag{5.177}$$

(note indeed that at first order in ξ one has $\Lambda^{\alpha}_{\bar{\beta}}\Lambda^{\bar{\beta}}_{\gamma} = \delta^{\alpha}_{\gamma}$).

4. So we have

$$g_{\bar{\mu}\bar{\nu}} = \Lambda^{\alpha}_{\bar{\mu}}\Lambda^{\beta}_{\bar{\nu}}(\eta_{\alpha\beta} + h_{\alpha\beta}) \tag{5.178}$$
$$= (\delta^{\alpha}_{\mu} - \xi^{\alpha}_{,\mu})(\delta^{\beta}_{\nu} - \xi^{\beta}_{,\nu})(\eta_{\alpha\beta} + h_{\alpha\beta}) \tag{5.179}$$
$$\approx \eta_{\mu\nu} + h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \tag{5.180}$$

where we discarded higher-order terms like $\xi\xi$ and ξh. Notice that it is correct to use η alone to raise and lower indexes, $\eta_{\alpha\beta}\xi^{\alpha}_{,\gamma} = \xi_{\beta,\gamma}$ since using the full $g_{\alpha\beta}$ would introduce only higher order terms. Similarly,

$$h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu} \tag{5.181}$$
$$h^{\alpha}_{\beta} = \eta^{\alpha\mu}h_{\mu\beta} \tag{5.182}$$
$$h = h^{\alpha}_{\alpha} \tag{5.183}$$

5. The components of $h^{\mu\nu}$ are then identical to those of $h_{\mu\nu}$. Notice however that if we write $g^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu}$, then $\tilde{h}^{\mu\nu}$ has the same components as $-h_{\mu\nu}$, since $\delta^{\mu}_{\nu} = g^{\mu\nu}g_{\nu\sigma} = \eta^{\mu\nu}\eta_{\nu\sigma} + \tilde{h}^{\mu\nu}\eta_{\nu\sigma} + \eta^{\mu\nu}h_{\nu\sigma} + \tilde{h}^{\mu\nu}h_{\nu\sigma}$ which reduces to $\eta^{\mu\nu}\eta_{\nu\sigma} + \tilde{h}^{\mu\nu}h_{\nu\sigma} \approx \delta^{\mu}_{\sigma}$ (i.e., the identity matrix at first order) if $\tilde{h}^{\mu\nu} = -h_{\mu\nu}$. So $\tilde{h}^{\mu\nu} \neq \eta^{\alpha\mu}\eta^{\beta\nu}h_{\alpha\beta}$. In fact, we should simply write $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\alpha\beta}$.

6. Eq. (5.180) shows that we can always change the perturbation $h_{\alpha\beta}$ so that

$$h^{(new)}_{\alpha\beta} = h^{(old)}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{5.184}$$

for any choice of $\vec{\xi}$. This is called *gauge freedom*. By a suitable choice of $\vec{\xi}$ (which amount to choosing four functions of space and time) one can simplify a lot the equations.

7. Inserting the perturbed metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ inside the Riemann tensor, and taking the limit of small h, one gets at first order

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) \tag{5.185}$$

It is not difficult to show that this expression is gauge-invariant, i.e. does not change under any gauge choice $\vec{\xi}$.

8. Now we define the Trace reverse

$$\bar{h}^{\alpha\beta} \equiv h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h \tag{5.186}$$
$$\bar{h} \equiv \bar{h}^{\alpha}_{\alpha} = -h \tag{5.187}$$

(notice that $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \eta_{\alpha\beta}\bar{h}/2$). In terms of $\bar{h}$, the Einstein equations at first order become

$$G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{,\mu} - \bar{h}_{\beta\mu,\alpha}{}^{,\mu}] \tag{5.188}$$

9. Now we can use a gauge transformation to simplify this expression. We choose the so-called *Lorentz gauge*, such that

$$\bar{h}^{\mu\nu}_{,\nu} = 0 \tag{5.189}$$

(that is, we choose four functions ξ^{α} such that the $\bar{h}$ is transformed into a new $\bar{h}$ that obeys the Lorentz gauge; notice that we do not need to specify $\vec{\xi}$, just show that it exists).

10. The gauge condition (5.184) can be written for $\bar{h}$ as

$$\bar{h}^{(new)}_{\alpha\beta} = \bar{h}^{(old)}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^{;\sigma}_{\sigma} \quad (5.190)$$

11. We now differentiate Eq. (5.190) wrt β

$$(\bar{h}^{(new)}_{\alpha\beta})^{,\beta} = (\bar{h}^{(old)}_{\alpha\beta})^{,\beta} - \xi^{,\beta}_{\alpha,\beta} - \xi^{,\beta}_{\beta,\alpha} + \eta_{\alpha\beta}\xi^{;\sigma\beta}_{\sigma} \quad (5.191)$$

$$= (\bar{h}^{(old)}_{\alpha\beta})^{,\beta} - \xi^{,\beta}_{\alpha,\beta} \quad (5.192)$$

The new metric is the one we want to obey the Lorentz gauge, so the lhs vanishes. We see then that the condition on $\vec{\xi}$ to satisfy the Lorentz gauge is

$$\Box\xi^{\alpha} = \bar{h}^{\alpha\beta}_{\ \ ,\beta} \quad (5.193)$$

Notice that we can always add to $\vec{\xi}$ another vector $\vec{\eta}$ such that $\Box\eta^{\alpha} = 0$ and still satisfy the Lorentz gauge.

12. In this case, all terms in (5.188) cancel except the first one,

$$G_{\alpha\beta} = -\frac{1}{2}\bar{h}^{\ \ ,\mu}_{\alpha\beta,\mu} \equiv -\frac{1}{2}\Box\bar{h}_{\alpha\beta} \quad (5.194)$$

where we employed the box operator

$$\Box\phi \equiv \phi^{;\alpha}_{;\alpha} \quad (5.195)$$

When using our background metric, η, the covariant derivatives and the ordinary ones coincide.

13. Finally, the weak-field Einstein equations in the Lorentz gauge ("linearized theory") are

$$\Box\bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta} \quad (5.196)$$

14. The energy-momentum tensor for a perfect fluid is already linear in the metric. For a non-relativistic fluid we can now neglect all the entries except $T^{00} = \rho$. In fact, in general, the components T^{0i} are of order v, and the components T^{ij} are of order v^2, while ρ is of order 1 (i.e., goes like c^2).

15. Then the equations in flat space and small velocities, i.e. in the Newtonian limit, are just one,

$$\Box\bar{h}_{00} = -16\pi T_{00} = -16\pi\rho \quad (5.197)$$

The other components $\bar{h}_{ij}$ will be then much smaller than $\bar{h}_{00}$. Therefore

$$\bar{h} \approx \bar{h}^{0}_{0} \quad (5.198)$$

Moreover, we can assume a static metric (as we have seen, the perturbed metric is proportional to the gravitational potential Φ, so we are assuming a static potential), and therefore

$$\Box \to \nabla^2_{(3)} \quad (5.199)$$

(standard 3D Laplacian), and therefore

$$\nabla^2_{(3)}\bar{h}_{00} = -16\pi\rho \quad (5.200)$$

which, compared to Poisson equation Eq. (5.161) gives

$$\bar{h}_{00} = -4\Phi \quad (5.201)$$

Finally, since

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h} \approx \bar{h}_{00} - \frac{1}{2}\bar{h}_{00} = \frac{1}{2}\bar{h}_{00} \quad (5.202)$$

we have

$$h_{00} = -2\Phi \tag{5.203}$$

and also

$$h_{xx} = h_{yy} = h_{zz} = \frac{1}{2}\bar{h}_{00} = h_{00} = -2\Phi \tag{5.204}$$

This shows that the linearized metric is indeed

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) \tag{5.205}$$

as we know already. This confirms that $\kappa = 8\pi G/c^4 \to 8\pi$ is indeed the right choice.

16. How large is Φ? For the Earth, Sun and Galaxy we have

$$\Phi_{Earth} = \frac{GM_{Earth}}{R_{Earth}c^2} \approx 10^{-9} \tag{5.206}$$

$$\Phi_{Sun} = \frac{GM_{Sun}}{R_{Sun}c^2} \approx 10^{-6} \tag{5.207}$$

$$\Phi_{Galaxy} = \frac{GM_{Galaxy}}{R_{Galaxy}c^2} \approx 10^{-4} \tag{5.208}$$

That is, the weak field approximation is always very good, except near compact stars and black holes.

5.14 Far-field solution

1. Far from any source, $\rho \approx 0$. If we also impose a static field, then the GR equations are

$$\nabla^2_{(3)}\bar{h}^{\mu\nu} = 0 \tag{5.209}$$

Now these are just the standard Poisson equation in vacuum, except applied to every component of $\bar{h}$. Far from a source, the distance to every point of the source can be approximated just by the distance to its center r.

2. The Laplacian of a function ϕ in radial form can be obtained from Eq. (5.30) with $V^\mu = \phi^{;\mu} = \phi^{,\mu}$, noting that the determinant of the metric in spherical coordinates is $g = -r^4\sin^2\theta$, and assuming that ϕ only depends on r

$$\nabla^2_{(3)} = \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} = 0 \tag{5.210}$$

so that we have now

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\bar{h}^{\mu\nu} = 0 \tag{5.211}$$

whose solution is

$$\bar{h}^{\mu\nu} = \frac{A^{\mu\nu}}{r} \tag{5.212}$$

where $A^{\mu\nu}$ are constants (a second solution, $\bar{h}^{\mu\nu} = const$, should be discarded since we must recover $\eta_{\mu\nu}$ at infinity).

3. However, the $A^{\mu\nu}$ are not independent, since we have imposed the Lorentz gauge, so they must obey the equation

$$0 = \bar{h}^{\mu\nu}_{,\nu} = \bar{h}^{\mu i}_{,i} = -A^{\mu j}n_j r^{-2} \tag{5.213}$$

where we defined the unit radial 3-vector $n_j = r^{-1}x_i$, (since $(1/r)_{,i} = -x_i/r^3$). It follows that

$$A^{\mu j} = 0 \tag{5.214}$$

(so only A^{00} remains) and since $\Phi = -\bar{h}^{00}/4$, we obtain

$$\Phi = -\frac{1}{4}\frac{A^{00}}{r} \tag{5.215}$$

Comparing with Newton's law, $\Phi = -M/r$, we see that

$$A^{00} = 4M \tag{5.216}$$

So, the gravitational mass can always be inferred in GR by looking at the far-field behavior of the metric around a static source.

5.15 Einstein-Hilbert Lagrangian

1. The Einstein equations can be derived from an Action, known as Einstein-Hilbert action:

$$A = \int d^4x\sqrt{-g}\, R + 16\pi \int d^4x\sqrt{-g}\, L_m \tag{5.217}$$

where L_m is the Lagrangian for the matter sector.

2. To obtain Einstein's equation, one finds the variation of A with respect to $g^{\mu\nu}$. Since $R = g^{\mu\nu}R_{\mu\nu}$, and considering only the gravitational part, one has

$$\delta A = \int d^4x[\delta(\sqrt{-g})\, R + \sqrt{-g}(\delta g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}] + 16\pi \int d^4x[\delta(\sqrt{-g}\, L_m)] \tag{5.218}$$

3. Now, from (5.25) we have

$$\delta(\sqrt{-g}) = \left(\frac{\partial}{\partial g^{\mu\nu}}\sqrt{-g}\right)\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{5.219}$$

4. We now show that the $g^{\alpha\beta}\delta R_{\alpha\beta}$ part is a total divergence, and therefore gives rise to a constant boundary term, which does not influence the equations of motion. One has (see Eq. 5.106)

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{\alpha\beta}[(\delta\Gamma^{\mu}_{\alpha\beta})_{;\mu} - (\delta\Gamma^{\mu}_{\alpha\mu})_{;\beta}] \tag{5.220}$$
$$= [(g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta}) - (g^{\alpha\mu}\delta\Gamma^{\nu}_{\nu\alpha})]_{;\mu} \tag{5.221}$$
$$= V^{\mu}_{;\mu} \tag{5.222}$$

where in the last step we see that the term can be written as the covariant divergence of a vector. By Gauss' theorem, this is equivalent to a boundary term, which as usual can be taken to be zero on a surface very far from the source of the gravitational field.

5. Assuming now that L_m depends only on $g_{\mu\nu}$, and not on, e.g., its derivatives, we have

$$\delta A = \int d^4x[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\, R + \sqrt{-g}R_{\mu\nu}]\delta g^{\mu\nu} + 16\pi \int d^4x[\frac{\partial(\sqrt{-g}\, L_m)}{\partial g^{\mu\nu}}]\delta g^{\mu\nu} \tag{5.223}$$
$$= \int d^4x\sqrt{-g}[-\frac{1}{2}g_{\mu\nu}\, R + R_{\mu\nu} + 16\pi\frac{\partial(\sqrt{-g}\, L_m)}{\sqrt{-g}\,\partial g^{\mu\nu}}]\delta g^{\mu\nu} \tag{5.224}$$

In order for the variation to vanish, $\delta A = 0$, for any $\delta g_{\mu\nu}$, one requires the term inside square bracket to vanish. This gives Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R = 8\pi T_{\mu\nu} \tag{5.225}$$

where we have defined

$$T_{\mu\nu} = -2\frac{\partial(\sqrt{-g}L_m)}{\sqrt{-g}\partial g^{\mu\nu}} \tag{5.226}$$

6. As we have seen in Sec. 4.7, the torsion tensor is defined as

$$T^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha} \tag{5.227}$$

If we do not assume it to vanish, then the connection is no longer given as function of the metric and does not necessarily vanish even in a flat manifold. In fact, one has then a new connection

$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + K^{\gamma}_{\alpha\beta} \tag{5.228}$$

where Γ is the usual Levi-Civita connection, and K is the contorsion tensor, defined as

$$K^{\gamma}_{\alpha\beta} = \frac{1}{2}(T^{\gamma}_{\alpha\beta} - T_{\alpha\ \beta}^{\ \gamma} - T_{\beta\ \alpha}^{\ \gamma}) \tag{5.229}$$

Now the commutator of covariant derivatives of a scalar field is no longer zero

$$[\nabla_\alpha, \nabla_\beta]\phi = K^{\gamma}_{\alpha\beta}\phi_{,\gamma} \tag{5.230}$$

The Riemann and Ricci tensors can be written as is the standard case, but with $\tilde{\Gamma}$ instead of Γ.

7. We will not make further use of the Lagrangian formalism in this course.

Chapter 6

Gravitational waves

6.1 Wave equation

1. We go back to Eq. (5.196) now, and we solve this equation in empty space, i.e.

$$\Box\bar{h}_{\alpha\beta} = \bar{h}_{\alpha\beta;\gamma}^{\ \ \ \ ;\gamma} = (-\frac{\partial^2}{\partial t^2} + \nabla^2_{(3)})\bar{h}_{\alpha\beta} = 0 \tag{6.1}$$

   Contrary to the cases seen above, we now keep the time derivative. This is a wave equation and the solutions are called *gravitational waves* (GW). They exist (also) in empty space and propagate, as we see in a moment, with the speed of light. As well-known, they have been first directly measured in 2015 by the LIGO interferometer.

2. We now verify that a solution is a *plane wave*

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{ik_\alpha x^\alpha} \tag{6.2}$$

   where $A^{\alpha\beta}$ are constants and $\vec{k}$ is a null vector. In fact (remember the background is Minkowski, so all derivatives here are standard derivatives)

$$\bar{h}^{\alpha\beta}_{,\mu} = A^{\alpha\beta}\frac{\partial}{\partial x^\mu}e^{ik_\alpha x^\alpha} = iA^{\alpha\beta}k_\alpha\frac{\partial x^\alpha}{\partial x^\mu}e^{ik_\alpha x^\alpha} \tag{6.3}$$

$$= iA^{\alpha\beta}k_\mu e^{ik_\alpha x^\alpha} \tag{6.4}$$

   and similarly

$$\bar{h}^{\alpha\beta,\mu}_{,\mu} = -A^{\alpha\beta}k_\mu k^\mu e^{ik_\alpha x^\alpha} = 0 \tag{6.5}$$

   This shows that $k_\mu k^\mu = 0$, i.e. is a null vector. This implies that the GWs travel along the same paths as light.

3. Since $\vec{k}$ is a null vector, we can write its component as $\vec{k} \to (\omega, \boldsymbol{k})$ with $\omega^2 = |\boldsymbol{k}|^2$, and therefore

$$ik_\alpha x^\alpha = ik^\alpha x^\beta \eta_{\alpha\beta} = -i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}) \tag{6.6}$$

   which becomes $-i\omega(t-x)$ for propagation along x. This, again, shows that the velocity of propagation is c.

4. Now, Eq. (6.1) is linear and therefore any linear combination of plane waves is a solution. For a realistic case, the linear combination has to produce a real number, not a complex one, for instance

$$\frac{1}{2}A^{\alpha\beta}(e^{ik_\alpha x^\alpha} + e^{-ik_\alpha x^\alpha}) = A^{\alpha\beta}\cos(k_\alpha x^\alpha) \tag{6.7}$$

   We need to consider only a single plane wave. A realistic GW signal will be the sum of many independent plane waves with different $\vec{k}$'s.

5. We still have to impose the Lorentz gauge. This gives immediately

$$A^{\alpha\beta}k_\beta = 0 \tag{6.8}$$

i.e., the matrix $A^{\alpha\beta}$ is orthogonal to the propagation vector $\vec{k}$.

6. We are still free to modify $\bar{h}_{\alpha\beta}$ by adding a gauge vector $\vec{\xi}$ such that $\Box\vec{\xi} = 0$. Now we choose this vector as

$$\xi_\alpha = B_\alpha e^{ik_\mu x^\mu} \tag{6.9}$$

Then, as a consequence, the perturbed metric changes as

$$h^{new}_{\alpha\beta} = h^{old}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{6.10}$$

and

$$\bar{h}^{new}_{\alpha\beta} = \bar{h}^{old}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi^\mu_{,\mu} \tag{6.11}$$

This shows that we can write

$$A^{new}_{\alpha\beta} = A^{old}_{\alpha\beta} - iB_\alpha k_\beta - iB_\beta k_\alpha + i\eta_{\alpha\beta}B^\mu k_\mu \tag{6.12}$$

Taking the trace we obtain

$$A^{new,\alpha}_{\alpha} = A^{old,\alpha}_{\alpha} + 2iB^\mu k_\mu \tag{6.13}$$

So if we choose B^μ in such a way that

$$-A^{old,\alpha}_{\alpha} = 2iB^\alpha k_\alpha \tag{6.14}$$

we produce a matrix that is traceless,

$$A^\alpha_\alpha = 0 \tag{6.15}$$

and therefore $\bar{h} = 0$.

7. Similarly, taking $\vec{k} \to \{\omega, 0, 0, \omega\}$, we can write Eqs. (6.12) as

$$A^{new}_{00} = A^{old}_{00} - i\omega(B_0 + B_3) \tag{6.16}$$
$$A^{new}_{11} = A^{old}_{11} - i\omega(B_0 - B_3) \tag{6.17}$$
$$A^{new}_{22} = A^{old}_{22} - i\omega(B_0 - B_3) \tag{6.18}$$
$$A^{new}_{01} = A^{old}_{01} - i\omega B_1 \tag{6.19}$$
$$A^{new}_{02} = A^{old}_{02} - i\omega B_2 \tag{6.20}$$
$$A^{new}_{12} = A^{old}_{12} \tag{6.21}$$

and we can easily choose coefficients B^μ such that

$$A^{new}_{00} = A^{new}_{01} = A^{new}_{02} = 0 \tag{6.22}$$

(for instance, $B_1 = -iA^{old}_{01}/\omega$, $B_2 = -iA^{old}_{02}/k$ **etc.**). Finally, $A^{\alpha\beta}k_\beta = 0$ implies $A^{\mu 0} = A^{\mu 3}$ and therefore $A^{00} = A^{03} = A^{30} = A^{33} = 0$ (all for the new A). These conditions together are called traceless-transverse gauge (TT). These are additional to the Lorentz gauge. The conditions on B^μ are equivalent to choosing $A^{\mu\nu}$ such that

$$A_{\alpha\beta}U^\beta = 0 \tag{6.23}$$

where $\vec{U}$ is any constant 4-velocity time-like vector (for instance, $\vec{U} \to (1, 0, 0, 0)$).

8. So in the TT gauge we have

$$\bar{h}^{TT}_{\alpha\beta} = h_{\alpha\beta} \tag{6.24}$$

9. The only non-zero components are then A_{xx}, A_{xy}, A_{yy} (as every metric, $h_{\alpha\beta}$ is symmetric). However, the traceless condition says that $A_{xx} = -A_{yy}$ so finally a GW has only two independent non-zero components, A_{xx}, A_{xy} (also called *degrees of freedom*)

$$\bar{h}^{TT}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\alpha x^\alpha} \tag{6.25}$$

To summarize, this expression applies to a GW propagating along the z direction, seen in the frame in which $\vec{U}$ is the timelike rest-frame 4-velocity, in the Lorentz TT gauge. The complex factor can be replaced by the real part $\cos\omega(t-z)$ in any practical application. It is clear then that ω is the wave frequency, and the wavelength is $\lambda = 2\pi/\omega$. The wavelength of course depends on the generation mechanism: two binary stars, for instance, will typically produce waves of wavelength that depends on their orbital radius.

10. Initially, the metric has 10 degrees of freedom. Four of these can be erased by the gauge transformation, and four more by the extra freedom to choose ξ^α such that $\Box\xi^\alpha = 0$, so we are left with two. This extra freedom however only works in vacuum, where $\Box\bar{h}_{\alpha\beta} = 0$ as well.

6.2 GWs on particles

1. Consider a particle initially at rest in a frame, so that its 4-velocity is $\vec{U} \to (1, 0, 0, 0)$. Its geodesic equation is then

$$\frac{d}{d\tau}U^\alpha + \Gamma^\alpha_{\mu\nu}U^\mu U^\nu = 0 \tag{6.26}$$

and, when at rest, one has that a GW with metric $h_{\alpha\beta}$ will act on the particle as

$$\frac{d}{d\tau}U^\alpha = -\Gamma^\alpha_{00} = \frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}) \tag{6.27}$$

In the TT gauge, $h^{TT}_{\beta 0} = 0$, so it would seem that the particle remains at rest. However, in reality this only means that it remains at rest in a particular frame, that is, the coordinates that describe the particle remain the same. This would also be true if one uses the geodesic deviation equation (5.89), which just gives the coordinate separation between nearby geodesics. What we should do is looking for how the *proper separation* between two particles change when a GW hits the system.

2. Let us consider two particles at positions $\boldsymbol{x}_1 = (0, 0, 0)$ and $\boldsymbol{x}_2 \to (\varepsilon, 0, 0)$ and assume they can move only along x. Their space-time interval is

$$ds = \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta} = \sqrt{g_{xx}dx^2} = \sqrt{g_{xx}}\varepsilon \tag{6.28}$$

For the GW metric $\eta_{\alpha\beta} + h_{\alpha\beta}$ this amounts to

$$ds = \sqrt{g_{xx}}\varepsilon = (1 + \frac{1}{2}h^{TT}_{xx})\varepsilon \tag{6.29}$$

So we see that the proper distance (the distance measured by an observer that has both events on its axis of simultaneity) changes when a GW hits the particles.

3. For realistic GWs emitted by astrophysical sources (see later) one has

$$h_{xx} \approx 10^{-21} \tag{6.30}$$

which means the particles will move by only an infinitesimal distance. Over 1 km, the change in separation is around $10^{-9}nm$!

4. Following the derivation in Maggiore, *Gravitational Waves* I, or in Hobson, Efstathious, Lasenby, *General Relativity*, the invariant way to describe the effect of an impinging GW is to work out the variation of the proper distances (all in TT gauge but we suppress the TT subscript for now)
$$ds^2 = (\delta_{ij} + h_{ij})dx^i dx^j = \varepsilon^2 + h_{ij}dx^i dx^j \tag{6.31}$$
where ε is the initial separation. At first order, this gives
$$s = \varepsilon + \frac{1}{2}h_{ij}dx^i dx^j$$
Then if the initial separation vector is ε^i, the proper distance vector at any given time is $s^i(t) = \varepsilon^i + \xi^i(t)$ with $\xi^i \ll \varepsilon^i$ (because ξ is proportional to h, as intuitive and shown below) and we have
$$s^2 = \xi^2 + \varepsilon^2 + 2\xi_i\varepsilon^i$$
and, at first order
$$s = \varepsilon + \xi_i n^i$$
where $n^i = \varepsilon^i/\varepsilon$. Therefore, at first order
$$\xi_i n^i = s - \varepsilon = \frac{1}{2}h_{ij}n^i\varepsilon^j \tag{6.32}$$
or
$$n^i(\xi_i - \frac{1}{2}h_{ij}\varepsilon^j) = 0 \tag{6.33}$$
Since this is to be valid for any n^i, the expression in parentheses must vanish.

5. Differentiating wrt t (which coincides with τ when the motion of the particles is non-relativistic) we obtain
$$\ddot{\xi}_i = \frac{1}{2}\ddot{h}_{ij}(t)\varepsilon^j \tag{6.34}$$

6. We can now write the equation for the proper distance along x as
$$\ddot{\xi}_x = \frac{1}{2}\ddot{h}_{xj}(t)\varepsilon^j = \frac{1}{2}\ddot{h}_{xx}(t)\varepsilon^x + \frac{1}{2}\ddot{h}_{xy}(t)\varepsilon^y \tag{6.35}$$
and, repeating along y and using $h_{yy} = -h_{xx}$,
$$\ddot{\xi}_y = \frac{1}{2}\ddot{h}_{yj}(t)\varepsilon^j = \frac{1}{2}\ddot{h}_{yx}(t)\varepsilon^x - \frac{1}{2}\ddot{h}_{xx}(t)\varepsilon^y \tag{6.36}$$

7. Let us now consider separately a wave composed only by the diagonal terms in the TT gauge, $h_{xx}^{TT} = -h_{yy}^{TT}$ and a wave with only the off-diagonal ones, h_{xy}^{TT}. The equations for the two waves become
$$\ddot{\xi}_x = \frac{1}{2}\varepsilon^x\ddot{h}_{xx}^{TT} \tag{6.37}$$
$$\ddot{\xi}_y = -\frac{1}{2}\varepsilon^y\ddot{h}_{xx}^{TT} \tag{6.38}$$
and
$$\ddot{\xi}_y = \frac{1}{2}\varepsilon^x\ddot{h}_{xy}^{TT} \tag{6.39}$$
$$\ddot{\xi}_x = \frac{1}{2}\varepsilon^y\ddot{h}_{xy}^{TT} \tag{6.40}$$

8. If one first assumes that $h_{xy}^{TT} = 0$, then one sees that the four particles oscillate forming a "+", because the oscillation is proportional to the separation along the same direction. If instead $h_{xx}^{TT} = 0$, then the four particles oscillate forming a "×", because the oscillation is now orthogonal to the separation (see Fig. 6.1).

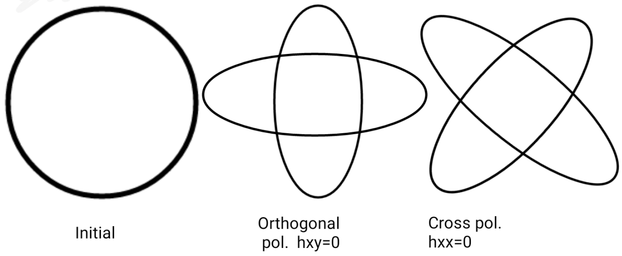

Figure 6.1: GW polarization states.

9. This shows that it makes sense to write

$$h_{\mu\nu}^{TT} = h_{\mu\nu(+)}^{TT} + h_{\mu\nu(\times)}^{TT} = \begin{pmatrix} h_{xx} & 0 \\ 0 & -h_{xx} \end{pmatrix} + \begin{pmatrix} 0 & h_{xy} \\ h_{xy} & 0 \end{pmatrix} \tag{6.41}$$

i.e. as the sum of two GWs ("polarization states") called orthogonal and cross polarization, respectively. A generic GW will of course be the sum of both states.

10. Electromagnetic waves, in contrast, can be decomposed into polarization states that are orthogonal to each other. Ultimately, this depends on the fact that EM waves are spin 1 (vector field), while GW are spin 2 (tensor field).

6.3 Production of GWs

1. Let's go back to the weak field equations

$$\Box\bar{h}_{\alpha\beta} = (-\frac{\partial^2}{\partial t^2} + \nabla_{(3)}^2)\bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta} \tag{6.42}$$

and let us apply this to a particular source that varies in time, for instance a binary star. For simplicity, we take as source a sinusoidal signal with frequency Ω

$$T_{\mu\nu} = S_{\mu\nu}(x^i)e^{-i\Omega t} \tag{6.43}$$

and assume that $S_{\mu\nu} = 0$ outside a small region of size $\varepsilon \ll 2\pi c/\Omega$. This means that the binary stars move with velocity $\approx \Omega\varepsilon \ll 1$, i.e. non-relativistically (in this particular case, ε would be the binary star orbit radius).

2. The GR equations are then solved by

$$\bar{h}_{\alpha\beta} = B_{\mu\nu}(x^i)e^{-i\Omega t} \tag{6.44}$$

where the coefficients obey the equations

$$(\Omega^2 + \nabla_{(3)}^2)B_{\mu\nu}(x^i) = -16\pi S_{\mu\nu} \tag{6.45}$$

Notice that in a radial system,

$$\nabla^2_{(3)} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) \tag{6.46}$$

3. Outside the source $S_{\mu\nu} = 0$, so in this region it is easy to see by direct substitution in the equation that we have the general homogeneous solution

$$B_{\mu\nu} = \frac{A_{\mu\nu}}{r}e^{i\Omega r} + \frac{Z_{\mu\nu}}{r}e^{-i\Omega r} \tag{6.47}$$

Only the first term however represents outward moving waves, i.e. emitted waves (the full solution would in fact be $h \sim e^{i(\Omega r - \Omega t)}$, whose phase is constant when $r = t$; we put then $Z_{\mu\nu} = 0$). The external solution represents then a GW outside the source. At large r, the behavior within a region $\Delta r \ll r$ (as is the case for a GW detection) is just like we found already in Eq. (6.2).

4. So we need now to find $A_{\mu\nu}$ as a function of $S_{\mu\nu}$ matching the external solution with the internal one at ε. Let us integrate the equations within the source volume $4\pi\varepsilon^3/3$:

$$\int d^3x(\Omega^2 + \nabla^2_{(3)})B_{\mu\nu}(x^i) = -16\pi\int d^3x S_{\mu\nu} \tag{6.48}$$

The first term on the lhs gives

$$\Omega^2\int d^3x B_{\mu\nu}(x^i) \leq \Omega^2 B_{\mu\nu,max}\frac{4\pi\varepsilon^3}{3} \tag{6.49}$$

which is negligible because $\Omega\varepsilon \ll 1$. Using Gauss' theorem, the second term gives (3D vectors here)

$$\int d^3x\nabla^2_{(3)}B_{\mu\nu}(x^i) = \oint \vec{n}\cdot\vec{\nabla}B ds \tag{6.50}$$

where $\vec{n}$ is the outward unit normal to the spherical surface of radius ε. Now the radial unit normal is

$$n^i = \frac{\partial r}{\partial x^i} = \frac{x^i}{r} \tag{6.51}$$

and since $B_{\mu\nu}$ depends only on r (because of symmetry) we have

$$\vec{\nabla}B_{\mu\nu} \to \frac{\partial}{\partial x^i}B_{\mu\nu} = \frac{\partial r}{\partial x^i}\frac{\partial}{\partial r}B_{\mu\nu} = n_i\frac{\partial}{\partial r}B_{\mu\nu} \tag{6.52}$$

Since $n^i n_i = 1$, this implies

$$n^i B_{\mu\nu,i} = \frac{\partial B_{\mu\nu}}{\partial r} \tag{6.53}$$

Now, the spherical area element is $dS = r^2\sin\theta d\theta d\phi$. At the surface S, or immediately outside, $B_{\mu\nu}$ is given by the external solution and therefore

$$\oint \vec{n}\cdot\vec{\nabla}B ds = \int \frac{\partial B_{\mu\nu}}{\partial r}r^2\sin\theta d\theta d\phi \tag{6.54}$$

$$\approx 4\pi[(-\frac{A_{\mu\nu}}{r^2}r^2)e^{i\Omega r}]_{r=\varepsilon} \approx -4\pi A_{\mu\nu} \tag{6.55}$$

Here we have neglected the small contribution due to the derivative of $e^{i\Omega r}$ at $r = \varepsilon$ and finally approximated $e^{i\Omega\varepsilon} \approx 1$ because $\Omega\varepsilon \ll 1$

5. So finally we have shown that

$$\int d^3x(\Omega^2 + \nabla^2_{(3)})B_{\mu\nu}(x^i) = -4\pi A_{\mu\nu} = -16\pi\int S_{\mu\nu}d^3x \equiv -16\pi J_{\mu\nu} \tag{6.56}$$

where the last member defines the source matrix $J_{\mu\nu} \equiv \int S_{\mu\nu}d^3x$. This shows that

$$A_{\mu\nu} = 4J_{\mu\nu} \tag{6.57}$$

$$\bar{h}_{\mu\nu} = 4J_{\mu\nu}\frac{e^{i\Omega(r-t)}}{r} \tag{6.58}$$

6. We still have to find $J_{\mu\nu}$. We have
$$J_{\mu\nu}e^{-i\Omega t} = \int T_{\mu\nu}d^3x \tag{6.59}$$
from which, differentiating wrt time,
$$-i\Omega J^{\mu 0}e^{-i\Omega t} = \int T^{\mu 0}_{,0}d^3x = -\int T^{\mu k}_{,k}d^3x = -\oint T^{\mu k}n_k ds \tag{6.60}$$
If the surface of the last integral is just outside the source, $T^{\mu\nu} = 0$, so
$$J^{\mu 0} = 0 \rightarrow \bar{h}^{\mu 0} = 0 \tag{6.61}$$

7. Let us define now the source quadrupole moment $I^{\ell m}$
$$I^{\ell m} \equiv \int T^{00}x^{\ell}x^m d^3x \tag{6.62}$$
(since we are here concerned only with spatial indexes in a Minkowski background, we can raise and lower them without worrying about signs nor position of indexes; repeated indexes are always understood to be summed over). Since $T^{00} = \rho e^{-i\Omega t}$, we have
$$I^{\ell m} = e^{-i\Omega t}\int \rho x^{\ell}x^m d^3x \tag{6.63}$$
We need now a result that will be derived in an exercise, namely the *tensor virial theorem*
$$\frac{\partial^2}{\partial t^2}\int T^{00}x^{\ell}x^m d^3x \equiv \frac{\partial^2}{\partial t^2}(I^{\ell m}) = 2\int T^{\ell m}d^3x \tag{6.64}$$
Now since
$$\frac{1}{2}\frac{\partial^2}{\partial t^2}(I^{\ell m}) = -\frac{1}{2}\Omega^2(\int \rho x^{\ell}x^m d^3x)e^{-i\Omega t} = -\frac{1}{2}\Omega^2 I^{\ell m} \tag{6.65}$$
by the virial theorem we get
$$\int T^{\ell m}d^3x = -\frac{1}{2}\Omega^2 I^{\ell m} \tag{6.66}$$

8. Inserting this in Eq. (6.58) and using Eq. (6.59) we obtain the GW emitted by a source of a given quadrupole 3D-tensor I_{ij}
$$\bar{h}_{ij} = \frac{4}{r}e^{i\Omega r}J_{ij}e^{-i\Omega t} = \frac{4}{r}e^{i\Omega r}\int T_{ij}d^3x = -2\Omega^2 I_{ij}\frac{e^{i\Omega r}}{r} \tag{6.67}$$

9. The *reduced* quadrupole is defined as
$$\bar{I}_{\ell m} = I_{\ell m} - \frac{1}{3}\delta_{\ell m}I \tag{6.68}$$
where I is the trace of $I_{\ell m}$. Clearly, the trace of $\bar{I}_{\ell m}$ is zero. A spherical source has vanishing reduced quadrupole. One can show this by writing in spherical coordinate $d^3x = r^2\sin\theta d\theta d\phi$, and $x = rf^x(\theta,\phi) = r\sin\theta\cos\phi$ etc, and therefore
$$I^{\ell m} \equiv \int T^{00}(r,t)x^{\ell}x^m d^3x = \int T^{00}(r,t)r^4 dr\int f^{\ell}(\theta,\phi)f^m(\theta,\phi)\sin\theta d\theta d\phi \tag{6.69}$$
It is easy to see by direct calculation that the last integral vanishes for any $\ell \neq m$ and is $4\pi/3$ for any $\ell = m$. Explicitly, e.g. for $\ell = m = 1$, we have
$$\int(\sin\theta d\theta d\phi)^2\sin\theta d\theta d\phi = \int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}\cos^2\phi d\phi = \frac{4\pi}{3} \tag{6.70}$$
Therefore, since $T^{00} = \rho(r)e^{-i\Omega t}$, we have
$$I^{\ell m} \equiv \frac{4\pi}{3}e^{-i\Omega t}\delta^{\ell m}\int \rho r^4 dr \tag{6.71}$$
and $\bar{I}_{\ell m} = 0$.

10. To go now to the TT gauge, we define the *projection operator*
$$P_{ij} \equiv \delta_{ij} - \hat{n}_i \hat{n}_j \tag{6.72}$$
where $\hat{n}$ is a unit vector in the direction of wave propagation. P_{ij} is called *projector* because $P_{ij}P_{jk} = P_{ik}$, i.e. when applied twice it gives back itself. Then the TT gauge can be obtained from $\bar{h}_{ij}$ as
$$\bar{h}^{TT}_{lm} \equiv \bar{h}_{ij}\Lambda_{ij,\ell m} \tag{6.73}$$
where
$$\Lambda_{ij,\ell m} \equiv \left(P_{il}P_{jm} - \frac{1}{2}P_{ij}P_{lm} \right) \tag{6.74}$$
(the comma here is not a derivative, it's just to separate the pairs of indexes!). One can easily check that 1) Λ is a projector itself; 2) Λ is traceless in ij and, separately, in ℓm; 3) Λ is normal to $\hat{n}_i$. Finally, since $\Box\bar{h}_{ij} = 0$,then also $\Box\bar{h}^{TT}_{ij} = 0$. This corresponds exactly to the properties of the TT gauge. So applying the projector Λ to h_{ij} we automatically implement the TT.

11. We now implement the TT gauge on $I_{\ell m}$. Because of Eq. (6.73), we have
$$I^{TT}_{\ell m} \equiv I_{ij}\Lambda_{ij,\ell m} \tag{6.75}$$
and, choosing $z-$propagation, i.e. $\hat{n}_i = \delta^z_i$, we can derive the relations
$$I^{TT}_{xx} = -I^{TT}_{yy} = \frac{1}{2}(I_{xx} - I_{yy}) \tag{6.76}$$
$$I^{TT}_{xy} = I^{TT}_{yx} = I_{xy} \tag{6.77}$$
$$I^{TT}_{zz} = 0 = I^{TT}_{xz} = I^{TT}_{yz} \tag{6.78}$$

12. Finally, since $I_{xx} - I_{yy} = \bar{I}_{xx} - \bar{I}_{yy}$ and $I_{xy} = \bar{I}_{xy}$, we write the result as
$$\bar{h}^{TT}_{xx} = -\bar{h}^{TT}_{yy} = -\Omega^2(\bar{I}_{xx} - \bar{I}_{yy})\frac{e^{i\Omega r}}{r} \tag{6.79}$$
$$\bar{h}^{TT}_{xy} = -2\Omega^2 \bar{I}_{xy}\frac{e^{i\Omega r}}{r} \tag{6.80}$$
An exactly spherical source has zero reduced quadrupole and therefore cannot emit GWs. Similarly, a static source (i.e. $\Omega \to 0$) does not emit GWs. Although we derived these two results only at first order in $1/r$, they are valid at any order. Remember that $\bar{I}_{\ell m}$ depends on time as $e^{-i\Omega t}$, so our solutions propagate as expected as $e^{-i\Omega(t-r)}$.

13. The mass monopole corresponds to the energy of a source. Since this is conserved, there is no GW emission. Similarly, the mass dipole corresponds to the center of mass and its first derivative to momentum. Since momentum is conserved for an isolated system, again there is no GW emission.

14. Let us now produce a simple order-of-magnitude estimate for the GW from an astrophysical source. The quadrupole of a source of mass M and size R has to be of the order of MR^2 . Then
$$|h| \approx MR^2\frac{\Omega^2}{r} = v^2\frac{M}{r} \tag{6.81}$$
where $v = \Omega R$ is the typical orbital velocity of the source. By the virial theorem, the velocity squared approximates the gravitational potential at the source,
$$v^2 \approx |\Phi_0| \tag{6.82}$$
while $M/r = |\Phi_r|$ is the potential at the observer. Then
$$|h| \approx |\Phi_0\Phi_r| \tag{6.83}$$
For hypothetical GWs emitted by the Sun ($\Phi_0 \approx 10^{-6}$ at Sun's radius R_S) and observed on Earth (at a distance D_S from the Sun), this would be $h \approx 10^{-6} \cdot 10^{-6} \cdot \frac{R_S}{D_S} = 10^{-15}$: but this is of course a gross overestimate, since the Sun is a spherical source and has very little quadrupole. For a binary system of solar mass at a typical distance of 10pc$= 2\cdot 10^6 AU$, one has an amplitude 10^6 times smaller, i.e. $h \approx 10^{-21}$, as we estimated previously.

6.4 Examples of GW sources

1. Let us consider two oscillating, equal point particle, each of mass m, separated initially by ℓ_0, lying on the x line (y, z constant). Their movement is described by the equations

$$x_1(t) = -\frac{1}{2}\ell_0 - A\cos(\omega t) \quad (6.84)$$

$$x_2(t) = \frac{1}{2}\ell_0 + A\cos(\omega t) \quad (6.85)$$

We can describe the energy density of the particles using the Dirac δ_D as $\rho_{1,2} = m\delta_D(x - x_{1,2})$, since then

$$m = \int \rho_{1,2} d^3x \quad (6.86)$$

The quadrupole tensor has then a single component

$$I_{xx} = \int \rho x^2 d^3x = \sum_i m x_i^2 \quad (6.87)$$

$$= m[(-\frac{1}{2}\ell_0 - A\cos(\omega t))^2 + (\frac{1}{2}\ell_0 + A\cos(\omega t))^2] \quad (6.88)$$

$$= const + mA^2\cos(2\omega t) + 2m\ell_0 A\cos\omega t \quad (6.89)$$

all the others being zero. Since any constant component disappears under time differentiation, we can discard the constant term.

2. The quadrupole shows two components of frequency ω and 2ω. Since the problem is entirely linear (because we linearized it!), we can estimate the contribution of the two components separately. We now complexify the problem, i.e. $\cos\omega t \to e^{-i\omega t}$ and if needed we go back to real values at the end.

3. For the ω component we find

$$\bar{I}_{xx} = \frac{2}{3}I_{xx} = \frac{4}{3}m\ell_0 A e^{-i\omega t} \quad (6.90)$$

$$\bar{I}_{yy} = \bar{I}_{zz} = -\frac{1}{3}I_{xx} = -\frac{2}{3}m\ell_0 e^{-i\omega t} \quad (6.91)$$

4. Therefore, for the z direction of GW propagation, we have

$$\bar{h}_{xx}^{TT} = -\bar{h}_{yy}^{TT} = -\Omega^2(\bar{I}_{xx} - \bar{I}_{yy})\frac{e^{i\Omega r}}{r} = -\frac{\omega^2}{r}(2m\ell_0 A)e^{i\omega(r-t)} \quad (6.92)$$

$$\bar{h}_{xy}^{TT} = 0 \quad (6.93)$$

i.e., a linearly polarized GW. For the y-component, one gets the same result.

5. For the x direction of GW propagation, instead, we obtain

$$\bar{h}_{zz}^{TT} = -\bar{h}_{yy}^{TT} = -\Omega^2(\bar{I}_{zz} - \bar{I}_{yy})\frac{e^{i\Omega r}}{r} = 0 \quad (6.94)$$

$$\bar{h}_{zy}^{TT} = 0 \quad (6.95)$$

that is, no emission along the x-direction!

6. The second component of the source is a quadrupole that has frequency 2ω and amplitude mA^2. So we can just repeat the calculation replacing in (6.92) ω with 2ω and the factor $2m\ell_0 A$ with mA^2. Then we obtain

$$\bar{h}_{xx}^{TT} = -\bar{h}_{yy}^{TT} = -4\frac{\omega^2}{r}mA^2 e^{2i\omega(r-t)} \quad (6.96)$$

$$\bar{h}_{xy}^{TT} = 0 \quad (6.97)$$

The total radiation field is then the real part of the sum of the two components:

$$\bar{h}^{TT}_{xx} = -\bar{h}^{TT}_{yy} = -[2m\omega^2 \ell_0 A\cos(\omega(r-t)) + 4m\omega^2 A^2\cos(2\omega(r-t))]r^{-1} \quad (6.98)$$

$$\bar{h}^{TT}_{xy} = 0 \quad (6.99)$$

7. If we insert "laboratory values", e.g. $m = 10^3$kg, $\ell_0 = 1$m, $A = 10^{-4}$ m, $\omega = 10^4 s^{-1}$, we get $h \approx 10^{-34}/r$. To make this calculation, one could either express lengths, times and masses in meters or reinsert the constants G, c and obtain for the dimensionless h the value

$$\frac{Gm}{rc^4}[2\omega^2\ell_0 A\cos(\omega(r-t)) + 4\omega^2 A^2\cos(2\omega(r-t))] \quad (6.100)$$

6.5 Binary stars

1. The second system we study is two stars of equal mass m orbiting around each other, with an orbital radius $r = \ell_0/2$ and frequency ω, located on the $z = const$ plane (see Fig. 6.2). Equaling the gravitational force and the centrifugal force we obtain

$$\frac{m^2}{\ell_0^2} = m\omega^2\frac{\ell_0}{2} \quad (6.101)$$

from which Kepler's law follows,

$$\omega = \left(\frac{2m}{\ell_0^3}\right)^{1/2} \quad (6.102)$$

2. The coordinates of the stars are described by

$$x_1(t) = \frac{1}{2}\ell_0\cos\omega t, \quad x_2(t) = -x_1(t) \quad (6.103)$$

$$y_1(t) = \frac{1}{2}\ell_0\sin\omega t, \quad y_2(t) = -y_1(t) \quad (6.104)$$

3. So we have ($\cos 2\theta = 2\cos^2\theta - 1, \sin 2\theta = 2\cos\theta\sin\theta$)

$$I_{xx} = m\sum(x_1^2 + x_2^2) = \frac{m}{4}\ell_0^2(2\cos^2\omega t) = \frac{m}{4}\ell_0^2\cos 2\omega t + const \quad (6.105)$$

(as previously, we neglect any constant because of time differentiation). Similarly,

$$I_{yy} = -\frac{m}{4}\ell_0^2\cos 2\omega t + const \quad (6.106)$$

$$I_{xy} = \frac{m}{4}\ell_0^2\sin 2\omega t + const \quad (6.107)$$

and $I_{zz} = I_{zy} = I_{zx} = 0$. Now for simplicity and to connect with the previous formulae, we complexify again the quadrupole, i.e. $\cos 2\omega t \to e^{-2i\omega t}$ and $\sin 2\omega t = -\cos(2\omega t + \pi/2) \to -e^{-(2i\omega t + \pi/2)} = ie^{-2\omega t}$. Then, since the trace $I = I_{xx} + I_{yy} + I_{zz} = 0$, we obtain

$$\bar{I}_{xx} = -\bar{I}_{yy} = \frac{m}{4}\ell_0^2 e^{-2i\omega t} \quad (6.108)$$

$$\bar{I}_{xy} = i\frac{m}{4}\ell_0^2 e^{-2i\omega t} \quad (6.109)$$

So we obtain finally, for the z component (orthogonal to the source plane)

$$\bar{h}^{TT}_{xx} = -2\frac{m}{r}\ell_0^2\omega^2 e^{2i\omega(r-t)} \quad (6.110)$$

$$\bar{h}^{TT}_{xy} = -2i\frac{m}{r}\ell_0^2\omega^2 e^{2i\omega(r-t)} \quad (6.111)$$

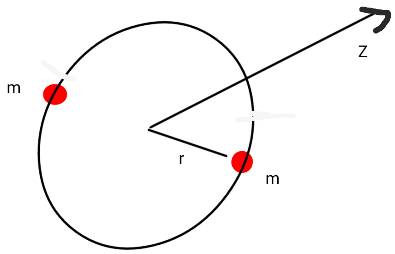

Figure 6.2: GW from a binary system.

This is a circularly polarized GW. Along y, instead, we find a linear polarization,

$$\bar{h}^{TT}_{zz} = -\Omega^2(\bar{I}_{zz} - \bar{I}_{xx})\frac{e^{i\Omega r}}{r} = \frac{m}{r}\ell_0^2\omega^2 e^{2i\omega(r-t)} \tag{6.112}$$

$$\bar{h}^{TT}_{xz} = -2\Omega^2\bar{I}_{zx}\frac{e^{i\Omega r}}{r} = 0 \tag{6.113}$$

and analogously along x.

4. Inserting the values for the pulsar PSR1913+16, orbital period 7h, distance $r = 5$kpc , $m = 1.4M_{\odot}$, we obtain $h \approx 10^{-20}$ at Earth. However, this is a pulsar system and we can measure the variation of the signal due to the stars losing energy, as we see next.

6.6 The pulsar PSR1913+16 and the detection of GWs

1. This section is very sketchy, refer to Schutz's book for a complete treatment. The flux of energy associated to a GW is

$$F = \frac{1}{32\pi}\Omega^2\langle\bar{h}^{TT}_{\mu\nu}\bar{h}^{\mu\nu TT}\rangle \tag{6.114}$$

(see the proof in Schutz, Chapter 9, or Maggiore, Chapter 3), where the brackets mean average over one period.

2. Inserting the solution (6.79) we find

$$F = \frac{1}{32\pi r^2}\Omega^6\langle 2\bar{I}^2_{xx} + 2\bar{I}^2_{yy} - 4\bar{I}_{xx}\bar{I}_{yy} + 8\bar{I}^2_{xy}\rangle \tag{6.115}$$

3. This result can be generalized to propagation in any direction n^i as

$$F = \frac{1}{16\pi r^2}\Omega^6\langle 2\bar{I}_{ij}\bar{I}^{ij} - 4n^j n^k\bar{I}_{ji}\bar{I}^i_k + n^i n^j n^k n^\ell\bar{I}_{ij}\bar{I}_{k\ell}\rangle \tag{6.116}$$

4. Integrating over a sphere, and using the identities

$$\int n^j n^k \sin\theta d\theta d\phi = \frac{4\pi}{3}\delta^{jk} \tag{6.117}$$

$$\int n^i n^j n^k n^\ell \sin\theta d\theta d\phi = \frac{4\pi}{15}(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}) \tag{6.118}$$

we obtain the luminosity of a GW-emitting source

$$L = \int Fr^2 \sin\theta d\theta d\phi = \frac{1}{5}\Omega^6 \langle \bar{I}_{ij}\bar{I}^{ij}\rangle \tag{6.119}$$

5. The factor Ω^6 comes from time derivatives, so in general

$$L = \frac{1}{5}\langle \dddot{\bar{I}}_{ij}\dddot{\bar{I}}^{ij}\rangle \tag{6.120}$$

6. As an order-of-magnitude estimate, we can write $\bar{I} \approx MR^2$ and $v \approx \Omega R$, so $L \approx M^2R^4\Omega^6 \approx \phi_0^2 v^6$, so the luminosity is extremely sensitive to the orbital velocity of the sources. That's why the prime target for detection were GW from fast inspiraling compact bodies, as in fact happened.

7. For the binary pulsar PSR1913+16, observed by Hulse and Taylor in the 80s, we can use the results of the previous section (6.109) for the emission along z to find

$$\langle \dddot{\bar{I}}_{ij}\dddot{\bar{I}}^{ij}\rangle = \Omega^6\langle \bar{I}_{ij}\bar{I}^{ij}\rangle = 2\Omega^6\langle \bar{I}_{xx}^2 + \bar{I}_{xy}^2\rangle = \Omega^6\frac{m^2}{8}\ell_0^4\langle\cos^2(2\omega t) + \sin^2(2\omega t)\rangle = \Omega^6\frac{m^2}{8}\ell_0^4 \tag{6.121}$$

which gives (here $\Omega = 2\omega$)

$$L = \frac{8}{5}m^2\ell_0^4\omega^6 \tag{6.122}$$

and, inserting Eq. (6.102)

$$L = \frac{64}{5}\frac{m^5}{\ell_0^5} \approx 4(m\omega)^{10/3}$$

Now, the luminosity is energy per unit time. Energy is mass per velocity squared, so to go back to SI units, we have to multiply energy by c^2/G, to convert mass, and again by c^2 to convert the velocity squared, and divide time by c, so in total a factor of c^5/G. For the pulsar we have $m \approx 1.4M_{\odot}$, $r = 5$kpc, and a orbital period of $P = 2\pi/\omega \approx 7$h (the rotation period of the pulsar is much shorter, of order 50 μs), which gives

$$L = \frac{c^5}{G}(1.7\cdot 10^{-29})\text{Watt} \tag{6.123}$$

8. Emitting this energy, the binary system becomes more compact. The energy of the pulsar system is the sum of the kinetic energy of the two stars and the potential energy ($r = \ell_0/2$). With Eq. (6.102), we can express it in function of the binary period P

$$E = 2\times\frac{1}{2}m\omega^2r^2 - \frac{m^2}{2r} \tag{6.124}$$

$$= -2^{-4/3}m^{5/3}\omega^{2/3} \tag{6.125}$$

$$= const \times P^{-2/3} \tag{6.126}$$

(negative because the system is gravitationally bound). So the relative energy and period changes are related by

$$\frac{1}{E}\frac{dE}{dt} = -\frac{2}{3}\frac{1}{P}\frac{dP}{dt} \tag{6.127}$$

The output luminosity L represents the change in energy, $L = -dE/dt$. Inserting the numbers, we obtain finally a theoretical prediction $\dot{P} \approx -2\cdot 10^{-13}$. However, the large ellipticity of the actual pulsar orbit makes the real value 12 times larger, so we get

$$\dot{P} = \frac{3}{2}\frac{P}{E}L \approx -2.4\cdot 10^{-12} \tag{6.128}$$

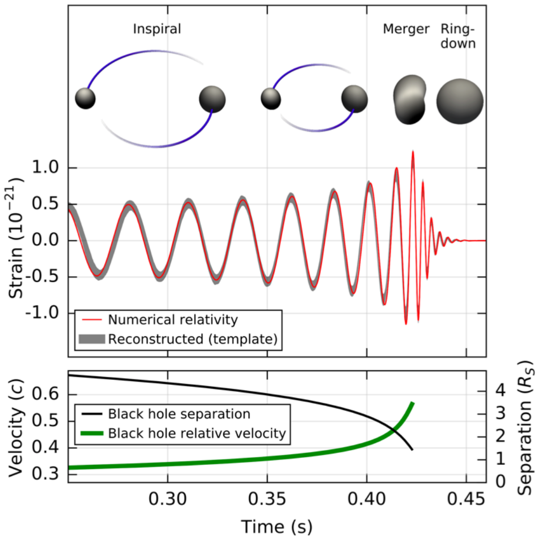

Figure 6.3: GW "chirp" signal detected in 2016 by the LIGO collaboration (from www.ligo.org, credit: LIGO).

or, roughly, a period decrease (angular velocity increase) of 10^{-4} s/yr, which is in perfect agreement with observations. This results earned Hulse and Taylor the Nobel Prize in 1993 as the first clear evidence of the existence of GWs.

9. In the final phase, the angular velocity increases to relativistic speed and the stars or black holes finally merge. The signal is then no longer a simple sinusoid, but a more complex signal called a *chirp*, with fast increasing amplitude and shorter and shorter period, lasting overall less than a second, with an abrupt halt. This chirp from two merging black holes has been first detected in 2016 by the LIGO collaboration (Fig. 6.3).

10. In 2017, the LIGO-Virgo collaboration detected also GWs from a system of two merging neutron stars. During the final phase, the two stars emitted also a burst of γ-ray radiation, detected by the Fermi space telescope. For the first time ever, an event was observed both in GWs and in electromagnetic radiation. Since the signals arrived almost simultaneously, this confirms GR prediction that GWs have the same velocity as light.

Chapter 7

Spherically symmetric systems

7.1 Spherical metric

1. We proceed now to solving the GR equations when the space-time is assumed spherically symmetric. In a flat space, this implies that the space part of the metric, when seen by an observer at rest with the system in the center of symmetry, has to depend equally on x, y, z, i.e. has to depend on

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.1}$$

In this case, the usual result that a circumference is $2\pi r$ and the surface of a sphere is $4\pi r^2$, where r is the radius of the sphere, holds true. The element of surface is

$$dS = r^2 \sin\theta d\theta d\phi \tag{7.2}$$

In a curved space, however, this is no longer the case. In other words, one can define two different "radii", one, R, that gives the line element for fixed θ, ϕ; the other, r, that gives the surface element for a fixed R. In this case the space part of the metric is

$$dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.3}$$

The coordinate r is called curvature coordinate. Now the circumference is $2\pi r$ and surface area is $4\pi r^2$, as previously, but the sphere has radius R.

2. Now, we choose θ, ϕ so that they are always orthogonal to r (in other words, if I move radially, I remain at the same values of θ, ϕ). This amounts to

$$g_{r\theta} = \vec{e}_r \cdot \vec{e}_\theta = 0 \tag{7.4}$$

$$g_{r\phi} = \vec{e}_r \cdot \vec{e}_\phi = 0 \tag{7.5}$$

So the space-time interval can be written as

$$ds^2 = g_{00}dt^2 + 2g_{0r}drdt + 2g_{0\theta}dtd\theta + 2g_{0\phi}dtd\phi + g_{rr}dr^2 + r^2d\Omega^2 \tag{7.6}$$

where we put for brevity

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{7.7}$$

Now the terms

$$2dt(g_{0\theta}d\theta + g_{0\phi}d\phi) = 2dt(V_i dx^i) \tag{7.8}$$

with $i = \theta, \phi$ and $V_i = \{g_{0\theta}, g_{0\phi}\}$ select a particular direction, $\vec{V}$. This is not acceptable however in a spherically symmetric system, and therefore we must assume $\vec{V} = 0$, i.e. $g_{0\theta}, g_{0\phi} = 0$. Then we obtain

$$ds^2 = g_{00}dt^2 + 2g_{0r}drdt + g_{rr}dr^2 + r^2d\Omega^2 \tag{7.9}$$

3. We now focus on static metrics, that is, time independent, and also we assume symmetry under time inversion, t into $-t$. The last condition forces $g_{0r} = 0$, otherwise $ds^2(t)$ would not be equal to $ds^2(-t)$. Then finally we obtain the most general static spherical metric (in the frame at rest at the center of the system)

$$\begin{aligned} ds^2 &= g_{00}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2 \quad (7.10) \\ &= -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \quad (7.11) \end{aligned}$$

4. We wish to apply this metric to isolated systems like stars and black holes. Therefore, we impose from the start that far from the source we recover a Minkowskian metric,

$$\lim_{r\to\infty} \phi(r), \Lambda(r) = 0 \quad (7.12)$$

5. The proper radial distance of the metric is obtained for $dt = d\phi = d\theta = 0$, so

$$\ell = \int_{r_1}^{r_2} e^{\Lambda} dr \quad (7.13)$$

   This is in general completely unrelated to the surface area $4\pi r^2$, as anticipated.

7.2 Gravitational redshift

1. Since the metric does not depend on t, we know that $p_0 = -E$ is a conserved quantity along a geodesic. However, this is not the energy measured in the LIF at distance r of a particle with momentum p (a LIF is momentarily at rest with the particle). In fact, in the LIF one has $U^i = 0$ and therefore $U^\mu U_\mu = g_{\mu\nu}U^\mu U^\nu = -1$ (always valid by definition) gives

$$U^0 = \frac{1}{\sqrt{-g_{00}}} = e^{-\phi(r)} \quad (7.14)$$

   Then

$$E_{LIF} = -\vec{U} \cdot \vec{p} = e^{-\phi(r)} E \quad (7.15)$$

   The two quantities, E_{LIF}, E, coincide at infinity ($\phi(\infty) = 0$). So we can write that the energy of a particle at r is

$$E_{LIF} = e^{-\phi(r)} E_{inf} \quad (7.16)$$

2. Imagine now a photon propagating from a star (i.e., emitted at a distance r from the center) towards infinity, where it is observed. At infinity, it has energy $E_{inf} = h\nu_{inf} = E_{LIF}e^{\phi(r)}$, where E_{LIF} is the energy measured in the LIF at distance r from the center of the star. If we write $E_{LIF} = h\nu_{em}$ for the emitted energy, we see that the observed one at infinity is

$$\nu_{obs} = \nu_{em} e^{\phi(r)} \quad (7.17)$$

3. This shift in frequency implies a shift in wavelength as well. So since $\lambda \sim 1/\nu$ we have

$$\lambda_{obs} = \lambda_{em} e^{-\phi(r)} \quad (7.18)$$

   and a *gravitational redshift*

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{\nu_{em}}{\nu_{obs}} = \frac{1}{\sqrt{-g_{00}}} = e^{-\phi(r)} \quad (7.19)$$

   This is then the redshift that is observed at infinity when a photon is emitted in a gravitational potential at distance r from the center of a source. (Of course we assume we know ν_{em} because this frequency will be identical to the frequency of a particular emission line measured in laboratory.)

4. Since light wavelength is what is used to synchronize clocks, this shows that clocks in a gravitational field do not remain synchronized. In particular, clocks in the gravitational field of a star run slower than at infinity (i.e. $\nu_{obs} > \nu_{em}$) because, as we should have expected and will be shown shortly, $\phi(r) \approx -M/r$.

5. In full generality, the redshift can be written in an invariant way as follows:

$$1 + z = \frac{(\vec{U} \cdot \vec{p})_{em}}{(\vec{U} \cdot \vec{p})_{obs}} \tag{7.20}$$

7.3 Gravitational equations

1. We proceed now to derive the gravitational equations in a spherically symmetric system. We have (a prime means d/dr)

$$G_{00} = \frac{1}{r^2} e^{2\phi} \frac{d}{dr}[r(1 - e^{-2\Lambda})] \tag{7.21}$$

$$G_{rr} = -\frac{1}{r^2} e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\phi' \tag{7.22}$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}[\phi'' + (\phi')^2 + \frac{\phi'}{r} - \phi'\Lambda' - \frac{\Lambda'}{r}] \tag{7.23}$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta} \tag{7.24}$$

All the other components vanish.

2. As we have seen, for an observer at rest with the star, $U_0 = -e^{\phi}$ and $U^0 = e^{-\phi}$, and $U^i = 0$. Therefore the EMT has components

$$T_{00} = (\rho + p)U_0 U_0 + p g_{00} = \rho e^{2\phi} \tag{7.25}$$

$$T_{rr} = p e^{2\Lambda}, \quad T_{\theta\theta} = r^2 p, \quad T_{\phi\phi} = \sin^2\theta T_{\theta\theta} \tag{7.26}$$

3. Now we need to add information about the kind of matter that forms the star. In particular, we need an *equation of state* (EOS) that links pressure with energy density and, if necessary, other thermodynamical quantities like temperature. We assume now that $p = p(\rho)$, i.e. that the pressure depends only on the energy density. For a ideal gas in equilibrium, indeed, the pressure depends only on ρ (since the temperature is constant).

4. The conservation equations tell us that $T^{\mu\nu}_{;\mu} = 0$. These vanish for all components except $\nu = r$, when we get

$$(\rho + p)\frac{d\phi}{dr} = -\frac{dp}{dr} \tag{7.27}$$

i.e. the pressure gradient is in equilibrium with the gravitational force.

5. The Einstein equation $G_{00} = 8\pi T_{00}$ is now

$$\frac{1}{r^2} e^{2\phi} \frac{d}{dr}[r(1 - e^{-2\Lambda})] = 8\pi\rho e^{2\phi} \tag{7.28}$$

from which

$$\frac{d}{dr}[r(1 - e^{-2\Lambda})] = 8\pi\rho r^2 \tag{7.29}$$

If we define now the gravitational mass

$$m(r) \equiv \frac{1}{2} r(1 - e^{-2\Lambda}) \tag{7.30}$$

we have that

$$g_{rr} = e^{2\Lambda} = [1 - \frac{2m(r)}{r}]^{-1} \tag{7.31}$$

and we obtain a familiar result, namely

$$\frac{dm(r)}{dr} = 4\pi\rho r^2 \tag{7.32}$$

or

$$m(r) = 4\pi \int \rho r^2 dr \tag{7.33}$$

This shows that indeed $m(r)$ is a mass (density times volume).

6. The rr Einstein equation is

$$\phi' = \frac{m(r) + 4\pi p r^3}{r[r - 2m(r)]} \tag{7.34}$$

Now we have all the equations we need: one for $m(r)$, one for ϕ', the EOS, and the conservation equation (7.27), for four variables, ρ, ϕ, Λ, p.

7.4 Schwarzschild metric

1. Outside the star, $\rho = p = 0$. In this case we get the so-called exterior solution, or Schwarzschild metric. We have now

$$m' = 0, \tag{7.35}$$
$$\phi' = \frac{m}{r(r - 2m)} \tag{7.36}$$

from which $m(r) = M = const$ and

$$\phi = \int \frac{M}{r(r - 2M)} dr \tag{7.37}$$
$$= \frac{1}{2}\log(1 - \frac{2M}{r}) \tag{7.38}$$

(we imposed $\phi = 0$ at infinity).

2. The Schwarzschild metric is therefore

$$ds^2 = -dt^2(1 - \frac{2M}{r}) + dr^2(1 - \frac{2M}{r})^{-1} + r^2 d\Omega^2 \tag{7.39}$$

For large r, this can be approximated as

$$ds^2 = -dt^2(1 - \frac{2M}{r}) + dr^2(1 + \frac{2M}{r}) + r^2 d\Omega^2 \tag{7.40}$$

and also as

$$ds^2 = -dt^2(1 - \frac{2M}{r}) + (1 + \frac{2M}{r})(dr^2 + r^2 d\Omega^2) \tag{7.41}$$

by a redefinition of the radial coordinate (still only for large r)

$$r \to r(1 + \frac{M}{r}) \tag{7.42}$$

This last form of ds^2 coincides indeed with the weak field limit we already obtained in Eq. (5.128) if $\Phi = -M/r$. This justifies again the use of M as the mass of the star.

3. The value $R = 2M$ is called *Schwarzschild radius*. For the Sun it is 3 km, for the Earth is 9 mm. Except for black holes, all objects in the Universe are much larger than their Schwarzschild radius.

4. *Birkhoff theorem* states that the Schwarzschild metric is the *unique* solution of the Einstein equations with spherical symmetry in vacuum, even if the source is not static (but maintains spherical symmetry, that is, collapsing or expanding but not rotating). This means that there are no GWs for spherical source at all orders, not just at first order as we have demonstrated explicitly.

5. There are several equivalent forms of the Schwarzschild metric, all related by simple coordinate transformations. For instance, with the transformation

$$r = \bar{r}(1 + \frac{M}{2\bar{r}})^2 \tag{7.43}$$

one finds the *isotropic form*

$$ds^2 = -\left(\frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\phi^2) \tag{7.44}$$

(isotropic because the space part can be written as $dx^2 + dy^2 + dz^2$ with suitably redefined coordinates). This shows once again that the meaning of the coordinates is not absolute; they are just labels on the manifold. Physical (i.e., measurable) quantities should be related to invariants like $d\tau$ or $\vec{p} \cdot \vec{U}$.

7.5 Interior solution

1. Inside the radius R of the star, we have $\rho, p \neq 0$. Combining Eq. (7.27) with (7.34) we obtain the Tolman-Oppenheimer-Volkov (TOV) equation,

$$p' = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)} \tag{7.45}$$

We need now to choose boundary conditions. We impose then $m(r = 0) = 0$ and $p(r = 0) = p_c$. The solution will therefore depend on the central pressure p_c. Moreover, we put the pressure at the surface of the star to vanish, since it must match the conditions in vacuum. Also, by continuity

$$m(R) = M = 4\pi \int_0^R \rho r^2 dr \tag{7.46}$$

However, one has to notice that the proper volume of the star is

$$\int \sqrt{-g_3} d^3x = \int e^\Lambda r^2 \sin\theta dr d\theta d\phi = 4\pi \int e^\Lambda r^2 dr \tag{7.47}$$

which differs from $4\pi \int r^2 dr$. So the mass we measure outside, M, is different from the total energy stored inside the star, which is given by $\hat{M} = 4\pi \int \rho e^\Lambda r^2 dr$. The difference gives the energy in the gravitational field itself.

2. Eq. (7.45) can be integrated if $\rho = const$. Then $m(r) = 4\pi\rho r^3/3$ and

$$p' = -\frac{4\pi}{3} r^3 \frac{(\rho + p)(\rho + 3p)}{r(r - 2m)} \tag{7.48}$$

so that

$$\int \frac{dp}{(\rho + p)(\rho + 3p)} = -\frac{4\pi}{3} \int \frac{r dr}{1 - \frac{8}{3}\pi r^2 \rho} = \frac{1}{4\rho} \log(1 - \frac{8}{3}\pi r^2 \rho) + const \tag{7.49}$$

The left-hand-side gives

$$\int \frac{dp}{(\rho + p)(\rho + 3p)} = \frac{1}{2\rho} \log \frac{3p + \rho}{3(p + \rho)} + const$$

The integration constant should be fixed by the condition $p(r = 0) = p_c$. Finally this gives

$$\frac{\rho + 3p}{\rho + p} = \frac{\rho + 3p_c}{\rho + p_c} (1 - \frac{2m}{r})^{1/2} \tag{7.50}$$

3. At $r = R$ we have $p = 0$ and $m = M = 4\pi\rho R^3/3$ so

$$1 = \left(\frac{\rho + 3p_c}{\rho + p_c}\right)^2 (1 - \frac{2M}{R}) \tag{7.51}$$

from which

$$R^2 = \frac{3}{8\pi\rho}[1 - \left(\frac{\rho + 3p_c}{\rho + p_c}\right)^{-2}] \tag{7.52}$$

or equivalently

$$p_c = \rho\frac{1 - (1 - \frac{2M}{R})^{1/2}}{3(1 - \frac{2M}{R})^{1/2} - 1} \tag{7.53}$$

which gives the central pressure as a function of the parameters M, R (see Fig. 7.2).

4. As an explicit function of M, R alone, the internal pressure is then

$$p = \rho\frac{\sqrt{1 - \frac{2M}{R}} - \sqrt{1 - \frac{2Mr^2}{R^3}}}{\sqrt{1 - \frac{2Mr^2}{R^3}} - 3\sqrt{1 - \frac{2M}{R}}} \tag{7.54}$$

5. When the denominator goes to zero, $p_c \to \infty$. This happens for

$$\frac{M}{R} = \frac{4}{9} \tag{7.55}$$

This result, called Buchdahl's theorem, can be extended also to more general cases. It is an upper bound for M/R for any star. Beyond this value, no central pressure can support a star.

6. For the Sun, $M_{\odot} \approx 1.5$ km, one has $R \approx 10^6$ km, so ordinary stars are very far from this limit.

7. We can obtain the internal metric by using now Eq. (7.27):

$$(\rho + p)\frac{d\phi}{dr} = -\frac{dp}{dr} \tag{7.56}$$

from which

$$\phi = -\int \frac{dp}{\rho + p} \tag{7.57}$$

Using the boundary value $\phi(R) = \frac{1}{2}\log(1 - \frac{2M}{R})$, we obtain finally

$$e^{\phi(r)} = \frac{3}{2}(1 - \frac{2M}{R})^{1/2} - \frac{1}{2}(1 - \frac{2M}{R^3}r^2)^{1/2} \tag{7.58}$$

for $r \leq R$. The interior and exterior solutions are represented in Fig. (7.1).

7.6 White dwarfs and quantum pressure degeneracy

1. White dwarfs are stars supported not by radiation pressure (as ordinary stars) but by the pressure exerted by a relativistic electron gas. We evaluate now the equation of state for a gas of charged particles (electrons).

2. Due to Heisenberg uncertainly, $\Delta p \Delta x \geq h$, in a volume V, the minimal momentum of a quantum wave is

$$\Delta p \geq hV^{-1/3} \tag{7.59}$$

(in this section p is momentum and P is pressure). Therefore, for fermions, the number of states in $p, p+dp$ is

$$dN = 2 \times 4\pi\frac{p^2 dp}{(\Delta p)^3} = 8\pi p^2\frac{dp}{h^3}V \tag{7.60}$$

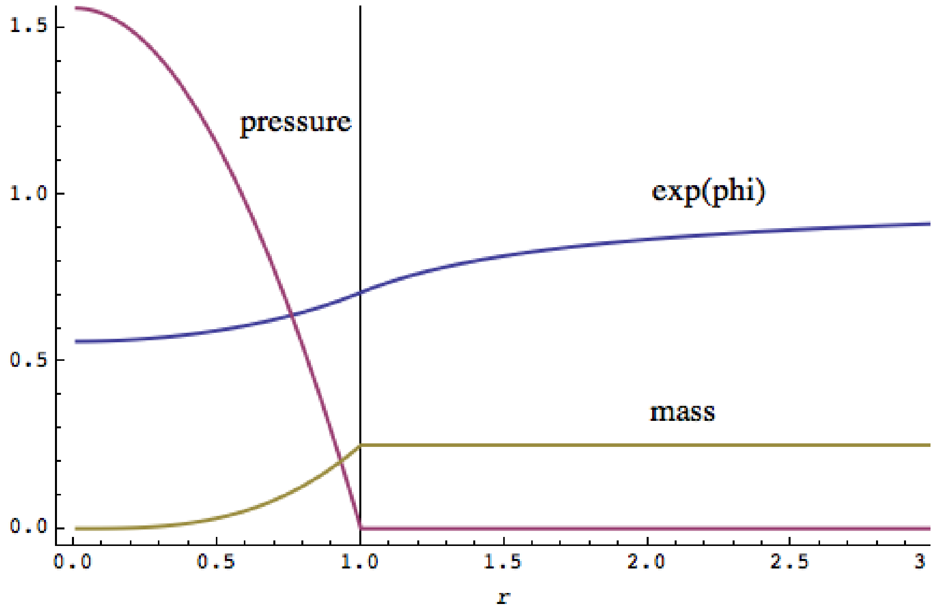

Figure 7.1: Internal and external Schwarzschild solution for a star of radius 1, mass 0.25 (the pressure is multiplied by 100).

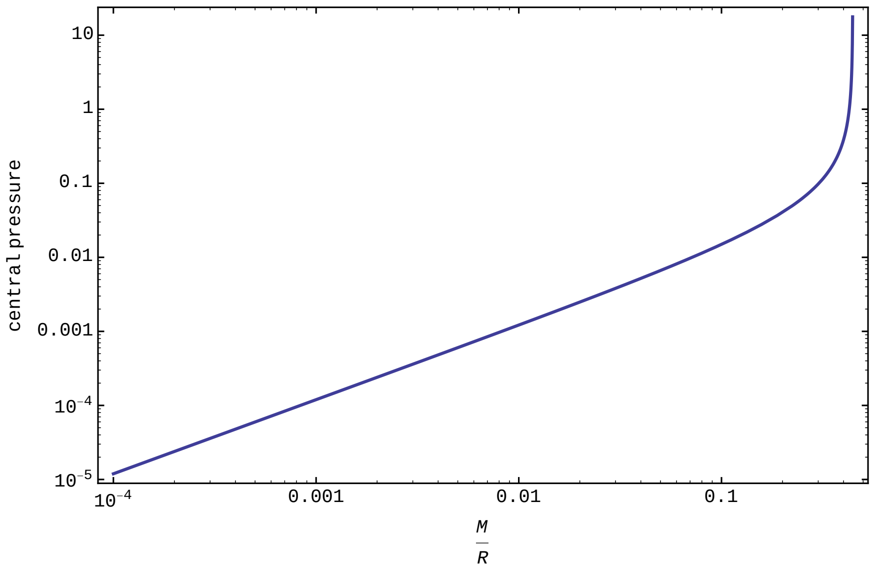

Figure 7.2: Central pressure as a function of M/R, showing the singularity for $M/R \to 4/9$.

where the extra factor of 2 takes into account the two spin states. The total density of states for N particles is then

$$\frac{N}{V} = \int_0^p 8\pi\bar{p}^2 \frac{d\bar{p}}{h^3} = \frac{8\pi p^3}{3h^3} \tag{7.61}$$

3. If we cool a gas of electrons as much as possible (*degenerate state*), they will fill all the N states from the lowest energy to some maximal p_f (Fermi momentum), so that N will be the total number of particle, $n = N/V$ the number density and p_f is given by

$$n = \frac{N}{V} = \frac{8\pi p_f^3}{3h^3} \tag{7.62}$$

so that the highest momentum that particles in a gas of density n can achieve is

$$p_f = (\frac{3h^3}{8\pi})^{1/3} n^{1/3} \tag{7.63}$$

independent of the particle masses. For a star of mass similar to the Sun and radius similar to the Earth (typical for white dwarfs), one finds $p_f \sim 1$ MeV, which is much larger that the electron mass but much smaller than the nucleon masses. Most of the electrons are therefore relativistic.

4. With each electron state is associated an energy $E = (p^2 + m^2)^{1/2}$ so the total energy density is

$$\rho = \frac{E_{tot}}{V} = \frac{1}{V}\int_0^{p_f} (p^2 + m^2) dN = \int_0^{p_f} 8\pi p^2 (p^2 + m^2)^{1/2} \frac{dp}{h^3} \tag{7.64}$$

This tends to

$$\rho(p \gg m) = \frac{2\pi p_f^4}{h^3} \tag{7.65}$$

in the relativistic limit.

5. Finally, the pressure exerted by this gas will be

$$P = -\frac{dE}{dV} \tag{7.66}$$

since the star is an isolated system. Assuming $N = const$, we have that $n \sim 1/V$ and also $p_f \sim V^{-1/3}$ so we have $(V/p_f)dp_f/dV = -1/3$ and therefore, differentiating also wrt the integration boundary,

$$P = -\frac{dE}{dV} = -V\frac{8\pi p_f^2}{h^3}(p_f^2 + m^2)^{1/2}\frac{dp_f}{dV} - \int_0^{p_f} 8\pi p^2 (p^2 + m^2)^{1/2} \frac{dp}{h^3} \tag{7.67}$$

or

$$P = -\frac{dE}{dV} = \frac{8\pi p_f^3}{3h^3}(m^2 + p_f^2)^{1/2} - \rho \tag{7.68}$$

Again in the limit $p_f \gg m$ this is

$$P = \frac{2\pi}{3h^3} p_f^4 = \frac{1}{3}\rho \tag{7.69}$$

just like for radiation. Note that this gas of electrons is at the same time relativistic but "cold" (in the minimum state), due to Pauli exclusion principle.

6. Now white dwarfs (WD) are stars that are too cold to activate nuclear fusion, so they are not supported by radiation pressure but by electron degeneracy pressure. They contain e^- (relativistic) and n, p (non relativistic); moreover, the density of electrons n_e equals the density of protons, since WD are neutral. Most mass is in n, p, so $\rho_m = \mu m_p n_e$, (here μ takes into account the number of neutrons per proton, an order of unity factor) while most pressure is in e^-.

7. So we have now a pressure
$$P \sim p_f^4 \sim n_e^{4/3} \sim \rho_m^{4/3} \tag{7.70}$$
or $P = k\rho_m^{4/3}$ (such power-law relations between pressure and density are called *polytropic* equations of state), where
$$k = \frac{2\pi}{3h^3}\left(\frac{3h^3}{8\pi\mu m_p}\right)^{4/3} \tag{7.71}$$

8. From the equation (we can safely assume $\rho \gg P$)
$$\frac{dP}{dr} = -\rho_m \frac{m(r)}{r^2} \tag{7.72}$$
we can estimate as an order of magnitude that (putting back G)
$$\frac{P}{R} \approx \rho_m \frac{GM}{R^2} \tag{7.73}$$
where R is the size of the star and M its mass. Finally we see that $GM = PR/\rho_m$ and replacing $R = (3M/4\pi\rho_m)^{1/3}$ and $P = k\rho^{4/3}$,
$$M = G^{-3/2}\left(\frac{3k^3}{4\pi}\right)^{1/2} = \frac{G^{-3/2}}{32\mu^2 m_p^2}(\frac{6h^3}{\pi})^{1/2} \approx \frac{M_P^3}{m_p^2} \tag{7.74}$$
(where in the last expression we used units such that $c = h = 1$ and therefore $G = M_P^{-2}$, see Eq. (5.169)). It is remarkable that the ratio M_P^3/m_p^2 is of the order of a solar mass. Inserting $\mu = 2$ (appropriate for a star with a large Helium content) one finds
$$M \approx 0.32 M_{\odot} \tag{7.75}$$

9. A more exact calculation gives
$$M \approx 1.4 M_{\odot} \tag{7.76}$$
a value called *Chandrasekhar mass*. This is the largest mass that the Fermi pressure of electrons can support. Beyond this, the WD will collapse into a neutron star or a black hole. The typical size of a WD is similar to the Earth.

10. In a more accurate treatment, one should consider that the typical energies are close to the electron mass, so some electrons are relativistic, some are not. In the non-relativistic case, the density ρ_m will not simplify out, and there will be a range of masses/radii that can be in equilibrium with pressure degeneracy (Fig. 7.3). Then it will appear that the Chandrasekhar mass is actually a maximal value, at which electron move with light speed. Since they cannot exert more pressure than this, more mass (from e.g. a companion star) will make the star unstable to further collapse, increased temperature, fusion of heavy elements, and supernovae (type Ia) explosion.

11. Much more details on compact stars, including neutron stars, can be found for instance in Ferrari et al., *General Relativity and its applications.*

7.7 Particle's motion

1. Let's go back to the Schwarzschild metric (SM)
$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2 \tag{7.77}$$

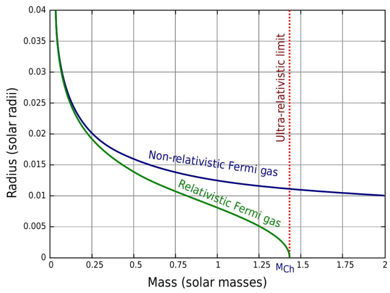

Figure 7.3: Relation mass-radius for white dwarfs. (Credit By AllenMcC., Public Domain, https://commons.wikimedia.org/w/index.php?curid=9383959)

2. The value (we put back c for a moment)

$$\frac{2GM}{c^2 r} = 1 \tag{7.78}$$

is of course particularly significant. In Newtonian mechanics, the escape velocity reaches c when

$$\frac{c^2}{2} = \frac{GM}{R} \tag{7.79}$$

so exactly the same condition. However, while in Newtonian physics light can go out but then, somehow, should fall back on the star, as we will see, in GR light cannot escape at all.

3. We want to study now the evolution of particles in the SM, i.e. around a star. We will treat the particles as test particles, i.e. they do not perturb the Schwarzschild metric but just move on the geodesics (of course in absence of any external non-gravitational force).

4. Since the SM is independent of time, the energy of a particle moving in it, $p_0 = -E$, is conserved. For $m \neq 0$, let us define now a specific energy, $\bar{E} = -p_0/m = E/m$.

5. The SM is also independent of ϕ, so there is conservation of the orbital angular momentum

$$p_\phi = mU_\phi = mg_{\phi\phi}U^\phi = mr^2\frac{d\phi}{d\tau}\sin^2\theta \approx mr^2\frac{d\phi}{dt} \tag{7.80}$$

where in the last step we put ourselves on the equatorial plane, $\theta = \pi/2$ and we approximated $d\tau$ with dt for non-relativistic speed.

6. Let us define also a specific momentum, $\bar{L} = p_\phi/m$. Also, we have

$$p^r = m\frac{dr}{d\tau} \tag{7.81}$$

For a photon we use instead just the momentum, $L = p_\phi$.

7. Because we are in spherical symmetry, there is no force acting in the direction θ, so

$$p_\theta = 0 \tag{7.82}$$

That is, if a particle begins its motion on a plane, it will remain in that plane, i.e. $\theta = const$. From now on, we take that plane to be the equator of the system, $\theta = \pi/2$.

8. The contravariant components of the momentum are

$$p^0 = m\bar{E}(1 - \frac{2M}{r})^{-1} \tag{7.83}$$

$$p^r = m\frac{dr}{d\tau} \tag{7.84}$$

$$p^\phi = \frac{m}{r^2}\bar{L} \tag{7.85}$$

for a massive particle, and

$$p^0 = E(1-\frac{2M}{r})^{-1} \quad (7.86)$$

$$p^r = \frac{dr}{d\lambda} \quad (7.87)$$

$$p^\phi = \frac{L}{r^2} \quad (7.88)$$

for a photon whose trajectory is parametrized by λ. (The parameter λ ultimately can be eliminated from the equations in favor of some other coordinate, e.g. time, using one of the geodesic equations.)

9. For a massive particle $\vec{p}\cdot\vec{p} = -m^2$ and therefore

$$(p^0)^2 g_{00} + (p^r)^2 g_{rr} + (p^\phi)^2 g_{\phi\phi} = -m^2 \quad (7.89)$$

This can be written as

$$\left(\frac{dr}{d\tau}\right)^2 = \bar{E}^2 - (1-\frac{2M}{r})(1+\frac{\bar{L}^2}{r^2}) \quad (7.90)$$

and similarly for a photon

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - (1-\frac{2M}{r})\frac{L^2}{r^2} \quad (7.91)$$

These are the fundamental equations to analyze. Notice that we have not used explicitly the geodesic equations themselves: this is because the high symmetry of the system allowed us to find immediately the conserved quantities. Notice that, as required by the equivalence principle, the particle mass m drops out of the equations: two particles of different masses with the same initial four-velocity will have the same geodesic. If that were not the case, then in a LIF (a free-falling frame) we would have seen two particles initially at rest moving under the action of different forces, thereby revealing that phenomena in a LIF are different than in Special Relativity.

10. The equation of motion can now be written as

$$r'^2 = \bar{E}^2 - \bar{V}^2(r) \quad (7.92)$$

where the prime is the derivative wrt τ or λ and

$$\bar{V}^2(r) = (1-\frac{2M}{r})(1+\frac{\bar{L}^2}{r^2}) \quad (7.93)$$

for massive particles and

$$V^2(r) = (1-\frac{2M}{r})\frac{L^2}{r^2} \quad (7.94)$$

for photons (in this case $\bar{E}$ must be replaced by E).

11. Differentiating Eq. (7.92) wrt to τ or λ we obtain

$$2\frac{dr}{d\tau}\frac{d^2r}{d\tau^2} = -\frac{d\bar{V}^2}{dr}\frac{dr}{d\tau} \quad (7.95)$$

i.e.,

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{d\bar{V}^2}{dr} \quad (7.96)$$

which is qualitatively analogous to $ma = -\Phi_{,r}$, so $\bar{V}^2$ acts as an effective potential over which the particle rolls down. The analogy is stronger for non-relativistic speeds, when we can always approximate τ with t. The potential is fixed once we know the mass of the star and the initial angular momentum of the particle (which remains constant throughout).

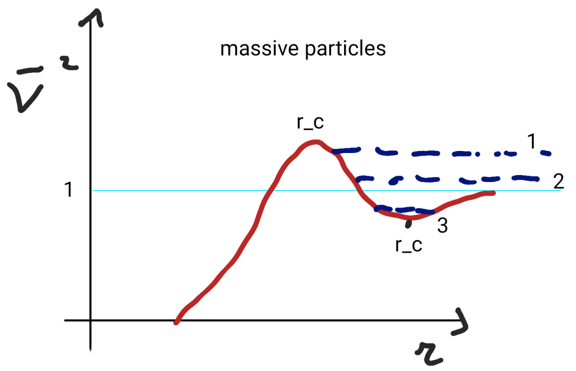

Figure 7.4: Potential for massive particles for $\bar{L}^2 > 12M^2$. Trajectories 1 and 2 are unbounded, trajectory 3 is bounded. The values r_c represent circular orbits.

12. A circular orbit is possible only at distances at which $\bar{V}_{,r} = 0$. For massive particles, there are two such points (if $\bar{L}^2 > 12M^2$) and zero otherwise, while for photon always one:

$$r_c = \frac{\bar{L}^2}{2M}\left[1 \pm (1 - \frac{12M^2}{\bar{L}^2})^{1/2}\right], \quad \text{particle} \tag{7.97}$$

$$r_c = 3M, \quad \text{photon} \tag{7.98}$$

13. It is not difficult to realize, by looking at the potential, that when there are two circular orbits, one is stable, the other is unstable. For the photon, instead, the circular orbit is always unstable. The smallest stable circular orbit is obtained for $\bar{L}^2 = 12M^2$ and lies at

$$r_c = 6M \tag{7.99}$$

14. When the orbits are not circular, they are *almost* hyperbolic or elliptic: the particle comes from infinity and returns to infinity, or moves along an elliptic orbit between a minimum and a maximum distance, just like in the Newtonian case.

15. In the Newtonian case, $M/\bar{L} \ll 1$, we obtain from (7.97) the stable orbit

$$r = \frac{\bar{L}^2}{M} \tag{7.100}$$

Since $\bar{L} = r^2\dot{\phi}$, we see that the frequency of the orbit is

$$\omega = \dot{\phi} = \sqrt{\frac{M}{r^3}} \tag{7.101}$$

i.e., Kepler's law (remember that M and time have dimensions of length).

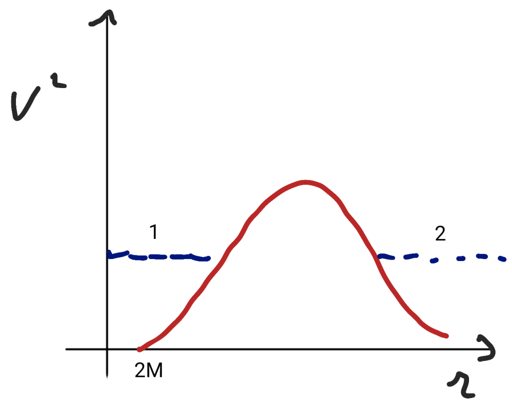

Figure 7.5: Potential for massless particles. Trajectory 1 represents a photon captured by the star; trajectory 2 a photon coming from infinity, reaching a minimum distance and then leaving again to infinity.

7.8 Perihelion shift

1. Let us first recall that the equation $r(\theta)$ of an ellipse with semi-major and semi-minor axes a, b, respectively, in polar coordinates with the origin on a focus is

$$r = \frac{a(1-e^2)}{1 \pm e\cos(k\phi)} \tag{7.102}$$

where $e = (1 - b^2/a^2)^{1/2}$ is the eccentricity. If $k = 1$, the value of r is the same after an interval $\Delta\phi = 2\pi$: in this case, a planet orbiting on the ellipse returns always at the same place after each orbit. Otherwise, it takes an interval $\Delta\phi = 2\pi/k$ for r to go back to the same value: this is called precession of the perihelion (i.e. of the shortest distance from the Sun). We will show that this formula holds also to a first approximation in GR.

2. On a circular stable orbit, inverting Eq. (7.97) we get

$$\bar{L}^2 = \frac{Mr}{1 - \frac{3M}{r}} \tag{7.103}$$

and since $\bar{E} = \bar{V}$ for a circular orbit, we also have

$$\bar{E}^2 = \frac{(1 - \frac{2M}{r})^2}{1 - \frac{3M}{r}} \tag{7.104}$$

Moreover, we have the following relations

$$\frac{d\phi}{d\tau} = U^\phi = \frac{p^\phi}{m} = \frac{\bar{L}}{r^2} = \left(\frac{Mr}{1 - \frac{3M}{r}}\right)^{1/2} \frac{1}{r^2} \tag{7.105}$$

$$\frac{dt}{d\tau} = U^0 = \frac{p^0}{m} = \frac{\bar{E}}{1 - \frac{2M}{r}} = \frac{1}{\sqrt{1 - \frac{3M}{r}}} \tag{7.106}$$

Therefore the orbital angular velocity is

$$\frac{d\phi}{dt} = \frac{\bar{L}}{r^2} \frac{1 - \frac{2M}{r}}{\bar{E}} = \left(\frac{M}{r^3}\right)^{1/2} \tag{7.107}$$

so the period (in coordinate time, i.e. for the observer at the center of the system, not proper time) is

$$P = 2\pi \frac{\Delta t}{\Delta \phi} = 2\pi \left(\frac{r^3}{M}\right)^{1/2} \tag{7.108}$$

the same result as for the Newtonian case.

3. A typical orbit of planets is just slightly eccentric, so can be treated as a small perturbation of a circular orbit. It is found that Mercury has a perihelion shift of 43 arcsec per century that cannot be explained by perturbations from other planets (see Fig. 7.6). Let us compare it to GR predictions.

4. Combining Eq. (7.90) and (7.105) one gets

$$(\frac{dr}{d\phi})^2 = \frac{\bar{E}^2 r^4 - (r^2 - 2Mr)(r^2 + \bar{L}^2)}{\bar{L}^2} \tag{7.109}$$

which, defining $u = 1/r$ can be written as

$$(\frac{du}{d\phi})^2 = \frac{\bar{E}^2}{\bar{L}^2} - (1 - 2Mu)(\frac{1}{\bar{L}^2} + u^2) \tag{7.110}$$

$$= \frac{\bar{E}^2}{\bar{L}^2} - \frac{1 - 2Mu}{\bar{L}^2} - u^2 + 2Mu^3 \tag{7.111}$$

We can begin by assuming that the last term is small wrt the preceding one because for the Sun

$$Mu = \frac{M}{r} \ll 1 \tag{7.112}$$

5. In a circular orbit, as we have seen, $r_c = \bar{L}^2/M$ in the Newtonian limit. We define now a "small circularity parameter" y, that vanishes for circular orbits in the Newtonian limit

$$y = u - \frac{M}{\bar{L}^2} \tag{7.113}$$

We rewrite then Eq. (7.111) as

$$\left(\frac{dy}{d\phi}\right)^2 = \frac{\bar{E}^2 - 1}{\bar{L}^2} + \frac{M^2}{\bar{L}^4} - y^2 \tag{7.114}$$

which can be directly integrated. The solution is

$$y = \frac{1}{r} - \frac{M}{\bar{L}^2} = \left[\frac{\bar{E}^2 + \frac{M^2}{\bar{L}^2} - 1}{\bar{L}^2}\right]^{1/2} \cos(\phi + \phi_0) \tag{7.115}$$

where ϕ_0 is an arbitrary initial phase. In terms of r, this gives the equation of an ellipse in polar coordinates, with respect to a focus, without perihelion shift: we are in fact in the Newtonian limit.

6. If we insert back the $2Mu^3$ term, we perturb this solution. We rewrite then Eq. (7.111) with y instead of u:

$$(y')^2 = \frac{\bar{E}^2}{\bar{L}^2} - (1 - 2My - 2\frac{M^2}{\bar{L}^2})(\frac{1}{\bar{L}^2} + \frac{M^2}{\bar{L}^4} + y^2 + 2\frac{M}{\bar{L}^2}y) \tag{7.116}$$

(where the prime stands for $d/d\phi$) and neglect the y^3 term, which means we assume slight deviation from circularity, since in a slightly non-circular, Newtonian orbit one has

$$y = u - \frac{M}{\bar{L}^2} \ll 1 \tag{7.117}$$

We obtain

$$(y')^2 = \frac{\bar{E}}{\bar{L}^2}^2 + \frac{M^2}{\bar{L}^4} - \frac{1}{\bar{L}^2} + \frac{2M^4}{\bar{L}^6} + \frac{6M^3}{\bar{L}^2}y - (1 - \frac{6M^2}{\bar{L}^2})y^2 \tag{7.118}$$

which has the form $(y')^2 = A + By - Cy^2$ with $C > 0$. The solution is then

$$\int \frac{dy}{\sqrt{A + By - Cy^2}} = \int d\phi \tag{7.119}$$

The solution can be found by writing $A + By - Cy^2 = (A + \frac{B^2}{4C}) - (y\sqrt{C} - \frac{B}{2\sqrt{C}})^2$ and then substituting $z = y\sqrt{C} - \frac{B}{2\sqrt{C}}$. The final solution is an ellipse with precession,

$$y = y_0 + a\cos(k\phi + \phi_0) \tag{7.120}$$

or, more explicitly,

$$r = \frac{\bar{L}^2\left(M - \bar{L}^2 y_0\right)^{-1}}{1 + \frac{a\bar{L}^2}{M - \bar{L}^2 y_0}\cos(k\phi + \phi_0)} \tag{7.121}$$

where

$$k = \left(1 - 6\frac{M^2}{\bar{L}^2}\right)^{1/2}, \quad y_0 = \frac{3M^3}{k^2\bar{L}^2} \tag{7.122}$$

$$a = \frac{1}{k}\left[\frac{\bar{E}^2 + \frac{M^2}{\bar{L}^2} - 1}{\bar{L}^2} + \frac{2M^4}{\bar{L}^6} - y_0^2\right]^{1/2} \tag{7.123}$$

This is an orbit that oscillates around y_0, not around $y = 0$, i.e. $u = M/\bar{L}^2$. The amplitude is also different from the Newtonian case and finally, since $k \neq 1$, the orbit does not return to the same y, i.e. r, after $\Delta\phi = 2\pi$, but rather after $k\Delta\phi = 2\pi$, i.e. for

$$\Delta\phi = \frac{2\pi}{k} = 2\pi(1 - \frac{6M^2}{\bar{L}^2})^{-1/2} \approx 2\pi(1 + \frac{3M^2}{\bar{L}^2}) \tag{7.124}$$

The extra term $\delta\phi = \Delta\phi - 2\pi$ is the amount of the perihelion shift. For $M/\bar{L} \ll 1$ we are back to the Newtonian case.

7. For the circular orbit we know that $\bar{L}$ is given by Eq. (7.103), which at first order can be approximated as $\bar{L}^2 = Mr$. Then we see that

$$\delta\phi = 6\pi\frac{M^2}{\bar{L}^2} \approx 6\pi\frac{M}{r} \tag{7.125}$$

We could have expected that, at first order, the shift should have been proportional to the Newtonian potential $\Phi(r)$.

8. For Mercury the effect is the largest, due to the small r. It turns out that $\delta\phi = 5 \cdot 10^{-7}$ radians per orbit. Since each orbit takes 0.24 yr, the total effect is exactly $0.43''$/yr, or 43 arcseconds per century. This was historically the first proof of GR (1915). Notice that the observed perihelion of Mercury is much larger (around 5600 arcsec per century) but most of it can be explained by perturbations due to other planets and by the imperfect sphericity of the Sun.

7.9 Gravitational deflection of light

1. For light propagation near a spherical body, the relevant equations are

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - (1 - \frac{2M}{r})\frac{L^2}{r^2} \tag{7.126}$$

$$p^\phi = \frac{d\phi}{d\lambda} = \frac{L}{r^2} \tag{7.127}$$

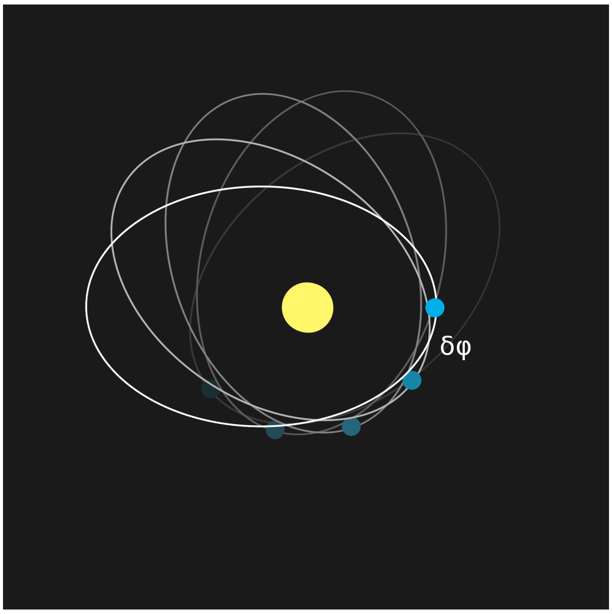

Figure 7.6: Perihelion shift (from Wikimedia Commons, author Rainer Zenz).

Then we have

$$\frac{d\phi}{dr} = \pm\frac{L}{r^2}\left[E^2 - (1-\frac{2M}{r})\frac{L^2}{r^2}\right]^{-1/2} \tag{7.128}$$

$$= \pm\frac{1}{r^2}\left[\frac{E^2}{L^2} - (1-\frac{2M}{r})\frac{1}{r^2}\right]^{-1/2} \tag{7.129}$$

$$= \pm\frac{1}{r^2}\left[\frac{1}{b^2} - (1-\frac{2M}{r})\frac{1}{r^2}\right]^{-1/2} \tag{7.130}$$

where we define $b = L/E$, the *impact parameter.* We see that $b = 0$ for a radial orbit, $L = 0$; it is also the minimal value of r, i.e. the closest distance to the star, in the Newtonian limit (see Fig. 7.7).

2. Replacing as previously $u = 1/r$ we find (we choose now the positive branch)

$$\frac{d\phi}{du} = \left(\frac{1}{b^2} - u^2 + 2Mu^3\right)^{-1/2} \tag{7.131}$$

3. As previously, the Newtonian limit is obtained by neglecting u^3. In this case is easy to see that the solution is

$$b = r\sin(\phi - \phi_0) \tag{7.132}$$

where ϕ_0 is the initial azimuth at infinity as seen from the center of the star. This is the equation of a straight line in polar coordinate: in this limit, the photon is not deflected by the gravitational field.

4. To move to the next order, we define

$$y = u(1 - Mu) \tag{7.133}$$

from which $du = dy(1 + 2My)$ (at first order in Mu), so

$$\frac{d\phi}{du} = \frac{d\phi}{dy(1+2My)} = \left(\frac{1}{b^2} - y^2(1+My)^2 + 2My^3(1+My)^3\right)^{-1/2} \tag{7.134}$$

and finally, again to first order in Mu,

$$\frac{d\phi}{dy} = (1+2My)\left(\frac{1}{b^2} - y^2\right)^{-1/2} + \Theta((Mu)^2) \tag{7.135}$$

5. The solution is
$$\phi - \phi_0 = \arcsin(yb) + 2M \int \frac{ydy}{(b^{-2} - y^2)^{1/2}} \tag{7.136}$$
or finally
$$\phi = \phi_0 + \frac{2M}{b} + \arcsin(yb) - 2M(b^{-2} - y^2)^{1/2} \tag{7.137}$$
At infinity, $y = 0$ and ϕ_0 is again the initial direction of the incoming photon.

6. The closest point to the star is at $dr/d\lambda = 0$, i.e. for $y = E/L = 1/b$. This minimal distance occurs for
$$\phi_{min} = \phi_0 + \frac{2M}{b} + \frac{\pi}{2} \tag{7.138}$$
All this is on the "positive branch" of the solution, i.e. from infinity to closest distance. Then the photon continues on the negative branch, from the closest point to infinity on the other side, and acquires another shift by $2M/b + \pi/2$. The total change of azimuth for a straight line would have been π. Here we see the total change $2(\phi_{min} - \phi_0)$ is π plus a relativistic correction
$$\delta\phi = \frac{4M}{b} \tag{7.139}$$
This expresses the gravitational deflection due to GR.

7. For the Sun, if $b = R_{\odot}$ (as close as possible to Sun's surface), we have
$$\delta\phi = 4\frac{1.5\text{km}}{7 \cdot 10^5\text{km}} = 8.4 \cdot 10^{-6}\text{rad} = 1''.74 \tag{7.140}$$
This gravitational deflection has been observed in 1919 during a solar eclipse, so as to measure the position of stars near the solar limb. For Jupiter, the effect is $0''.013$.

8. One could make a naive Newtonian argument replacing the formula for deflection for particles with velocity v for small deflections,
$$\Delta\phi = \frac{2M}{v^2 b} \tag{7.141}$$
with $v = c$; in this case, one obtains half the prediction of GR.

9. The same deflection effect, more generally called *gravitational lensing*, is nowadays observed using galaxies as lenses and distant quasars or galaxies as sources. In general, the original images are both distorted (or even multiplied) and magnified.

7.10 Black holes

1. We consider now the apparent singularity of the SM when $r = 2M$, a surface called *event horizon.* Let us assume that a star is so compact that this surface is outside the star itself: this is called a *black hole.* A particle falling into it in a radial trajectory (so $p_\phi = L = 0$) obeys the equation
$$\left(\frac{dr}{d\tau}\right)^2 = \bar{E}^2 - 1 + \frac{2M}{r} \tag{7.142}$$
from which
$$d\tau = -\frac{dr}{(\bar{E}^2 - 1 + \frac{2M}{r})^{1/2}} \tag{7.143}$$
(the minus sign implies the particle is infalling, not escaping). If $\bar{E} \geq 1$, the particle falls from infinity and reaches infinity again (unless absorbed by the star). In this case, τ remains finite and there is no singularity.

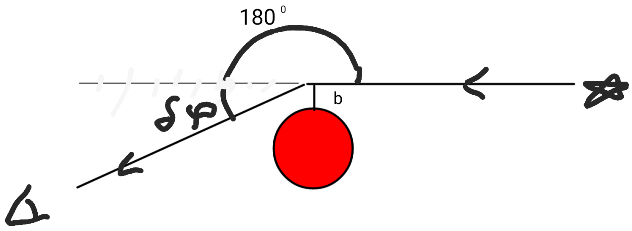

Figure 7.7: Gravitational deflection of light.

2. If $\bar{E} < 1$, the particle is bound to stay within r smaller than

$$r < \frac{2M}{1-\bar{E}^2} \tag{7.144}$$

so again there is no singularity. So in all cases a particle can cross the boundary $r = 2M$ without problem. The apparent singularity at $r = 2M$ appears only because of our choice of coordinates (in other coordinates in fact it disappears, see below). The elements of the Riemann tensor remain always regular for $r \neq 0$. Since we are in vacuum, $R_{\mu\nu} = 0$, but another invariant, $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ (called Kretschmann scalar) equals $48M/r^6$ and confirms that $r = 2M$ is not a singularity, while $r = 0$ is.

3. In terms of coordinate time, i.e. the time t measured by an observer at infinity at rest with the black hole, it takes indeed an infinite time for the particle to fall. In fact, one has

$$\frac{dt}{d\tau} = g^{00}U_0 = -g^{00}\bar{E} = \frac{\bar{E}}{1-\frac{2M}{r}} \tag{7.145}$$

and therefore for $\bar{E} = 1$

$$dt = -\frac{\bar{E}dr}{(1-\frac{2M}{r})(\bar{E}^2-1+\frac{2M}{r})^{1/2}} = -\frac{dr}{(1-\frac{2M}{r})(\frac{2M}{r})^{1/2}} \tag{7.146}$$

which is singular for $r \to 2M$. This means that for an observer at infinity it takes an infinite time for an object to enter the horizon.

4. Solving for $ds = 0$ the Schwarzschild metric along a radial path, we find

$$\frac{dt}{dr} = \pm(1-\frac{2M}{r})^{-1} \tag{7.147}$$

This shows that the light cones $t(r)$ become very narrow for $r \to 2M$ and take the usual 45^0 angle for $r \to \infty$. To analyze better the behavior near the horizon we move to the Kruskal-Szekeres coordinates u, v, defined as

$$u = \left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M}\cosh\frac{t}{4M} \tag{7.148}$$

$$v = \left(\frac{r}{2M}-1\right)^{1/2} e^{r/4M}\sinh\frac{t}{4M} \tag{7.149}$$

for $r > 2M$ and

$$u = \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M} \tag{7.150}$$

$$v = \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M} \tag{7.151}$$

for $r < 2M$. The SM metric becomes then

$$ds^2 = -32\frac{M^3}{r}e^{-r/2M}(dv^2 - du^2) + r^2 d\Omega^2 \tag{7.152}$$

where $r(u, v)$ is a function obtained by inverting the relation

$$(\frac{r}{2M} - 1)e^{r/2M} = u^2 - v^2 \tag{7.153}$$

so constant r means constant curves $u^2 - v^2$, that is, hyperbolae in the u, v plane (see Fig. 7.8). The time t depends on v, u through

$$t = 4M \tanh^{-1} \frac{v}{u} \tag{7.154}$$

for $r > 2M$ and

$$t = 4M \tanh^{-1} \frac{u}{v} \tag{7.155}$$

for $r < 2M$, so lines of constant t are straight lines in the u, v plane.

5. Now there is no singularity for $r = 2M$. There still is of course a singularity for $r = 0$. Notice that v plays the role of time (i.e. the factor in front of dv^2 has a negative sign). Now radial null cones are always 45^0, $dv = \pm du$

6. In Kruskal coordinates one sees that once a particle passes inside the Schwarzschild radius, its light cone points inward, so the particle cannot escape any longer and has to hit the $r = 0$ singularity.

7. Region IV (see Fig. 7.8) is called a "white hole": it represents a point in space in the asymptotic past from which particles might emerge "from nowhere". This is just the time-reverse of the fall "into nowhere" when particles reach the black hole singularity. This is a valid mathematical solution but it seems to contradict basic postulates of physics.

8. Region III is a region of space-time that appears completely disconnected from our accessible region I that lies outside the Schwarzschild radius. However, if we move along $\nu = 0$ (just mathematically, because physically this is a space-like line and no particle can run along it), we see that Eq. (7.153) becomes

$$(\frac{r}{2M} - 1)e^{r/2M} = u^2 \tag{7.156}$$

If we put $r = \sqrt{x^2 + y^2}$ (forgetting about z) we obtain a figure $u = u(x, y)$ as in Fig. 7.9. The hole in the middle occurs for $r = 2M$. Smaller r are not real solutions of Eq. (7.156). Moving up or down from the hole we reach $u = \pm\infty$ and $r \to +\infty$. In both cases we are in an asymptotic Minkowski space-time. So the hole, called Einstein-Rosen bridge, is a sort of passage between two distant regions of flat space-time. Such constructions are called wormholes. The Schwarzschild wormhole is, as already remarked, not traversable by real particles, but there are speculations that in more complicated space-times wormholes might be traversable and even give rise to closed time curves (i.e. travels in time!).

7.11 Kerr black holes

1. We study now a rotating black hole, with angular momentum J. For a solid sphere of size R, constant density ρ and angular velocity Ω, the angular momentum is the moment of inertia I times Ω, i.e.

$$J = \Omega I = \Omega \int \rho r^2 \sin^2 \theta r^2 dr d(\cos\theta) d\phi = \frac{8\pi}{15}\rho R^5 \Omega = \frac{2}{5} M R^2 \Omega \tag{7.157}$$

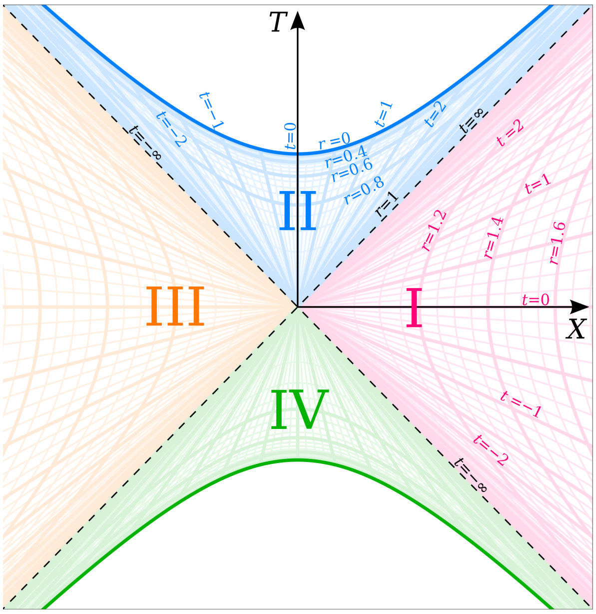

Figure 7.8: Kruskal coordinates (from Wikimedia Commons, author Dr Greg). Here, $X = u$ and $T = v$. The coordinate r is expressed in unit of the Schwarzschild radius.

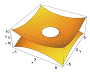

Figure 7.9: The Einstein-Rosen bridge.

We define the specific angular momentum

$$a = \frac{J}{M} \tag{7.158}$$

Then the Kerr metric for a rotating black hole in empty space is (Boyer–Lindquist coordinates)

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2} dt d\phi && (7.159) \\ &+ \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 && (7.160) \end{aligned}$$

where

$$\begin{aligned} \Delta &\equiv r^2 - 2Mr + a^2 && (7.161) \\ \rho^2 &\equiv r^2 + a^2\cos^2\theta && (7.162) \end{aligned}$$

(this ρ is not the density, is a new parameter). Now of course there is no longer spherical symmetry: the fact that $g_{t\phi} \neq 0$ implies rotation (one can show that there is no choice of frame such that $g_{\phi t} = 0$). For $r \to \infty$ however the rotation effects disappear. For $a \to 0$ we are back to SM.

2. Remarkably, the Kerr metric shows that a black hole is fully characterized by two parameters, mass and spin. A charged BH has a third parameter, the total charge (Reissner-Nordström metric).

3. Let us consider now a particle falling onto the Kerr BH with a trajectory that is initially radial, i.e. $p_\phi = 0$. Since the metric does not depend on ϕ, we know that $p_\phi = const$. Then we have

$$\begin{aligned} p^\phi &= g^{\phi t} p_t && (7.163) \\ p^t &= g^{tt} p_t && (7.164) \end{aligned}$$

and therefore

$$\frac{p^\phi}{p^t} = \frac{d\phi}{dt} = \frac{g^{\phi t}}{g^{tt}} \equiv \omega(r,\theta) \tag{7.165}$$

is the angular velocity. One can show that it has the same sign as a. This means that a particle that falls on an initially radial trajectory will be "dragged along" with the rotating metric and acquires a non-zero angular velocity in the same direction as the hole rotates. This phenomenon is called *frame dragging* or *gravitomagnetism*. Another consequence is that a gyroscope precesses around a rotating star (Lense-Thirring effect). This has indeed been measured even on Earth.

4. Now let us consider a photon emitted initially on tangential direction, $dr = 0$ and on the equatorial plane, $d\theta = 0$. In this case we have

$$ds^2 = 0 = g_{tt}dt^2 + 2g_{\phi t}d\phi dt + g_{\phi\phi}d\phi^2 \tag{7.166}$$

from which

$$\frac{d\phi}{dt} \equiv \Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \left[\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}\right]^{1/2} \tag{7.167}$$

Now on the surface on which $g_{tt} = 0$, one has two solutions,

$$\begin{aligned} \Omega_1 &= 0 && (7.168) \\ \Omega_2 &= -2\frac{g_{t\phi}}{g_{\phi\phi}} && (7.169) \end{aligned}$$

This implies that photons can orbit the rotating BH in two directions, in one being co-rotating with the star with angular velocity Ω_2, in the opposite one being "at rest" when seen by an external observer. That is, the dragging effect is so strong that all particles slower than photons rotate with the star, and photons emitted opposite to the rotating hole have no angular momentum.

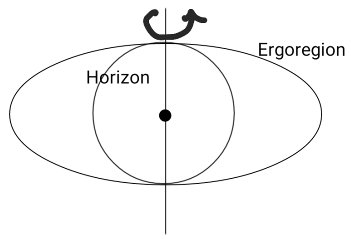

Figure 7.10: Ergoregion in a Kerr black hole.

5. All this happens when $g_{tt} = 0$, i.e. for

$$r^2 - 2Mr + a^2 - a^2 \sin^2\theta = 0 \tag{7.170}$$

so when

$$r = M \pm \sqrt{M^2 - a^2\cos^2\theta} \tag{7.171}$$

The region below the upper value of r is called *ergoregion.* Everything inside the ergoregion ($g_{tt} > 0$) is dragged along with the BH and remains bound to it.

6. The horizon is obtained for $g_{rr} = \infty$ (as in SM) i.e. for (we need consider only the upper value of r)

$$r_{HOR} = M + \sqrt{M^2 - a^2} \tag{7.172}$$

and is therefore contained within the ergoregion (and is tangent to it at the poles, $\theta = 0$). In principle, for $a > M$ the horizon seems to disappear: this could be a way to obtain a "naked singularity", but the issue is controversial. When $a = M$, the Kerr black hole is called *extremal.*

7. There is no known interior solution that matches the Kerr metric (as we have instead for the Schwarzschild metric), so the Kerr metric cannot be employed to describe the exterior metric of a rotating star, but only of black holes.

8. A complete introduction to the Kerr metric can be found in Hobson et al., *General Relativity. An introduction for physicists.*

Chapter 8

Cosmology

For an introduction to cosmology, see my lecture notes on Cosmology. Here we only discuss some preliminary concepts.

8.1 Homogeneity and isotropy

1. Measurements of galaxy density and clustering, and of the cosmic backgrounds, show that the universe is homogeneous and isotropic to a large extent. This is commonly called the *Cosmological Principle.* In order to maintain homogeneity and isotropy along with the cosmic expansion, the Hubble-Lemaitre law (or Hubble law for shortness) need to be verified at every point and at every time,

$$\vec{v} = H_0 \vec{d} \tag{8.1}$$

where $\vec{v}$ is the relative velocity between any two points separated by $\vec{d}$. Only in this case in fact the expansion law between any two points remains the same (since it is linear in $\vec{v}$ and $\vec{d}$). The Hubble constant H_0 has been measured to be

$$H_0 = 70 \pm 4 \; km/sec/Mpc \tag{8.2}$$

From this value we can derive a distance and a time lengths:

$$\frac{c}{H_0} = 3000 \; Mpc/h \tag{8.3}$$

$$\frac{1}{H_0} = 10 \; Gyr/h \tag{8.4}$$

where $h = H_0/(100 km/sec/Mpc) \approx 0.7$.

2. In reality, homogeneity and isotropy only hold when averaging out the fluctuations of matter density over distances larger than roughly 50 Mpc. That is, averaging over volumes of this size, the density of galaxies is approximately the same everywhere (as long as we do not go too far away so that we see galaxies into the remote past, when the average density was higher than today due to the cosmic expansion).

3. In order for the expansion to be the same everywhere, the space interval

$$d\ell^2 = [h_{ij} dx^i dx^j] R^2(t) \tag{8.5}$$

must depend on time only through an overall factor, denoted as $R^2(t)$. We are free to choose distance units such that $R(today) = 1$. R is called *scale factor.*

4. So the most general form of the metric so far is

$$ds^2 = -dt^2 + g_{0i} dt dx^i + R^2(t) h_{ij} dx^i dx^j \tag{8.6}$$

The first term should have been $g_{00}(t)dt^2$ but we can always redefine the time coordinate such that $dt^2 = -g_{00}(t')(dt')^2$. As for the Schwarzschild case, however, we require g_{0i} to vanish on account of isotropy. We need then to find h_{ij}.

5. We already know the most general h_{ij} that produces an isotropic (i.e., spherically symmetric) metric

$$d\ell^2 = e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \tag{8.7}$$

Now we must impose a further condition, namely that the space curvature is the same everywhere, because of homogeneity. Since there is only one unknown, Λ, we need a single condition. The space curvature is the curvature of $t = const$ hypersurfaces, so we need that the curvature scalar R or, equivalently G is constant, i.e.

$$G^i_i \equiv e^{-2\Lambda}G_{rr} + r^{-2}G_{\theta\theta} + r^{-2}\sin^{-2}\theta G_{\phi\phi} = \kappa \tag{8.8}$$

From Eqs. (7.21) we see that this becomes

$$-\frac{1}{r^2}[r(1-e^{-2\Lambda})]' = \kappa \tag{8.9}$$

which is solved by

$$g_{rr} = e^{2\Lambda} = \frac{1}{1 + \frac{\kappa r^2}{3} - \frac{A}{r}} \tag{8.10}$$

where A is an initial condition parameter. Assuming flatness around $r = 0$, we require $A = 0$. Finally, absorbing a factor $-1/3$ into κ we write

$$g_{rr} = e^{2\Lambda} = \frac{1}{1 - \kappa r^2} \tag{8.11}$$

6. This gives the famous Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right] \tag{8.12}$$

The parameter κ can take any value but we can always redefine r to absorb its absolute value, so from a qualitative point of view there are only three values of κ

$$\kappa = 0, \text{ flat} \tag{8.13}$$
$$\kappa = +1, \text{ spherical} \tag{8.14}$$
$$\kappa = -1, \text{ hyperbolic} \tag{8.15}$$

where the names refer to the geometry of the space surfaces at constant t. These three cases are the only homogeneous and isotropic geometries of space.

7. If $k = 0$, defining a new time $d\tau = dt/a$, one can write

$$ds^2 = a^2(-d\tau^2 + dr^2 + r^2 d\Omega^2) \tag{8.16}$$

i.e. a metric conformally flat (proportional to Minkowski). This form is often convenient for calculations.

8. For $\kappa = 1$, we can rewrite the metric by defining

$$d\chi^2 = \frac{dr^2}{1-r^2}, \to r = \sin\chi \tag{8.17}$$

so that

$$d\ell^2 = R^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{8.18}$$

which is the element of a surface of a 3-sphere with equation

$$x^2 + y^2 + z^2 + w^2 = r^2 \tag{8.19}$$

embedded into a 4D-space, with coordinates

$$w = r\cos\chi \tag{8.20}$$
$$z = r\sin\chi\cos\theta \tag{8.21}$$
$$y = r\sin\chi\sin\theta\cos\phi \tag{8.22}$$
$$x = r\sin\chi\sin\theta\sin\phi \tag{8.23}$$

This represents a space of finite volume,

$$V = \int \sqrt{-g_3}d^3x = 4\pi\int\frac{r^2dr}{(1-r^2)^{1/2}} = 4\pi\int_0^\pi \sin^2\chi d\chi = 2\pi^2 \tag{8.24}$$

While the "angle" χ varies from 0 to π, the radial coordinate r goes from 0 to 1 and back to 0 again. That is, traveling radially from a point one returns to the same point (assuming the expansion is frozen).

9. Similarly, for $k = -1$ one can transform

$$d\chi^2 = \frac{dr^2}{1+r^2}, \to r = \sinh\chi \tag{8.25}$$

and

$$d\ell^2 = R^2(t)[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{8.26}$$

Now the space is open and of infinite volume.

8.2 Redshift

1. The coordinates r, ϕ, θ are *comoving coordinates*, i.e. they remain fixed when the universe expands. This means that for a radial light ray the quantity

$$\chi = \int\frac{dt}{R} \tag{8.27}$$

must remain constant during the expansion. This in turn implies that the ratio $\Delta t/R$ taken at a time t_0 (today) when a photon is observed and at the time t_1 when that photon is emitted, is itself constant:

$$\frac{\Delta t_0}{R_0} = \frac{\Delta t_1}{R_1} \tag{8.28}$$

Suppose now the time interval Δt corresponds to the time it takes for two subsequent wave crests to pass at a given point; then $\Delta t = \lambda/c$. This gives

$$\frac{\lambda_0}{\lambda_1} = \frac{R_0}{R_1} \tag{8.29}$$

and the redshift is then

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{\lambda_0}{\lambda_1} - 1 = \frac{1}{R_1} - 1 \tag{8.30}$$

(we always put $R_0 = 1$). That is, the redshift of distant sources is related to the scale factor at the epoch of emission. In this way, the scale factor evolution becomes observable.

2. Multiplying the comoving coordinates by $R(t)$ one gets the physical coordinates at time t, that expand with the Universe. In the following we also use the notation

$$a(t) \equiv \frac{R(t)}{R_0} \tag{8.31}$$

so that $a(today) = a_0 = 1$. With this normalization, comoving distances r and physical distances ar coincide at the present time. This means that a galaxy that today is at a distance of, say, 100 Mpc, is said to be at 100 Mpc comoving distance at all times.

3. The proper distance is obtained with $dt, d\Omega = 0$ and is therefore

$$d = R\chi \tag{8.32}$$

Since χ is contant, the time derivative is

$$\dot{d} = v = \frac{\dot{R}}{R} d \tag{8.33}$$

which is again Hubble's law. We define the Hubble function or expansion rate

$$H(t) \equiv \frac{\dot{R}}{R} = \frac{\dot{a}}{a} \tag{8.34}$$

4. Therefore

$$a(t) = a_0 \exp \int H dt \tag{8.35}$$

Now we can expand around today's value H_0

$$H(t) = H_0 + \dot{H}|_0 (t - t_0) + ... \tag{8.36}$$

and thus

$$a(t) = a_0[1 + H_0(t - t_0) + \frac{1}{2}(H_0^2 + \dot{H}|_0)(t - t_0)^2 + ...] \tag{8.37}$$

$$= a_0[1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + ...] \tag{8.38}$$

where

$$q_0 = -(1 + \frac{\dot{H}}{H^2})_0 = -\frac{a\ddot{a}}{\dot{a}^2}|_0 \tag{8.39}$$

is the deceleration parameter (positive for a decelerated expansion, negative for an accelerated one).

8.3 Cosmological distances

1. The present proper radial distance to a source can be evaluated putting $dt = d\Omega = 0$ as

$$d = \int_0^r \frac{dr}{(1 - kr^2)^{1/2}} = \frac{1}{\sqrt{\Omega_k H_0^2}} \int_0^r \frac{d(\sqrt{\Omega_k H_0^2} r)}{(1 + \Omega_k H_0^2 r^2)^{1/2}} = \frac{1}{\sqrt{\Omega_k H_0^2}} \sinh^{-1} r\sqrt{\Omega_k H_0} \tag{8.40}$$

where $\Omega_k = -k/H_0^2$. If $\Omega_k < 0$ (spherical spaces), the sinh function converts into a sin. If $\Omega_k = 0$, then $d = r$.

2. On an incoming light ray path we have $ds = 0$ and therefore

$$\frac{1}{\sqrt{\Omega_k H_0^2}} \sinh^{-1} r\sqrt{\Omega_k H_0^2} = -\int_{t_0}^{t} \frac{dt}{a(t)} = -\int_{R_0}^{R} \frac{da}{H(t)a^2(t)} = \int_0^z \frac{dz}{H(z)} \tag{8.41}$$

since $dz = -dR/R^2$. That is, we find a relation between coordinate r and redshift,

$$r(z) = \frac{1}{H_0\sqrt{\Omega_k}} \sinh\left[H_0\sqrt{\Omega_k}\int_0^z \frac{dz}{H(z)}\right] \tag{8.42}$$

3. In laboratory, the relation between the flux (energy per area per second) received by an observer and the luminosity of a source (energy output per second) at distance d is given by

$$f = \frac{L}{4\pi d^2} \tag{8.43}$$

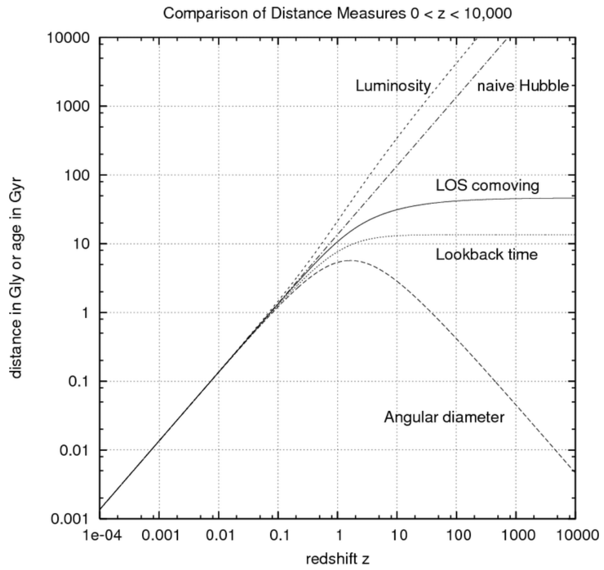

Figure 8.1: Distances in cosmology for a flat FLRW metric with standard cosmological parameters. The distance labeled "LOS comoving" is $r(z)$. The "lookback time" is $cT(z)$. The "naive Hubble" is cz/H_0 (By Wesino at English Wikipedia - Public Domain, https://commons.wikimedia.org/w/index.php?curid=18249846)

4. In a FLRW metric, the photons emitted are redshifted by $1+z$ and therefore their energy $h\nu$ decreases by the same amount; moreover, the time it takes for the emission, Δt_{em}, is dilated by another factor $1+z$. Therefore the flux is diminished by a factor $(1+z)^{-2}$, and the distance is $d = R_0 r$. Then the flux can be written as

$$f = \frac{L}{4\pi d^2 (1+z)^2} \tag{8.44}$$

5. So we define a luminosity distance such that $d_L^2 = L/4\pi f$, that is

$$d_L = d(1+z) \tag{8.45}$$
$$= r(z)(1+z) \tag{8.46}$$

If we know the absolute luminosity L we can therefore measure the flux and the redshift of many sources and obtain their coordinates $r(z)$. Comparing with Eq. (8.42) we can infer $H(z)$.

6. We can also define the proper area of a small source as seen projected on the sky, i.e. at $dt = dr = 0$,

$$A = D^2 = a^2 r^2 (d\Omega)^2 \tag{8.47}$$

From this, we can define another important astronomical distance, the angular-diameter distance of a source at coordinate r that subtends an angle $d\Omega$,

$$d_A = \frac{D}{d\Omega} = ar = \frac{d_L}{(1+z)^2} \tag{8.48}$$

Notice that $d_A \to 0$ for $z \to \infty$ (see Fig. 8.1). All the distances coincide for $z \ll 1$; in the same limit, $r(z) \to z/H_0$, i.e. the Hubble-Lemaitre law, independently of the cosmological model.

7. The relation $d_L = d_A(1+z)^2$ is called *Etherington relation.* Although the distance so far defined are valid only in a FLRW metric (and should therefore be generalized in, e.g., inhomogeneous models), the Etherington relation is valid in any cosmology.

8.4 Evolution of the FLRW metric

1. In a homogeneous and isotropic universe matter is at rest in comoving coordinates, that is, it just expands staying at constant r, θ, ϕ. So $U^\mu = (1, 0, 0, 0)$. We now solve the GR equations with the FLRW metric assuming a perfect fluid EMT.

2. Let us now write down the metric equations in a homogeneous and isotropic space, i.e. in the FLRW metric:
$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{8.49}$$
For $k = 0$ the Christoffel symbols are all vanishing except (it is easier here to perform the calculations using the Cartesian form $dr^2 + r^2 d\Omega^2 \to dx^2 + dy^2 + dz^2$)
$$\Gamma^i_{j0} = \Gamma^i_{0j} = H\delta^i_j \quad , \quad \Gamma^0_{ij} = a\dot{a}\delta_{ij}$$
We have then
$$R_{00} = \Gamma^\mu_{00,\mu} - \Gamma^\mu_{0\mu,0} + \Gamma^\mu_{\sigma\mu}\Gamma^\sigma_{00} - \Gamma^\mu_{\sigma 0}\Gamma^\sigma_{\mu 0} = -3\dot{H} - H^2\delta^i_j\delta^j_i = -3(\dot{H} + H^2) = -3\frac{\ddot{a}}{a}$$
and the trace
$$R = \frac{6}{a^2}(\dot{a}^2 + a\ddot{a} + k) = 6\dot{H} + 12H^2 + 6ka^{-2} \tag{8.50}$$

3. The energy momentum tensor for a perfect fluid in comoving expansion can be simply evaluated as
$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{8.51}$$

4. Let us now consider the $(0,0)$ component and the trace component of the Einstein equations:
$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}R &= 8\pi T_{00} \\ R &= -8\pi T \end{aligned}$$
From the first equation and by combining the two we obtain the two *Friedmann equations*:
$$H^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{8.52}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) \tag{8.53}$$
to be complemented by the conservation equation $T^{0\mu}_{;\mu}$ (all the other equations vanish identically)
$$\dot{\rho} + 3H(\rho + p) = 0 \tag{8.54}$$
Notice that this can be interpreted as a simple conservation law $dE = -pdV$ in an expanding space, by putting the total energy as $E = V\rho$ and $V = V_0 a^3$. Then one has in fact
$$\frac{d(a^3\rho)}{dt} = -p\frac{da^3}{dt} \tag{8.55}$$
which gives Eq. (8.54).

5. The Friedmann equations and the conservation equations are however not independent. By differentiating eq. (8.52) and inserting eq. (8.54) we obtain the other Friedmann equation. Let us define now the critical density
$$\rho_c = \frac{3H^2}{8\pi G}$$

and the density parameter

$$\Omega = \frac{\rho}{\rho_c}$$

so that eq. (8.52) becomes

$$1 = \Omega - \frac{k}{a^2 H^2} \tag{8.56}$$

This shows that $k = 0$ corresponds to a universe with density equal to the critical one, that is $\Omega = 1$. Spaces with $k = +1$ correspond instead to $\Omega > 1$, spaces with $k = -1$ to $\Omega < 1$. We can also define a curvature component

$$\Omega_k \equiv -\frac{k}{a^2 H^2} \tag{8.57}$$

(which implies the definition $\rho_k = -3k/8\pi a^2$). At every epoch we have then

$$1 = \Omega(a) + \Omega_k(a)$$

As we will see, this relation extends directly to models with several components.

8.5 Non relativistic component

1. Let us consider now a fluid with zero pressure

$$p = 0$$

Such a fluid approximates "dust" matter (like e.g. galaxies) or a gas composed by non-interacting particles with non-relativistic velocities (like e.g. cold dark matter). In fact, the pressure of a free-particle fluid with mean square velocity v is $p = nmv^2$, much smaller than $\rho = nmc^2$ for non relativistic speeds. Then we have from (8.54) that

$$\dot{\rho}/\rho = -3\dot{a}/a$$

or

$$\rho \sim a^{-3}$$

Every time we write a relation of this kind we mean a power law normalized to an arbitrary instant a_0 (here assumed to be the present epoch). We mean then

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3$$

As a function of redshift we have (the subscript NR or m denotes the pressureless non-relativistic component)

$$\rho = \rho_0(1+z)^3 = \rho_c \Omega_{NR}(1+z)^3 \tag{8.58}$$

where from now on, except where otherwise denoted, Ω_i represents the *present* value for the species i and ρ_c is the *present* critical density.

2. Let us assume now a flat space $k = 0$. The present density ρ_0 is linked to the Hubble constant by the relation

$$H_0^2 = \frac{8\pi}{3}\rho_0$$

Then we have (defining distances such that $a_0 = 1$)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0 a_0^3 a^{-3} = H_0^2 a^{-3}$$

from which integrating

$$a \sim t^{2/3}$$

3. Since we measure a present Hubble constant

$$H_0 = 100 h km/sec/Mpc$$

where $h = 0.70 \pm 0.04$, (roughly according to the recent estimates of the Hubble Space Telescope and of the Planck satellite), and since $1Mpc = 3 \cdot 10^{19} km$ and $G = 6.67 \cdot 10^{-8} cm^3 g^{-1} sec^{-2}$ we have the present critical density

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \approx 2 \cdot 10^{-29} h^2 g cm^{-3}$$

The matter density currently measured is indeed close to $\rho_{c,0}$. More exactly, it is found

$$\Omega_m \approx 0.3 \pm 0.1 \tag{8.59}$$

8.6 Relativistic component

1. A photon gas distributed as a black body has a pressure per unit volume equal to

$$p = \frac{1}{3}\rho$$

(notice that for radiation the energy-momentum trace vanishes, $T = 0$; this can be seen also from the form of the electromagnetic tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right)$$

whose trace vanishes). Then we have from (8.54) that

$$\dot{\rho} = -4H\rho$$

from which (the subscript R or γ denotes the relativistic or radiation component)

$$\rho_R \sim a^{-4} = \rho_c \Omega_R (1+z)^4 \tag{8.60}$$

The radiation density dilutes as a^{-3} because of the volume expansion and as a^{-1} because of the energy redshift.

2. To evaluate the present radiation density we'll remind that a photon gas in equilibrium with matter (black body) has energy density

$$\rho_\gamma = \frac{g}{2\pi^2}\int \frac{E^3 dE}{e^{E/T}+1} = \frac{g\pi^2}{30}\frac{(k_B T)^4}{\hbar^3 c^3}$$

where T is expressed in energy units and g are the degrees of freedom of the relativistic particles ($g = 2$ for the photons, $g \approx 3.36$ including 3 massless neutrino species). Notice that since $\rho_\gamma \sim a^{-4}$ the radiation temperature scales as

$$T \sim \frac{1}{a} \tag{8.61}$$

Since today we measure $T \approx 3K$, we have

$$\rho_\gamma = g \cdot 2.5 \cdot 10^{-34} g cm^{-3}$$

which is much smaller than the present matter density. The present epoch is denoted therefore *matter dominated epoch* (MDE). (Notice: in Planck units ($\hbar = G = c = 1$) $T_{3K} = 10^{-32}E_P$; so that $T_{3K}^4 = 10^{-128}E_P^4 = 10^{-128}M_P L_P^{-3} = 10^{-128}10^{-5}10^{99}$ g/cm^3. Thus $T_{3K}^4 \approx 10^{-34}$g/cm^3.)

3. From the matter and radiation trends

$$\rho_m = \rho_{m,0} a^{-3} \tag{8.62}$$
$$\rho_\gamma = \rho_{\gamma,0} a^{-4} \tag{8.63}$$

we can define the equivalence epoch a_e for which $\rho_\gamma = \rho_m$:

$$a_e = \frac{\rho_{\gamma,0}}{\rho_{m,0}} = \frac{\Omega_\gamma}{\Omega_m} \tag{8.64}$$

Since we have

$$\Omega_\gamma = \frac{\rho_\gamma}{\rho_{crit}} \simeq 4.3 \cdot 10^{-5} h^{-2} \tag{8.65}$$

it follows that the equivalence occurred at a redshift

$$1 + z_e = a_e^{-1} = \left(4.3 \cdot 10^{-5}\right)^{-1} \Omega_m h^2 = 23,000 \Omega_m h^2 \tag{8.66}$$

Putting $\Omega_c = 0.3$ and $h = 0.7$ we obtain $z_e \approx 3450$.

8.7 General component

1. It is clear now that any fluid with equation of state

$$p = w\rho$$

goes like

$$\rho \sim a^{-3(1+w)}$$

In the case $k = 0$ and if the fluid is the dominating component in the Friedmann equation, the scale factor grows like

$$a \sim t^{2/3(1+w)}$$

2. We can now write the Friedmann equation as

$$H^2 = \frac{8\pi}{3}(\rho_m a^{-3} + \rho_\gamma a^{-4} + \rho_k a^{-2}) = H_0^2(\Omega_m a^{-3} + \Omega_\gamma a^{-4} + \Omega_k a^{-2}) \tag{8.67}$$

where as already noted Ω_i denotes the present density of species i, so that $\sum_i \Omega_i = 1$. Every other hypothetical component can be added to this Friedmann equation when its behavior with a is known.

8.8 Qualitative trends

1. In all the cases seen so far we always had $\rho + 3p > 0$. Then from (8.53) it follows $\ddot{a} < 0$, that is, a decelerated trend at all times. From this it follows that 1) the scale factor must have been zero at some time t_{sing} in the past; and 2) the trajectory with $\dot{a} = const, \ddot{a} = 0$ is the one with minimal velocity in the past (among the decelerated ones) . From $\dot{a} = const$ it follows the law

$$a(t) = a_0 + \dot{a}_0(t - t_0)$$

and one can derive that the time

$$T = t_0 = a_0/\dot{a}_0 = H_0^{-1}$$

it takes for the expansion to go from $a = 0$ to $a = a_0$ is the maximal one. H_0^{-1} is then the maximal age of the universe for all models with $\rho + 3p > 0$. Note that

$$H_0^{-1} = \frac{sec \cdot Mpc}{100h\,km} = 9.97 Gyr/h$$

This extremal model is called Milne's universe and can be obtained from 8.67 for

$$\Omega_m = \Omega_\gamma = 0$$

so that $\Omega_k = 1$. Then we have $H^2 = H_0^2 a^{-2}$ and thus $\dot{a} = H_0$.

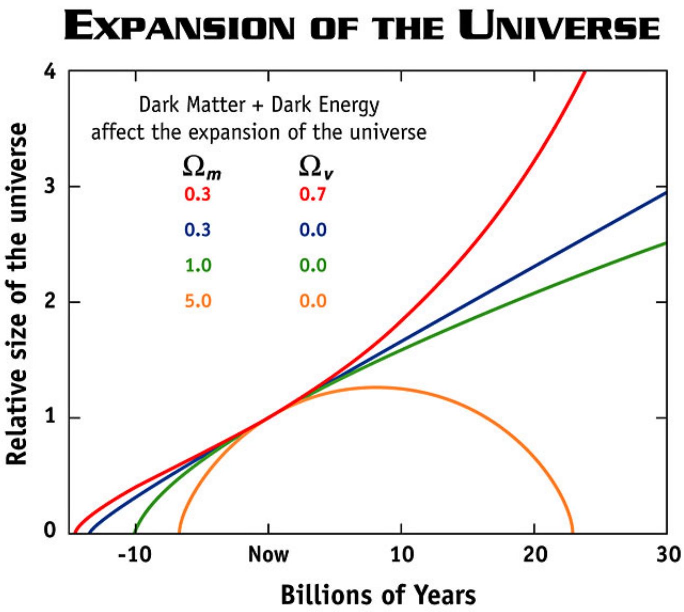

Figure 8.2: Evolution of the scale factor $a(t)$ in various cosmological models. For $\Omega_\Lambda = 0$ (here denoted Ω_V), the models are decelerated: the blue curve has $k = -1$, the green curve has $k = 0$ and the orange curve has $k = +1$. The red curve is accelerated at the present time and in the future. (Credits: NASA)

2. For a general case with non vanishing matter we have instead, by integrating the Friedmann equation,

$$H = \frac{da}{adt} = H_0(\Omega_m a^{-3} + \Omega_k a^{-2})^{1/2} \equiv H_0 E(a)$$

that

$$H_0 T = \int_0^{a_0} \frac{da}{aE(a)} = \int_0^{a_0} \frac{da}{\sqrt{\Omega_m a^{-1} + 1 - \Omega_m}}$$

For $\Omega_m = 1$ we get

$$T = \frac{2}{3H_0} = 6.7h^{-1} Gyr$$

an age too short to accommodate the oldest stars in our Galaxy, unless h is smaller than 0.5. The age corresponding to a given redshift z can be obtained by integrating from $a = 0$ to $a = (1+z)^{-1}$, or

$$t = \frac{2}{3} H_0^{-1} (1+z)^{-3/2}$$

For instance, $z_{dec} = 1100$ corresponds to an age $t = 200,000h^{-1}yr$.

3. Finally, in the case $k = 1$, i.e. a closed spherical geometry, we can see that H vanishes for $\rho = 3/8\pi a^2$ or $\Omega_m a^{-3} = -\Omega_k a^{-2}$ i.e. (when only matter is present and obviously for $\Omega_m > 1$) when

$$a_{max} = \frac{\Omega_m}{\Omega_m - 1}$$

(remember that $\Omega_k = 1 - \Omega_m$). At this epoch, expansion stops and a contraction phase with $H < 0$ begins. This phase will end in a *big crunch* after an interval equal to the one needed to reach the maximum a_{max}. See Fig. (8.2) for an illustration.

8.9 Cosmological constant

1. To obtain a cosmic age larger than H_0^{-1} it is necessary to violate the so-called "strong energy condition" $\rho + 3p > 0$. The most important example of this case is the vacuum energy or cosmological constant.

2. As we have briefly mentioned, Einstein's eqations remain second-order in the metric and satisfy Bianchi identities if we modify them in the following way

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{8.68}$$

3. Let us consider the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu} \tag{8.69}$$

   This expression holds for observers that are comoving with the expansion. Every other observer will see a different content of energy/pressure. There exists however a case in which every observer sees exactly the same energy-momentum tensor, regardless of the 4-velocity u_μ: this occurs when $\rho = -p$: in such a case in fact $T_{\mu\nu} = -\rho g_{\mu\nu}$. The conservation condition then implies $\rho_{,\mu} = 0$ or $\rho = const$. It follows that if we identify the cosmological constant term with an energy-momentum tensor such that

$$T_{\mu\nu(\Lambda)} = -\frac{\Lambda}{8\pi} g_{\mu\nu} \tag{8.70}$$

   where Λ is the *cosmological constant*, then the tensor is independent of the observer motion. This condition indeed characterizes an empty space, i.e. a space without real particles. $T_{\mu\nu(\Lambda)}$ is denoted then *vacuum energy*. We have now

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi} \tag{8.71}$$

   which corresponds to the equation of state $w = p/\rho = -1$. If the Λ term is dominating, one sees from Eq. (8.53) that the expansion would be accelerated.

4. Present observations using supernovae Type Ia as standard candles (i.e., objects with constant absolute luminosity) show that the present value of the density fraction Ω_Λ is

$$\Omega_\Lambda = \frac{8\pi\rho_\Lambda}{3H_0^2} \approx 0.7 \pm 0.1 \tag{8.72}$$

   Together with $\Omega_m = 0.3 \pm 0.1$, one sees that

$$\Omega_k = 1 - \Omega_m - \Omega_\Lambda \approx 0 \tag{8.73}$$

   That is, the current Universe appears spatially flat or Euclidian.